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Density Estimation and Option-Implied Risk Aversion of the Nikkei 225: Evidence Before and After the Subprime Crisis

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Abstract

This paper examined the relationship between an option-implied risk-neutral density and a real-world density derived from historical returns of the Japanese stock market from 2006 to 2010. In particular, I estimated one-month-ahead empirical relative risk aversion (RRA) of a representative agent implied from risk-neutral and real-world densities. The level of the Nikkei 225 index peaked at 18, 262 points and bottomed at 7, 055 points during the sampling period. The non-parametric method that requires a large amount of data and assumes stationary underlying asset prices over a lengthy past may not be appropriate. To obtain one-month-ahead real-world densities, the use of an overlapping return series is subject to autocorrelation problems, whereas a shorter sampling window decreases the number of observed non-overlapping return series that can result in unreliable estimation. I solved this dilemma by using a daily GJR(1,1) model based on daily returns data a year before the observation date to predict one-month-ahead distribution parameters. The risk-neutral densities for the same observation date are derived from a cross section of 30-days-to-maturity Nikkei 225 options based on Heston (1993) density, generalized beta density of the second kind (GB2), and mixture of two log-normal densities. Akaike information criterion (AIC) statistics show that GJR (1, 1) with normally distributed innovations is superior to a t-distribution. Option-implied risk-neutral density based on GB2 model is the preferred model according to the log-likelihood value and likelihood ratio (LR) statistic from Berkowitz (2001). Overall, risk-neutral densities are more negatively skewed and have fatter tails than real-world densities. The RRA functions are downward sloping across options' moneyness and show a U-shape pattern around at-the-money (ATM) level that can be explained by option mispricing as suggested by Jackwerth (2000). I found that although the average level of RRA decreases during a crisis period, it increases during the post-crisis period.

1. Introduction

This paper chiefly aims to study the empirical relative risk aversion (RRA) of the representative investor in the Japanese stock market to determine how the subprime loan crisis affected the measure of risk aversion. In particular, this paper examined the shape of and relationship between a risk-neutral density that is derived from cross sections of Nikkei 225 options data during 2006 to 2010 and a real-world density that is derived from time series of historical returns. The relationship between the two world densities as used by Ait-Sahalia and Lo (2000) and Jackwerth (2000) is then used as a basis to estimate the implied risk aversion measure. I found that the level of RRA rose when the crisis began to build up; however, it fell below that of the pre-crisis period when Lehman Brothers' shock hit the Japanese stock market. During the post-crisis period, the RRA level then rose again to a higher level than during the pre-crisis period.

Previous studies that used option prices to derive empirical risk aversion measures can be divided into two broad groups. The first group assumes a representative agent's utility function in the form of power and exponential functions that have one parameter as a constant relative risk aversion measure. Studies on this group usually begin with an estimation of risk-neutral density from option prices, and then use the risk aversion coefficients as a risk preference adjustment to derive real-world densities, for example, refer to Liu et al. (2007), Bliss and Panigirtzoglou (2004), and Anagnou et al. (2002). The risk aversion measures can be estimated conveniently using a closed-form solution based on the assumed utility function. However, the estimated risk aversion measures are either time-invariant or constant across the level of underlying assets because of the pre-assumed utility function.

The second group of previous studies does not specify a representative agent's utility functional form but rather derives a risk aversion measure directly from the relationship between risk-neutral and real-world densities, such as Ait-Sahalia and Lo (2000) and Jackwerth (2000). The risk aversion measure estimated in this group is flexible and time-variant, which may better reflect current aggregate risk aversion at any point in time than the measures estimated from the first group. However, the estimated risk aversion measures in general shows a U-shape pattern across the level of underlying assets and turned negative for certain asset levels.

This paper can be categorized into the latter group. The risk aversion measures of the Japanese stock market shows a downward sloping trend across a range of underlying indexes; however, they also show a U-shape pattern and fall into a negative region. This ill-behaved risk aversion measure contradicts economic theory that a representative agent is risk averse and has a concave utility function. Jackwerth's (2000) explanation to this ill-behavior is credible; that is options used to derive risk-neutral densities are mispriced. In addition, the results of Ait-Sahalia et al. (2001) and Bondarenko (2003) support this mispricing explanation.

This paper further aims to propose a simple approach to solve a dilemma in the estimation of real-world density between the validity of a stationary assumption and the reliability of estimated parameters. Real-world density estimation usually requires a long historical return series to estimate reliable distribution parameters. On the contrary, risk-neutral density parameters are estimated from a cross section of option data at a single point in time. Ideally, the same period of data should be used to estimate both risk-neutral and real-world densities to derive the exact implied risk aversion measure. Moreover, the use of a long past sampling period assumes that the return series are stationary over the entire period, which is questionable in the real world. In addition, a non-overlapping return series should be used in density estimation to avoid the autocorrelation problem. With a 30-day target horizon, one year data will give only 12 non-overlapping monthly return observations. Consequently, a dilemma arises regarding whether to use a non-overlapping return series observed over a long period, or to use overlapping returns from the shortest possible sampling period that is subject to the autocorrelation problem.

To address this dilemma, I proposed the use of a parametric method instead of a non-parametric to reduce the need for the data required to obtain reliable parameter estimation. Moreover, instead of using a target multi-period return series, I proposed the use of a single-period model to forecast multi-period distribution parameters. In particular, I estimate real-world density using the Glosten et al. (1993) Glosten−Jagannathan−Runkle Model (GJR (1, 1)). Daily Nikkei 225 returns were used in parameter estimation of a daily GJR (1, 1) model and then the 30-day-ahead conditional mean and variance of the distribution were forecasted using the estimated model. Therefore, with only past one-year data, the number of returns is increased to over 200 observations. The multi-period conditional mean and variance are estimated based on the simple law of expectations that does not alter the properties of the assumed model.

According to Akaike information criterion (AIC) statistics, the GJR (1, 1) models with normal innovations outperform t-distributed innovations. This contradicts the common belief that student's t-distribution is preferred in modeling equity returns because the t-distribution has a fat-tailed property compared with normal distribution.

With regard to risk-neutral density estimation, I choose a parametric method over a non-parametric method. Ait-Sahalia and Lo (2000) and Ait-Sahalia et al. (2001) used pooled cross sections of option data over the observation period to non-parametrically estimate risk-neutral density. However, as Bliss and Panigirtzoglou (2004) pointed out, pooling cross sections of options data over the observation period assumed that risk-neutral density is stationary across periods, which is questionable because the Nikkei 225 varied substantially during the sampling period and the number of available strikes used in estimation changes every observation day. Even if the stationary assumption is not violated, Rosenberg and Engle (2002) suggested that the obtained risk-neutral density is an average of risk-neutral density over the observation period, not the risk-neutral density that reflects the current preference of the representative agent.

I used three competing models to estimate risk-neutral density, the Heston (1993) density, the generalized beta density of the second kind (GB2), and the mixture of two-lognormal densities. The likelihood ratio (LR) statistics of Berkowitz (2001) indicate that GB2 density performs better than the other two competing models but the differences among the three were minimal.

Section 2 discusses the estimation method in detail. Section 3 then describes the data and section 4 presents an empirical estimation of the results. Finally, I conclude in section 5.

2. Methodology

2.1. Risk Aversion Measures

This study builds on the assumptions that there exists a representative investor who is rational and risk averse or has a concave utility function, and other general characteristics such as a complete and frictionless market. With these assumptions, Ait-Sahalia and Lo (2000) showed that market-wide or aggregate risk-neutral density $q(S_T)$ and real-world density $p(S_T)$ are related through the representative investor's utility function $U(S_T)$ as follows:

$$\frac{q_t(S_T)}{p_t(S_T)} = \lambda \frac{U'(S_T)}{U'(S_t)} = \xi_t(S_T)$$

where $\xi_t(S_T)$ is a pricing kernel function and λ is a constant independent of underlying asset level (S_T). Jackwerth (2000) and Ait-Sahalia and Lo (2000) illustrated further that instead of using a utility function to derive a risk-aversion measure, which requires knowledge of the λ value, the pricing kernel can be written as

$$\xi_t'(S_T) = \lambda \frac{U''(S_T)}{U'(S_t)}$$

Then, the Arrow-Pratt relative risk aversion measure can be derived from

$$\mathrm{RRA}_t(S_T) = -S_T \frac{U''(S_T)}{U'(S_T)} = -S_T \frac{\xi_t'(S_T)}{\xi_t(S_T)} = S_T \left[\frac{p_t'(S_T)}{p_t(S_T)} - \frac{q_t'(S_T)}{q_t(S_T)}\right] . \qquad (1)$$

To obtain the Arrow-Pratt relative risk aversion measure according to (1), risk-neutral and real-world densities are required. The measure in (1) is also called option-implied risk aversion because estimation of risk-neutral density is usually based on observed option data. This relationship was used in previous studies such as Jackwerth (2000), Ait-Sahalia and Lo (2000), and Bliss and Panigirtzoglou (2004). The differences among the three studies are the approaches used to estimate risk-neutral and real-world densities. Jackwerth (2000) fitted option-implied volatility with the volatility from his model to derive risk-neutral density, and uses a Gaussian kernel to estimate real-world density. Ait-Sahalia and Lo (2000) used a non-parametric model with a Gaussian kernel to estimate both risk-neutral and real-world densities. Moreover, Bliss and Panigirtzoglou (2004) used a spline function involving fitting implied volatilities of observed options to estimate risk-neutral density. However, instead of estimating real-world density to obtain implied risk aversion function, they assume a utility function form to derive a risk-aversion coefficient that in turn is used to estimate real-world density.

2.2. Risk-Neutral Density Estimation

I used three competing parametric models to estimate risk-neural density, Heston (1993) density, GB2, and mixture of two log-normal densities (MLN).

With regard to Heston (1993) density, I followed the method applied by Gatheral (2006). Under Heston density, stock prices follow the process:

$$dS_t = \mu_t S_t dt + \sqrt{v_t} S_t dZ_1,$$

$$dv_t = -\lambda\,(v_t - \bar{v})\,dt + \eta\sqrt{v_t} dZ_2,$$

$$\langle dZ_1\, dZ_2 \rangle = \rho dt$$

where v_t is a conditional variance at time t and λ is the speed of reversion of v_t to its long-term mean $\bar{v}$. The return density under the Heston density can be written as

$$f_{Hest}(k_i|\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\varphi \cdot \exp\{\Omega(\varphi,\tau)\bar{v} + \Psi(\varphi,\tau)v - i\varphi k_i\} \qquad (2)$$

where φ is an integral variable, $\Omega(\varphi,\tau)$ and $\Psi(\varphi,\tau)$ are functions of model parameters, and k is defined as $\log(X_i/S_t)$For details on the parameters and derivation of the model, refer to Gatheral (2006) or the original Heston (1993).

With regard to the GB2 introduced by Bookstaber and MacDonald (1987), I follow the estimation method used by Liu et al. (2007). The GB2 density is defined as

$$f_{GB2}(X_i|a,b,p,q) = \frac{aX_i^{ap-1}}{b^{ap}\,B(p,q)[1+(X_i/b)^a]^{p+q}}\ , \qquad X_i > 0 \qquad (3)$$

where a, b, p, q are parameters of distribution and $B(p,q)$ is a beta function with parameters p and q. The risk neutrality imposes a condition on the forward price of the underlying asset to be equal to

$$F_{t,\tau} = \frac{bB(p+1/a, q-1/a)}{B(p,q)}\ .$$

The mixture of two-lognormal densities also follows the method of Liu et al. (2007), defined as

$$f_{MLN}(X_i|\theta) = wf_{LN,1}(X_i|\theta_1) + (1-w)f_{LN,2}(X_i|\theta_2) \qquad (4)$$

where $\theta = (F_1, F_2, \sigma_1, \sigma_2, w)$ is a set of distribution parameters. θ_j is a set of parameters associated with the individual lognormal density $f_{LN,j}$ defined as

$$f_{LN,j}\left(X_i\middle|\theta_j\right) = \frac{1}{X_i\sqrt{2\pi\sigma_j^2}}exp\left(-\frac{\left[lnX_i - lnF_j + 0.5\sigma_j^2\tau\right]^2}{2\sigma_j^2\tau}\right).$$

A European call option under each assumed risk-neutral density is computed based on

$$C_i^\theta(X_i,\tau) = \exp\left(r_f\tau\right)\int_0^\infty (S_T - X_i)^+ f(S_T|\theta)\,dS_T \qquad (5)$$

where $C_i^\theta(X_i,\tau)$ is a theoretical European call option price at strike $X_i = K_i$ with time to maturity $\tau = T - t$ computed under assumed density models $f(S_T|\theta)$ in (2), (3), and (4) given set of parameters θ. r_f is the risk-free rate and S_T is an underlying asset price at maturity date T. All assumed densities

$f(S_T|\theta)$ in (5) satisfied the relationship found by Ross (1976), Breeden and Litzenberger (1978), and Banz and Miller (1978):

$$f(S_T|\theta) = \left.\frac{\partial^2 C_i^\theta(X_i,\tau)}{\partial X_i^2}\right|_{X_i=S_T} .$$

A closed-form solution for theoretical option prices in (5) under the Heston density in (2) is based on Gatheral (2006), whereas the GB2 density in (3) and the MLN density in (4) are based on Liu et al. (2007).

The least-squares method is used to estimate parameters of densities in (2), (3), and (4) by minimizing the sum of the squared differences between the theoretical price $C_i^\theta(X_i,\tau)$ and the observed market price $C_i^{mkt}(X_i,\tau)$ at all strike prices $X_i = K_i$ according to the equation

$$lsq(\theta^*,\tau) = \min_\theta \sum_i^N \left[C_i^\theta(X_i,\tau) - C_i^{mkt}(X_i,\tau)\right]^2 .$$

Although Ait-Sahalia and Lo (2000) applied a non-parametric method in estimating risk-neutral density, their method is subject to the assumption that representative investor preference is constant across observation periods and the derived risk-neutral density reflects average density over multiple periods rather than the current period density as highlighted by Bliss and Panigirtzoglou (2004) and Rosenberg and Engle (2002).

In Japan, Nakamura and Shiratsuka (1999) and Shiratsuka (2001) used the finite difference method as documented in Breeden and Litzenberger (1978) and Neuhaus (1995) to directly derive probability density at each option strike price.

Although parametric approach offers many candidate models to choose from, previous studies seem to be in favor of variation of spline functions estimation, MLN, generalized beta density, and Heston (1993) density. Examples of relevant studies are Shackleton et al. (2010), Liu et al. (2007), Moodley (2005), Anagnou et al. (2002), and Melick and Thomas (1997).

To be comparable with previous studies, I chose the Heston density as a representative of the risk-neutral density based on the stochastic volatility model. The mixture of two lognormal densities should be more flexible than one lognormal density, while mixture of three-lognormal densities requires a larger number of minimum cross section of option data to obtain reliable parameter estimation. The GB2 requires the lowest minimum number of cross section option data to estimate the parameters. Moreover, GB2 is relatively quick to estimate compared with the other two models, with Heston as the most time-consuming for estimation.

2.3. Real-World Density Estimation

Ideally, an estimation of real-world density should use the shortest possible observation period of past returns series because risk-neutral density can be estimated using cross sections of option data during the current observation period. However, estimation of real-world density in general requires using a long period of an observed past return series, which in turn may not reflect the true distribution of assets at the current observation date. Consequently, the risk aversion measure derived from risk-neutral and real-world densities based on different observation periods may be inaccurate.

However, Rosenberg and Engle (2002) suggested that real-world density based on a stochastic volatility model should incorporate the return series over all observation periods to reflect the past events that affect a representative investor's expectations over a future period's returns distribution.

Nonetheless, according to efficient and complete market assumptions, all information regarding a representative investor's past experience and expectation of a future returns distribution should be incorporated in the current asset price, and hence, reflected in the current period return.

Any researcher can increase the sample of observed returns by using an overlapping returns series. However, the overlapping returns series will be plagued with autocorrelation problems that, in turn, result in unreliable parameter estimation. To estimate reliable real-world density parameters using the shortest possible observation period, I proposed estimation using a daily model under the parametric method to forecast multi-period distribution parameters. Therefore, a daily returns series can be used in model parameter estimation that, in turn, results in an instant increase in observation data with a minimum autocorrelation problem. In particular, I use the estimation process summarized in the following three steps.

1. Estimate parameters of the daily GJR (1, 1) model based on a daily returns series over the past one year from the current observation date.
2. Use estimated parameters to forecast multi-period distribution parameters matching the target maturity horizon of risk-neutral density.

3. Adjust the location parameter based on a non-overlapping return series over the arbitrary observation periods.

In the first step, daily stock returns are assumed to follow an asymmetric GARCH structure proposed by Glosten et al. (1993), known as GJR (1, 1). Under the GJR (1, 1) model, the daily returns of the Nikkei 225 index follow the process:

$$r_t = \mu_t + h_t^{1/2} z_t, \qquad z_t \sim iid(0,1),$$
$$h_t = \omega + (\alpha_1 + \alpha_2 d_{t-1})\varepsilon_{t-1}^2 + \beta h_{t-1} \text{ with } d_{t-1} = 1 \text{ if } r_{t-1} < \mu, \text{ otherwise } d_{t-1} = 0 \tag{6}$$

where r_t is daily return $\ln(S_t/S_{t-1})$ at time t. μ_t is assumed to be either constant or follow the AR (1) process automatically selected according to the AIC statistic criteria. h_t is conditional variance at time t and d_{t-1} is an asymmetric dummy variable. The innovation process z_t is assumed to either follow a normal distribution or a t-distribution. Under step 1, the parameters $\omega, \alpha_1, \alpha_2, \beta$ and degree of freedom υ in the case of a t-distribution were estimated.

Next, the estimated parameters are used to forecast the next single-period conditional mean $\mu_{t+1}^{(1)}$ and variance $h_{t+1}^{(1)}$. The single-period conditional mean and variance of the t + 2 period are then forecasted based on $\mu_{t+1}^{(1)}$ and $h_{t+1}^{(1)}$. The process is repeated until the forecast period reaches the target horizon, in this case, 30 days ahead. All of the 30-single-period-forecasts are then summed to obtain the multi-period conditional mean $\mu_{T-t}^{(30)} = \sum_{i=1}^{29} \mu_{t+i}^{(1)}$ and conditional variance $h_{T-t}^{(30)} = \sum_{i=1}^{29} h_{t+i}^{(1)}$. The relationship between single-period and multi-period parameters is based on properties of expectation. The proof is provided in the appendix.

In the final steps, if the conditional mean μ_t in (6) was estimated with a constant mean model, I replaced the forecast multi-period conditional mean with the sample average of the non-overlapping 30-day returns series for the past one year to better reflect the real-world returns distribution. However, if the AR (1) model is selected, the forecasted multi-period conditional mean is used.

3. Data

This study applies the Nikkei 225 Index from January 2006 to December 2010 as it represents the broad index of the Japanese stock market. The option data on the Nikkei 225 Index obtained from NEEDS-Financial QUEST 2.0 is used in estimation of risk-neutral density. Options on the Nikkei 225

Index are European style options that can be exercised on the second Friday of the expiration month. The settlement prices are based on special quotation (SQ) calculated from total opening prices of component stocks of the Nikkei 225 on the business day following the last trading day. Strike prices are multiples of ¥500 integer intervals based on the Nikkei 225 but for the nearest three expiration months, and the strike prices are multiples of ¥250 integer intervals. On any trading day, strike prices were set such that there are a total of 17 strike prices for any maturity month, eight below and above the at-the-money (ATM) strike price. A total of 128,076 call and put options exist for the sampling period, and only 80,276 options passed the screening process.

The screening process began by eliminating options with maturities less than seven days. Then for each maturity on each day, a pair of ATM call and put options is identified by the pair with the least differences in the closing transaction price. If a pair of ATM options cannot be identified, the options in that maturity are entirely disregarded. An implied forward index level F_t is calculated during this step using closing transaction prices of ATM put and call options based on the relationship presented in Ait-Sahalia and Lo (2000) as

$$F_t = \exp(r_f \tau)(C_{ATM} - P_{ATM}) + K_{ATM} \qquad (7)$$

where r_f is the risk-free rate, τ is the remaining time-to-maturity of the options, and C_{ATM} and P_{ATM} are prices of the ATM call and put options, respectively. K_{ATM} is the ATM strike price.

For maturity that ATM options are identified, only options with existing closing bid and ask prices, and ask prices within two times the bid price are retained. Next mid prices are calculated for all options. The estimations of risk-neutral density are based on out-of-the-money option data with more liquidity than in-the-money options, and hence, are less subject to pricing error. Out-of-the-money put data were used instead of in-the-money call data. However, the put data were converted to call prices using put-call parity. Only options data that implied volatility can be estimated and only values less than 200% per year were retained. Table 1 shows the summary statistics of the screened options. There were on average four brackets of maturities and a total of 15 strikes for each maturity on each observation date that passed the screening process. An ATM option is defined as a call option with strike price directly below the implied forward index level estimated from (7). By definition, this option is regarded as an in-the-money call option. Consequently, the implied call price from out-of-the-money put option prices were used instead of the real in-the-money call option price at the ATM strike price.

The estimation of real-world density on any observation date requires prior one-year index data to calculate daily returns series. Consequently, the Nikkei 255 Index daily closing data from January 2005 were used for the estimation. This study also estimated risk-neutral densities and real-world densities at the end of each month during the sampling period to obtain non-overlapping one-month empirical RRA measures. Because the RRA measures estimated at the end of each month are in fact forward looking measures for the next 30-day period, the observation period also includes data at the end of December 2005, totaling 61 months of observations.

[Table 1 around here]

4. Empirical Results

4.1. Real-World Density Estimation

At the end of each month from December 2005 to December 2010, two real-world densities were estimated based on GJR (1, 1) with normal and t-distributed innovations. The parameters were estimated from daily returns series data from one year before the observation date. The averages of the estimated parameters by observation period are presented in Table 2.

The observation periods were divided into three sub-periods, the pre-crisis period from December 2005 to June 2007, the crisis period from July 2007 to March 2009, and the post-crisis period from April 2009 to December 2010. I divided the periods based on the observed movement of the Nikkei 225 Index. Although the Lehman shock had actually hit the Japanese stock market in October 2008, the Nikkei 225 index continuously declined since July 2007 and did not pick up until April 2009.

[Table 2 around here]

On each observation date, the estimation was designed to automatically choose between constant mean model and AR (1) conditional mean model separately for GJR with normal and t-distributed innovations based on AIC. Under the GJR-normal innovation model, in none of the periods the AR (1) conditional mean model was preferred over the constant mean model. On the contrary, under the GJR-t innovation model, AR (1) conditional mean models were selected in 4 of the 61 observation periods. The AR (1) model was selected twice during the pre-crisis period and twice during the crisis period but not at all selected during the post-crisis period.

The sum of GARCH, ARCH, and leverage parameters under variance stationary restriction $\beta + \alpha_1 + 0.5\alpha_2 < 1$ implies a volatility persistence level within the 0.91−0.95 range on average for each observation period. The Ljung-Box Q-test for residual autocorrelation at 1, 5, 10, 15, and 20 lags did not reject the null hypothesis at the 5% significant level at which the residuals series showed no autocorrelation. P-values for all test lags were well-above 60%.

The unexpected finding from Table 2 is that GJR (1, 1) with normal innovation dominate GJR (1, 1) with t-distributed innovation for all observation periods as evidenced by consistently lower AIC statistics. On average, from all 61 observation months, GJR with a normal distribution outperforms the t-distribution counterpart by 5.7 points. This finding contrasts with the general results in the finance literature that the t-distribution performs better than the normal distribution in modeling stock market returns because of its fat-tailed characteristic.

4.2. <u>Risk-Neutral Density Estimation</u>

At the end of each observation month, a cross section of the options data that passed the screening process described in section 3 was used to estimate the 30-days-ahead risk-neutral density. The option series with the maturity nearest to +/–15 days of 30 days to maturity is used as a proxy for 30-day maturity options. However, if two option series were within +/–15 days of the 30-day to maturity, risk-neutral densities were estimated from both near-term and next-term option series and then interpolated in the same manner as the Chicago Board Options Exchange (CBOE) interpolates the VIX index. Before the interpolation, each density is normalized by the respective implied forward level of its days to maturity.

The number of available strikes must meet the minimum strikes requirement of each density model to ensure the estimation of density parameters. Because the density of Heston (1993) has five parameters, I had set the required minimum number of strikes equal to eight. The GB2 has four parameters with one parameter restricted to be equal to the forward index level under risk neutrality. The minimum number of strikes for the GB2 density is set to seven. Because the mixture of two-log normal densities has five parameters, the minimum number of strikes required is set to 10. However, if the minimum number of strikes for the Heston and GB2 densities is not attained, the cross-section option data of the previous trading day is used instead and the mixture of two-log normal reduces to a lognormal density with minimum strike requirement of five. If any of the models does not pass the minimum strike requirement, the remaining models are forced to use the same previous trading day data for comparison purposes.

Figure 1 shows the effective strike range as introduced by Andersen et al. (2011) to measure the coverage of options used in computation of the volatility index (VIX). The effective range on any observation date t is defined as

$$ER_t = \left[\frac{\ln(K_{1,t}/F_t)}{\hat{\sigma}_{BS,t}\sqrt{T}}, \frac{\ln(K_{n,t}/F_t)}{\hat{\sigma}_{BS,t}\sqrt{T}}\right], \qquad (8)$$

where $\hat{\sigma}_{BS,t}$ is the Black-Scholes implied volatility of ATM options and F_t is the forward level of the underlying asset price. $K_{1,t}$ and $K_{n,t}$ are the lowest and highest, respectively, strikes observed. The effective range can be considered as the range of highest and lowest returns at the future date *T*.

[Figure 1: Effective strike range]

The shaded area in Figure 1 indicates the crisis period. The minus region represents coverage of out-of-the-money put data as $K_1 < F$. The positive region shows coverage of out-of-the-money call data.

To compare the performance of the three competing models of risk-neutral density, I use the log likelihood value and LR statistic from Berkowitz (2001), which provides a reliable test of the entire distribution content even with a small sample size of forecasted distributions. I can compare the LR statistic of Berkowitz (2001) across different estimated densities because it is robust against the functional form densities. For any assumed function form density, the following transformed variable should be iid N (0, 1):

$$z_t = \Phi^{-1}\left[\int_{-\infty}^{y_t} f(u)du\right] \qquad (9)$$

where $f(u)$ is an ex-ante forecasted density function of the realization y_t. In this study, y_t is a realization of the 30-days-ahead Nikkei 225 index level that is the actual index level on observation day t + 30. On each observation date t, the z_t value is calculated and then the series $z_1, z_2, \dots z_N$ over the entire sampling period is used to estimate the AR (1) model with constant mean and variance as follows.

$$\mathrm{z}_t - \mu = \rho(\mathrm{z}_{t-1} - \mu) + \epsilon_t \quad . \qquad (10)$$

The obtained $\hat{\mu}$, $\hat{\sigma}$, and $\hat{\rho}$ are used to calculate the log likelihood value of (10). The likelihood ratio test statistic can now be given with a restricted likelihood value obtained by setting $\hat{\mu} = 0, \hat{\sigma}^2 = 1$, and $\rho = 0$. Specifically, the LR statistic is in the form

$$\mathrm{LR} = -2[L(0,1,0) - L(\hat{\mu}, \hat{\sigma}^2, \hat{\rho})] \ . \qquad (11)$$

Table 3 shows the LR statistic and log likelihood values of Berkowitz (2001) obtained from the three competing models of risk-neutral densities. In addition, I included statistics estimated from real-world densities under the GJR (1, 1) model with both normal innovation and t-distributed innovation as a benchmark and to cross check against AIC criteria.

LR statistics confirm the AIC criterion that GJR (1, 1) with normal innovation performs better than GJR (1,1) with t-distributed innovation in forecasting 30-days-ahead Nikkei 225 index returns density. The p-value of the GJR-normal cannot be rejected that the transformed variable z_t is iid N (0, 1). Among the three competing risk-neutral density models, the GB2 is selected with the smallest LR statistic. However, the differences in LR statistics and log likelihood values are not substantial.

[Table 3 about here]

4.3. Risk-Aversion Measure Estimation

With both risk-neutral density $q_t(S_T)$ and real-world density $p_t(S_T)$, the relative risk aversion measure $\mathrm{RRA}_t(S_T)$ on each observation month can be estimated using (1). The following results include only RRA measures that is calculated based on real-world density from GJR (1, 1)-normal as it is better than GJR (1, 1)-t whether through AIC criteria or the LR criteria of Berkowitz (2001). Nevertheless, all of the RRA measures based on risk-neutral densities from three competing models will be presented because there was no obvious winner.

The RRA measures defined by (1) were estimated only over the observable strike ranges on each sampling date. Consequently, RRA measures are defined over a different domain of underlying assets on each observation month. To compare the RRA measures across observation periods, I normalized the domain by using moneyness defined as a ratio of $K = S_T$ over the forward index level F_t instead of the gross level of the underlying index.

Figure 2 shows commonly observed 30-days-ahead option-implied risk-neutral densities and the matching horizon real-world density estimated from the daily GJR (1, 1) model with normal innovation during each observation period. The three models of risk-neutral density include the Heston (1993) density, the GB2, and mixture of two log-normal densities. In each panel a, b, and c, the upper rows show common shapes of densities found during the pre-crisis period, the crisis period, and the post-crisis period, respectively. The lower rows indicate the associated relative risk aversion functions $RRA_t(X_i)$ estimated from each pair of real-world and risk-neutral densities using (1).

[Figure 2 around here]

During the pre-crisis period from December 2005 to June 2007, 90% of 19 observation months, the mode of real-world densities (labeled GJR-N) is located to the right of the mode of risk-neutral densities (labeled Hest, GB2, and MLN). A well-behaved downward sloping RRA function in line with economic theory can be observed when the mode of the real-world density locates further to the right and higher than the mode of the risk-neutral density (Panel a, middle). A U-shaped pattern can be observed around the ATM level as found in previous studies, for example, Jackwerth (2000) and Ait-Sahalia and Lo (2000). The RRA functions flatten as the mode of the real-world density falls below the mode of the risk-neutral density. The general shapes of the relative risk-aversion function during this period are either downward sloping curves (Panel a, left) or flat U-shapes (Panel a, right).

During the crisis period, the risk-neutral densities become more left skewed, showing expectation of further declines in future asset prices and the relationships between real-world and risk-neutral densities are the opposite of those during the pre-crisis period. All modes of real-world densities across observation dates locate to the left of risk-neutral densities' modes. When the Lehman Brothers' shock hit the Japanese market in October 2008, the upward sloping risk aversion functions implied the perverse risk preference of a representative investor (Panel b, right). In other words, the representative investor became risk seeking instead of risk averse.

Bliss and Panigirtzoglou (2004) explained that the risk preference of a representative investor might actually change during a period of high volatility in underlying assets. The mix of market participants changed because investors with greater risk aversion left the market during the high volatile period. However, Jackwerth (2000) provided another explanation that this anomaly is the result of option mispricing. I found that option mispricing might be a more plausible explanation. During October 2008, daily returns varied from +13% to −13%, and Figure 1 shows that the effective range of options shrank substantially. Options are subject to a high degree of mispricing during such a tumultuous period. In fact, the cross section of option data used in risk-neutral density estimation for October 2008 observations were data from the previous two trading days, not the precise last trading days of the month. The estimation failed to obtain valid parameters from the data on the previous and the precise last trading day because of an anomaly in the availability of data and pricing.

During the post-crisis period, the mode of real-world density began to move right toward the mode of risk-neutral density. However, the modes of real-world densities are found on the right of risk-neutral densities' modes during only 2 out of the 21 observation months. A downward sloping risk aversion function is again observed.

All of the sub-periods show that the RRA functions, on average, are downward sloping, in line with economic theory. However, the degree of the U-shape pattern around the ATM level becomes more severe as the mode of the real-world density becomes lower than mode of the risk-neutral density.

Figure 3 shows average relative risk aversion measures $RRA(X_i)$ at each level of moneyness defined by X_i/F_t. The range of moneyness was limited to the 0.95−1.05 levels, and the level of risk aversion was limited to between +/−20 to be comparable with Jackwerth (2000). The results for full, pre-crisis, crisis, and post-crisis periods are shown in Panels a, b, c, and d respectively. Across the observation periods, the RRA function did not show a severe U-shape pattern as found in Jackwerth (2000). However, all curves are upward sloping and inconsistent with economic theory. Moreover, except for the pre-crisis period, all curves fell into the negative region.

[Figure 3 around here]

The U-shape pattern found in Figure 2 and the upward sloping curves found in Figure 3 suggested that, after underlying asset prices increase to certain level, a representative investor's risk aversion level again increases. Instead, of welcoming the higher returns from rising asset prices, the representative investor prefers lower returns. As Jackwerth (2000) suggested, the U-shape pattern may be the result of option mispricing. Bondarenko (2003) studied the anomaly of the so-called "overpriced puts puzzle" of S&P 500 put options and explained that investors' excessive weight of the probability of negative S&P 500 returns results in biased subjective future returns density, and hence, biased risk-neutral distribution. Jackwerth (2000) and Ait-Sahalia et al. (2001) showed that an option trading strategy can earn positive returns, which supports the mispricing explanation.

4.4. Relative Risk Aversion Overtime

To study the structure of risk aversion over time, the definition of relative risk aversion in (1) is modified. On each observation date, I calculate the sample average of $RRA_t(K_i)$ across levels of strike prices K to obtain a representative relative risk aversion level $\overline{RRA_t}$. However, when encountered with a negative value of $RRA_t(K_i)$, I interpolate between the two positive values of $RRA_t(K_{i\,-})^+$ and $RRA_t(K_{i\,+})^+$, where K_{i-} and K_{i+} represent strike prices below and above, respectively, the current observed strike price K_i. If only one nearest positive value exists, that value is used to calculate the sample average. Table 4 shows summary statistics of $\overline{RRA_t}$ over various observation periods. The average levels of $\overline{RRA_t}$ measures estimated from risk-neutral density under a GB2 and mixture of two log-normal densities constantly increase from the pre-crisis period to the post-crisis period. The average level of $\overline{RRA_t}$ measures estimated from the Heston risk-neutral density show the opposite trend. Nevertheless, the LR statistic of Berkowitz (2001) presented in Table 3 suggests that the Heston density performs the worst among the three competing models of risk-neutral density; therefore, I disregarded the Heston density results.

[Table 4 around here]

Figure 4 shows the movement of relative risk aversion measure $\overline{RRA}_t$ over the observation periods. The shaded area represents the crisis period. Although the average levels of relative risk aversion measures $\overline{RRA}_t$ during the crisis period are slightly less than the post-crisis level, $\overline{RRA}_t$ vary substantially from a peak of approximately 43 points in September 2007 to the bottom of only one point in September 2008. The peak of $\overline{RRA}_t$ level in September 2007 may be the result of the bankruptcy of distressed financial institutions in the U.S. and Europe.

However, contrary to the general belief that the risk aversion level will skyrocket when the external shock hits the stock market the hardest, the risk aversion level bottomed out in September 2008 when Lehman Brothers filed for bankruptcy protection on September 15, and the Nikkei 225 closed 5% lower than the previous day. Even in October 2008, when the bankruptcy of Daiwa Life Assurance sent the Nikkei 225 Index down by 10% in one day, the RRA measure was still well below the non-crisis level. Bliss and Panigirtzoglou (2004) provided an explanation that the mixture of market participants changed because investors with higher risk aversion left the market during the highly volatile period, only investors with a high degree of risk tolerance remained in the market. Although their explanation is rational and perhaps accurate, they did not formally test this hypothesis. Nevertheless, option mispricing is a more plausible explanation, as evidenced from the lower panel of Figure 4, which shows that the effective range of out-of-the-money put options contracted substantially during September and October 2008. With highly volatile underlying assets, options are subject to a high degree of mispricing.

[Figure 4 around here]

5. Conclusion

This study extended the study of Shiratsuka (2001) by supporting the use of density implied from options data from the Japanese market.

A simple approach that uses a daily model to forecast multi-period distribution parameters worked well in estimating real-world density, as evidenced by the LR test of Berkowitz (2001). However, as opposed to expectations that a t-distribution is better in modeling stock returns, both the AIC statistic and Berkowitz (2001) preferred a model with normal innovations. With regard to risk-neutral density estimation, there was no clear winner among competing models. Though GB2 has the lowest LR statistic, its performance was very close to mixture of the two lognormal models. In contrast, the Heston density was a comparatively poor performer.

The relative risk aversion in the Japanese market in general shows downward sloping characteristics consistent with the economic theory that a representative agent is risk averse. However, the Japanese market also shares the ill-behaved risk aversion function as documented in previous studies on S&P 500 Index. Through an examination of densities' shapes and associated shapes of the RRA function, I showed that the desired relationship between the densities of both worlds occurs when the mode of real-world density $p_t(S_T)$ stays further to the right and higher than the mode of the risk-neutral density $q_t(S_T)$.

When RRA measures at each level for the underlying assets were averaged across observation periods in the same manner as Jackwerth (2000), the U-shape pattern found in the Japanese market was less severe. However, the RRA curves were upward sloping and stayed in the negative region at almost all levels of moneyness during the crisis period and the post-crisis period.

On each observation date, the market-wide level of relative risk aversion measure $\overline{RRA_t}$, which is an average of $RRA_t(K_i)$ over underlying asset levels, showed that, contrary to the general belief, risk aversion levels dropped below the normal level during market turmoil characterized by high return volatility. To explain both the U-shape pattern and the sinking RRA level during the crisis period, previous literature preferred the option mispricing explanation and showed that a simple option trading strategy can result in abnormal positive returns. I agree with the mispricing explanation particularly for the bottomed-out phenomenon as evidenced from a very volatile effective strike range of cross sections of option data. However, it is questionable that if options were constantly mispriced, some arbitragers should be able to exploit this mispricing, and hence, correct prices. Further research should be conducted on the explanation of mispricing in the Japanese market.

Appendix: Inference of multi-period mean and variance parameters from single-period parameters

In Section 2.3, I proposed a simple approach to forecast multi-period distribution parameters based on a daily GJR (1, 1) model. The multi-period mean and variance can be obtained based on single-period means and variances using the linearity property of expectations without altering the properties of the assumed model.

First, define a single-period return as

$$r_t^{(1)} = \log\left(\frac{P_t}{P_{t-1}}\right). \tag{A. 1}$$

Campbell et al. (1997) showed that a multi-period return from single-period return in (A. 1) can be derived from

$$r_t^{(k)} = r_t^{(1)} + r_{t-1}^{(1)} + \cdots + r_{t-k+1}^{(1)}. \tag{A. 2}$$

In the same manner as (A. 2), from the future period time $t + 3$, a three-period return can be written as

$$r_{t+3}^{(3)} = r_{t+3}^{(1)} + r_{t+2}^{(1)} + r_{t+1}^{(1)}. \tag{A. 3}$$

At current time t, take the expectation of (A. 3) then use the linearity property of expectations, and the mean of multi-period return at time $t + 3$ can be written as a sum of a single-period mean as

$$E_t\left[r_{t+3}^{(3)}\right] = E_t\left[r_{t+3}^{(1)}\right] + E_t\left[r_{t+2}^{(1)}\right] + E_t\left[r_{t+1}^{(1)}\right]. \tag{A. 4}$$

Under the GJR (1, 1) model in (6) with constant mean, (A. 4) is reduced to $E_t\left[r_{t+3}^{(3)}\right] = 3\mu^{(1)}$.

With regard to the variance of $r_{t+3}^{(3)}$, it can be written as

$$V_t\left[r_{t+3}^{(3)}\right] = E_t\left[\left(r_{t+3}^{(3)} - E_t\left[r_{t+3}^{(3)}\right]\right)^2\right]. \tag{A. 5}$$

Using (A. 3) and (A. 4) assuming the constant mean model of (6) and through the linearity property of expectations, (A. 5) can be rewritten as

$$V_t\left[r_{t+3}^{(3)}\right] = V_t\left[r_{t+3}^{(1)}\right] + V_t\left[r_{t+2}^{(1)}\right] + V_t\left[r_{t+1}^{(1)}\right] + 2E_t\left[\sum_{i\neq j}\left(r_{t+i}^{(1)} - \mu^{(1)}\right)\left(r_{t+j}^{(1)} - \mu^{(1)}\right)\right]. \tag{A. 6}$$

Under the assumed single-period GJR (1, 1) model in (6), each $r_{t+i}^{(1)} - \mu^{(1)}$ in the cross term of (A. 6) is equal to $\sqrt{h_{t+i}^{(1)}} z_{t+i}^{(1)}, z_{t+i}^{(1)} \sim iid(0, 1)$, which makes all of the cross terms equal to zero. It follows that (A. 6) is reduced to

$$V_t\left[r_{t+3}^{(3)}\right] = V_t\left[r_{t+3}^{(1)}\right] + V_t\left[r_{t+2}^{(1)}\right] + V_t\left[r_{t+1}^{(1)}\right]. \qquad \text{(A. 7)}$$

With regard to GJR (1, 1) with the AR (1) conditional mean model, the constant mean $\mu^{(1)}$ expression can be replaced with a forecasted conditional mean $\mu_{t+i}^{(1)}$ and proceed in the same manner as the proof here, and eventually the results in (A. 7) are obtained.

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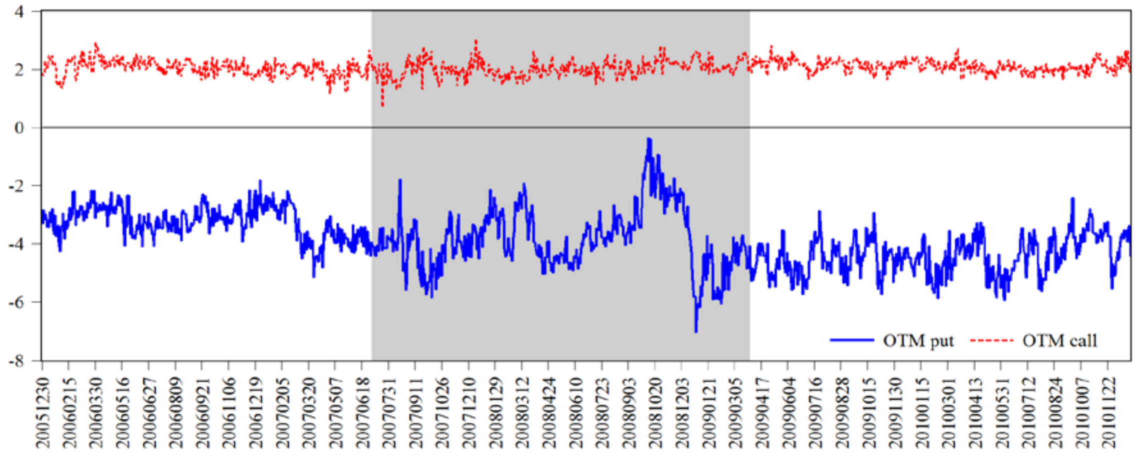
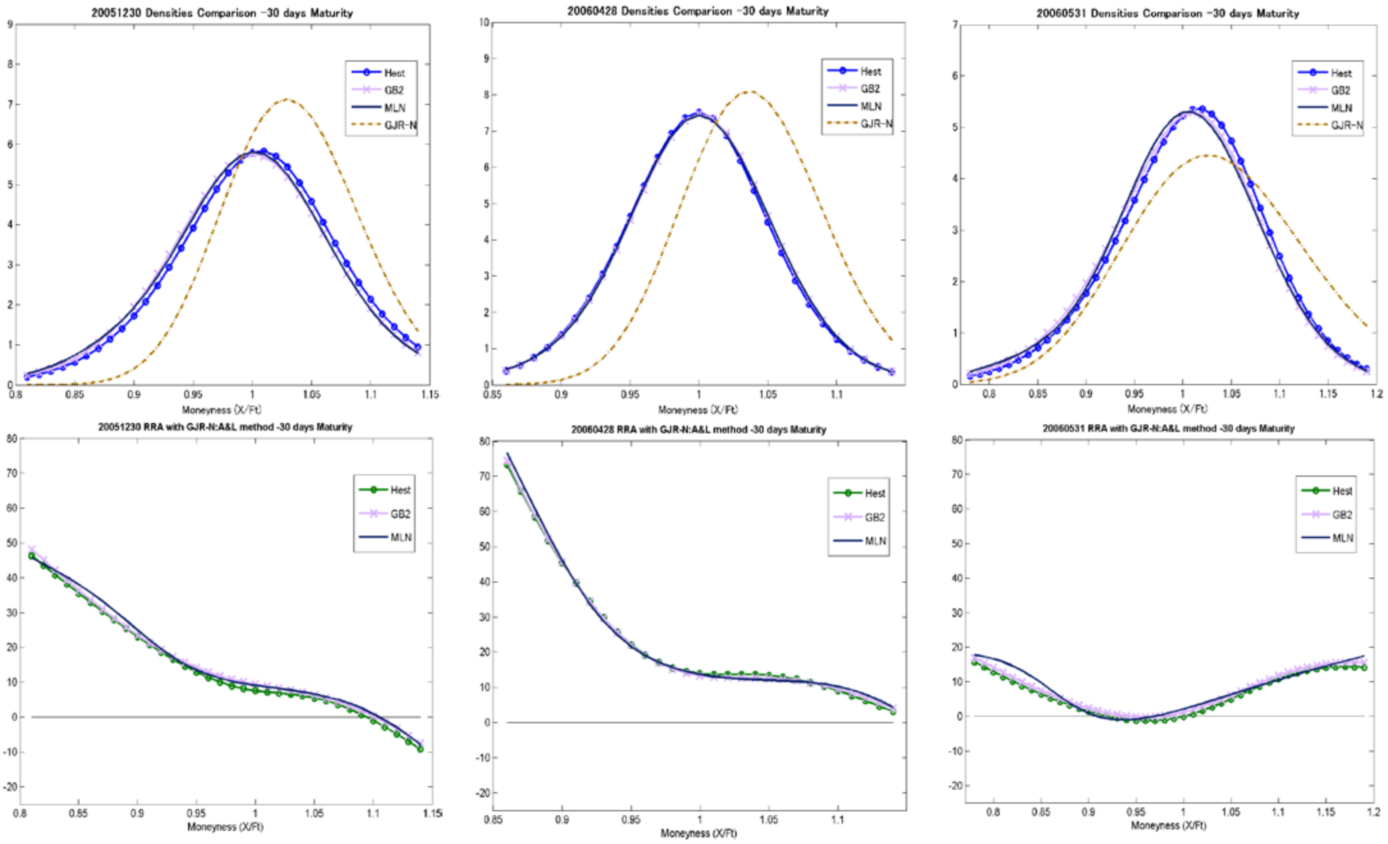


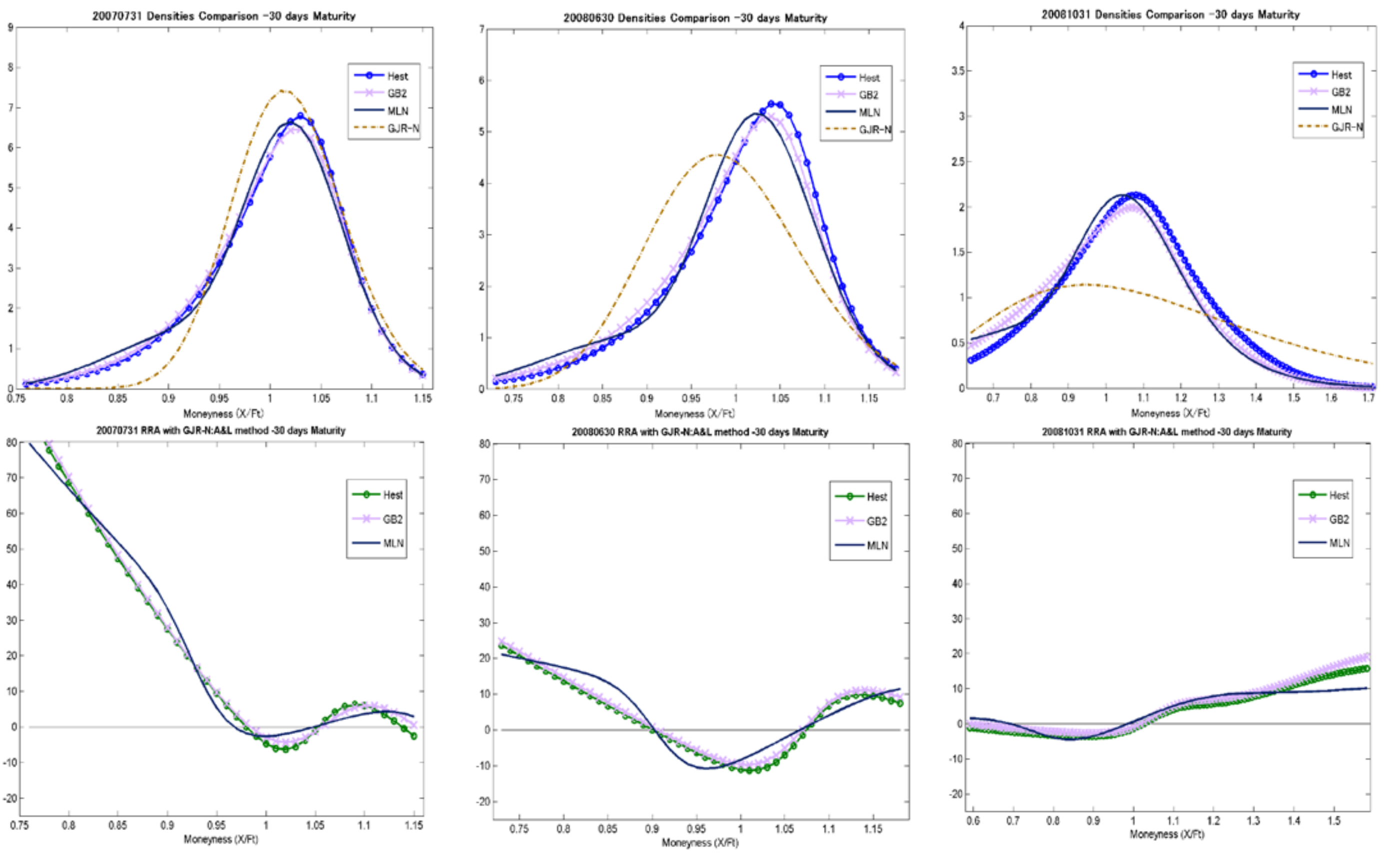
Figure 1: Effective strike range

The effective strike range on each observation date ER_t is defined in (8). Andersen et al. (2011) introduced the effective range to measure the coverage of options used in the volatility index (VIX) calculation.

Panel a: Common shapes of densities and RRA functions during pre-crisis period from 2005/12 to 2007/06



Panel b: Common shapes of densities and RRA functions during crisis period from 2007/07 to 2009/03



Panel c: Common shapes of densities and RRA functions during post-crisis period from 2009/04 to 2010/12

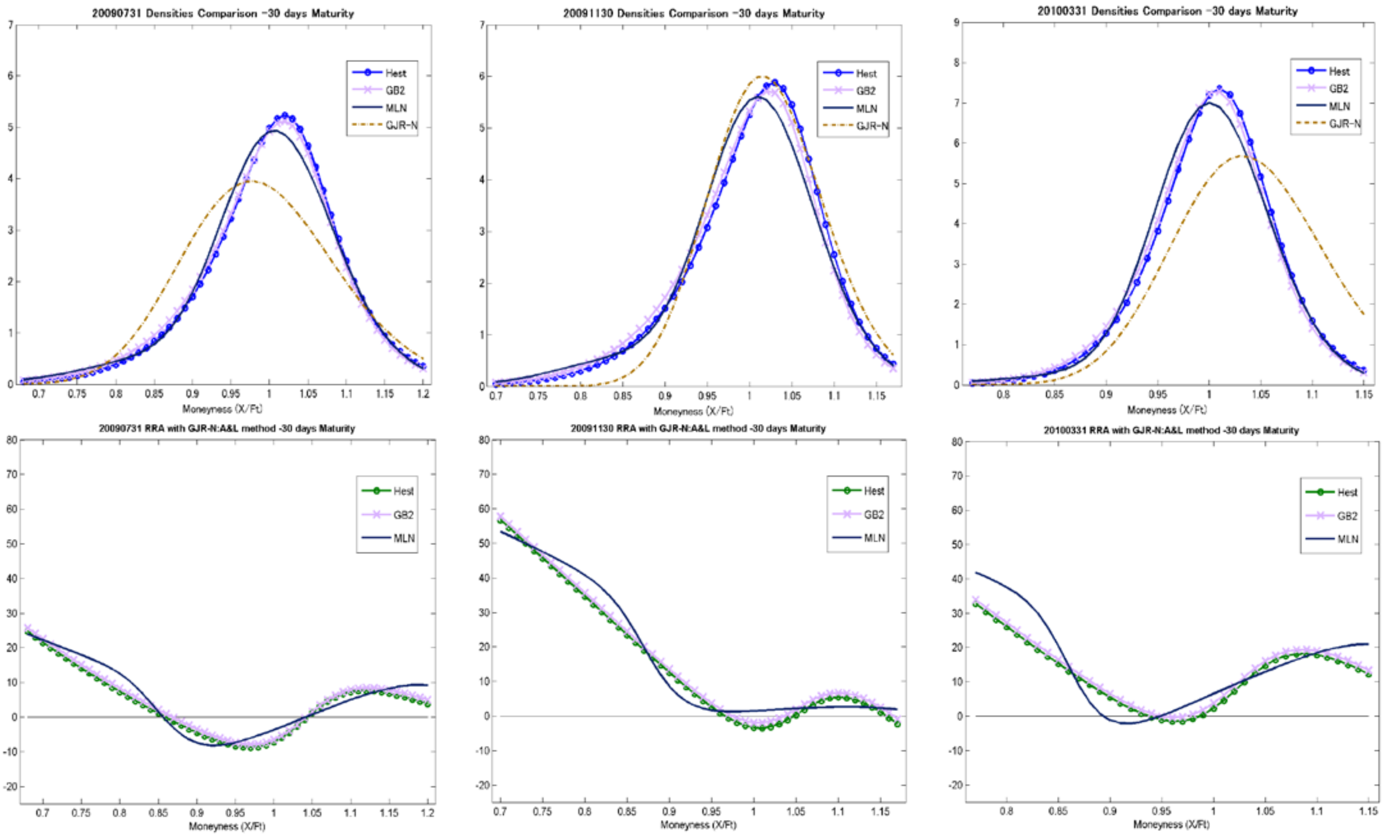


Figure 2: Density estimation and associated RRA functions

The upper row in each panel shows common shapes found from estimated real-world density $p_t(S_T)$ under GJR (1, 1) with normal innovations and risk-neutral densities $q_t(S_T)$ under assumed models, Heston in (2), GB2 in (3), and MLN in (4). The lower rows show relative risk aversion functions $RRA_t\,(X_i)$ estimated from each pair of real-world and risk-neutral densities using (1).

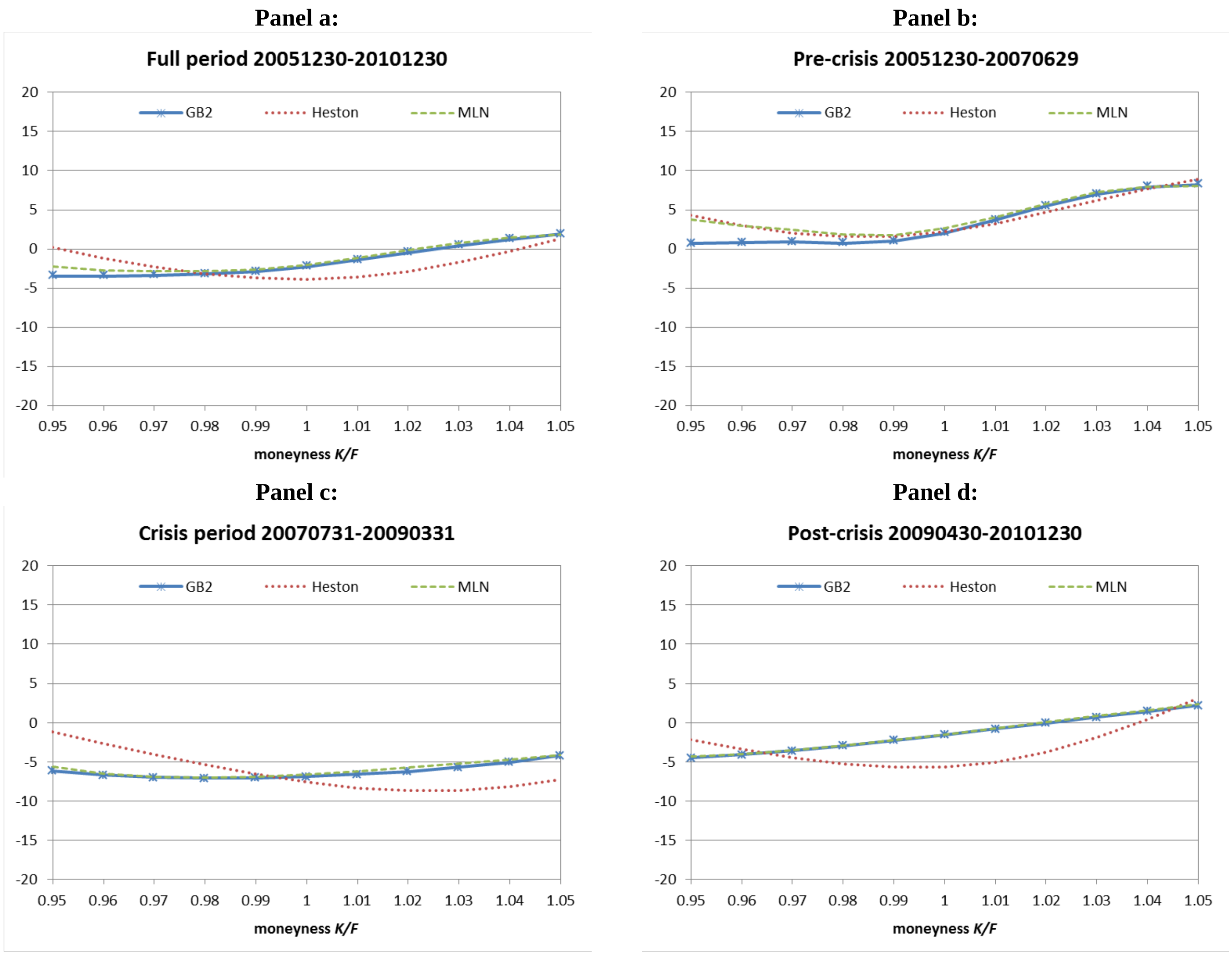


Figure 3: Average RRA at each underlying asset level.

Curves labeled GB2, Heston, and MLN are average relative risk aversion measures $RRA(X_i)$ at each level of moneyness defined by X_i/F_t calculated from risk-neutral densities under assumed models and real-world density under GJR (1, 1) with normal innovations.

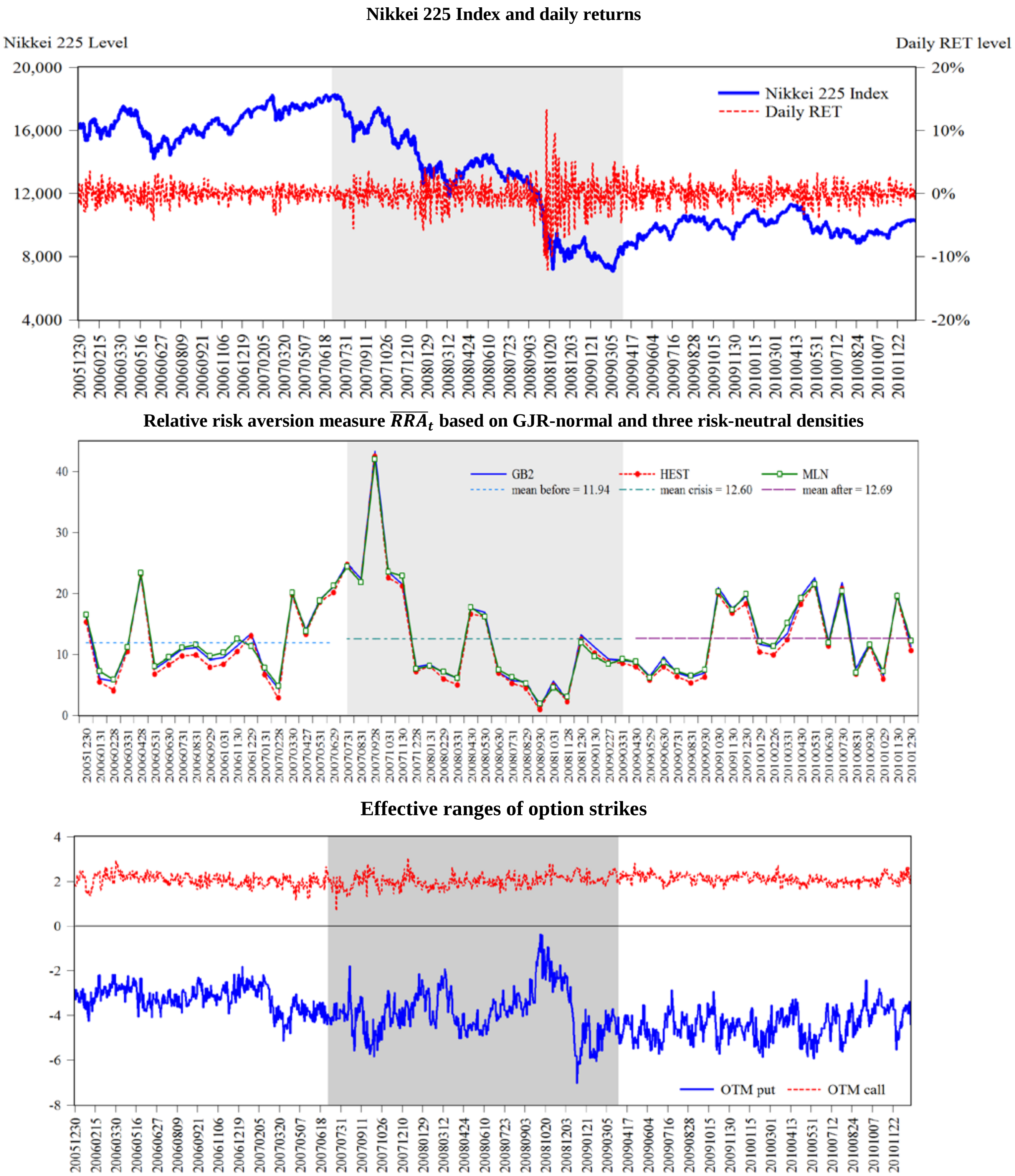


Figure 4: Average relative risk aversion measure $\overline{RRA}_t$ and Nikkei 225 Index

The middle chart shows $\overline{RRA}_t$ across observation periods. $\overline{RRA}_t$ is a sample average of relative risk aversion $RRA_t(X_i)$ across levels of moneyness defined as X_i/F_t on each observation date *t*.

Table 1: Summary statistic of Nikkei 225 options that passed screening process.

	Total Number	Average Options per day	Min per day	Max per day
OTM call	32,972	27	7	63
OTM put	47,301	39	9	77
Total	80,273	65	23	140

Out-of-the-money is abbreviated as OTM. Total number refers to the total number of options that passed the screening process on a daily basis from December 2005 to December 2010. Average options per day is the average number of options remaining after the screening process on any observation date. Min and max per day is the minimum and maximum, respectively, number of options that passed daily screening processes.

Table 2: GJR (1, 1) parameter estimation and AIC statistics

GJR (1,1) - normal innovations

	Observation Period start		end	μ_t	ϕ	ω	α_1	α_2	β
Full	20051230	:	20101230	−0.0004	N/A	0.0000	0.0203	0.1666	0.8305
Before	20051230	:	20070629	0.0008	N/A	0.0000	0.0337	0.1705	0.8245
Crisis	20070731	:	20090331	−0.0013	N/A	0.0000	0.0004	0.1957	0.8440
After	20090430	:	20101230	−0.0005	N/A	0.0000	0.0280	0.1340	0.8225

GJR (1,1)–t-distributed innovations

	Observation Period start		end	μ_t	ϕ	ω	α_1	α_2	β	υ
Full	20051230	:	20101230	−0.0003	−0.0372	0.0000	0.0141	0.1948	0.8259	49.3922
Before	20051230	:	20070629	0.0007	−0.0222	0.0000	0.0149	0.3087	0.7789	60.3605
Crisis	20070731	:	20090331	−0.0011	−0.0522	0.0000	0.0007	0.1614	0.8682	9.5606
After	20090430	:	20101230	−0.0004	N/A	0.0000	0.0268	0.1252	0.8261	79.3000

	Observation Period			AIC statistic		Number of Observations
	start		end	GJR (1,1) -normal	GJR (1,1) -t	
Full	20051230	:	20101230	1,167.73	1,173.40	199
Before	20051230	:	20070629	1,286.07	1,291.35	201
Crisis	20070731	:	20090331	1,152.15	1,160.43	201
After	20090430	:	20101230	1,076.23	1,079.67	196

The GJR (1, 1) model is estimated using (6). The mean equation parameter is μ_t, which is automatically selected to be a constant mean or an AR(1) mean model basing on AIC statistics. If the AR(1) model is selected $\mu_t = c + \phi r_{t-1}$, where c is a constant, ϕ is a first-order autoregressive coefficient and r_{t-1} is the lag return of the current return. The constant mean model implies $\phi = 0$. ω is a constant of the conditional variance equation, β is the GARCH coefficient, α_1 is the ARCH coefficient, and α_2 is the leverage coefficient. In the case of t-distributed innovations, degree of freedom υ is an additional parameter.

Table 3: Berkowitz LR statistics and log likelihood value

	Risk-neutral density			Real-world density	
	Hest	GB2	MLN	GJR-t	GJR-n
LR	13.9922	12.4513	12.6003	9.4963	3.8527
P-value	0.0029	0.0060	0.0056	0.0234	0.2778
LL unrestricted	−64.0912	−65.3569	−65.1563	−68.8402	−77.8509
LL restricted	−71.0873	−71.5825	−71.4564	−73.5883	−79.7773

Berkowitz LR statistics are calculated using (11). LL unrestricted is a maximized log likelihood value from model (10) without restriction and LL restricted is a maximized log likelihood value under model (10) with restriction $\mu = 0, \sigma^2 = 1,$ and $\rho = 0$.

Table 4: Average RRA $\overline{RRA}_t$ based on GJR (1, 1) - normal innovation real-world density

Observation Period					Risk-neutral density models		
	start		end		Heston	GB2	MLN
Full	20051230	:	20101230	Ave	11.8887	12.6953	12.6872
				Min	0.9387	1.4330	1.9416
				Max	42.5894	43.3625	42.1114
Before	20051230	:	20070629	Ave	11.2983	12.1172	12.3963
				Min	2.8794	4.3056	4.8349
				Max	23.1650	23.1282	23.4516
Crisis	20070731	:	20090331	Ave	12.2285	12.8951	12.6750
				Min	0.9387	1.4330	1.9416
				Max	42.5894	43.3625	42.1114
After	20090430	:	20101230	Ave	12.0832	13.0186	12.9626
				Min	5.3011	6.3172	6.1907
				Max	21.4451	22.6504	21.5426

The numbers under each column labeled Heston, GB2, and MLN are average levels of $\overline{RRA}_t$ across observation periods under assumed risk-neutral density models. $\overline{RRA}_t$ is a sample average of relative risk aversion $RRA_t(X_i)$ across the level of moneyness defined as X_i/F_t on each observation date *t*.