Labor Markets and Business Cycles

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Osaka University
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Overview of Lectures

1. The Labor Wedge
2. Benchmark Search Model
3. Capital
4. Rigid Wages
The Labor Wedge
Representative Agent Model

- representative household
  - consumes
  - supplies labor
  - owns firms and government bonds

- representative firm
  - produces output (single consumption and capital good)
  - demands labor
  - buys and sells capital

- government: sets taxes, transfers, and spending, and issues bonds

- perfect competition
States and History

- time is \( t = 0, 1, 2, \ldots \)
- state of the economy at \( t \) is \( s_t \)
- history of the economy at \( t \) is \( s^t \equiv \{s_0, s_1, \ldots, s_t\} \)
  - productivity
  - government spending
  - distortionary tax rates
  - :
- \( \Pi(s^t) \) is time-0 probability of history \( s^t \)
Household Problem

- household chooses \( \{c(s^t), h(s^t)\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \frac{\gamma \varepsilon}{1 + \varepsilon} h(s^t)^{\frac{1+\varepsilon}{\varepsilon}} \right),
\]

subject to

\[
a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t)(c(s^t) - (1 - \tau(s^t))w(s^t)h(s^t) - T(s^t))
\]

taking \( a_0 \) and \( \{q_0(s^t), w(s^t), \tau(s^t), T(s^t)\} \) as given
firm chooses $\{h^d(s^t), k(s^{t+1})\}$ to maximize

$$\sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t)(z(s^t)k(s^t)^\alpha h^d(s^t)^{1-\alpha} + (1 - \delta)k(s^t) - k(s^{t+1}) - w(s^t)h^d(s^t))$$

taking $k_0 = k(s^0)$ and $\{q_0(s^t), w(s^t)\}$ as given

call the value of the firm $J(s^0)$
government faces a budget constraint

\[ b_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( \tau(s^t)w(s^t)h(s^t) - g(s^t) - T(s^t) \right) \]
Market Clearing

- labor market clearing: \( h(s^t) = h^d(s^t) \) for all \( t \)

- goods market clearing:

\[
k(s^{t+1}) = z(s^t)k(s^t)^\alpha h^d(s^t)^{1-\alpha} + (1 - \delta)k(s^t) - c(s^t) - g(s^t)
\]

for all \( t \)

- capital market clearing: \( a(s^t) = J(s^t) + b(s^t) \) for all \( t \)

  - \( a(s^t) \): household assets in history \( s^t \)
  - \( J(s^t) \): value of firm in history \( s^t \)
  - \( b(s^t) \): value of government debt in history \( s^t \)
  - this is implied by the other equations
Equilibrium

$a_0, b_0, k_0$ and \{c(s^t), h(s^t), h^d(s^t), k(s^t), q_0(s^t), w(s^t), \tau(s^t), T(s^t), g(s^t)\} s.t.:

- household problem is solved
- firm problem is solved
- government budget constraint is satisfied
- labor and goods markets clear
Marginal Rate of Substitution Equals Wage

☐ household solves

\[
\max_{\{c(s^t), h(s^t)\}} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \frac{\gamma \varepsilon}{1 + \varepsilon} h(s^t)^{\frac{1+\varepsilon}{\varepsilon}} \right)
\]

s.t. \[ a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( c(s^t) - (1 - \tau(s^t)) w(s^t) h(s^t) - T(s^t) \right) \]

☐ first order conditions:

\[ \nabla \quad c(s^t): \quad \frac{\beta^t \Pi(s^t)}{c(s^t)} = \lambda q_0(s^t) \]

\[ \nabla \quad h(s^t): \quad \beta^t \Pi(s^t) \gamma h(s^t)^{\frac{1}{\varepsilon}} = \lambda q_0(s^t) (1 - \tau(s^t)) w(s^t) \]

\[ w(s^t) = \frac{\gamma c(s^t) h(s^t)^{\frac{1}{\varepsilon}}}{1 - \tau(s^t)} \]
firm solves

\[
\sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( y(s^t) + (1 - \delta) k(s^t) - k(s^{t+1}) - w(s^t) h^d(s^t) \right)
\]

where \( y(s^t) = z(s^t) k(s^t)^\alpha h^d(s^t)^{1-\alpha} \)

first order conditions:

\( h^d(s^t): q_0(s^t) \left( (1 - \alpha) z(s^t) k(s^t)^\alpha h^d(s^t)^{-\alpha} - w(s^t) \right) = 0 \)

\[
w(s^t) = \frac{(1 - \alpha) y(s^t)}{h^d(s^t)}
\]
Labor Market Clearing

\[ h(s^t) = h^d(s^t) \]
\[ \tau(s^t) = 1 - \frac{\gamma}{1 - \alpha} \left( \frac{c(s^t)}{y(s^t)} \right) h(s^t)^{\frac{1+\varepsilon}{\varepsilon}} \]
Measurement
Measurement

- consumption-output ratio
  - consumption: nominal PCE including nondurables and services
  - output: nominal GDP

- hours:
  - Prescott, Ueberfeldt, and Cociuba (2008)
  - total hours worked from CPS, plus estimate of military
    - number of civilians at work times average hours worked
    - estimate 40 hours per week for the military
  - deflate by noninstitutional population aged 16 to 64
Consumption-Output Ratio

percent deviation from trend

Year


"Labor Markets and Business Cycles"
Hours Worked

percent deviation from trend

Year

Hours Worked

![Graph showing hours worked with annual growth rate and percent deviation from trend over years from 1955 to 2005.](image)
Comovement of $c/y$ and $h$

<table>
<thead>
<tr>
<th></th>
<th>s.d. $c/y$</th>
<th>s.d. $h$</th>
<th>relative s.d.</th>
<th>correlation</th>
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<td>detrended annual growth</td>
<td>0.010</td>
<td>0.013</td>
<td>1.32</td>
<td>−0.663</td>
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<tr>
<td>annual growth</td>
<td>0.015</td>
<td>0.018</td>
<td>1.25</td>
<td>−0.571</td>
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Measuring the Labor Wedge

\[
\tau(s^t) = 1 - \frac{\gamma}{1 - \alpha} \left( \frac{c(s^t)}{y(s^t)} \right) h(s^t)^{\frac{1+\varepsilon}{\varepsilon}}
\]

- use data for \(c/y\) and \(h\)
- use different values of \(\varepsilon\)
- set \(\gamma/(1 - \alpha)\) so \(\tau = 0.4\) on average
Labor Wedge

$\varepsilon = \infty$
$\varepsilon = 4$
$\varepsilon = 1$
$\varepsilon = \frac{1}{2}$

“Labor Markets and Business Cycles”
Labor Wedge

Year

Deviation from Trend


-20 -10 0 10 20
Labor Wedge

```
Year  | Annual Growth Rate
1955  | -20
1960  | -15
1965  | 10
1970  | 5
1975  | -10
1980  | 0
1985  | 15
1990  | 20
1995  | 25
2000  | 30
2005  | 30
```

“Labor Markets and Business Cycles”
## Summary

### detrended

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 1$</th>
<th>$\varepsilon = 4$</th>
<th>$\varepsilon = \infty$</th>
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<tr>
<td>s.d.</td>
<td>0.055</td>
<td>0.031</td>
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<td>0.014</td>
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<tr>
<td>$c/y$</td>
<td>0.010</td>
<td>0.338</td>
<td>0.278</td>
<td>0.049</td>
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<tr>
<td>$h$</td>
<td>0.013</td>
<td>-0.795</td>
<td>-0.835</td>
<td>-0.745</td>
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</table>

### annual growth rate

<table>
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<tr>
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<th>$\varepsilon = 0.5$</th>
<th>$\varepsilon = 1$</th>
<th>$\varepsilon = 4$</th>
<th>$\varepsilon = \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.d.</td>
<td>0.079</td>
<td>0.045</td>
<td>0.027</td>
<td>0.022</td>
</tr>
<tr>
<td>$c/y$</td>
<td>0.015</td>
<td>0.256</td>
<td>0.163</td>
<td>-0.088</td>
</tr>
<tr>
<td>$h$</td>
<td>0.018</td>
<td>-0.803</td>
<td>-0.835</td>
<td>-0.733</td>
</tr>
</tbody>
</table>
Possible Resolutions
The Issue

- hours are 50 percent more volatile than consumption/output
- their correlation is $-0.6$

$\Rightarrow$ the labor wedge is countercyclical
Possible Resolutions

- shocks to the marginal disutility of work $\gamma$

“Sargent (1976) has attempted to remedy this fatal flaw by hypothesizing that the persistent and large fluctuations in unemployment reflect merely corresponding swings in the natural rate itself. In other words, what happened to the United States in the 1930’s was a severe attack of contagious laziness! I can only say that, despite Sargent’s ingenuity, neither I nor, I expect, most others at least of the nonmonetarists’ persuasion are quite ready yet to turn over the field of economic fluctuations to the social psychologist” — Franco Modigliani (1977, p. 6)

“Alternatively, one could explain the observed pattern without a procyclical real wage by positing that tastes for consumption relative to leisure vary over time. Recessions are then periods of ‘chronic laziness.’ As far as I know, no one has seriously proposed this explanation of the business cycle” — N. Gregory Mankiw (1989, p. 82)
Why Search?

- model structure is reasonable for business cycles
  - focus on employment rather than hours
  - focus on unemployment rather than nonemployment
Why Search?

- model structure is reasonable for business cycles
  - focus on employment rather than hours
  - focus on unemployment rather than nonemployment
- plausible environment for analyzing wage rigidities
  - Barro (1977) critique of implicit contract models
- caveat: search frictions naturally reduce volatility in employment
Baseline Search Model
Baseline Search Model

- representative household
  ▶ consumes
  ▶ supplies labor
  ▶ owns firms

- representative firm
  ▶ produces consumption good using labor
  ▶ recruits workers using labor

- government: constant tax, time-varying transfer, no debt

- wages are bargained by workers and firms
time is \( t = 0, 1, 2, \ldots \)

state of the economy at \( t \) is \( s_t \)

history of the economy at \( t \) is \( s^t \equiv \{s_0, s_1, \ldots, s_t\} \)

- productivity \( z(s^t) \)

\( \Pi(s^t) \) is time-0 probability of history \( s^t \)
Two Technologies

- divide \( n(s^t) = \ell(s^t) + v(s^t) \) workers between two technologies

- constant-returns-to-scale production: \( y(s^t) = z(s^t)\ell(s^t) \)

- constant-returns-to-scale recruiting:

  \[
  n(s^{t+1}) = (1 - x)n(s^t) + v(s^t)\mu(\theta(s^t))
  \]

  - \( x \): employment exit probability
  - \( \theta \): recruiter-to-unemployment ratio
  - \( \mu(\theta) \): new hires per recruiter
    - continuous and nonincreasing on \((0, \infty)\)
    - \( \mu(0) = \infty \) and \( \mu(\infty) = 0 \)
    - \( f(\theta) \equiv \mu(\theta)\theta \) is nondecreasing
firm chooses \( \{ \nu(s^t) \} \), where \( \nu(s^t) \equiv v(s^t)/n(s^t) \), to maximize

\[
J(s^0, n_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) n(s^t) (z(s^t)(1 - \nu(s^t)) - w(s^t))
\]

s.t. \( n(s^{t+1}) = n(s^t)(1 - x + \nu(s^t)\mu(\theta(s^t))) \)

taking \( n_0 = n(s^0) \) and \( \{ q_0(s^t), w(s^t), \theta(s^t) \} \) as given

- \( q_0(s^t) \): price of an Arrow-Debreu security
- \( w(s^t) \): wage in history \( s^t \) (in units of history-\( s^t \) consumption)

firm’s value is linear in \( n_0 \),

\[
J(s^0, n_0) = \bar{J}(s^0)n_0
\]
Individual Preferences

- A representative household with many individual members \( i \in [0, 1] \)
  - \( i \) has time-separable preferences over consumption and leisure
    - felicity \( \log c_i - \gamma \) if employed and consuming \( c_i \)
    - felicity \( \log c_i \) if unemployed and consuming \( c_i \)

- The household maximizes the sum of its members’ utility
  - \( c_i(s^t) \) is the same for all \( i \)

- Standard trick for getting the complete markets allocation
  - Merz (1995)
household acts as if it has preferences

$$\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \gamma n(s^t) \right) ,$$

- $c(s^t)$: consumption in history $s^t$
- $n(s^t)$: employment rate in history $s^t$
- $\beta \in (0, 1)$: discount factor
- $\gamma$: disutility of working
Household Constraints

single lifetime budget constraint

\[ a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t)\left(c(s^t) - (1 - \tau)w(s^t)n(s^t) - T(s^t)\right) \]

- \( a_0 \): initial assets
- \( \tau \): constant labor income tax rate
- \( T(s^t) \): lump-sum transfer in history \( s^t \)

sequence of employment constraints:

\[ n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t)) \]
Household Problem

Household chooses \( \{c(s^t)\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \gamma n(s^t) \right)
\]

s.t. \( a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( c(s^t) - (1 - \tau) w(s^t) n(s^t) - T(s^t) \right) \)

and \( n(s^{t+1}) = (1 - x) n(s^t) + f(\theta(s^t))(1 - n(s^t)) \),

taking \( a_0, n_0 = n(s^0) \), and \( \{q_0(s^t), w(s^t), \theta(s^t), \tau, T(s^t)\} \) as given
balanced budget: $T(s^t) = \tau w(s^t)n(s^t)$

note that Ricardian equivalence holds
define two objects:

- $\tilde{J}_n(s^t, w)$: value of paying a worker $w$ in $s^t$, $w(s^{t'})$ thereafter, compared to not employing the worker.
- $\tilde{V}_n(s^t, w)$: value of having a worker paid $w$ in $s^t$, $w(s^{t'})$ thereafter, compared to having the worker unemployed.

- evaluated at the equilibrium level of assets and employment.

the wage satisfies the Nash bargaining solution:

$$w(s^t) = \arg \max_w \tilde{V}_n(s^t, w)^{\phi} \tilde{J}_n(s^t, w)^{1-\phi},$$

where $\phi$ is workers’ bargaining power.
goods market clearing: \( c(s^t) = z(s^t)n(s^t)(1 - \nu(s^t)) \) for all \( t \)

definition of \( \theta \): \( \theta(s^t) = \frac{\nu(s^t)n(s^t)}{1-n(s^t)} \)

capital market clearing: \( a(s^t) = J(s^t, n(s^t)) \) for all \( t \)

\( a(s^t) \): household assets in history \( s^t \)

\( J(s^t, n(s^t)) \): value of firm in history \( s^t \)

this is implied by the other equations
Equilibrium

- $a_0$, $n_0$, and $\{c(s^t), n(s^t), \nu(s^t), \theta(s^t), q_0(s^t), w(s^t), T(s^t)\}$ such that:
  - firm problem is solved
  - household problem is solved
  - government budget constraint is satisfied
  - wages satisfy the Nash bargaining solution
  - goods market clears
  - $\theta$ is the ratio of recruiters to unemployed
Firm’s Problem

- firm’s value is linear in employment, \( J(s^t, n) = \bar{J}(s^t)n \)

- express per-worker value recursively

\[
\bar{J}(s^t) = \max_{\nu} \left[ z(s^t)(1 - \nu) - w(s^t) + (\nu \mu(\theta(s^t)) + 1 - x) \sum_{s^{t+1}|s^t} q_t(s^{t+1})\bar{J}(s^{t+1}) \right]
\]
Firm’s Problem: Main Results

- first order condition for $\nu$:

$$z(s^t) = \mu(\theta(s^t)) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \tilde{J}(s^{t+1})$$

firms are indifferent between production and recruiting

- simplify Bellman equation:

$$\tilde{J}(s^t) = z(s^t) \left( 1 + \frac{1 - x}{\mu(\theta(s^t))} \right) - w(s^t)$$

value of job is current output plus saved recruiting minus wage

- the value of paying a worker $w$ is

$$\tilde{J}_n(s^t, w) = w(s^t) - w + \tilde{J}(s^t)$$
express worker's problem recursively

$$V(s^t, a, n) = \max_{\{a(s^{t+1})\}} \left( \log c - \gamma n + \beta \sum_{s^{t+1}|s^t} \frac{\Pi(s^{t+1})}{\Pi(s^t)} V(s^{t+1}, a(s^{t+1}), n') \right),$$

where $c = a + (1 - \tau)w(s^t)n + T(s^t) - \sum_{s^{t+1}|s^t} q_t(s^{t+1})a(s^{t+1})$

and $n' = (1 - x)n + f(\theta(s^t))(1 - n)$

first order condition for future assets:

$$\frac{q_t(s^{t+1})}{\tilde{c}(s^t, a, n)} = \beta \frac{\Pi(s^{t+1})}{\Pi(s^t)} V_a(s^{t+1}, a(s^{t+1}), n')$$

envelope condition for current assets:

$$V_a(s^t, a, n) = 1/\tilde{c}(s^t, a, n)$$
Worker’s Problem: Main Results

- intertemporal Euler equation

\[ q_t(s^{t+1}) = \beta \frac{\Pi(s^{t+1})c(s^t)}{\Pi(s^t)c(s^{t+1})} \]

- envelope condition for current employment:

\[ V_n(s^t, a(s^t), n(s^t)) = \frac{(1 - \tau)w(s^t)}{c(s^t)} - \gamma \]

\[ + \beta(1 - x - f(\theta(s^t))) \sum_{s^{t+1}|s^t} \frac{\Pi(s^{t+1})}{\Pi(s^t)} V_n(s^{t+1}, a(s^{t+1}), n(s^{t+1})) \]

- value of having a worker paid \( w \) rather than unemployed is

\[ \tilde{V}_n(s^t, w) = \frac{(1 - \tau)(w - w(s^t))}{c(s^t)} + V_n(s^t, a(s^t), n(s^t)) \]
**Wage Setting: Main Result**

- wage solves $w(s^t) = \arg \max_w \tilde{V}_n(s^t, w) \phi \tilde{J}_n(s^t, w)^{1-\phi}$

- from previous expressions, this implies implies

$$
\frac{(1 - \tau)w(s^t)}{c(s^t)} = \phi \frac{(1 - \tau)z(s^t)(1 + \theta(s^t))}{c(s^t)} + (1 - \phi)\gamma
$$

- after-tax wage (in utils) is weighted average of

  - after-tax output (in utils) produced by
    1. the worker, $(1 - \tau)z(s^t)$
    2. other workers freed from recruiting, $(1 - \tau)z(s^t)\theta(s^t)$

  - marginal rate of substitution between consumption and leisure
Equilibrium

- consumption, wage, and firm value are proportional to $z$
  - $c(s^t) = \bar{c}z(s^t)$
  - $w(s^t) = \bar{w}z(s^t)$
  - $\bar{J}(s^t) = \bar{J}z(s^t)$

- recruiters/unemployed, employment, and worker value are constant:
  - $\theta(s^t) = \bar{\theta}$
  - $n(s^t) = \bar{n}$
  - $V_n(s^t, a(s^t), n(s^t)) = \bar{V}_n$

- constant measured labor wedge $\hat{\tau}$

- note that this equilibrium requires $n_0 = \frac{f(\bar{\theta})}{f(\bar{\theta}) + x}$
Explanation

- productivity shock does not effect the efficiency of recruiting
- income and substitution effects offset
planner chooses \( \{\nu(s^t), \theta(s^t)\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log \left( z(s^t)n(s^t)(1 - \nu(s^t)) \right) - \gamma n(s^t) \right)
\]

s.t. \( n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t)) \)

and \( \theta(s^t) = \frac{\nu(s^t)n(s^t)}{1 - n(s^t)} \)

- productivity terms \( z(s^t) \) are additively separable
- the planner’s solution coincides with equilibrium if
  - \( \tau = 0 \): no distortionary taxes
  - \( \phi = 1 - \frac{\theta f'(\theta)}{f(\theta)} \): Mortensen-Hosios condition holds
Extensions
household chooses \( \{c(s^t), u(s^t), n(s^{t+1})\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \gamma_n n(s^t) - \gamma_u u(s^t) \right)
\]

subject to \( n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))u(s^t) \) and budget constraint

express household problem recursively:

\[
V(s^t, a, n) = \max_{\{a(s^{t+1})\}, u \in [0,1-n]} \left( \log c - \gamma_n n - \gamma_u u \\
+ \beta \sum_{s^{t+1}|s^t} \frac{\Pi(s^{t+1})}{\Pi(s^t)} V(s^{t+1}, a(s^{t+1}), n') \right)
\]

where \( c = a + (1 - \tau)w(s^t)n + T(s^t) - \sum_{s^{t+1}|s^t} q_t(s^{t+1})a(s^{t+1}) \)

and \( n' = (1 - x)n + f(\theta(s^t))u \)
Euler equation, marginal value of employment unchanged

First order condition for $u$:

$$
\gamma_u = \beta f(\theta(s^t)) \sum_{s^{t+1}|s^t} \frac{\Pi(s^{t+1})}{\Pi(s^t)} \left( \frac{(1 - \tau)w(s^{t+1})}{c(s^{t+1})} - \gamma_n + \frac{\gamma_u(1 - x)}{f(\theta(s^{t+1}))} \right)
$$

Firm’s problem is unchanged

Wage is unchanged

Constant equilibrium employment, unemployment, and labor wedge
Variable Hours

- household chooses \( \{c(s^t)\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \frac{\gamma \varepsilon}{1 + \varepsilon} n(s^t) h(s^t)^{1+\varepsilon} \right)
\]

subject to the evolution of \( n \) and a budget constraint

- firm chooses \( \{\nu(s^t)\} \) to maximize

\[
J(s^0, n_0) = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) n(s^t) h(s^t) (z(s^t)(1 - \nu(s^t)) - w(s^t))
\]

s.t. \( n(s^{t+1}) = n(s^t)(1 - x + h(s^t)\nu(s^t)\mu(\theta(s^t))) \)
Variable Hours: Household Problem

- marginal value of employed worker:

\[
V_n(s^t, a(s^t), n(s^t)) = \frac{(1 - \tau)w(s^t)h(s^t)}{c(s^t)} - \frac{\gamma \varepsilon}{1 + \varepsilon} h(s^t)^{1+\varepsilon} \\
+ \beta(1 - x - f(\theta(s^t))) \sum_{s^{t+1}|s^t} \frac{\Pi(s^{t+1})}{\Pi(s^t)} V_n(s^{t+1}, a(s^{t+1}), n(s^{t+1}))
\]

- value of worker paid \( w \) and working \( h \) instead of unemployed:

\[
\tilde{V}_n(s^t, w, h) = \frac{(1 - \tau)(wh - w(s^t)h(s^t))}{c(s^t)} \\
- \frac{\gamma \varepsilon}{1 + \varepsilon} \left( h^{1+\varepsilon} - h(s^t)^{1+\varepsilon} \right) + V_n(s^t, a(s^t), n(s^t))
\]
firm indifferent between recruiting and producing:

$$z(s^t) = \mu(\theta(s^t)) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \bar{J}(s^{t+1})$$

value of job is current output plus saved recruiting minus wage

$$\bar{J}(s^t) = z(s^t) \frac{1 - x + h(s^t)\mu(\theta(s^t))}{\mu(\theta(s^t))} - w(s^t)h(s^t)$$

value of worker paid $w$ and working $h$

$$\tilde{J}_n(s^t, w, h) = (z(s^t) - w)h - (z(s^t) - w(s^t))h(s^t) + \bar{J}(s^t).$$
Variable Hours: Closing Model

\[(w(s^t), h(s^t)) = \arg \max_{w,h} \tilde{V}_n(s^t, w, h) \phi \tilde{J}_n(s^t, w, h)^{1-\phi}\]

- \(h(s^t)\) set to maximize joint surplus

\[h(s^t) = \left(\frac{(1 - \tau)z(s^t)}{\gamma c(s^t)}\right)^\varepsilon.\]

- \(w(s^t)\) set to divide the surplus

\[w(s^t) = \left(\phi \left(1 + \frac{\theta(s^t)}{h(s^t)}\right) + (1 - \phi)\frac{\varepsilon}{1 + \varepsilon}\right)z(s^t)\]

- constant equilibrium employment, hours, and labor wedge
unemployed workers get an after-tax benefit $Bw(s^t)$

- tied to current wage, not past wage

- government budget constraint: $T(s^t) = \tau w(s^t)n(s^t) - Bw(s^t)(1 - n(s^t))$

- constant equilibrium employment and labor wedge (prove it)
government spends $g(s^t)$

preferences $\sum_{t=0}^{\infty} \sum_s \beta^t \Pi(s^t)(\log c(s^t) - \gamma n(s^t) + \psi(g(s^t)))$

government budget constraint $T(s^t) + g(s^t) = \tau w(s^t)n(s^t)$

resource constraint $c(s^t) + g(s^t) = z(s^t)n(s^t)(1 - \nu(s^t))$

if $g(s^t) = \bar{g}z(s^t)$, constant employment and labor wedge (prove it)

is this reasonable?

- balanced budget requirement
- optimal if $\psi(g) \equiv \log(g)$
Capital
Firm Problem

d a firm chooses \{\nu(s^t), n(s^{t+1}), k(s^{t+1})\} to maximize

\[ \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( z(s^t) k(s^t)^\alpha (n(s^t)(1 - \nu(s^t)))^{1-\alpha} \right. \]

\[ \left. + (1 - \delta) k(s^t) - k(s^{t+1}) - w(s^t)n(s^t) \right), \]

where firm growth satisfies

\[ n(s^{t+1}) = n(s^t)(1 - x + \nu(s^t)\mu(\theta(s^t))) \],

taking \( k_0 = k(s^0), n_0 = n(s^0) \), and \{q_0(s^t), w(s^t), \theta(s^t)\} as given

call the value of the firm \( J(s^0, n_0, k_0) \)

\( \triangleright \) homogeneous of degree 1 in \((n_0, k_0)\)
express the firm’s problem recursively:

\[
J(s^t, n, k) = \max_{\nu, k'} \left( z(s^t) k^\alpha (n(1 - \nu))^{1-\alpha} + (1 - \delta) k - k' - n\omega(s^t) \right. \\
+ \left. \sum_{s^{t+1}|s^t} q_t(s^{t+1}) J(s^{t+1}, n(\nu \mu(\theta(s^t)) + 1 - x), k') \right)
\]
Firm’s Problem: Main Results

- marginal value of worker is her and recruiters’ output minus wage:

\[ J_n(s^t, n(s^t), k(s^t)) = (1 - \alpha)z(s^t) \left( \frac{k(s^t)}{n(s^t)(1 - \nu(s^t))} \right)^{\alpha} \left( 1 + \frac{1 - x}{\mu(\theta(s^t))} \right) - w(s^t) \]

- firms are indifferent between production and recruiting:

\[ (1 - \alpha)z(s^t) \left( \frac{k(s^t)}{n(s^t)(1 - \nu(s^t))} \right)^{\alpha} = \mu(\theta(s^t)) \sum_{s^{t+1}|s^t} q_t(s^{t+1}) J_n(s^{t+1}, n(s^{t+1}), k(s^{t+1})) \]

- firms are indifferent about purchasing capital:

\[ 1 = \sum_{s^{t+1}|s^t} q_t(s^{t+1}) \left( \alpha z(s^{t+1}) \left( \frac{k(s^{t+1})}{n(s^{t+1})(1 - \nu(s^{t+1}))} \right)^{\alpha-1} + 1 - \delta \right) \]
Worker’s Problem

- intertemporal Euler equation

\[
q_t(s^{t+1}) = \beta \frac{\Pi(s^{t+1})c(s^t)}{\Pi(s^t)c(s^{t+1})}
\]

- marginal value of an employed worker (in utils) is

\[
V_n(s^t, a(s^t), n(s^t)) = \frac{(1 - \tau)w(s^t)}{c(s^t)} - \gamma
\]

\[
+ \beta(1 - x - f(\theta(s^t))) \sum_{{s^{t+1}|s^t}} \frac{\Pi(s^{t+1})}{\Pi(s^t)} V_n(s^{t+1}, a(s^{t+1}), n(s^{t+1}))
\]
Nash bargaining solution implies

\[(1 - \tau)w(s^t) = \phi \frac{(1 - \tau)(1 - \alpha)z(s^t)k(s^t)^{\alpha}}{(n(s^t)(1 - \nu(s^t)))^{\alpha}}(1 + \theta(s^t)) + (1 - \phi)\gamma c(s^t)\]

after-tax wage is weighted average of

- after-tax marginal product of labor produced by
  1. the worker and
  2. the \(\theta(s^t)\) other workers freed from recruiting
- marginal rate of substitution between consumption and leisure
goods market clearing:

\[ c(s^t) + k(s^{t+1}) = z(s^t)k(s^t)^\alpha n(s^t)(1 - \nu(s^t))^{1-\alpha} + (1 - \delta)k(s^t) \]

law of motion for employment:

\[ n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t)) \]

definition of \( \theta \):

\[ \theta(s^t) = \frac{\nu(s^t)n(s^t)}{1 - n(s^t)} \]
Balanced Growth

- Suppose \( \log z(s^{t+1}) = \log z(s^t) + \bar{s} \)

- Consumption, capital, and wage grow at rate \( \bar{s}/(1 - \alpha) \)
  
  \[ c(s^t) = \bar{c}z(s^t)^{\frac{1}{1-\alpha}} \]
  \[ k(s^t) = \bar{k}z(s^t)^{\frac{1}{1-\alpha}} \]
  \[ w(s^t) = \bar{w}z(s^t)^{\frac{1}{1-\alpha}} \]

- Recruiters/unemployed and employment are constant:
  
  \[ \theta(s^t) = \bar{\theta} \]
  \[ n(s^t) = \bar{n} \]
Deterministic Trend

- Suppose \( \log z(s^t) = \bar{st} + s_t \), where \( s_t \) is persistent.
  - First order Markov process, transition matrix \( \pi(s_{t+1} | s_t) \).

- Define relative consumption, capital, and wage:
  - \( c(s^t) = \tilde{c}(s^t)e^{\bar{st}(1 - \alpha)} \)
  - \( k(s^t) = \tilde{k}(s^t)e^{\bar{st}(1 - \alpha)} \)
  - \( w(s^t) = \tilde{w}(s^t)e^{\bar{st}(1 - \alpha)} \)

  These three variables are stationary.

- So are recruiters/unemployed \( \theta(s^t) \) and employment \( n(s^t) \).
Stationary Version of Key Equations

- firms indifferent about purchasing capital

\[ e^{\frac{s}{1-\alpha}} = \beta \sum_{s_{t+1}|s_t} \pi(s_{t+1}|s_t) \frac{\tilde{c}(s_t)}{\tilde{c}(s_{t+1})} \left( \alpha e^{s_{t+1}} \left( \frac{\tilde{k}(s_{t+1})}{n(s_{t+1})(1-\nu(s_{t+1}))} \right)^{\alpha-1} + 1-\delta \right) \]

- interior condition for recruiting, wage equation:

\[ (1-\alpha)e^{s_t} \left( \frac{\tilde{k}(s_t)}{n(s_t)(1-\nu(s_t))} \right)^\alpha = \beta \mu(\theta(s_t)) \sum_{s_{t+1}|s_t} \pi(s_{t+1}|s_t) \frac{\tilde{c}(s_t)}{\tilde{c}(s_{t+1})} \left( \frac{(1-\phi)\gamma\tilde{c}(s_{t+1})}{1-\tau} - \right) + (1-\alpha)e^{s_{t+1}} \left( \frac{\tilde{k}(s_{t+1})}{n(s_{t+1})(1-\nu(s_{t+1}))} \right)^\alpha \left( \frac{1-x}{\mu(\theta(s_{t+1}))} + 1-\phi - \phi\theta(s_{t+1}) \right) \]
Stationary Version of Key Equations

- **resource constraint:**

\[
\tilde{k}(s^{t+1}) e^{\frac{s}{1-\alpha}} = e^{s_t} \tilde{k}(s^t)^\alpha (n(s^t)(1 - \nu(s^t)))^{1-\alpha} + (1 - \delta)\tilde{k}(s^t) - \tilde{c}(s^t)
\]

- **unemployment rate:**

\[
n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t))
\]

- **relationship between \(\nu\) and \(\theta\):**

\[
\theta(s^t) = \frac{\nu(s^t)n(s^t)}{1 - n(s^t)}
\]

use this to eliminate \(\nu\) from previous equations

- **log linearize around steady state** \(s = 0, \ n = \bar{n}, \) and \(\tilde{k} = \bar{k}\)
Log-Linearization

posit

\[ \Theta(s, n, \tilde{k}) \equiv \log \bar{\theta} + \theta_s s + \theta_n (\log n - \log \bar{n}) + \theta_k (\log \tilde{k} - \log \bar{k}) \]
\[ C'(s, n, \tilde{k}) \equiv \log \bar{c} + c_s s + c_n (\log n - \log \bar{n}) + c_k (\log \tilde{k} - \log \bar{k}) \]

▷ eliminate \( \theta(s^t) \) and \( c(s^t) \) using these approximations
▷ eliminate \( n(s^{t+1}) \) and \( \tilde{k}(s^{t+1}) \) using their laws of motion

reduces to an equation of the form \( T(s, n, \tilde{k}) = 0 \)

impose \( T(0, \bar{n}, \bar{k}) = T_s(0, \bar{n}, \bar{k}) = T_n(0, \bar{n}, \bar{k}) = T_k(0, \bar{n}, \bar{k}) = 0 \)

solve for the unknown constants
Calibration

- **discount factor** $\beta = 0.996$
- **employment exit probability** $x = 0.034$
- **average productivity growth** $\bar{s} = 0.0012$
- **productivity shocks** $s_{t+1} = 0.98s_t + 0.005\nu_{t+1}$
- **capital share** $\alpha = 0.033$
- **depreciation rate** $\delta = 0.0028$: $k/y = 3.2$ in stochastic steady state
- **tax rate** $\tau = 0.4$
- **bargaining power** $\phi = 0.5$
- **matching function** $f(\theta) = 2.32\theta^{1/2}$
- **disutility** $\gamma = 0.471$: 5% unemployment rate in stochastic steady state
Log-Linearized System

- policy functions

\[ \log \theta = \log 0.078 + 7.387 s - 0.480 \log(n/0.95) - 2.779 \log(\tilde{k}/218.2), \]
\[ \log \tilde{c} = \log 4.696 + 0.250 s + 0.014 \log(n/0.95) + 0.603 \log(\tilde{k}/218.2) \]

- state equations

\[ \log n_{+1} = \log 0.95 + 0.126 s + 0.312 \log(n/0.95) - 0.047 \log(\tilde{k}/218.2), \]
\[ \log \tilde{k}_{+1} = \log 218.2 + 0.020 s + 0.019 \log(n/0.95) + 0.991 \log(\tilde{k}/218.2) \]

- write state as \( m \equiv \{s, \log(n/\bar{n}), \log(\tilde{k}/\bar{k})\} \), so \( m_+ = Am + Dv_+ \)

- local stability iff eigenvalues of \( A \) lie in unit circle

\[ \text{here they are } 0.99, 0.98, \text{ and } 0.31 \]
using $m_+ = Am + Dv_+$, variance-covariance matrix is

$$\Sigma = \mathbb{E}(m_+m_+') = \mathbb{E}((Am + Dv_+)(m'A' + v'D')) = A\Sigma A' + DD'$$

here

$$\Sigma = \begin{pmatrix} 25.253 & 3.175 & 19.505 \\ 3.175 & 0.560 & 0.469 \\ 19.505 & 0.469 & 46.550 \end{pmatrix} \varsigma^2.$$

standard deviation of employment is $\varsigma \sqrt{0.560} = 0.004$

construct other variables $\tilde{m} = \tilde{A}m$

for example, linearize labor wedge:

$$\hat{\tau}(s^t) = 1 - \frac{\hat{\gamma}}{1 - \alpha} \left( \frac{\tilde{c}(s^t)}{e^{s_t\tilde{k}(s^t)\alpha}(n(s^t) - \theta(s^t)(1 - n(s^t)))^{1-\alpha}} \right) n(s^t)$$
### detrended data ($\varepsilon = \infty$)

<table>
<thead>
<tr>
<th></th>
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<tbody>
<tr>
<td>$c/y$</td>
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<td>−0.131</td>
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### model

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Growth Rates

compute comovements of $i$-period growth rates

\[ m_{+i} = A^i m + \sum_{j=0}^{i-1} A^j Dv_{+(i-j+1)} \]

▷ proof by induction

\[ \mathbb{E}\left((m_{+i} - m)(m_{+i} - m)\right)' = (A^i - I)\Sigma(A^i - I)' + \sum_{j=0}^{i-1} A^j DD'(A^j)' \]

▷ proof by induction

\[ \tilde{m} = \tilde{A} m \Rightarrow \mathbb{E}\left((\tilde{m}_{+i} - \tilde{m})(\tilde{m}_{+i} - \tilde{m})\right)' = \tilde{A}\mathbb{E}\left((m_{+i} - m)(m_{+i} - m)\right)' \tilde{A}' \]
## Labor Wedge

### annual growth rate data ($\varepsilon = \infty$)

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Impulse Response

output $\tilde{y}$

consumption $\tilde{c}$

employment $n$

capital $\tilde{k}$

labor wedge $\hat{\tau}$

productivity $s$

$\bar{\mu} = 2.32 \text{ v.s. } \bar{\mu} = 1$
Stochastic Trend

- suppose \( \log z(s_{t+1}) = \log z(s_t) + s_{t+1} \), where \( s_{t+1} \) is Markov

- define relative consumption, capital, and wage:
  
  - \( c(s_t) = \tilde{c}(s_t) z(s_t) \frac{1}{1-\alpha} \)
  - \( k(s_t) = \tilde{k}(s_t) z(s_t) \frac{1}{1-\alpha} \)
  - \( w(s_t) = \tilde{w}(s_t) z(s_t) \frac{1}{1-\alpha} \)

  these three variables are stationary

- so are recruiters/unemployed \( \theta(s_t) \) and employment \( n(s_t) \)
Calibration

- change stochastic process

- \[ s_{t+1} = 0.0012 + 0.4(s_t - 0.0012) + 0.00325v_{t+1} \]

- other calibration targets are unchanged
Log-Linearized System

- **Policy functions**
  \[
  \log(\theta/0.078) = 1.548(s - 0.0012) - 0.480 \log(n/0.95) - 2.779 \log(\tilde{k}/218.2),
  \]
  \[
  \log(\tilde{c}/4.696) = 0.381(s - 0.0012) + 0.014 \log(n/0.95) + 0.603 \log(\tilde{k}/218.2)
  \]

- **State equations**
  \[
  \log(n_{+1}/0.95) = 0.026(s - 0.0012) + 0.312 \log(n/0.95) - 0.047 \log(\tilde{k}/218.2),
  \]
  \[
  \log(\tilde{k}_{+1}/218.2) = -0.605(s - 0.0012) + 0.019 \log(n/0.95) + 0.991 \log(\tilde{k}/218.2)
  \]

- **Eigenvalues** 0.99, 0.4, and 0.31

- **Only response to** \( s \) **changes**
### Labor Wedge

**detrended data** ($\varepsilon = \infty$)

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**model**

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<tr>
<td>n</td>
<td>0.002</td>
<td>0.975</td>
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## Labor Wedge

### Annual growth rate data ($\varepsilon = \infty$)

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### Model


Impulse Response

output $y$

consumption $c$

employment $n$

capital $k$

labor wedge $\hat{\tau}$

productivity growth $\bar{s}$

$\bar{\mu} = 2.32$ v.s. $\bar{\mu} = 1$
Other Shocks

- combination of stochastic/deterministic trend
- shocks to the employment exit probability $x$
- investment-specific technological change
- government spending shocks
- preference shocks and wage markup shocks are off the table
two-shock model, deterministic trend

\[
\log x(s^t) = \log \bar{x} + s_{x,t} \quad \text{where} \quad s_{x,t+1} = \rho_x s_{x,t} + \varsigma_x u_{x,t+1}
\]

\[
\log z(s^t) = \bar{st} + s_{z,t} \quad \text{where} \quad s_{z,t+1} = \rho_z s_{z,t} + \varsigma_z u_{z,t+1} - \varsigma_{zx} u_{x,t+1}
\]

calibration:

\[
\begin{align*}
\rho_x &= 0.83 \\
\varsigma_x &= 0.034 \\
\rho_z &= 0.98 \\
\varsigma_z &= 0.0037 \\
\varsigma_{zx} &= 0.0034: \text{negative correlation between } f(\theta(s^t)) \text{ and } x(s^t)
\end{align*}
\]

interpret as aggregate and reallocation shock
### Labor Wedge

- **detrended data** \((\varepsilon = \infty)\)

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- **model**

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<td></td>
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<td>0.021</td>
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## Labor Wedge

### annual growth rate data ($\varepsilon = \infty$)

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Summary

- with either a deterministic or stochastic trend
  - employment is not very volatile
    - in absolute terms
    - relative to the consumption-output ratio
  - measured labor wedge is positively correlated with employment

- “reallocation shocks” do not change the conclusion
Rigid Wage Model
Wage Indeterminacy

- firm is willing to pay some \( w > w(s^t) \), if \( \tilde{J}_n(s^t, w) \geq 0 \)

\[
(1 - \alpha)z(s^t) \left( \frac{k(s^t)}{n(s^t) - \theta(s^t)(1 - n(s^t))} \right)^\alpha \left( 1 + \frac{1 - x}{\mu(\theta(s^t))} \right) \geq w.
\]

- worker is willing to work at some \( w < w(s^t) \), if \( \tilde{V}_n(s^t, w) \geq 0 \)

\[
w \geq \gamma c(s^t) \left( 1 - \tau \right) - \frac{\phi}{1 - \phi} (1 - \alpha)z(s^t) \left( \frac{k(s^t)}{n(s^t) - \theta(s^t)(1 - n(s^t))} \right)^\alpha \left( 1 - x - f(\theta(s^t)) \right) \left( \frac{1 - x - f(\theta(s^t))}{\mu(\theta(s^t))} \right).
\]

- this indeterminacy is irrelevant in existing matches

- it is critical for firms’ incentive to recruit

- the wage bands may be quite large

- balanced growth path: \( 0.88w(s^t) \leq w \leq 1.12w(s^t) \)
Firm Problem

firm chooses \( \{\nu(s^t), n(s^{t+1}), k(s^{t+1})\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) \left( z(s^t) k(s^t)^\alpha (n(s^t)(1 - \nu(s^t)))^{1-\alpha} \right.
\]

\[
+ (1 - \delta) k(s^t) - k(s^{t+1}) - w(s^t)n(s^t),
\]

where firm growth satisfies

\[
n(s^{t+1}) = n(s^t)(1 - x + \nu(s^t)\mu(\theta(s^t))),
\]

taking \( k_0 = k(s^0), n_0 = n(s^0) \), and \( \{q_0(s^t), w(s^t), \theta(s^t)\} \) as given
Household Problem

The household chooses \( \{c(s^t)\} \) to maximize

\[
\sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pi(s^t) \left( \log c(s^t) - \gamma n(s^t) \right)
\]

subject to

\[
a_0 = \sum_{t=0}^{\infty} \sum_{s^t} q_0(s^t) (c(s^t) - (1 - \tau)w(s^t)n(s^t) - T(s^t))
\]

and

\[
n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t)),
\]

taking \( a_0, n_0 = n(s^0) \), and \( \{q_0(s^t), w(s^t), \theta(s^t), \tau, T(s^t)\} \) as given.
government budget constraint: \( T(s^t) = \tau w(s^t)n(s^t) \)

goods market clearing:

\[
k(s^{t+1}) = z(s^t)k(s^t)^\alpha \left(n(s^t)(1 - \nu(s^t))\right)^{1-\alpha} + (1 - \delta)k(s^t) - c(s^t)
\]

law of motion for employment:

\[
n(s^{t+1}) = (1 - x)n(s^t) + f(\theta(s^t))(1 - n(s^t))
\]

definition of \( \theta \):

\[
\theta(s^t) = \frac{\nu(s^t)n(s^t)}{1 - n(s^t)}
\]
Backward-Looking Wages

- Target wage:

\[ w^*(s^t) = \arg \max_w \tilde{V}_n(s^t, w)^\phi \tilde{J}_n(s^t, w)^{1-\phi} \]

- \( \tilde{J}_n(s^t, w) \): value of paying a worker \( w \) in \( s^t \), \( w(s^{t'}) \) thereafter, compared to not employing the worker

- \( \tilde{V}_n(s^t, w) \): value of having a worker paid \( w \) in \( s^t \), \( w(s^{t'}) \) thereafter, compared to having the worker unemployed

- Actual wage:

\[ w(s^t) = rw(s^{t-1})e^{\frac{\bar{s}}{1-\alpha}} + (1 - r)w^*(s^t) \]

- \( r \in [0, 1] \) indicates extent of wage rigidity

- \( \bar{s} \) is average productivity growth
Wage Behavior

forward-looking equation for the target wage:

\[ w^*(s^t) = \phi(1 - \alpha)z(s^t) \left( \frac{k(s^t)}{n(s^t)(1 - \nu(s^t))} \right)^\alpha (1 + \theta(s^t)) + (1 - \phi) \frac{\gamma_c(s^t)}{1 - \tau} + (1 - x - f(\theta(s^t))) \sum_{s^{t+1}|s^t} q_t(s^{t+1})(w^*(s^{t+1}) - w(s^{t+1})) \]

backward-looking equation for the actual wage:

\[ w(s^t) = rw(s^{t-1})e^{\bar{s}(1-\alpha)} + (1 - r)w^*(s^t) \]

saddle-path dynamics, downward-sloping saddle path
Balanced Growth

- Suppose \( \log z(s^{t+1}) = \log z(s^t) + \bar{s} \)

- Consumption, capital, and actual and target wages grow at rate \( \frac{\bar{s}}{1 - \alpha} \)

  - \( c(s^t) = \bar{c}z(s^t)^{\frac{1}{1-\alpha}} \)
  - \( k(s^t) = \bar{k}z(s^t)^{\frac{1}{1-\alpha}} \)
  - \( w(s^t) = \bar{w}z(s^t)^{\frac{1}{1-\alpha}} \)
  - \( w^*(s^t) = \bar{w}^*z(s^t)^{\frac{1}{1-\alpha}} \)

- Recruiters/unemployed and employment are constant:

  - \( \theta(s^t) = \bar{\theta} \)
  - \( n(s^t) = \bar{n} \)

- This implies \( w(s^t) = w^*(s^t) \), so no distortions from rigidity
Deterministic Trend
Calibration

- new parameter is wage rigidity \( r = 0.95 \)
- other parameters are unchanged
Policy Functions

linearize system around steady state

\[
\log \theta = \log 0.078 + 40.825s - 0.630 \log(n/0.95) \\
\quad + 10.441 \log(\tilde{k}/218.2) - 38.184 \log(\tilde{w}_{-1}/4.017),
\]

\[
\log \tilde{c} = \log 4.696 + 0.259s + 0.014 \log(n/0.95) \\
\quad + 0.607 \log(\tilde{k}/218.2) - 0.023 \log(\tilde{w}_{-1}/4.017),
\]

\[
\log \tilde{w}^* = \log 4.017 + 2.974s - 0.215 \log(n/0.95) \\
\quad + 1.146 \log(\tilde{k}/218.2) - 2.321 \log(\tilde{w}_{-1}/4.017).
\]

compare to flexible wage model \((r = 0)\)

\[
\log \theta = \log 0.078 + 7.387s - 0.480 \log(n/0.95) - 2.779 \log(\tilde{k}/218.2),
\]

\[
\log \tilde{c} = \log 4.696 + 0.250s + 0.014 \log(n/0.95) + 0.603 \log(\tilde{k}/218.2)
\]
State Equations

- rigid wages $(r = 0.95)$

\[
\log n_{+1} = \log 0.95 + 0.694s + 0.309 \log(n/0.95) \\
+ 0.178 \log(\tilde{k}/218.2) - 0.649 \log(\tilde{w}_{-1}/4.017),
\]
\[
\log \tilde{k}_{+1} = \log 218.2 + 0.018s + 0.019 \log(n/0.95) \\
+ 0.990 \log(\tilde{k}/218.2) + 0.003 \log(\tilde{w}_{-1}/4.017),
\]
\[
\log \tilde{w} = \log 4.107 + 0.149s - 0.011 \log(n/0.95) \\
+ 0.057 \log(\tilde{k}/218.2) + 0.834 \log(\tilde{w}_{-1}/4.017).
\]

- compare to flexible wage model $(r = 0)$

\[
\log n_{+1} = \log 0.95 + 0.126s + 0.312 \log(n/0.95) - 0.047 \log(\tilde{k}/218.2),
\]
\[
\log \tilde{k}_{+1} = \log 218.2 + 0.020s + 0.019 \log(n/0.95) + 0.991 \log(\tilde{k}/218.2)
\]

- eigenvalues $k : 0.99, s : 0.98, \tilde{w} : 0.85,$ and $n : 0.29.$
Impulse Response

output $\tilde{y}$

consumption $\tilde{c}$

employment $n$

capital $\tilde{k}$

labor wedge $\hat{\tau}$

productivity $s$

$r = 0.95$ v.s. $r = 0$
### Labor Wedge

#### detrended data ($\varepsilon = \infty$)

<table>
<thead>
<tr>
<th></th>
<th>s.d.</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>c/y</td>
<td>0.010</td>
<td>-0.131</td>
</tr>
<tr>
<td>n</td>
<td>0.010</td>
<td>-0.633</td>
</tr>
</tbody>
</table>

#### model

<table>
<thead>
<tr>
<th></th>
<th>s.d.</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>c/y</td>
<td>0.020</td>
<td>-0.938</td>
</tr>
<tr>
<td>n</td>
<td>0.010</td>
<td>0.690</td>
</tr>
</tbody>
</table>
## Labor Wedge

### annual growth rate data ($\varepsilon = \infty$)

<table>
<thead>
<tr>
<th></th>
<th>s.d.</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c/y$</td>
<td>0.015</td>
<td>-0.260</td>
</tr>
<tr>
<td>$n$</td>
<td>0.014</td>
<td>-0.597</td>
</tr>
</tbody>
</table>

### model

<table>
<thead>
<tr>
<th></th>
<th>s.d.</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c/y$</td>
<td>0.019</td>
<td>-0.881</td>
</tr>
<tr>
<td>$n$</td>
<td>0.012</td>
<td>0.670</td>
</tr>
</tbody>
</table>
Sensitivity to Rigidity

\[ \text{corr}(n, \hat{\tau}) \]

\[ \text{corr}(c/y, \hat{\tau}) \]

wage rigidity \( r \)
Sensitivity to Rigidity $r$

\begin{align*}
\text{correction in annual growth rates} & \\
\text{wage rigidity } r & \\
\text{corr}(n, \hat{r}) & \\
\text{corr}(c/y, \hat{r}) &
\end{align*}
Stochastic Trend
Calibration

- new parameter is wage rigidity \( r = 0.95 \)
- other parameters are unchanged
Policy Functions

- linearize system around steady state

$$\log(\theta/0.078) = 81.417(s - 0.0012) - 0.630 \log(n/0.95)$$
$$+ 10.441 \log(\tilde{k}/218.2) - 38.184 \log(\tilde{w}_1/4.017),$$
$$\log(\tilde{c}/4.696) = 0.436(s - 0.0012) + 0.014 \log(n/0.95)$$
$$+ 0.607 \log(\tilde{k}/218.2) - 0.023 \log(\tilde{w}_1/4.017),$$
$$\log(\tilde{w}^*/4.017) = 5.096(s - 0.0012) - 0.215 \log(n/0.95)$$
$$+ 1.146 \log(\tilde{k}/218.2) - 2.321 \log(\tilde{w}_1/4.017).$$

- compare to flexible wage model ($r = 0$)

$$\log(\theta/0.078) = 1.548(s - 0.0012) - 0.480 \log(n/0.95) - 2.779 \log(\tilde{k}/218.2),$$
$$\log(\tilde{c}/4.696) = 0.381(s - 0.0012) + 0.014 \log(n/0.95) + 0.603 \log(\tilde{k}/218.2)$$
State Equations

 sö rigid wages \((r = 0.95)\)

\[
\log(n_{+1}/0.95) = 1.384(s - 0.0012) + 0.309 \log(n/0.95) \\
+ 0.178 \log(\tilde{k}/218.2) - 0.649 \log(\tilde{w}_{-1}/4.017),
\]

\[
\log(\tilde{k}_{+1}/218.2) = -0.612(s - 0.0012) + 0.019 \log(n/0.95) \\
+ 0.990 \log(\tilde{k}/218.2) + 0.003 \log(\tilde{w}_{-1}/4.017),
\]

\[
\log(\tilde{w}/4.017) = -1.163(s - 0.0012) - 0.011 \log(n/0.95) \\
+ 0.057 \log(\tilde{k}/218.2) + 0.834 \log(\tilde{w}_{-1}/4.017).
\]

 sö compare to flexible wage model \((r = 0)\)

\[
\log(n_{+1}/0.95) = 0.026(s - 0.0012) + 0.312 \log(n/0.95) - 0.047 \log(\tilde{k}/218.2),
\]

\[
\log(\tilde{k}_{+1}/218.2) = -0.605(s - 0.0012) + 0.019 \log(n/0.95) + 0.991 \log(\tilde{k}/218.2)
\]

 sö eigenvalues \(k: 0.99, s: 0.98, \tilde{w}: 0.85, \text{ and } n: 0.29.\)
Impulse Response

output $y$

consumption $c$

employment $n$

capital $k$

labor wedge $\hat{\tau}$

productivity growth $s$

$r = 0.95$ v.s. $r = 0$
### Labor Wedge

#### detrended data \( (\varepsilon = \infty) \)

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<tr>
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<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s.d.</td>
<td>0.014</td>
</tr>
<tr>
<td>( c/y )</td>
<td>0.010</td>
<td>-0.131</td>
</tr>
<tr>
<td>( n )</td>
<td>0.010</td>
<td>-0.633</td>
</tr>
</tbody>
</table>

#### model

<table>
<thead>
<tr>
<th></th>
<th>s.d.</th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>s.d.</td>
<td>0.008</td>
</tr>
<tr>
<td>( c/y )</td>
<td>0.011</td>
<td>0.040</td>
</tr>
<tr>
<td>( n )</td>
<td>0.013</td>
<td>-0.449</td>
</tr>
</tbody>
</table>
### Labor Wedge

**annual growth rate data** \( (\varepsilon = \infty) \)

<table>
<thead>
<tr>
<th></th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.d.</td>
<td>0.023</td>
</tr>
<tr>
<td>( c/y )</td>
<td>0.015</td>
</tr>
<tr>
<td>( n )</td>
<td>0.014</td>
</tr>
</tbody>
</table>

**model**

<table>
<thead>
<tr>
<th></th>
<th>( \tau )</th>
</tr>
</thead>
<tbody>
<tr>
<td>s.d.</td>
<td>0.009</td>
</tr>
<tr>
<td>( c/y )</td>
<td>0.012</td>
</tr>
<tr>
<td>( n )</td>
<td>0.016</td>
</tr>
</tbody>
</table>
Sensitivity to Rigidity $r$

The diagram illustrates the correlation in levels between wage rigidity and labor market variables. The red line represents $\text{corr}(n, \hat{r})$, and the blue line represents $\text{corr}(c/y, \hat{r})$. As wage rigidity increases, the correlation with labor market variables changes, reflecting the sensitivity of these correlations to wage rigidity.
Sensitivity to Rigidity $r$

The diagram illustrates the correlation in annual growth rates as a function of wage rigidity $r$.

- The blue line represents $\text{corr}(c/y, \hat{r})$.
- The red line represents $\text{corr}(n, \hat{r})$.

As wage rigidity increases from 0.90 to 1.00, the correlation $\text{corr}(c/y, \hat{r})$ increases, while $\text{corr}(n, \hat{r})$ decreases.
Looking Ahead
Speculation

- testable predictions (hours margin)
- finite sample properties of the model
- policy analysis and exploration of other shocks (gov’t spending)
- microfoundations of bargaining (Hall-Milgrom, Gertler-Trigari)
- micro-measurement of wage rigidity (Pissarides, Haefke et al)
- alternatives to search frictions (mismatch)
- other markets (housing, financial markets)