Discussion Paper No. 591

# COMPETITIVE EQUILIBRIA: CONVERGENCE, CYCLES OR CHAOS

Anjan Mukherji

July 2003

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

# Competitive Equilibria:

# Convergence, Cycles or Chaos

**Anjan Mukherji**[1]

**Centre for Economic Studies and Planning**

**Jawaharlal Nehru University**

**New Delhi**

**and**

**ISER, Osaka University**

**PRELIMINARY**

**This Version: July 11, 2003**

[1]Comments welcome. Email: anjan_mukherji@yahoo.com

# Contents

# Preface

The title of this monograph could have been "What does one do if Anything Goes"; a friend suggested that I should use it as a sub-title instead of the more prosaic one that I have used.

There are two basic "Anything Goes " type of results which influence the role of dynamics in economic theory. The first is the Sonnenschein-Debreu-Mantel set of results which indicate that excess demand functions which satisfy only Walras law and the Homogeneity of degree zero postulate do not imply too many restrictions since almost any set of functions could be taken to excess demand functions; the second set of results are due to Boldrin and Montruchhio which show that any dynamical system (continuous or discrete) can describe the time evolution of the optimal paths of an infinite horizon discounted concave maximization problem subject to stationarity constraints. Thus loosely speaking, any dynamical system can be rationalized as occurring in the context of some maximization problem. These two sets of results have had a profound impact on economic theory since they seem to indicate that economic theory is unable to be definitive.

A third type of "Anything Goes" result arise from the theory of dynamical systems itself, which economists do not perhaps refer to as much, is due to Smale. To describe the content of this result some notation becomes necessary. Let $X$ be any $C^1$ vector field in the unit simplex of dimension $n-1$, $\Delta^{n-1}$; then there exists a $C^1$ vector field $F = (F_i)$ in $\Re^n$ satisfying $F_i = x_i M_i(x)$, $M_{ij}(x) < 0, j \neq i$ such that $F|\Delta^{n-1} = X$ and $\Delta^{n-1}$ is an attractor. Thus for any $n > 2$, it would appear that anything goes on account of dynamical systems alone.

Thus not only the theory of stability of competitive equilibrium, but the study of growth processes and even the study of dynamical systems and their long term behavior need to be handled and understood carefully. General results would be difficult to obtain; any result will require special and some times, what may appear, to be ad-hoc conditions. As the three sets of results mentioned above seems to indicate, we do not have any choice in the matter and we should thus address ourselves to the nature of conditions which might provide us with results of some interest. This monograph is directed towards this objective. Another aspect that we shall be concerned with is the robustness of results obtained. In economic theory we sometimes know signs of some terms; their magnitude is some thing beyond our grasp. Accordingly we need to worry about our results if they are dependant upon magnitudes of parameters. Or if this is some thing which we cannot overcome, we should look for some ways of obtaining such information. There should be thus better cooperation between researchers in economic theory and applied economics. Finally, it should not be surprising given Smale's results that we have devoted special attention to models where the dynamics involve motion on the plane. It turns out however that even here a rich variety of situations may be exhibited. We hope that this sets out the reason why such a study is being attempted.

Currently, the study has been divided up into five chapters. Chapters 1 and 2 contain the basic tools of analysis; the first refers to continuous time processes whereas the second refers to discrete time processes. These chapters contain a summary of definitions and results and some applications of these results. The chapters are by no means a comprehensive account of non-linear dynamic systems; they are there to keep the study self-contained. Chapters 3 and 4 contain an analysis of Stability of Competitive Equilibria; the first refers to Walrasian Tatonnement processes while the second to Non-Walrasian or Non-tatonnement processes. We have tried to make the analysis in these chapters as exhaustive as possible, so that readers may understand and appreciate the different aspects of this problem. In short, we examine the workings of the so-called 'Invisible Hand' and obtain conditions when the Invisible Hand is also successful in carrying out the tasks that we usually assume that it is capable of. Chapter 5, currently the last chapter, is devoted to processes of economic growth in one sector models. The aspect studied in some detail is the approach to the question of unemployment cycles. This study will be complete only when discussions on optimal growth processes and multi-sector growth models are included. The references to the bibliography needs to be added to; some items may still be missing and details for some items may be absent. I apologize for this and the other shortcomings; I hope that I will be able to attend to these matters soon.

The present version was completed during a sabbatical (August 2002- July 2003) spent at the Institute of Social and Economic Research (ISER) of Osaka University; the scholarship and the facilities provided are very gratefully acknowledged. Clearly without the support received from ISER, progress would have been impossible. Comments on the material will be deeply appreciated.

# 1 Continuous time processes

## 1.1 Definitions and Propositions

Let $x \in R^n$ where $R^n$ denotes the n-dimensional Euclidean space, $F : R^n \to R^n$. Consider the system

$$\dot{x}(t) = F(x(t)) \tag{1.1}$$

where $t \in [0, \infty) = I$, $\dot{x}(t) = (\dot{x_i}(t), i = 1, 2 \cdots n)$ and

$$\dot{x_i}(t) = \frac{dx_i}{dt}.$$

The equation ( 1.1 ) is referred to as an **autonomous system** of first order differential equations.

We shall say that $x^* \in R^n$ is an **equilibrium point** of ( 1.1 ) if

$$F(x^*) = 0$$

Let $(t_1, t_2) \in I$; $\phi(t_1, t_2) \to R^n$ is said to be a **solution** of ( 1.1 ) if

$$(i)\ \phi(t) \text{ is differentiable on } (t_1, t_2)$$

and

$$(ii)\ \dot{\phi}(t) = F(\phi(t)) \text{ on } (t_1, t_2).$$

Let $x^o \in R^n$; we shall be interested in a solution to ( 1.1 beginning from $x^o$ ( the **initial point)**, i.e., a solution $\phi(t)$ such that $\phi(0) = x^o$.

To fix the implications of the above definitions, consider an autonomous differential equation on $R$ given by

$$\dot{x} = \alpha x(t), x(t) \in R, \alpha \text{ some constant} \tag{1.2}$$

Let $\phi(t) = C.e^{\alpha.t}$ where C is some constant; it is easy to check that the function $\phi(t)$ is differentiable and $\in R$ $\forall$ t; further

$$\dot{\phi}(t) = \alpha\phi(t)$$

so that $\phi(t)$ is a solution of ( 1.2 ) for any $(t_1, t_2) \in I$.

Let us say we have some $r \in R$. Note that by specifying that

$$\phi(0) = r \Rightarrow C = r;$$

consequently, the solution to ( 1.2 ) with r as initial point is given by $r.e^{\alpha t}$.

Another point which may be exhibited by this example is the **uniqueness of the solution with respect to the initial point.** Suppose that $s \in R$; then the solution to ( 1.2 ) with s as initial point is $\psi(t) = s.e^{\alpha.t}$. Suppose further that s is such that for some $\bar{t} > 0$ we have $r.e^{\alpha.\bar{t}} = s$. Then $\psi(t) = r.e^{\alpha.\bar{t}}.e^{\alpha.t} = r.e^{\alpha(t+\bar{t})} = \phi(t+\bar{t})$. Thus $\psi(t)$ coincides with $\phi(t+\bar{t})$.

## 1.2 Existence of Solution

The first problem, then, is to determine conditions under which the system of equations ( 1.1 ) has a solution with some $x^o$ as initial point. This is done through the following proposition:

**Proposition 1.1** *: Let F(.) be continuous on $R^n$ with continuous first order partial derivatives ( $\dfrac{\partial F_i}{\partial x_j}$ ) on $R^n$ and $x^o \in R^n$. Then there exists a function $\phi : [0,t) \rightarrow R^n$ for some $t$ such that $\phi(.)$ is continuous on $[0,t)$ with $\phi(0) = x^o$ such that $\phi(t)$ is a solution of the system ( 1.1 ) which is continuous and unique with respect to the intial point $x^o$.*

**Remark 1** Instead of the existence and continuity of partial derivatives $\dfrac{\partial F_i}{\partial x_j}$, one may require instead that the function $F(.)$ satisfy a **Lipschitz condition** i.e., there exists a constant $k > 0$ such that $|F(x^1) - F(x^2)| < k.|x^1 - x^2|\ \forall x^1, x^2 \in R^n$.

**Remark 2** It should be pointed out that Proposition 1.1 guarantees the existence of solution on $[0,t)$; ideally we would like the solution to exist on $[0,\infty)$; in other words, we would like to be able to **continue** the solution beyond $[0,t)$; this is possible if the function $F(.)$ satisfies some additional restrictions. A condition which permits this type of continuation is that the function $F(.)$ be **bounded** on $R^n$.

**Remark 3** An extension of Proposition 1.1 to cover cases where the function $F(.)$ may have a discontinuity at a boundary point is also available. This refers to systems of differential equations defined as below [2]

$$\dot{x}_i(t) = \begin{cases} F_i(x(t)) & \text{provided } x_i(t) > 0 \\ Max(0, F_i(x(t)) & \text{otherwise} \end{cases}$$

[2] See for instance, Champsaur et. al (1977).

## 1.3 Stability

We shall write the solution to ( 1.1 ) with $x^o$ as initial point, as $\phi(t, x^o)$ or alternatively as $x(t, x^o)$. Let $x^*$ be an equilibrium point of ( 1.1 ). We shall say that the **solution** $x(t, x^*)$ **(or the equilibrium** $x^*$**) is stable** if for every $\epsilon > 0$ there is $\delta > 0$ such that for all solutions $x(t, x^o)$ for which $|x^o - x^*| \leq \delta$, $|x(t, x^o) - x(t, x^*)| \leq \epsilon \ \forall t \geq 0$.

The solution $x(t, x^*)$ (or the equilibrium $x^*$) is said to be **locally asymptotically stable** if it is stable and there exists $\alpha > 0$ such that $|x^o - x^*| \leq \alpha \Rightarrow \lim_{t\to\infty} |x(t, x^o) - x(t, x^*)| = 0$.

The solution $x(t, x^*)$ (or the equilibrium $x^*$) is said to be **globally asymptotically stable** if it is stable and if $\lim_{t\to\infty} |x(t, x^o) - x(t, x^*)| = 0$ for any $x^o \in R^n$.

To consider the **stability of the system** ( 1.1 ) consider the following:

**Claim 1.3.1** *Let $x(t, x^o)$ be a solution to ( 1.1 ). If $\lim_{t\to\infty} x(t, x^o) = \overline{x}$ for some finite $\overline{x} \in R^n$ then $\overline{x}$ is an equilibrium of ( 1.1 ).*

Proof: Suppose this is not so; i.e., $F_k(\overline{x}) \neq 0$ for some $k$. By hypothesis, note that $F_k(x(t, x^o)) \to F_k(\overline{x})$. Consequently, given any $\epsilon > 0$, there is $T$ such that

$$F_k(\overline{x}) - \epsilon < F_k(x(t, x^o)) < F_x(\overline{x}) + \epsilon, \text{ for all } t > T$$

Let $|F_k(\overline{x})| = 2\eta > 0$; then $\epsilon$ can be so chosen that $|F_k(x(t, x^o))| > \eta$ for all $t > T$. Thus either $\dot{x}_k(t) > \eta$ or $\dot{x}_k(t) < -\eta$ for all $t > T$.

Integrating from $t$ to $t + h$, for $t > T$ and any $h > 0$, we have

$|x_k(t + h) - x_k(t)| > \eta.h$ for all $t > T$, and any $h > 0$:

which contradicts the convergence of $x_k(t)$. This establishes the claim. •

The above claim allows us to define the **system ( 1.1 ) as globally stable,** if for all $x^o$ in the domain of definition of the function $F(.)$, $x(t, x^o)$ converges.

Next, consider solutions $x(t, x^o)$ which are contained in some compact set $C \subset R^n$ for all $t > 0$. Then $\exists$ a subsequence of the solution, $x(t_s, x^o)$ such that $\lim_{s\to\infty} x(t_s, x^o) = \overline{x}$ for some $\overline{x} \in C$. $\overline{x}$ is called a **limit point of the solution** $x(t, x^o)$**.** Further, $x(t, \overline{x})$ is called a **limit path** of the system ( 1.1 ). For limit paths, we have the following[3]:

**Remark 4** $x(t, \overline{x})$

$= x(t, \lim_{s\to\infty} x(t_s, x^o))$, *( from the definition of* $\overline{x}$*)*

$= \lim_{s\to\infty} x(t, x(t_s, x^o))$, *( from the continuity of the solution with respect to the initial point)*

$= \lim_{s\to\infty} x(t + t_s, x^o)$ *(from the property of uniqueness of the solution with respect to the initial point).*

*Thus* **every point of the limit path is a limit point of the solution.**

We can now define the process ( 1.1 ) as **quasi-globally stable** if for all $x^o$ in the domain of definition of the function $F(.)$, the limit points of $x(t, x^o)$ are equilibrium points of ( 1.1 ). Consequently, we have:

**Claim 1.3.2** *If the equilibrium points of ( 1.1 ) are* **isolated** , then quasi global stability of ( 1.1 )$\Rightarrow$ global stability of ( 1.1 ).

[3]See, Uzawa (1961).

Proof: If ( 1.1 ) is quasi globally stable, then every limit point of the solution $x(t, x^o)$ for any $x^o$ in the domain of definition of $F(.)$ is an equilibrium point for ( 1.1 ); for some $x^o$, let $x(t, x^o)$ have two distinct limit points $x^1, x^2$. Then these are equilibrium points for ( 1.1 ), and are consequently, isolated points. Thus there exist open neighborhoods $N(x^1)$ and $N(x^2)$ in $R^n$ such that $N^c(x^1) \cap N^c(x^2) = \emptyset$ where $N^c(.)$ denotes the closure of $N(.)$. Further, given that the equilibrium points of ( 1.1 ) are isolated, the neighborhoods $N(.)$ can be so chosen that $N^c(x^i) - x^i$ contain no other equilibria, $i = 1, 2$. Now let the subsequence $x(t_s, x^o)$ converge to $x^1$ as $s \to \infty$; since there is another distinct limit point, $x^2$, there must be another subsequence $x(t_s + h_s, x^o)$ such that $x(t, x^o) \in N(x^1)$ for all t satisfying $t_s \leq t < t_s + h_s$ with $x(t_s + h_s, x^o) \notin N(x^1)$, and $x(t_s + h_s, x^0) \in N_B^c(x^2)$ where $N_B^c(.)$ denotes the boundary of $N^c(.)$. Since $N_B^c(.)$ is a closed and bounded subset,$x(t_s + h_s, x^o)$ must have a limit point in $N_B^c(.)$ and this must be an equilibrium point of ( 1.1 ) which is distinct from $x^2$, given quasi global stability. But this contradicts the definition of the sets $N(x^i)$. Hence there cannot be two distinct limit points for the solution to ( 1.1 ): and the claim follows by virtue of Claim 1. •

The basic method for establishing quasi global stability or global stability for a process such as ( 1.1 ) is by way of Liapunov's second method and we describe this next. A function $V : R^n \to R$ is said to be a **Liapunov function** for ( 1.1 ), if the following conditions hold:

(i) $V(x)$ is a continuous function on $R^n$;

(ii) $V(x(t, x^o))$ converges as $t \to \infty$ for all $x^o$ in the domain of definition of $F(.)$;

(iii) $\dot{V}(x(t, x^o))$ exists and is zero if and only if $x(t, x^o)$ is an equilibrium for ( 1.1 ).

**Claim 1.3.3** *If a Liapunov function exists for ( 1.1 ), and if given any initial point, the solution to ( 1.1 ) is bounded, then ( 1.1 ) is quasi globally stable.*

Proof: Suppose, there is a Liapunov function for the system ( 1.1 ).Then by (ii) mentioned above, $\lim_{t\to\infty} V(x(t,x^o)) = V^*$ for some finite $V^*$, given some $x^o$. Further, since $x(t,x^o)$ is bounded, $\exists$ a subsequence $x(t_s,x^o)$ such that $x(t_s,x^o) \to x^*$ as $s \to \infty$ for some $x^*$. Consequently, by the property (i) of Liapunov functions, $V(x(t_s,x^o) \to V(x^*)$. Again by (ii) $V(x^*) = V^*$. Next, consider $x(t,x^*)$: a limit path and use Remark 5, to conclude that $V(x(t,x^*)) = V^*$ for all t. By property (iii), every point of $x(t,x^*)$ must be an equilibrium for ( 1.1 ), including its initial point viz.,$x^*$. Since this is true for any arbitrary limit point of $x(t,x^o)$, the claim follows. •

## 1.4 The Linear Case

We turn next to some special cases. Consider first of all, the case when $F(x) = A.x$ where $A$ is some constant $n \times n$ matrix. Our system ( 1.1 ), then reduces to the linear system

$$\dot{x} = A.x$$

The classical result in this connection, is contained in the following:

**Proposition 1.2** *The solution $x(t,0)$ is globally asymptotically stable if and only if the real parts of all characteristic roots of $A$ are negative.*

A demonstration of the sufficiency aspect of Proposition 2, by noting the following facts may be of some interest:

(i) All characteristic roots of $A$ have real parts negative $\iff$ the matrix equation $A^T.B + B.A = -Q$, for each real symmetric positive definite matrix $Q$, has as its solution B, a positive definite matrix (see Murata(1977),p.63, for example) $\iff$ the matrix $B$ defined by the matrix equation $A^T.B + B.A = -I$ is positive definite (see Bellman (1960),p. 245, for example). These are alternative statements of **Liapunov's Theorem** . It should be noted also that the system of equations $A^T.B + B.A = -Q$ has a unique solution for all $Q$ $\iff$ $\lambda_i + \lambda_j \neq 0$ for any pair of characteristic roots $\lambda_i, \lambda_j$ of the matrix $A$.

(ii) If $A$ has all characteristic roots with real parts negative, the linear system has a unique equilibrium $x = 0$; since $A.x = 0 \iff x = 0$ given the fact that A must be non-singular (i.e., cannot have zero as a characteristic root).

(iii) $V(x) = x^T.B.x$ ( where $B$ is any positive definite matrix such that $A^T.B + B.A$ is negative definite ) is a Liapunov function and any solution $x(t, x^o)$ remains bounded.

Using the above (i)-(iii) together with Claims 2 and 3, establishes the validity of the sufficiency part of Proposition 1.2.

The use of the linear system arises in a natural way when one tries to analyze solutions to ( 1.1 ) with the initial point $x^o$ close to an equilibrium $x^*$. For then, using a Taylor's series expansion of the functions $F_i(.)$ around the equilibrium $x^*$, we have the following:

$$\dot{x}_i \approx \sum_j F_{ij}(x^*)(x_j - x_j^*)$$

Next writing the matrix of partial derivatives $F_{ij}(x^*)$ as $A$ and a change of variables, $x_j - x_j^* = \theta_j$ we have the system $\dot{\theta} = A.\theta$ which is of course, the linear system. Consequently an immediate application of the Proposition 1.2 leads to the following[4]:

**Proposition 1.3** *For the system (1.1), a sufficient condition for local asymptotic stability of $x^*$ is that $A$ (as defined above) should have all characteristic roots with real parts negative.*

## 1.5 Motion on the Plane

The next special case is when instead of $\Re^n$ for arbitrary $n$, we confine attention to the case when $n = 2$: that is, we shall consider motion on the plane. Such situations are of great importance and it would be of some interest to collect together results available in this connection. Apart from being of interest independently, there is another very important reason why we must consider motion on the plane with great care. This is due to the following result of Smale (1976). Let $X$ be any $C^1$ vector field in the unit simplex of dimension $n-1$, $\Delta^{n-1}$; then there exists a $C^1$ vector field $F = (F_i)$ in $\Re^n$ satisfying $F_i = x_i M_i(x)$, $M_{ij}(x) < 0, j \neq i$ such that $F|\Delta^{n-1} = X$ and $\Delta^{n-1}$ is an attractor. Thus for any $n > 2$, it would appear that no general results are possible.

Recall the definition of a solution $x(t, x^o)$ to ( 1.1 ) and of a limit point to the solution. The term **$\omega$-limit set** is used to describe the set of limit points of $x(t, x^o)$. Formally, the

[4]Notice that this is a **sufficient** condition for local stability; this is not necessary since even with characteristic root zero, it may be possible to have stability. Consider for instance, $\dot{x} = -x^3$; note that $x = 0$ is the unique equilibrium and is globally stable; however when one linearizes around equilibrium, the linear system is $\dot{x} = 0$. It is for this reason that some authors have described this condition as being necessary and sufficient for **linear approximation stability** .

$\omega$-limit set of $x(t, x^o)$ is the set $L(x^o) = \{y : \exists$ a subsequence $x(t_s, x^o)$ such that $x(t_s, x^o) \rightarrow y$ as $s \rightarrow \infty\}$. Note that if $x^*$ is an equilibrium of ( 1.1 ), then $L(x^*) = x^*$. On the other hand, global stability would be equivalent to requiring $L(x^o) = x^*$ for any $x^o$. It should also be pointed out that if $\overline{x} \in L(x^o)$, then by virtue of Remark 5, above, $x(t, \overline{x}) \in L(x^o) \forall t$. A point $x^o$ is said to be in a **closed orbit** if $x(t, x^o) = x^o$ for some $t > 0$. Closed orbits are also called **cycles** . We shall try to characterize the nature of the the limit sets for systems such as ( 1.1 ) when $n = 2$. The classical *Poincaré-Bendixson Theorem*[5] achieves this:

**Proposition 1.4** *In case of ( 1.1 ) being a system in $R^2$ with the functions $F(.)$ continuously differentiable, then any nonempty compact limit set which contains no equilibrium point is a closed orbit.*

Thus whenever the solution is bounded, it either has an equilibrium as a limit point or it converges to a closed orbit, which is called **a limit cycle** in these circumstances. This is perhaps the most clear-cut result that can be obtained for the case $n = 2$.

For higher dimensions the nature of the limit sets are often quite complicated. We mention, in this connection, some recent results due to Hirsch ( Systems of Differential Equations which are competitive or cooperative, I, Siam Journal of Mathematics, 1982 ). Hirsch considers the case when $n = 3$ and considers the system ( 1.1 ) to be **cooperative** if $\dfrac{\partial F_i}{\partial x_j} \geq 0$ for $i \neq j$. If the above inequality is reversed, the system is said to be **competitive.** The result is *that for $n = 3$, if the system is cooperative and if $L$ is a compact $\omega-$limit set which contains no equilibrium, then $L$ is a closed orbit.*

[5]See, for example, Hirsch and Samle (1974), p. 248.

The treatise by Andronov et. al. (1966) contains an exhaustive account of the dynamics on the plane. To be more specific about the nature of $\omega-$limit sets we have the following:

**Proposition 1.5** *On the plane, the structure of* **non-empty** *$\omega$-limit sets is known to be one of the following*[6]*:*

*i. Consists of a single equilibrium or*

*ii. Consists of one closed orbit or*

*iii. an union of equilibria and paths tending to them.*

Notice that for convergence to equilibrium, we need to eliminate not only limit cycles but case (iii) as well which are the most complicated types. To enable us to understand the implications of the above, it would be of some help to note that first, **for the linear case**, Proposition 1.2, allows us to obtain a particularly simple necessary and sufficient condition for convergence.

**Remark 5** For the linear case, when $n = 2$, a necessary and sufficient condition for convergence is that both the following conditions should hold: (i) trace of the matrix should be negative and (ii) the determinant of the matrix should be positive.

**Remark 6** *For local stability analysis, where the functions $F(.)$ are linearized around $x^*$, an equilibrium, the conditions indicated above, reduce to the following:*

$$(i)\frac{\partial F_1(x^*)}{\partial x_1} + \frac{\partial F_2(x^*)}{\partial x_2} < 0$$

[6]See, for instance, Andronov et. al. (1966), p. 362.

*and*

$$(ii)\ det \begin{pmatrix} \frac{\partial F_1(x^*)}{\partial x_2} & \frac{\partial F_1(x^*)}{\partial x_2} \\ \frac{\partial F_2(x^*)}{\partial x_1} & \frac{\partial F_2(x^*)}{\partial x_2} \end{pmatrix} > 0.$$

The above conditions, if satisfied, guarantee that if the solution $x(t, x^o)$ has an initial point $x^o$ "close" to the equilibrium $x^*$, then the solution to the original system ( 1.1 ) will converge. In case this nearness to equilibrium initially cannot be guaranteed then these results are inapplicable, since the original system is, in general, non-linear. But the nature of the characteristic roots of the matrix in (ii), the Jacobian of $F(x_1, x_2)$ evaluated at equilibrium $x^*$ (which we shall denote by $\nabla F(x^*)$ ) would play some role in determining the nature of limit sets in the non-linear case as well. This so if equilibria or fixed points $x^*$ for the dynamic system (1.1) are **hyperbolic** or nondegenerate or simple[7] i.e., $\nabla F(x^*)$ has characteristic roots with real parts non-zero. Thus having equilibria which are hyperbolic or simple makes classification easier.

Consider the characteristic roots of $\nabla F(x^*)$. A **focus** is an equilibrium or fixed point where the characteristic roots are complex conjugates; the equilibrium is a stable focus when the real parts of these roots are negative; it is an unstable focus when the real parts are both positive; the equilibrium is called a **node** when these characteristic roots are both real and of the same sign; again it is a stable node if the real roots are both negative and an unstable node if the real roots are positive. Sometimes stable focii and nodes are called **sinks**; unstable nodes and focii are called **sources**. A **saddle-point** is an equilibrium when the characteristic roots are both real but of opposite sign. A **center** is an equilibrium when

[7] See, for instance, Guckenheimer and Holmes (1983), p. 13.

the characteristic roots are pure complex conjugates with real parts zero.

Returning to the Proposition 1.5 and assuming that equilibria are hyperbolic, consider the case (iii): it should be clear the equilibria in case (iii) can neither be sinks nor sources. Since points in the limit set are approached by the solution or trajectory, approaching a source arbitrary closely is not possible. Approaching a sink means that the trajectory cannot leave the neighborhood as well and we are in case (i) of Proposition 1.5. Thus the only possible equilibria to be in limit sets of the type case (iii), are saddle-points. Thus for convergence, if all equilibria are hyperbolic, we need to rule out not only cycles but saddle-point equilibria as well.

We first present a result which achieves this end[8]: Consider (1.1) where the functions $F_i$ are assumed to be continuously differentiable on the plane $\Re^2$ and the matrix $\nabla F(x)$, the Jacobian of $F(x)$. Assume that[9]:

- O1: There is an unique equilibrium $(\bar{x}_1, \bar{x}_2)$ to (1.1).
- O2: Trace of $\nabla F(x) < 0$ for all $x \in \Re^2$.
- O3: Determinant of $\nabla F(x) > 0$ for all $x \in \Re^2$
- O4: Either $F_{11}.F_{22} \neq 0$ for all $x \in \Re^2$ or $F_{12}.F_{21} \neq 0$ for all $x \in \Re^2$.

**Proposition 1.6** *Under the conditions O1 - O4, the unique equilibrium* $(\bar{x}_1, \bar{x}_2)$ *is globally asymptotically stable.*

[8] See Olech (1963) and Ito (1978).

[9] We shall denote by $F_{ij}(x)$ the partial derivative of $F_i$ with respect to $x_j$.

It is clear that we still need to satisfy a lot of conditions, even for the relatively simple case of motion on the plane, to achieve convergence. It may be of some interest to break down the implications of the various conditions employed in Olech's Theorem to get at exactly what is required. For this purpose we first consider the elimination of cyclical orbits.

**Remark 7** *Dulac's Criterion: If we can find a function $\theta(x_1, x_2)$ which is continuously differentiable on the region $S$ and for which*

$$\frac{\partial\theta(x_1,x_2)F_1(x_1,x_2)}{\partial x_1} + \frac{\partial\theta(x_1,x_2)F_2(x_1,x_2)}{\partial x_2}$$

*is not identically zero and is of constant sign on $S$*[10]*, then there is no closed orbit for the system (1.1) on the region $S$*[11]*.*

Notice therefore, the constancy of the sign of the Trace of $\nabla F(x)$, which is basically condition O2 implies that there cannot be cycles. The problem with the application of Dulac's Criterion is the same as that of the Liapunov Method we described above: there is no general method of obtaining the function $\theta(.)$.

To proceed, it may be worthwhile to take into account Poincaré's Indices[12]. Consider a simple closed curve $S$ which does not pass through equilibrium and $N$ any point on it; consider the tangent to the trajectory of (1.1) through $N$ $(F_1, F_2)$ and consider the rotation of the point $N$ on the closed curve $S$: the vector $(F_1, F_2)$ will rotate continuously and when

[10] The sign could thus be either $\geq 0$ on $S$ or $\leq 0$ on $S$ but not 0 everywhere in the region.

[11] See, for instance, Andronov et. al. (1966), p. 305; another condition, the Bendixson's Criterion, which serves the same purpose, is a special case of Dulac's Criterion when $\theta(x_1, x_2) = 1$.

[12] See, Andronov et. al. (1966), p. 300 for details.

we return to the point $N$ (since $S$ is a closed curve), the vector would have rotated through an angle $2\pi j$ where $j$ is an integer. It is shown that the integer $j$ is independent of the shape of the closed curve and is called the **index of the closed curve** $S$ with respect to the vector filed $(F_1, F_2)$. If the closed curve encircles an equilibrium then the index is determined by the nature of the equilibrium and is hence referred to as the **Poincaré index** of the equilibrium. In case the equilibrium $x^*$ is such that the determinant of the matrix $(\nabla F(x^*)$ does not vanish, it may be shown that the Poincaré indices for a node, focus or center are all +1 while for a saddle point it is -1[13]. Several other interesting conclusions may be drawn; we include a sample:

- i.The index of a closed curve not enclosing an equilibrium is zero.
- ii. The index of a closed curve surrounding a number of equilibrium points is equal to the sum of the indices of these points.
- iii. The index of a closed orbit is +1

Some important corollaries: first of all, if there is a single equilibrium inside a closed orbit, it must be a node or a focus; if inside a closed orbit there are many equilibria, then they must be odd in number with the number of saddle-points being one less than the number of nodes and focii.

We shall use these properties at a later stage to provide a more general form of Proposition 1.6.

[13]For these conclusions, the non-vanishing of the determinant of the Jacobian at equilibrium is crucial; if this is not satisfied, it is not necessary that indices be + or - 1; see Andronov et. al. (1966), p. 300-305.

### 1.5.1 Lotka-Volterra System of Equations

An important application of the theory of the last section is the Lotka-Volterra system of equations or alternatively, the Predator-Prey Models. We shall also find this to be a useful way of introducing another topic that we shall be concerned with later. This has to do with the theory of bifurcations. This is a study of how the dynamics of a system undergoes changes when some parameters change.

Consider an environment made up of two species of life-forms, one of which preys on the other: the prey and the predator. Let the population of the prey be designated by $x$ while that of the predator by $y$. The basic assumption is that in the absence of the predator, the population of the prey **grows** at a constant proportional rate $a$; and on the other hand, in the absence of the prey, the population of the predator **decays** at a constant proportional rate $b$ (here both $a$ and $b$ are assumed positive). In the presence of both the prey and predator, adjustments to this basic story have to be made and we have

$$\dot{x} = x(a - \alpha y) \text{ and } \dot{y} = y(\beta x - b) \tag{1.3}$$

where $\alpha, \beta$ are also assumed to be positive and are to be interpreted as the effect of the presence of one population on the other.

There are two equilibria for the above system of equations:

$$(x = 0, y = 0) \text{ Trivial Equilibrium or (TE)}$$

and

$$(x = b/\beta, y = a/\alpha) \text{ Non-Trivial Equilibrium or (NTE)}$$

We are interested in what happens to the solution, $z(t) = (x(t), y(t))$ to the system ( 1.3 ) beginning from an initial configuration $z^o = (x^o, y^o)$; we shall represent this solution by $z(t, z^o)$.

**Stability: Local and Global** We note first of all, the following local stability properties of the equilibria mentioned above:

**Claim 1.5.4** *For the system ( 1.3 ), TE is a saddle point while NTE is a center.*

Proof: The Jacobian of the rhs of the system ( 1.3 ) is given by:

$$\begin{pmatrix} a - \alpha y & -\alpha x \\ y\beta & \beta x - b \end{pmatrix}$$

It is then straight forward to check that at TE the characteristic roots are:

$$(a, -b);$$

while at NTE, the characteristic roots are purely imaginary:

$$(\imath\sqrt{a.b}, -\imath\sqrt{a.b}).$$

The claim follows. •

Next, we note that

**Claim 1.5.5** *With any $z^o = (x^o, y^o) > (0, 0)$ as initial point, the solution to the system ( 1.3 ), $\phi_t(z^o)$, is a closed orbit around NTE $(b/\beta, a/\alpha)$.*

Proof[14]: Consider

[14] Hirsch and Smale, (1974), p.262, Theorem 1.

$$V(t) = \{\beta x(t) - b \log x(t)\} + \{\alpha y(t) - a \log y(t)\}$$

and consider the derivative $\dot{V}$ along the solution $z(t) = (x(t), y(t))$ to the system( 1.3 ) and notice that

$$\dot{V} = (\beta x - b)\frac{\dot{x}}{x} + (\alpha y - a)\frac{\dot{y}}{y} = 0;$$

Thus the function $V$ remains constant along the solution to the system ( 1.3 ) the value of this constant is defined by the initial point. Thus $V(t) = V^o = (\beta x^o - b \log x^o) + (\alpha y^o - a \log y^o)$ for all $t$; also notice that $V(z) = V(x, y) = (\beta x - b \log x) + (\alpha y - a \log y)$ is strictly convex and attains a global minimum value, say $V^*$, at the NTE $(b/\beta, a/\alpha)$; thus if the initial point is not the NTE, we have $V^o > V^*$; consequently the trajectory cannot approach NTE; nor can it approach any other equilibrium, since along the solution $x(t), y(t)$ must remain positive as otherwise $V(t)$ would become unbounded. In addition, $x(t), y(t)$ must also remain bounded, since otherwise, $V(t)$ would become unbounded. Thus the $\omega$-limit set is non-empty, compact and can not contain any equilibrium; by the Poincaré-Bendixson Theorem, the $\omega$-limit set must be a closed orbit. Thus there are two possibilities: either the closed orbit is approached in the limit ( a limit cycle ) or the trajectory itself is a closed orbit.

In case of a limit cycle $\mathcal{L}$, it must be the case that for every $z \in \mathcal{L}$, $V(z) = V^o$; in addition for some neighborhood $\mathcal{N}$ of $z^o$, $z \in \mathcal{N} \Rightarrow \phi_t(z) \to \mathcal{L}$[15]. Consequently we must have $V(z) = V^o$ for all $z \in \mathcal{N}$: this cannot be since the function $V(z)$ is not constant over

[15]For this property of limit cycles, see, Hirsch and Smale (1974), p. 251.

open sets. So there cannot be a limit cycle. The only possibility then is that $\phi_t(z^o)$ is a closed orbit. •

The above claim may be seen from the following diagram.

FIGURE 1: Closed Orbits in a Lotka-Volterra Model

The above has been used to explain why the population of some species like the above constantly keep chasing one another and never settles down to any fixed values. We wish to point out what happens when we change the basic story somewhat. But in order to do that, we need some theoretical developments which we turn to next.

## 1.6 Bifurcations

An interesting area of research and study is the theory of bifurcations; the main point of enquiry is whether the qualitative properties of a system such as ( 1.1 ) change when any of the parameters which may define the function $F(.)$ alter. Notice that these parameters, if there are any, have been suppressed in what we have discussed so far. We make the dependence explicit below:

$$\dot{x} = F(x, \mu),\ x \in R^n\ ,\ \mu \in R \tag{1.4}$$

Consider first of all, $n = 1$. Assume that for $\mu = 0$, the above system has an equilibrium $x* = 0$, i.e., $F(0, 0) = 0$. It is known that if the multiplier $\lambda^o = \dfrac{\partial F(0,0)}{\partial x}$ is negative, then the equilibrium $(0, 0)$ is locally asymptotically stable. Note that at an equilibrium $(x^o, \mu^o)$ if $\lambda(x^o, \mu^o) \neq 0$, then by the implicit function theorem, in a neighborhood of $\mu^o$, we may express $x^o$ solving $F(x^o, \mu^o) = 0$ as a differentiable function $x(\mu)$ with $x^o = x(\mu^o)$; $x(\mu)$ is called a **branch of equilibrium.** If at a particular $(\tilde{x}, \tilde{\mu})$ several branches come together, then $(\tilde{x}, \tilde{\mu})$ is called a **bifurcation point.**

At a bifurcation point, we have necessarily, $\lambda(\tilde{x}, \tilde{\mu}) = 0$. The types of bifurcation are classified according to the signs of the other partial derivatives evaluated at $(\tilde{x}, \tilde{\mu})$[16]:

- We have a **fold bifurcation,** when on one side of $\tilde{\mu}$ there is no equilibrium, while on the other side, there are two equilibria $\{\frac{\partial F}{\partial \mu} \neq 0; \frac{\partial^2 F}{\partial x^2} \neq 0\}$.
- We may have a **transcritical bifurcation,** where on one side of $\tilde{\mu}$, the equilibrium $x(\mu)$ is stable and on the other side this becomes unstable and another stable equilibrium branch emerges $\{\frac{\partial^2 F}{\partial x \partial \mu} \neq 0; \frac{\partial^2 F}{\partial x^2} \neq 0\}$.
- We may have **pitchfork bifurcation,** where on one side of $\tilde{\mu}$, $x(\mu)$ is stable and on the other side two additional branches of stable equilibria emerge $\{\frac{\partial^2 F}{\partial x \partial \mu} \neq 0; \frac{\partial^3 F}{\partial x^3} \neq 0\}$.

Consider, for instance, the equation $\dot{x} = F(x, \mu) = \mu - x^2$; notice that equilibria exist only if $\mu > 0$; for $\mu > 0$, there are two branches of equilibria: $+\sqrt(\mu)$ and $-\sqrt(\mu)$; the former is a stable equilibrium while the latter is an unstable equilibrium. Notice that $F(x, \mu) = 0$ if and only if $\mu - x^2 = 0$ and $F_x(0,0) = 0; F_{xx}(0,0) = -2; F_\mu(0,0) = 1$, we have an example of a fold bifurcation at $\mu = 0$.

If $F(x, \mu) = \mu x - x^2$, then $F_x(0,0) = 0; F_{xx}(0,0) = -2; F_{x\mu}(0,0) = 1$; notice that for $\mu < 0$, the equilibria are $x = 0$ and $x = \mu$; the former is stable and the latter unstable; on the other hand for $\mu > 0$, the equilibria are $x = 0$ and $x = \mu$; but now the former is

[16] See Lorenz (1993), Chapter 3 for details.

unstable and the latter is stable. Thus there has been exchange of stability between the two and this is an example of a transcritical bifurcation at $\mu = 0$.

If $F(x, \mu) = \mu x - x^3$ then $F_x(0,0) = 0; F_{xxx}(0,0) = -6; F_{x\mu}(0,0) = 1$. Notice that equilibria are given by $x = 0$ or by $x = \pm\sqrt{(\mu)}$ when $\mu > 0$ and by $x = 0$ if $\mu < 0$; consider then that when $\mu > 0$ $x = 0$ is unstable, while the remaining two are stable; for $\mu < 0$ only $x = 0$ remains and it is stable. This is an example of a pitchfork bifurcation at $\mu = 0$. This is called **supercritical** since the two equilibria which appear are both stable; in case $F_{xxx}(0,0) > 0$ the two equilibria which would appear are unstable, we shall call the bifurcation **subcritical**.

When $n > 1$, we may still have bifurcations of the type described above; except, we must note that the multipliers become the eigenvalues (characteristic roots) of the Jacobian of $F(.)$ at the bifurcation point; and out of the $n$ eigenvalues, a single eigenvalue is zero and so on.

However when $n \neq 1$, a possibility which emerges is that characteristic roots (multipliers) are pure imaginary numbers; since these appear in pairs, we must have $n = 2$ at least. This brings us to what is called the **Hopf Bifurcation.** It is best to state a version of what is known as the[17] **Hopf Bifurcation Theorem:**

**Proposition 1.7** *Suppose that the system ( 1.4 ) has an equilibrium* $(\tilde{x}, \tilde{\mu})$ *at which the following conditions hold:*

*i. The Jacobian of* $F(., \tilde{\mu})$ *evaluated at* $\tilde{x}$ *has a pair of pure imaginary eigenvalues*

[17] See Lorenz(1993), p.96, for example

*$(\lambda(\tilde{\mu}), \overline{\lambda(\tilde{\mu})}$ and no other eigenvalues with zero real parts; and*

*ii.* $\dfrac{d(Re\lambda(\tilde{\mu}))}{d\mu} |_{\mu=\tilde{\mu}} > 0$

*then there exist periodic solutions bifurcating from $x(\tilde{\mu})$ at $\mu = \tilde{\mu}$ and the period of the solutions is close to $2\pi/(Im\lambda(\tilde{\mu}))$.*

Notice that the theorem tells us about the **existence** of periodic solutions only; no information is contained about either their **number** or their **stability.**To establish stability, we need to transform the system ( 1.4 ), say for $n = 2$, by a change of coordinates into the following:

$$\begin{pmatrix} \dot{x} \\ \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu & -\omega \\ \\ \omega & \mu \end{pmatrix} \cdot \begin{pmatrix} x \\ \\ y \end{pmatrix} + \begin{pmatrix} f(x, y) \\ \\ g(x, y) \end{pmatrix} \tag{1.5}$$

The matrix of linear terms in $x$ and $y$ on the rhs is in the **normal form;** and the fixed point of the original system ( 1.4 ) has been shifted to the origin and the terms have been sorted into the linear terms and nonlinear terms $f(x, y), g(x, y)$. Note that the eigenvalues of the Jacobian evaluated at the fixed point are $\mu \pm \imath\omega$; the real part of the complex root is a positive function of the bifurcation parameter. And we know that we have a Hopf bifurcation; to determine the stability of the periodic solutions, the nonlinear terms play a role. The expression

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) +$$

$$\frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})$$

is computed and if found positive (negative), the emerging limit cycles are stable (unstable, respectively).[18]

[18] In this connection, see also, C. Kind (1999).

### 1.6.1 Lotka-Volterra Model Reconsidered

It may be recalled that the standard Lotka-Volterra Model is given by

$$\dot{x} = x(a - by) \text{ and } \dot{y} = y(dx - c) \tag{1.6}$$

with $a, b, c, d > 0$. As we have noted earlier, in Claim 1.5.5, For any $(x, y) > (0, 0)$, the trajectory or solution to (1.6) originating from the point $(x, y)$, $\phi_t(x, y)$, is a closed orbit around NTE $(c/d, a/b)$.

We consider a perturbation of this model by allowing the rates of growth of population of preys and predators to depend on their own population as well[19]; this takes us to :

$$\dot{x} = x(a - \alpha x - by) \text{ and } \dot{y} = y(dx - \beta y - c) \tag{1.7}$$

Now for $\alpha = \beta = 0$ we are back in the system (1.6); we shall consider $\alpha = \beta = p$ and consider $p$ to be of small magnitude. Thus the system (1.7) may be thought of as a perturbation of the system (1.6). Figure 2 shows the orbits of this new system for the same values of parameters as in Figure 1 with the addition of a positive value for the term $p$: the convergence to an equilibrium could not have been more clearly seen. But we need an analytical argument and this is what we proceed with next.

FIGURE 2: Convergence in a Lotka-Volterra Model

For the system, (1.7) the equilibria are given by: $E_1(0, 0), E_2(a/p, 0)$ and

[19] Hirsch and Smale (1974), refer to this as the intordduction of social phenomenon; see, for example, p. 263.

$$E_3(\frac{ap+bc}{bd+p^2}, \frac{ad-pc}{bd+p^2}).$$

Note that for $E_3$ to exist in a meaningful way, we do need $ad - pc > 0$ which should be the case if the magnitude of $p$ is small enough, as we have said earlier. We note the following:

**Claim 1.6.6** *The characteristic roots of the Jacobian of the system (1.7) at the equilibrium $E_3$ are given by:*

$$\frac{1}{2(bd+p^2)}$$

$$\{-bcp-adp-p(ad-cp) \pm \surd((bcp+adp+ap^2-cp^2)^2-4(bd+p^2)(abcd-bc^2p+a^2dp-acp^2))\}$$

It is easy to check that for $p = 0$, the roots are purely imaginary; this is not surprising, since we had noted this property of the system (1.6) earlier and for $p = 0$, the system (1.7) reduces to (1.6). It is easy to check that for $p$ small, the *real part of the characteristic roots has the same sign as* $-p$. Recall that we have assumed that $ad - cp > 0$; consequently, local stability at $E_3$ is completely determined by the sign of $p:$ for positive $p$, the equilibrium $E_3$ is locally asymptotically stable; while for a negative $p$, the equilibrium is locally unstable. Thus with small $p$, the roots are still complex, and we either have a spiralling approach to $E_3$ (if $p > 0$) or a spiralling movement away from this equilibrium if the sign of $p$ is reversed. In particular, it is easy to check that the following holds.

**Claim 1.6.7** *Consider the system (1.7); there is an equilibrium $E_3\{(\frac{c}{d}, \frac{a}{b}); p = 0\}$ at which both conditions required for the Hopf Bifurcation Theorem hold.*

Consequently, there may still be some chance of obtaining cyclical behavior near $p = 0$. To consider these and other matters, we shall find it to consider the more general form of the system (1.7)[20]. We recall that the system (1.7) is

$$\dot{x} = x(a - \alpha x - by) \text{ and } \dot{y} = y(dx - \beta y - c)$$

where $a, b, c, d, \alpha, \beta$ are all positive. For this system $E_3$ is given by $(x^\star, y^\star)$ say where $x^\star, y^\star$ solve the following system of equations:

$$a - \alpha x - by = 0 \text{ and } dx - \beta y - c = 0.$$

Note that we shall maintain the restriction $\alpha c - ad < 0$ throughout to ensure that $E_3$ occurs in the positive orthant. This allows us to rewrite the system (1.7) as

$$\dot{x} = x(\alpha(x^\star - x) + b(y^\star - y)) \text{ and } \dot{y} = y(d(x - x^\star) - \beta(y - y^\star)) \tag{1.8}$$

We also notice that the function $f(x) = \theta_1 x - \theta_2 \log x$ with $\theta_i > 0, i = 1, 2$ is a **strictly convex function** of $x$ for all $x \in \Re_{++}$; in addition $f(x) \to \infty$ as $x \to 0$ or as $x \to \infty$[21].

FIGURE 3: The f-function

It may be noted too that a global minimum is attained at $x = \dfrac{\theta_2}{\theta_1}$. With these observations, we define the function[22]:

$$V(x, y) = \theta_1 x - \theta_2 \log x + \gamma_1 y - \gamma_2 \log y$$

[20] In other words, we do not assume that in (1.7), $\alpha = \beta$ necessarily.

[21] Observe that $f''(x) = \frac{\theta_2}{x^2} > 0$. See Figure 3.

[22] By virtue of what we have said about the function $f(.)$, it follows that $V(x, y)$ is strictly convex function in $x, y$ attaining a global minimum at $(x^\star, y^\star)$, by virtue of the restrictions we shall place on the coefficients.

where $\theta_i, \gamma_i > 0, i = 1, 2$ and in particular, we define:

$$\theta_2 = x^\star \theta_1 \ ; \ \gamma_2 = y^\star \gamma_1 \ ; \ b\theta_1 = d\gamma_1. \tag{1.9}$$

We claim that

**Claim 1.6.8** *The function $V(x, y)$ defined above is a Liapunov Function for the system (1.7) or (1.8).*

Proof. Notice that

$$\dot{V}(x, y) = (x\theta_1 - \theta_2).\frac{\dot{x}}{x} + (y\gamma_1 - \gamma_2).\frac{\dot{y}}{y}$$

so that using our definitions in (1.9) and the form (1.8), we have:

$$\dot{V}(x, y) = \theta_1(x - x^\star)\{\alpha(x^\star - x) + b(y^\star - y)\} + \gamma_1(y - y^\star)\{d(x - x^\star) - \beta(y - y^\star)\}$$

$$= -\alpha\theta_1(x - x^\star)^2 - \beta\gamma_1(y - y^\star)^2$$

by virtue of the restrictions placed by (1.9). It is clear therefore that along the solution to (1.7), we must have $\dot{V} < 0$ unless $x = x^\star$ **and** $y = y^\star$. This establishes our claim. •

By virtue of the above, it is possible to claim that[23]

**Claim 1.6.9** *Any solution to the system (1.7) originating from any $(x^o, y^o) > (0, 0)$ converges to the equilibrium $E_3 = (x^\star, y^\star)$.*

Returning to the Claim 1.6.7, when we considered $\alpha = \beta = p$ , we notice that the expression for $\dot{V}$ would reduce to:

$$-\frac{p}{d}\theta_1\{d(x - x^\star)^2 + b(y - y^\star)^2\}$$

[23]In contrast, consider the conclusion in Hirsch and Smale (1974), p. 264-265. For the system such as (1.7), they were unable to rule out the existence of a limit cycle.

It is clear that if $p < 0$[24] $V(x(t), y(t))$ would be nondecreasing along any solution to the system (1.7), when $\alpha = \beta = p$ ; in this situation, consider a solution originating from some point $(x^o, y^o)$, which remains bounded so that it has limit points such as $(\bar{x}, \bar{y})$; along any such solution, the function $V()$ must converge to some number say $\bar{V}$. It follows therefore that $\bar{V} = V(\bar{x}, \bar{y})$; it now follows that the solution originating from the limit point must have $V$ constant along it at the value $\bar{V}$; and hence the limit point must be $E_3$; but the function $V$ is known to attain a minimum at $E_3$ and it was nondecreasing all along which implies that the path must have originated from $E_3$. Thus the only solution which remains bounded is the one originating from $E_3$. We have therefore demonstrated the validity of the following:

**Claim 1.6.10** *For the system (1.7), when $\alpha = \beta = p < 0$, and $|p|$ is small, the only solutions which remain bounded are the ones originating from $E_3$.*

Consequently, the chances of cyclical behavior, which seemed to have been indicated by the conclusions of Claim 1.6.7 are limited **only** to the case $p = 0$. In fact, one may show that there are no cycles when $\alpha, \beta$ have the **same sign**:

**Claim 1.6.11** *For the system (1.7), there can be no cycles, when $\alpha, \beta$ are non-zero and have the same signs*[25]*.*

Proof: Assume that $\alpha, \beta$ have the same sign; consider the function $\theta(x, y) = \dfrac{1}{xy}$ and use it

[24]When $p < 0$ note that for a meaningful $E_3$, we have to have $ap + bc > 0$, which would be guaranteed for the case when $p$ is small in absolute magnitude.

[25]Thus $\alpha.\beta \geq 0$ is also admissible; in other words, one could be zero and the other non-zero.

to check Dulac's Criterion[26]:

$$\frac{\partial x(a-\alpha x-by).\theta(x,y)}{\partial x}+\frac{\partial y(dx-\beta y-c).\theta(x,y)}{\partial y}$$

=

$$\frac{-\alpha}{y}+\frac{-\beta}{x}$$

which has the sign of $-\alpha$ or $-\beta$ and hence the claim follows, by virtue of Dulac's Criterion.

•

Since cyclical behavior is also of interest to us, the above indicates:

**Claim 1.6.12** *In a system such as (1.7), for cyclical orbits to exist, we must have either $\alpha=\beta=0$ or $\alpha,\beta$ have to be of opposite signs in some part of the positive orthant.*

The above seems to indicate that the simple form of (1.7)) where $\dot{x}/x, \dot{y}/y$ are represented as affine functions with constant partial derivatives will not be suitable for generating cycles, unless of course, social phenomenon do not exist and $\alpha=\beta=0$. Notice that $\alpha=\dfrac{\partial(\dot{x}/x)}{\partial x}$; similarly, $\beta=\dfrac{\partial(\dot{y}/y)}{\partial y}$ and we need to consider some possibilities of variation in these partial derivatives to provide a chance for the existence of cyclical behavior. We provide, next, an example of such a behavior within the context of a Predator-Prey Model.

### 1.6.2 Robust Cyclical Behavior in a Lotka-Volterra Model

We consider a more complicated configuration which allows us to consider the rates of growth of the prey and predators as follows:

$$\dot{x}=x\{r(1-\frac{x}{A})-\frac{\beta y}{x+h}\} \text{ and } \dot{y}=y\{\frac{bx}{x+h}-c\} \tag{1.10}$$

[26]See, Remark 7.

where $r, A, \beta, b, c$ are all positive constants. The interpretation of the above system is as follows: in the absence of the predator, the prey $x$ thrives on some other food items (maybe grass, for example) and if the amount of grass is unlimited, the rate of growth of the prey would have been a constant $r$; this is the so-called birth-rate of the prey; the birth-rate has to be modified to arrive at a growth rate, since grass is not unlimited and given the availability of grass, $A$ is the maximum population of the prey which can be sustained[27]. The standard logistic law provides us with rate of growth of the prey population. However there is the predator too, which restricts the rate of growth of the population: this effect is a bit more subtle than what we have considered so far. In the previous sections, we took the loss on account of predators to be proportional to the product of the populations $x.y$ since this would in turn be proportional to the encounter rate; now we consider the loss to be proportional to $\frac{x.y}{x+h}$, where the denominator is in fact a saturation effect: if the population of the prey is plentiful, the predators being not so hungry, would not kill on every encounter; if for example $x$ is infinite, the fraction $x/x+h$ tends to unity and the loss on account of the presence of the predators is just proportional to the population of the predators $y$. For the population of the predators, the first term takes into account the birth-rate, $b$ being the natural birth-rate with plentiful prey and $c$ is the death-rate among the predators; notice that the saturation parameter $h$ is the same as before and is purely a matter of convenience for the algebra to follow.

[27] Alternatively, the death-rate of the prey needs to be taken into account.

Rewriting the system (1.10) as:

$$\dot{x} = xM(x, y) \text{ and } \dot{y} = yN(x, y)$$

we note that now $M(x, y) = r(1 - x/A) - \beta y/(x + h)$ and $N(x, y) = bx/(x + h) - c$. Notice that $M_y < 0, N_x > 0$; however $M_x$ could be either positive or negative. For the system (1.7), $M_x < 0$ which had allowed us to use Dulac's criterion to rule out the presence of cycles when admitting "social phenomenon". The non-trivial equilibrium for this system is given by:

$$x^\star = \frac{ch}{b - c} \text{ and } y^\star = r(1 - x^\star/A)(x^\star + h)/\beta$$

Thus for a meaningful non-trivial equilibrium, we must have $b > c$ and in addition, we must have $A(b - c) > ch$ and we take it that this is so. At this equilibrium the Jacobian of the system is given by:

$$\begin{pmatrix} x^\star M_x(x^\star, y^\star) & x^\star M_y(x^\star, y^\star) \\ x^\star N_x(x^\star, y^\star) & 0 \end{pmatrix}$$

Notice that given the signs indicated earlier, the determinant is **positive** while the trace is of ambiguous sign. Notice too that the trace is given by:

$$x^\star.M_x(.) = x^\star \frac{A\beta y^\star - r(x^\star + h)^2}{A(x^\star + h)^2}$$

After some simplification, it may be shown that the sign of the trace depends on the sign of

$$A - h - 2x^\star$$

and hence at the nontrivial equilibrium, either both of the characteristic roots have their real parts positive; or both have their real parts negative. Consider what happens when the trace happens to be positive: notice that $x^\star$ is independent of the parameter $A$ so, keeping all the other parameters fixed, we need to choose a large enough value for $A$ for this to happen. Clearly then, the non-trivial equilibrium is a source.

Consider then, the case when $A > h + 2x^\star$; in other words, $A(b-c) > (b+c)h$. The Poincaré-Bendixson Theorem, if applicable, would imply the existence of a closed orbit. We need to check whether any equilibrium can be approached and then check whether trajectories are bounded for this purpose. Notice that the only other equilibria for the system (1.10) are given by $(0,0)$ and $(A,0)$; computation of characteristic roots of the Jacobian of the system at these equilibria implies that $(0,0)$ is a saddle-point and if $b+c > 1$ then so is $(A,0)$. The following may be checked:

- If the initial point $(x^o, y^o) > (0,0)$ none of these equilibria can be approached.
- Trajectories remain bounded.

Consequently the only possibility is a limit cycle around the non-trivial equilibrium. Notice also that small perturbation of the system is unable to dislodge the cyclical behavior of trajectories. Thus to clinch matters we need to establish the validity of the items noted above.

For the first, it is best to consult the following figure, where we have considered the phase plane of the system (1.10) for appropriate values of the various parameters: $r = 12, \beta = 20, b = 4, c = 8/5, h = 4$ with appropriate units[28]. Notice that $A$ has been left unspecified. The case we are interested in, consists of requiring $A > 28/3$ and the situation

[28] All but the last are measured per year for example, whereas the last is a stock.

for $A = 10$ is captured below.

FIGURE 4: ROBUST PERIODIC BEHAVIOR

While the above is a computer generated figure and cannot be taken for an analytical proof, we note that the axes $x = 0$ and $y = 0$ are trajectories and cannot be crossed; and the only trajectories which approach the saddle-point equilibria are these trajectories. The non-trivial equilibrium being a source cannot be approached. Hence no trajectory, with a strictly positive initial point, can approach an equilibrium, as claimed.

To show that any trajectory beginning from an initial positive configuration remains bounded, notice the following: denoting the solution to (1.10) from an arbitrary $z^o = (x^o, y^o) > (0,0)$ by $\phi_t(z^o) = (\phi_{t_x}(z^o), \phi_{t_y}(z^o))$, $\phi_{t_x}(z^o) < A \forall t > 0$ since $\dot{x} = \dot{\phi}_{t_x}(z^o) < 0$ whenever $\phi_{t_x}(z^o) = A$; thus unbounded behavior, if possible, may arise only if $\phi_{t_y}(z^o) \to +\infty$. If this were to be the case, $\dot{x} \to -\infty$ and for all $t > T$, say, $\dot{x} < -\delta$ for some $\delta > 0$ $\Rightarrow \phi_{t_x}(z^o) \to -\infty$ which contradicts the fact that $\phi_{t_x}(z^o) > 0 \forall t$. This allows us to conclude that the solution $\phi_t(z^o)$ remains within a bounded region of the positive quadrant. Thus the $\omega$-limit set is non-empty and does not contain any equilibrium when $A(b-c) > (b+c)h$; an appeal to Poincaré-Bendixson Theorem establishes that there must be a limit cycle.

That this is an example of robust cyclical behavior may be gauged from the fact that small perturbation in parameter values would maintain the local properties of the three equilibria[29] and also maintain the crucial inequality $A(b-c) > (b+c)h$. Hence the limiting periodic behavior would also be maintained.

[29] Notice too that the bounded nature of trajectories follow regardless of parameter values.

In the example considered, there are points of bifurcation of course and to analyze them we consider the fixing of all parameters except $A$. For the sake of a meaningful value for $x^\star$, we maintain $b > c$ ( the birth-rate of predators is greater than their death-rate ). The three equilibria are $(0,0), (A,0)$ and $(x^\star, y^\star)$, the non-trivial equilibrium. Consider then the Jacobian of the system (1.10):

$$\begin{pmatrix} xM_x + M & xM_y \\ yN_x & N \end{pmatrix}$$

At $(0,0)$, the Jacobian reduces to:

$$\begin{pmatrix} M(0,0) = r & 0 \\ 0 & N(0,0) = -c \end{pmatrix}$$

Thus regardless of parameter values, $(0,0)$ is a saddle-point and the only trajectory approaching it is the one along the $y$-axis. At the maximum prey equilibrium with no predators present $(A,0)$, the Jacobian reduces to:

$$\begin{pmatrix} AM_x(A,0) = -r & AM_y(A,0) \\ 0 & N(A,0) = \dfrac{bA}{A+h} - c \end{pmatrix}$$

Notice that now the equilibrium is locally asymptotically stable whenever $A < \frac{ch}{b-c}$; notice that in this range, there is no non-trivial equilibrium; if this inequality is violated this equilibrium becomes a saddle-point loses stability and a new equilibrium, the non-trivial equilibrium emerges. To complete matters, the Jacobian evaluated at the non-trivial equilibrium may be recalled and as we had seen then, the determinant is always positive; the trace has the same sign as $A - h - 2x^\star$ which has the same sign as $A(b-c) - h(b+c)$. Consequently for the range

$$\frac{ch}{b-c} < A < \frac{h(b+c)}{b-c}$$

the non-trivial equilibrium is locally asymptotically stable; as we have seen above, when $A(b-c) > h(b+c)$ the non-trivial equilibrium loses stability and a limit cycle emerges. Thus there are two points of bifurcation: $A = A_1 = ch/(b-c)$ and $A = A_2 = (b+c)h/(b-c) = h + 2x^{\star}$. To the left of $A_1$ there are only two equilibria; one a saddle-point and the other stable; to the immediate right of $A_1$, the maximum prey equilibrium loses stability and a new stable equilibrium, the non-trivial equilibrium emerges. This thus is another example of a transcritical bifurcation.

To the right of $A_2$ no equilibrium is stable; further at $A = A_2$, the trace of the Jacobian, evaluated at the non-trivial equilibrium vanishes; the determinant is still positive however, and hence the characteristic roots are purely imaginary. It should be recalled that trace is given by the expression:

$$\frac{r(1-\frac{x^{\star}}{A})}{x^{\star}+h} - \frac{r}{A} = \frac{r}{x^{\star}+h}[1 - \frac{h+2x^{\star}}{A}]$$

Thus the derivative of the trace with respect to the parameter $A$ is positive and it follows that we have an example of a Hopf Bifurcation at $A = A_2$; at this value, trajectories are closed orbits; and as we have established for larger values of $A$ periodic behavior remains and hence the claim of a robust periodic behavior.

# 2 Discrete Processes

## 2.1 Preliminary Definitions

**Definition 1** *Let $f : X \to X$ be continuous; $X = [a, b]$; then $(X, f)$ is a* ***dynamical system*** *with $x_{n+1} = f(x_n)$.*

For any $x \in X, f^0(x) = x, f^n(x) = f(f^{n-1}(x)), n = 1, 2, ...; \gamma(x) = \{(f^n(x)\}_{n \geq 0}$: **trajectory through** $x$. If $f^k(x) = x$ and $f^r(x) \neq x \forall r < k, r > 0$ then $x$ is **periodic with period** $k$; and $\{x, f(x), ....f^k(x)\}$is the **cycle through** $x$ with period $k$. Let $z \in (a, b)$ be a **fixed point (or equilibrium point)** i.e., $f(z) = z$; thus a fixed point of the map $f$ is periodic with period unity. We are interested in characterizing the asymptotic behaviour of the trajectory through $x \in X : \gamma(x)$; this is described by the $\omega$-limit set $\omega(x) = \{y \in X : \lim_{r \to \infty} f^{n_r}(x) = y$ for some subsequence $\{f^{n_r}(x)\} \in \gamma(x)\}$. An **Attractor** for the map $f(.)$ is a closed set $F \subset X$ such that $\omega(x) = F$ for some $x$ in a nondegenerate interval in $X$. Thus attractors depict the long term behaviour of trajectories which begin from some non-negligible subsets of $X$ in a sense to be made precise soon.

We first note that if the trajectory converges, then it must converge to a fixed point of the system:

**Claim 2.1.1** *If $f^n(x) \to \overline{x}$ as $n \to \infty$ then $\overline{x} = z$: a fixed point.*

Suppose to the contrary that the claim is false and $f^n(x) \to \overline{x} \neq z$. Suppose that $|f(\overline{x}) - \overline{x}| = \delta > 0$. The from the covergence of $f^n(x)$ we have for any $\epsilon > 0$ a $n(\epsilon)$ such that for all $n > n(\epsilon)$

$|f^{n+1}(x) - \overline{x}| < \epsilon \forall n > n(\epsilon)$

or $\forall n > n(\epsilon)$, $|f(f^n(x)) - f(\overline{x}) + f(\overline{x}) - \overline{x}| < \epsilon$ .

Now note that $\epsilon$ is arbitrary and that in the last line, the first two terms move arbitrarily close due to the continuity of the function $f$ and we have arrived at a contradiction. Hence the Claim 2.1.1.

## 2.2 Stability of Periodic Points

We saw earlier that if the trajectory from some point $x \in X$ converges, then it can only do so to a fixed point of the dynamical system. To check whether any fixed point may be an attractor, it would be convenient to impose an additional restriction on the dynamical system $(X, f)$:

The map $f : X \to X$ is differentiable on $X$.

Unless stated to the contrary, we shall assume that the above is satisfied. We now have the following:

**Proposition 2.1** *Let $x^\star$ be an equilibrium (fixed point) for the dynamical system $(X, f)$. Then $x^\star$ is locally asymptotically stable $\Rightarrow f'(x^\star) \leq 1$; and $f'(x^\star) < 1 \Rightarrow x^\star$ is locally asymptotically stable.*

To see this, note that $|x_{n+1} - x^\star| = |f(x_n) - x^\star| = |f'(\overline{x_n})| \, |x_n - x^\star|$ where $\overline{x_n} = \lambda_n x_n + (1 - \lambda_n)x^\star$ for some $\lambda_n$, $0 \leq \lambda_n \leq 1$. So if $|f'(x^\star)| < 1$, and $x_n$ is sufficiently close to $x^\star$, there exists $k < 1$ such that $|x_{n+1} - x^\star| = k^{n+1} \, |x_0 - x^\star|$ and the right hand side tends to zero as $n$ becomes large.

As we defined above, the attractor could be some closed subset $F$ of the domain space $X$ such that $\omega(x) = F$ for $x$ belonging to some non-negligible subset in $X$. Specifically, the set $F$ could be a cycle $C$ through some point $y \in X$; the cycle $C$ or the trajectory through $y, \gamma(y)$ or the periodic point $y$ is asymptotically stable if there is some non-degenerate interval $V$ with $y \in V$ such that $\omega(x) = \gamma(y) \forall x \in V$. In this connection, we also note that if $\overline{x}$ is a periodic point of the system $(X, f)$ with period $r$, then $\overline{x}$ is a fixed point or equilibrium point of the system $(X, f^r)$. That is, we have distinct points $C = \{x_1, ...x_r\}$, forming a cycle, where $\{\overline{x} = x_1, x_2 = f(x_1), ..., x_{r-1} = f(x_{r-2}), x_r = f(x_{r-1}), x_1 = f(x_r)\}$. It is clear that $f^r(x_1) = x_1$; so applying the previous Proposition **??**

to the system $(X, f^r)$, we have that if $|f^{r\prime}(x_1)| < 1$ then $x_1$ is locally asymptotically stable for the system $(X, f^r)$; and hence for the system $(X, f)$, the cycle $C$ is locally asymptotically stable. It should be clear that $|f^{r\prime}(x_1)| = |f'(x_1).f'(x_2)...f'(x_r)|$ and hence we have the following:

**Claim 2.2.2** *If $C = \{x_1, ...x_r\}$ is a cycle of period $r$ for the dynamical system $(X, f)$ then $|f'(x_1).f'(x_2)...f'(x_r)| < 1 \Rightarrow C$ is locally asymptotically stable for the system $(X, f)$.*

### 2.2.1 The Logistic Map

Consider $f(x) = K.x(1 - x)$ For $1 \leq K \leq 4$, $f : X \to X$ where $X = [0, 1]$. For equilibrium or fixed points:

we need to locate the solution to the equation:

$$f(x) = x$$

or

$$Kx(1-x) = x;$$

thus the equilibria are:

$$x_1^\star = 0 \ ; \ x_2^\star = \frac{K-1}{K}$$

Further, for local stability, by virtue of Proposition **??**, we note that

$$f'(x^\star) = K(1-2x^\star)$$

Note then that the critical point of the logistic map $\overline{x}$ is given by $\overline{x} = 1/2$ and is independent of the value of the parameter $K$; further we note that the equilibrium $x_1^\star$ is locally unstable if $K > 1$ and is locally stable when $K < 1$; on the other hand, the equilibrium $x_2^\star$ is locally stable when $1 < K < 3$. Thus different values of K correspond to quite different dynamics.

We ask next, the question whether there are some two period cycles: i.e., do there exist distinct points $p, q$ such that they solve $p = f(q)$ and $q = f(p)$; since $f(.) = Kx(1-x)$, we need to solve the following equations:

$$p = K(q - q^2) \text{ and } q = K(p - p^2)$$

These imply that

$$p + q = \frac{K}{K+1} \text{ and } p.q = \frac{1+K}{K^2}$$

These equations imply that $p, q$ must be the roots of the quadratic

$$x^2.K^2 - (K+1)Kx + (K+1) = 0$$

In other words, $p, q$ are the roots

$$\frac{K(K+1) \pm \sqrt{K^2(K+1)(K-3)}}{2K^2}$$

Thus $p, q$ exist if and only if $K > 3$.

Now when are these two period cycles stable ? Again using the results of the last section, we need to compute $|f'(p).f'(q)|$ i.e., $|K^2(1-2p)(1-2q)| = |-K^2+2K+4| < 1$ for $3 < K < 3.5$. Thus just as $K$ crosses the value 3,the fixed point $x_2^\star$ loses stability and a stable 2-cycle appears.

## 2.3 Recurrence

Since we are interested in the long term behavior of a trajectory, we would also be interested in finding out the subsets of the domain space that the trajectory visits and whether there are subsets which are repeatedly visited by typical trajectories. To enable us to present these notions in a concrete manner, it is necessary for us to consider the following definitions.

**Definition 2** *A $\sigma$-algebra $\Sigma$ is a collection of subsets of the set $X$ such that*

*i.* $X \in \Sigma$

*ii.* $A \in \Sigma \Rightarrow X - A \in \Sigma$ *and*

*iii. If* $\{A_i : i = 1, 2, ...\}$ *is a countable collection of subsets in* $\Sigma$ *then* $\cup_i A_i \in \Sigma$.

**Definition 3** *Let $\Sigma$ be a $\sigma$-algebra on $X$ and let $\{A_i : i = 1, 2, ...\}$ be a collection of disjoint subsets in $\Sigma$. $\mu : \Sigma \to \Re$ where $\Re$ is the real line, is a measure if :*

*i.* $\mu(\emptyset) = 0$ *and ii.* $\mu(\cup_i A_i) = \sum_i \mu(A_i)$.

*If in addition, $\mu(X) = 1$, then the measure is called a probability measure. In this case, $(X, \Sigma, \mu)$ is called a probability space.*

One of the more familiar measures defined over subsets of $\Re$ is the Lebesgue measure, which we shall denote by $m(.)$; the $\sigma$-algebra is made up of intervals and the measure for any interval is its length. The probability space is said to **continuous** if $\mu(x) = 0$ for all points $x \in X$. We shall consider probability spaces from here onwards, unless specified to the contrary.

Consider next a continuous map $f : X \to X$ where $(X, \Sigma, \mu)$ is a probability space; i.e., $(X, f)$ is a dynamical system as defined earlier. $f : X \to X$ is said to be **measure-preserving** and the measure $\mu$ is said to be **invariant** if $\forall E \in \Sigma, \mu(f^{-1}(E)) = \mu(E)$.

**Exercise 1** *Let $X = [a, b] \subset \Re, b > a$; $\Sigma$ denotes the collection of all intervals of $X$ of the type $(c, d], [c, d)$ and their union where $a \leq c \leq d \leq b$ together with the set $X$ and the empty set. Show that $\Sigma$ is a $\sigma$-algebra. Now define $\mu$ thus: $\mu([c, d)) = \mu((c, d]) = \dfrac{d-c}{b-a}$. Show that $(X, \Sigma, \mu)$ is a probability space.*

**Exercise 2** *Let $(X, \Sigma, \mu)$, with $b = 1, a = 0$ be as defined for the above exercise. Define the map $f_M$ as follows:*

$$f_M(x) = \begin{cases} 2Mx & \text{for } x \in [0, \frac{1}{2}) \\ 2M(1-x) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

*Show that $f_1$ is measure preserving and $\mu$ is invariant.*

The map $f_M$ is the well known tent-map.

The advantage of the definitions introduced above lies in the well known result:

**Proposition 2.2** *(Poincaré Recurrence Theorem) Let $(X, \Sigma, \mu)$ be a probability space, $f : X \to X$ is measure preserving and $\mu$ invariant under $f$. Let $E \in \Sigma, \mu(E) > 0$. Then any trajectory beginning from almost all points of the set $E$ returns to $E$ infinitely often.*

Thus sets of positive measure are visited infinitely often; notice then that to understand long term behavior of trajectories we need to understand the structure of sets with positive measure. We need another property of the dynamical system:

**Definition 4** *Let $(X, \Sigma, \mu)$ be a probability space. The map $f : X \to X$ is said to be ergodic if for all $E \in \Sigma$, $f^{-1}(E) = E \Rightarrow \mu(E) = 0$ or $1$.*

For ergodic maps, thus the motion cannot be split up into two parts; for suppose that a map is non-ergodic i.e., there is a set $E$ such that $0 < \mu(E) < 1$ and $f^{-1}(E) = E$; thus trajectories originating from $E$ remain in $E$ ( and those from $X - E$ remain inside $X - E$). For ergodic maps, we have the following:

**Proposition 2.3** *(Birkhoff-von Neumann) Let $(X, \Sigma, \mu)$ be a probability space, $f : X \to X$ be mean preserving and ergodic and $g$ be any integrable function. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i(x)) = \int_X g d\mu$$

The above result is often stated as the time average being equal to the space average. Consider, for example, $E \in \Sigma, \mu(E) > 0$ and we are interested in finding out the time spent by some trajectory $\gamma(x)$ in the set $E$. For this purpose, we define the characteristic function of the set $E$:

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Consider $\chi_E(f^i(x))$; summing this over all the points of the trajectory, we should have the number of periods spent in E: according to the Poincare Recurrence Theorem, however, this should be infinite. Suppose however we consider the average time spent in the set $E$ for the first $n$-periods and look at the average as $n$ becomes large. Then we wish to consider,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_E(f^i(x))$$

By the Birkhoff-von Neumann Theorem, this is the same as

$$\int_E \chi_E(x) d\mu = \mu(E)$$

So a typical trajectory spends, on the average, as much time in a set as its measure. In this connection, it is clear that if the support of the measure is a large set, then the behaviour of the typical trajectory is complicated. To provide an idea of how complicated this behaviour can be, we need the notion of:

**Definition 5** *A subset $S \subset X$ is said to be* ***scrambled if:***

*(i) $x, y \in S, x \neq y \Rightarrow$*

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0; \text{ and } \liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$$

*(ii) $x \in S$ and $y$ a periodic point of $f(.)$ in $X \Rightarrow$*

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| > 0$$

Thus points of $S$ are not periodic and are not even asymptotically periodic. A sufficient condition for the existence of a " large" scrambled set is contained in the following:

**Proposition 2.4** *(Li-Yorke) Assume there is some* $\overline{x} \in X$ *such that*

$$\textit{either } f^3(\overline{x}) \geq \overline{x} > f(\overline{x}) > f^2(\overline{x})$$

$$\textit{or } f^3(\overline{x}) \leq \overline{x} < f(\overline{x}) < f^2(\overline{x})$$

*then*

*for the dynamical system* $(X, f)$*, there is an uncountable scrambled set* $S$ *and for every positive integer* $k$*, there is a cycle with period* $k$*. Moreover, there exists a continuous measure* $\mu$ *which is invariant and ergodic under* $f$*.*

There is, under the condition specified in Proposition 2.4,the measure $\mu$ is supported on an uncountable scrambled set. But how "large" is this set compared to the whole set $X$. For instance, consider the Logistic Map examined earlier: $f(x) = Kx(1-x), 1 \leq K \leq 4$. For a value of $K$ near 3.83, one may show that $f^3(\frac{1}{2}) = \frac{1}{2}$. We note that $\frac{1}{2} < f(\frac{1}{2})$; and that $f^2(\frac{1}{2}) < \frac{1}{2}$. Let $\overline{x} = f^2(\frac{1}{2})$; note that this allows us to conclude that the Li-Yorke condition is satisfied and hence the conclusions of Proposition 2.4 hold. Nevertheless, our earlier consideration of stability of periodic points also allows us to conclude that the 3-period cycle $\frac{1}{2}, f(\frac{1}{2}), f^2(\frac{1}{2})$ is asymptotically stable and a typical trajectory through almost any ( in the sense of Lebesgue measure) point of $X$ approaches this 3-period cycle. So there is little chance of observing any complicated behaviour. In this particualr example, although the scrambled set is uncountable, it has zero Lebesgue measure. For almost sure noticeable complicated behaviour, we need the the following:

**Definition 6** *A measure $\mu$ is absolutely continuous with respect to the Lebesgue measure $m$ if there is a Lebesgue-integrable function $g$ such that $\mu(E) = \int_E gdm$; $g$ is then the density function for $\mu$.*

Note that the existence of an absolutely continuous measure would imply that the problem raised earlier would disappear since now, if $\mu$ is absolutely continuous with respect to $m$, $m(A) = 0 \Rightarrow \mu(A) = 0$ and hence $m($ support of $\mu) \neq 0$.

The existence of an absolutely continuous invariant measure is guaranteed by the well known result:

**Proposition 2.5** *(Lasota and Yorke) Assume that $f : X \to X$ is piecewise $C^2$ ( i.e., $X = [a, b]$ is partitioned by points $a < x_1 < x_2 < .... < x_n = b$ and on each $(x_i, x_{i+1})$ $f$ is a $C^2$ function which is extendable to a $C^2$ function on $[x_i, x_{i+1}]$). If*

$$|f'(x)| \geq \lambda > 1 \text{ } m\text{-almost everywhere}$$

*then there exists an absolutely continuous invariant measure.*

A map satisfying the condition specified above is called **expansive.** There are other conditions which show the existence of absolutely continuous and invariant measure for maps which need not be expansive and we shall have an oppotunity to consider one such result later.

To sum up: we have considered situations under which trajectories for the dynamical system $(X, f)$ exhibit complicated behaviour. These situations were identified first by the existence of an invariant, continuous and ergodic measure. A sufficient condition under which such a measure existed was contained in the Li-Yorke condition (Proposition 2.4). Under this condition, an uncountable scrambled set may be shown to exist; further, for every positive integer, $k$, a cycle of period $k$ may be shown to exist as well. Sometimes, the existence of an uncountable scrambled set is taken to be the existence of chaos in the sense of Li-Yorke. We shall present in the next section a formal definition of **topological chaos:**it will then turn out that topological chaos implies chaos in the sense of Li-Yorke but the implication need not run the other way.

We also showed why we may need some thing stronger to guarantee that the unscrambled set is non-negligible: this we presented in the form of the existence of invariant ergodic and absolutely continuous measure. A more operational definition will be introduced next section. This stronger notion of chaos will be defined to be **ergodic chaos.**

### 2.3.1 Chaos and Unimodal Maps

To provide a formal definition of topological chaos, we need the following:

**Definition 7** *A finite subset $E$ of $X$ is said to be* $(n, \epsilon)$ **separated** $n = 1, 2, ...$ $\epsilon > 0$ in $X$ if for every $x, y \in X, x \neq y$, $\exists k, 0 \leq k < n$ such that

$$\left| f^k(x) - f^k(y) \right| \geq \epsilon$$

Let $s(n, \epsilon)$ denote the maximum cardinality of an $(n, \epsilon)$-separated set in $X$; $\Psi_\epsilon(f, X) = \limsup_{n\to\infty} \frac{1}{n} \log s(n, \epsilon)$; the **topological entropy** of the dynamical system $(X, f)$ is defined by the $\lim_{\epsilon\to 0} \Psi_\epsilon(f, X)$. If for some dynamical system $(X, f)$ the topological entropy is defined to be positive then the dynamical system is said to exhibit **topological chaos.**

An operational characterization of this property is contained in the following:

**Proposition 2.6** *(Misiurewicz) $(X, f)$ exhibits topological chaos if and only if there is a periodic point in $X$ whose period is not a power of* $2$.

How does one search for periodic points whose periods are not a power of 2? Since 3 is such a number, the following **necessary and sufficient condition** for the existence of a periodic point with period 3 is often found quite useful.

**Proposition 2.7** *(Li-Yorke) A necessary and sufficient condition for the dynamical system $(X, f)$ to have a periodic point with period 3 is that there is some $\overline{x} \in X$ such that*

$$\text{either } f^3(\overline{x}) \geq \overline{x} > f(\overline{x}) > f^2(\overline{x})$$

$$\text{or } f^3(\overline{x}) \leq \overline{x} < f(\overline{x}) < f^2(\overline{x})$$

We shall refer to the condition noted above as the Li-Yorke condition. Actually, Li-Yorke went on to demonstrate that the condition, was sufficient to guarantee a good deal more than just a three period periodic point (see, Proposition 2.4). Given the earlier proposition, the condition is **sufficient** for topological chaos. We next present another type of dynamical system which will exhibit topological chaos.

We shall study a type of dynamical system which has been called turbulent, in the literature:

**Definition 8** *Now $(X, f)$ is **turbulent if** $\exists a, b, c \in X$ such that $f(a) = a = f(b), f(c) = b$ and either $a > c > b$ or $a < c < b$.*

**Exercise 3** *Show that* $f(x) = Kx(1-x), 1 \leq K \leq 4$ *is turbulent if and only if* $K = 4$.

**Remark 8** *If the dynamical system* $(X, f)$ *is turbulent, for every positive integer* $k$ *there is periodic point of period* $k$ *and an uncountable scrambled set.*

To see the above: suppose that $a < c < b$; note that since the map $f$ is continuous and since $f(a) = a$ and $f(c) = b$ hence there must be $q \in [a, c]$ such that $f(q) = c$. Thus $q < f(q) = c < f^2(q) = f(c) = b$ and $f^3(q) = f(f(f(q))) = f(f(c)) = f(b) = a \leq q$ and we have the Li-Yorke condition being met; consequently the claim follows by virtue of the Li-Yorke result mentioned above.

**Remark 9** *The existence of a three period cycle is sufficient for topological chaos. Notice that if the dynamical system* $(X, f^2)$ *is turbulent then the map* $f^2$ *satisfies the Li-Yorke condition and hence by virtue of the Li-Yorke theorem,* $(x, f^2)$ *has a periodic point with period* 3*; note that then* $(X, f)$ *has periodic point with period* 6 *and by the Misiurewicz proposition, exhibits topological chaos.*

We shall confine our attention to the following type of dynamical systems (unimodal) in the notes below:

**Definition 9** $X = [a, b]$*,* $f : X \to X$ *is continuous; further, there is* $m \in [a, b]$ *such that the map* $f$ *is increasing in* $[a, m]$ *and decreasing in* $[m, b]$*; and* $f(a) \geq a, f(b) < b$ *and* $f(x) > x \forall x \in [a, m]$.

Below, we shall confine attention to the following three iterates of the point $m$: $f(m), f^2(m), f^3(m)$. For maps of the type defined in 9, these three iterates will reveal some information about the long term behavior of the trajectories.

Note first of all that by virtue of the properties mentioned in Definition 9 there is an unique $z \in (a, b)$ such that $f(z) = z$; $z$ is the unique **interior fixed point** . Also $m < z < b$. Further $f(m) > m$ and $f(m) \leq b$. Thus for the location of $f(m)$ relative to $m$:

$$b \geq f(m) > f(z) = z > m. \tag{2.1}$$

Using the above, we have immediately, the following condition for the location $f^2(m)$:

$$f^2(m) < f^2(z) = z = f(z) < f(m) \leq b \tag{2.2}$$

There are two possibilities that we need to consider $f^2(m) \geq m$ or $f^2(m) < m$; since we cannot rule out any one of the two cases.

**Claim 2.3.3** *In case $f^2(m) \geq m$, for any $x \in [a, b]$, $f^n(x) \in [m, f(m)]$ for some $n = \overline{n}$ and $f^n(x) \in [m, f(m)] \forall n \geq \overline{n}$.*

Note first of all that $x \in [m, f(m)] \Rightarrow m \leq x \leq f(m) \Rightarrow f(m) \geq f(x) \geq f^2(m)$ so that in case $f^2(m) \geq m$, $f(x) \in [m, f(m)]$. Thus to establish Claim 2.3.3 we need only establish that for any $x \in [a, b]$, $f^n(x) \in [m, f(m)]$ for some $n = \overline{n}$.

Suppose then to the contrary that, for some $x \in [a, b]$ there is no such $\overline{n}$. Then there are the following possibilities:

i. $f^r(x) < m \forall r > 0$

ii.$f^r(x) > f(m) \forall r > 0$

iii. There are subsequences $\{r_i\}, \{r_j\}$ such that $f^{r_i}(x) < m$ and $f^{r_j}(x) > f(m)$.

We consider the case iii first; in this case subsequences have to jump back and forth between the intervals $I_1 = [a, m]$ and $I_3 = [f(m), b]$ jumping across $I_2 = [m, f(m)]$. That this is not possible is easily seen by noting that in case $x \in I_1$, $f(x) < f(m)$ so that $f(x) \notin I_3$ so either $f(x) \in I_2$ or $f(x) \in I_1$: the first being ruled out by hypothesis, $I_1$ cannot be left. Thus jumping back and forth across $I_2$ is not possible. Thus we can only have case i or ii. In both of these cases the successive iterates form monotonic sequences and being bounded must converge. But the only possibility is convergence to the interior fixed point by virtue of Claim 2.1.1 $z$ which is neither in $I_1$ nor in $I_3$ (note that convergence to the fixed point $a$ in $I_1$ is not possible since in $I_1$ the iterates form a monotone increasing sequence) and hence neither of these possibilities is feasible too. This establishes the Claim 2.3.3. The full implication of what we have demonstrated may be seen from the following:

**Claim 2.3.4** *If there is a periodic point in $A = [m, f(m)]$, then the period is either 1 or 2.*

Suppose that there is a periodic point in $A$ with period $r > 1$; i.e., there are distinct points $\{x_1, ...., x_r\} \in A$ such that $f(x_i) = x_{i+1}$ and $f(x_r) = x_1$. Since the points are distinct, without loss of generality, assume that $x_r > x_1$. Now since $A \subset [m, b]$ and $f$ is decreasing on $A$, it follows that $x_1 < z < x_r$ and hence we have $x_2 > z > x_3$ and so on so that $\{x_1, x_3, x_5...\} \in [m, z)$ and $\{x_2, x_4, .....\} \in (z, f(m)]$. Thus we observe that $r$ is even and $= 2k$ say where $k > 1$. Suppose that $x_3 < x_1$ then we can say that we have the following: $x_{2k-1} < ... < x_3 < x_1$ and $x_2 < x_4 < ... < x_{2k}$ and so that we have $f(x_{2k}) = x_1 < f(x_{2k-2}) = x_{2k-1}$: which is a contradiction. A similar contradiction emerges if $x_3 > x_1$. Hence $x_3 = x_1$ and $k = 1$ and the claim is established. This allows us to conclude, using the result due to Misiurewicz that:

**Claim 2.3.5** *A necessary condition for topological chaos is that $f^2(m) < m$.*

This follows since with the violation of the condition one may have at best cycles with period of 1 or 2. We next claim the following:

**Claim 2.3.6** *$f^2(m) < m$ and $f^3(m) \leq m \Rightarrow (X, f)$ exhibits topological chaos.*

The claim follows by noting that in case $f^3(m) = m$, we have a three period cycle involving $\{m, f(m), f^2(m)\}$ and hence the Li-Yorke Theorem does the rest. In case $f^3(m) < m$, since $f(f^2(m)) < m$ and $f(m) > m$ there is $q \in [f^2(m), m]$ such that $f(q) = m$ and note that $f^3(q) = f^2(f(q)) = f^2(m) \leq q < m = f(q) < f(m) = f^2(q)$ so that the Li-Yorke condition is satisfied. We note next that a slightly weaker condition is also sufficient to ensure that the dynamical system $(X, f)$ exhibits topological chaos.

**Claim 2.3.7** *(Mitra (2001)) $f^2(m) < m$ and $f^3(m) < z \Rightarrow (X, f)$ exhibits topological chaos.*

The proof of this claim will be through the following steps: we shall show that under the stated conditions, the dynamical system $(X, f^2)$ is turbulent: consequently, the system has a three period cycle and hence the dynamical system $(X, f)$ has a six period cycle and, by virtue of the the Misiurewicz theorem, exhibits topological chaos. So we shall attempt to find three points in $X$ which meet the definition for turbulence for the system $(X, f^2)$. Crucial use is made of the continuity of the map $f$: First of all note that:

$$f(f^2(m)) < z \text{ and } f(m) > z$$

so there must exist $p \in (f^2(m), m)$ such that $f(p) = z$; it is also clear that $p < m < z$.

Next, observe

$$f^2(m) = f(f(m)) < p \text{ and } f(z) = z > p$$

thus there must exist $q \in (z, f(m))$ such that $f(q) = p$.

Next, since $z, f(m) \in [m, b]$ we have:

$$f(z) = z > m \text{ and } f(f(m)) < m$$

and hence there must be $r \in (z, f(m))$ such that $f(r) = m$. And further, $r < q$ since $f(r) = m, f(q) = p$ and $p < m$.

Finally, note that:

$$f^2(r) = f(f(r)) = f(m) > q \text{ and } f^2(z) = z < q$$

so that using the continuity of $f^2(.)$, there must exist $s \in (z, r)$ such that $f^2(s) = q$.

Collecting the above steps, we have $z, s, q$ such that $z < s < q$ and $f^2(z) = z, f^2(q) = f(f(q)) = f(p) = z, f^2(s) = q$; hence $f^2(.)$ is turbulent and the claim follows.

### 2.3.2 Unimodal Maps with Negative Scwartzian Derivatives

In the last section we identified conditions which could be used to check for topological chaos. We present here some results which are relevant for unimodal maps which satisfy the following:

Sf(x) (the Schwartzian derivative of the map h ) is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\{\frac{f''(x)}{f'(x)}\}^2$$

We require the unimodal maps to satisfy in addition: $Sf(x) < 0$ for all x $\neq m$ where $m$ is as noted above, the value of $x$ for which $f(x)$ attains a maximum.

Sometimes, we may speak of $m$ as being a critical point of $(X, f)$ (since if $f(x)$ is differentiable then $f'(m) = 0$). For unimodal maps (which we shall refer to as A1) with a negative Scwartzian derivative (except at the critical point)which we shall refer to as A2), there are some strong chacterizations possible and we present these. These relate to how the enire bifurcation process unfolds when parameters alter. Let $\{x_1, x_2, ...x_{k-1}\} =$ P be a cycle of period k. If the products of the derivatives of f(.) at each point of the cycle has an absolute value less than or equal to 1, we shall say that the cycle is **weakly stable**. Further, if one of the points of the cycle P is the critical point $m$, then the cycle P is said to be **superstable**.

Consider a family of maps $f_\alpha : [a_\alpha, b_\alpha] \rightarrow [a_\alpha, b_\alpha]$, where $\alpha \in [0, 1]$, say. Assume that each map $f_\alpha$ satisfies A1 and that $a_\alpha, b_\alpha$ depend continuously on $\alpha, f_\alpha$ and $f'_\alpha$. Let $\overline{x_\alpha}$ denote the unique critical point of each $f_\alpha$. The family is said to be **full** if (i) $f_0^2(b_0) > \overline{x_0}$ and (ii) $f_1^2(\overline{x_1}) < \overline{x_1}, f_1^3(\overline{x_1}) < \overline{x_1}$.

**Proposition 2.8** *Consider a full one parameter family of maps described above;*

*A. for any $k \geq 2$, the set of parameters $\alpha$ for which $f_\alpha$ has a superstable cycle of period k is non-empty and closed; given such an $\alpha$, there is an open interval V containing $\alpha$ such that for every $\beta \in V, f_\beta$ has a stable cycle of period k for every $\beta \in V$.*

*B. Let $a_k$ denote the first value of $\alpha$ for which there is a superstable cycle of period* $2^k, k \geq 1$. *Then the sequence $\{a_k\}$ increases with k and converges to some value $a_\infty$ as $k \rightarrow$*

$\infty$. *For each* $\alpha \in [0, a_\infty)$, *all cycles of the map* $f_\alpha$ *have a period which is a power of 2 or are fixed points.*

*C. If each member of the family* $f_\alpha$ *satisfies A2, then*

*(i) for any* $\alpha \in [0, a_\infty]$, *the map* $f_\alpha$ *has a unique weakly stable cycle;*

*(ii) there is an uncountable set of values of* $\alpha$ *in* $[a_\infty, 1]$ *for which* $f_\alpha$ *has no weakly stable periodic orbit.*(Theorems D.1.6 and D.1.7, Grandmont (1988))

These results indicate what will happen when the parameter changes and how the first the fixed point loses stability and stable cycles with periods which are powers of 2 successively appear; how ultimately topological chaos is shown to be present for parameter values in the range of $[a_\infty, 1]$.

How do we identify whether a dynamical system $(X, f)$ exhibits ergodic chaos ? Recall this means that there is an uncountable scrambled set with positive Lebesgue measure or a "large" (relative to the domain space) scrambled set. We had identified one such condition in terms of expansive maps. We consider here the case of unimodal maps which are of course not necessarily expansive. For such maps, we need to consider the following derivative: For such maps, a result due to Misiurewicz provides a condition which again relates to the iterates of the critical point $m$:

**Proposition 2.9** *Given that* $(X, f)$ *is such that* $f$ *is unimodal and satisfies the Scwartzian derivative condition, if there exists an integer* $k$ *such that either* $f^k(m) = x^\star$, *where* $x^\star$ *is a fixed point of* $f(.)$ *and* $|f'(x^\star)| > 1$ *or* $f^k(m) = x$ *where* $x$ *is a periodic point of an unstable cycle, then the map* $f(.)$ *exhibits ergodic chaos.*

# 3 Stability of Competitive Equilibrium

## 3.1 Introduction

The stability of competitive equilibrium is one area which attracted a lot of attention in the mid-fifties and the sixties. But thereafter, the interest some how shifted with the realization that unlike the question of the existence of competitive equilibrium, stability questions could not be resolved as satisfactorily. Thereafter, the stability question was never analyzed and neglected for reasons which are not really clear.

It may be of some interest to note what Malinvaud had to say in 1993 delivering the anniversary lecture on the eve of the Conference held to celebrate the 40th anniversary of the Arrow- Debreu-McKenzie contributions at the December 1952 Meetings of the Econometric Society: " Were there any failure of general nature ? Thinking about the question, I am identifying two defeciencies, which would be considered as failures. They may be called imperfect competition and price stability " and then later, " The second deficiency follows from an ambiguity in the teachings of general equilibrium theory about the performance of the market system...... Here I am not referring to the formal problem of how equilibrium is reached, a problem about which, by the way, we shall perhaps be a little too silent in this conference". More recently, McKenzie (1994), comments that " interest in this question (*of stability of the competitive equilibrium* ) had been revived by the contribution of Arrow and Hurwicz (1958).... Although interest has lasped in recent years, I do not regard this subject as completely obsolete".

The views of these two eminent scholars provide the first motivation for re-examining these issues more closely with the tools of non-linear dynamics. The views of McKenzie and Malinvaud not withstanding, there is a more basic and and fundamental reason in carrying out such a reexamination in the current context. During the late 20th Century, the overwhelming favorite economic principle was based on a complete belief in the powers of the market mechanism. Clearly this logically implied that equilibria were stable. Since if these were not so, then the equilibrium would be attained by serendipity rather than design. Yet on the theoretical side of things, at the same time, there were no indication that there was any design which could guarantee attainment of equilibrium. This is what Malinvaud refers to in the paragraph noted above. One of the things that we wish to do in the present chapter is to rexamine the working of market mechanism.

The foundations of the subject had rested on the role of income effects[1]. The belief, from the classical Samuelson (1947), Hicks (1946)contributions, had been that if the income effects could be overcome, the competitive equilibrium would at least be locally stable. One of the principal conditions for local stability has been that income effects cancel out at equilibrium. That some such restriction would be required became evident due to the contribution of Scarf (1960) and Gale (1963). Later contributions, which were interested in global stability primarily, [2] introduced other conditions such as gross substitutes, dominant diagonals and the weak axiom of revealed preference. The really restrictive nature of these conditions were never fully appreciated till the contributions of Sonnenschein (1973) and

[1]This has been referred rather picturesquely to as being part of the consciousness of the profession. As we hope to show in the pages below, many such items in our professional consciousness need to be reformulated.

[2]See Negishi (1962)or Hahn (1982) for surveys of these contributions.

Debreu (1974) established that excess demand functions could only be subjected to Walras Law and homogeneity of degree zero in the prices. With the Sonnenschein and Debreu contributions, and with the Scarf and Gale examples in the background, it was felt that anything could happen. Thus while the theorists among the profession felt that anything could happen, when they wrote in professional journals, the same theorists strangely remained quiet when the stability of equilibrium was routinely assumed and the virtues of competitive equilibria were extolled. Even as the present version is being prepared, the virtues of the competitive mechanism appear some what tarred but not on account of lack of stability [3]. Accordingly the issues concerning stability of competitive equilibrium need a thorough reexamination.

The above provides, it is hoped, a satisfactory reason to be engaged in looking once more at question of stability of competitive equilibrium, we need to first set out the context or the model, which shall be the main vehicle of our discussion.

In the sections below, we hope to reexamine these issues to provide an unified view. This unification, as we shall show, serves to clarify issues and reveal connections which we believe are of fundamental interest. We begin with the nature of the price adjustment process and investigate how the standard form of the *tatonnement* may be derived from the Walrasian hypothesis of price behavior in disequilibrium. The question of what the numeraire should be, is also considered. Thus what has been called the market mechanism

[3]This has more to do with the realization that the market mechanism may break down due to informational asymmetries. In other words, concerns were with the problems of existence of competitive equilibrium and not with its stability.

needs to be provided with some form which makes it amenable for analysis.

We hope to establish in the next stage, two things: first that it is the weak axiom of revealed preference which is the basic condition for stability of competitive equilibrium. We do this by first examining a *necessary* condition for local stability; it will be shown that this is related to the weak axiom; all the conditions mentioned above, the so-called *sufficient* conditions for local stability follow from the weak axiom. So far as global stability consideration are concerned, we try to pin down what the path of prices would be in the general situation. It turns out that although this path may be quite arbitrary, the weak axiom of revealed preference may again be used to provide some clues. The second thing that we hope to establish is that there are ways of determining stability conditions, in other words, place restriction on parameter choices, so that stability is obtained. Some well known models of instability are taken up for analysis in this context.

Apart from the above mentioned results being of interest on their own right, there is another aspect that we should point out. It is the recent contribution of Anderson et. al. (2002) in experimental economics. This is particularly important since one of the reasons for the neglect of this very fundamental area has been the feeling that the formalization of the market mechanism into the tatonnement has been inappropriate[4]. The experiment conducted by Messrs Anderson et. al., however, seems to indicate that the predictions of results by the tatonnement process are accurate even in a non-tatonnement experimental double auction situation. Thus the predictions of a tatonnement process may be important after all.

[4]Whether this was because the results were not quite satisfactory is not very clear.

Finally, it should be pointed out that on account of the theory of dynamical systems alone, we cannot expect very general conditions for convergence. This has been established relatively recently by Smale (1976). It may be worthwhile to consider what his result is. Let us denote by $\Delta^n$ the unit simplex in $\Re^n$ spanned by the unit vectors $e_i = (\delta_{ik}), \delta_{ik} = 0, k \neq i, \delta_{ii} = 1$. The Smale result is as follows: **Let $X$ be any $C^1$ vector field in $\Delta^{n-1}$; then there exists a $C^1$ vector field $F = (F_i)$ in $\Re^n$ satisfying $F_i = x_i M_i(x)$, $M_{ij}(x) < 0, j \neq i$ such that $F|\Delta^{n-1} = X$ and $\Delta^{n-1}$ is an attractor.** In other words, for $n > 2$, "anything goes" on account of dynamical systems alone. Convergence therefore for **any** dynamical system involving more than 2 variables would require special conditions. This justifies the need for analyzing what these stability conditions are. And even more so, if the more than 2 variables are concerned. In the pages below, we shall identify "stability conditions" and also pay close attention to what happens on the plane.

## 3.2 Excess Demand Functions

The economy is considered to be made up of households $h, h = 1, 2, \cdots, H$, each with a consumption possibility set $X_h \subset R^n$ where $X_h$ is **convex and bounded below.**Also each household $h$ has a **strictly quasi-concave, strictly increasing and continuously differentiable** utility function $U^h : X_h \to R$; further each household $h$ has an endowment $w^h \in IntX_h$ where $IntX_h$ denotes the interior of the set $X_h$ and further $w^h \neq 0$. Firm $j, j = 1, 2, \cdots, J$ possesses a production possibility set $Y^j \subset R^n$ which is assumed to be **strictly convex and bounded above;** also $Y^j \cap R^n_+ = \{0\}$; $Y^j \cap -Y^j = \{0\}$; $y \in Y^j, z \leq y \Rightarrow z \in Y^j$ are assumed to hold for every $j$. Finally, $\theta_{hj} \geq 0$ is the share of household $h$ in the share of firm $j$'s profit $\Pi^j$ with $\sum_h \theta_{hj} = 1$ for all $j$. Profits $\Pi^j$ are defined by the value of the following programme:

$$\text{Max p.y}$$

$$\text{subject to } y \in Y^j$$

Let $y^j(p)$ solve the above problem; note that, given our assumptions, this solution exists and is unique for all $p \in R^n_{++}$ (i.e., strictly positive prices). $y^j(p)$ is the **supply function** and the **profit function** $\Pi^j(p)$ is defined by $\Pi^j(p) = p.y^j(p)$. Household $h$ solves the programme:

$$\text{Max } U^h(x)$$

$$\text{subject to } p.x \leq p.w^h + \textstyle\sum_j \theta_{hj}\Pi^j,\ x \in X_h$$

Given our assumptions,a unique solution $x^h(p)$, **the demand function** exists to the above utility maximization exercise, for all $p \in R^n_{++}$. The **excess demand function** then is defined by:

$$Z(p) = \sum_h x^h(p) - \sum_j y^j(p) - \sum_h w^h = X(p) - Y(p) - W, say$$

where $X(p), Y(p)$ and $W$ respectively stand for the **aggregate demand, aggregate supply** and the **aggregate endowment** . The excess demand function, so derived, will be taken to satisfy the following properties:

P1. $Z(p)$ is a continuously differentiable function of $p$ which is bounded below for all $p \in R^n_{++}$.

P2. $p.Z(p) = 0$ for all $p \in R^n_{++}$(**Walras Law)**

P3. $Z(\lambda p) = Z(p)$ for any $\lambda > 0$ and for all $p \in R^n_{++}$.

P4. If $p^s, s = 1, 2, \cdots, p^s \in R^n_{++}, \|p^s\| \geq \delta > 0$ for some $\delta$ for all $s$ and if $p^s_k \to 0$ as $s \to \infty$ for some $k$, then $\sum_j Z_j(p^s) \to \infty$.

The above properties P1. - P4. are all standard properties[5]. The references provided below would also convince persons that the set $E = \{\ p \in R^n_{++}{:}Z(p) = 0\} \neq \emptyset$. Let $p^* = (p^*_1, \cdots, p^*_{n-1}, 1) \in E$. Unless otherwise stated, we shall choose all prices $p = (p_1, \cdots, p_{n-1}, 1)$ i.e., with good n as the **numeraire.** Define the set $K = \{\ p \in R^n_{++},\ p_n = 1 : p^*.Z(p) \leq 0\}$. It is relatively straightforward to see

**Claim 3.2.1** *$K$ is a nonempty and compact subset of $R^n_{++}$ and has a positive distance from the boundary of $R^n_+$.*

Proof. Note that $E \subset K$ and hence $K$ is nonempty, since $E$ is non-empty. The remaining part of the claim follows by virtue of the fact that if there is any sequence $p^s \in K$ such that $\|p^s\| \to \infty$ then clearly $p^s_k \to \infty$ as $s \to \infty$ for some $k$. Define $q^s = \dfrac{1}{p^s_k}.p^s$. Note that by virtue of P3., $Z(q^s) = Z(p^s)\forall s$; hence $p^*.Z(q^s) \leq 0 \forall s$. Note also that $q^s_k = 1 \forall s$ and further $q^s_n \to 0$ as $s \to \infty$. Consequently P4. applies and given the bounded below nature of excess demand functions, one may conclude that $p^*.Z(q^s) \to \infty$ as $s \to \infty$; thus, $p^*.Z(q^s) > 0 \forall s$ large enough. But then $\forall s$ large enough, $p^s \notin K$: a contradiction; so no such sequence exists and $K$ is bounded. The closure of $K$ follows from the definition. Next note that the

[5]See for instance Mukherji (1990), (2002), Neuefeind (1980).

distance of $K$ from the boundary of $R^n_+$ denoted by $B$, say is

$$d(K,B) = \inf_{x\in K, y\in B} d(x,y) = \alpha \text{ say}$$

where

$$d(x,y)^2 = \sum_i (x_i - y_i)^2$$

If $\alpha = 0$, then there is a sequence $p^s \in K$ $\forall s$ such that $p^s_j \to 0$ as $s \to \infty$. Since $\|p^s\| \geq 1$ $\forall s$, P4 applies and $\sum_j Z_j(p^s) \to \infty$ as $s \to \infty$; thus exactly as argued above, $p^s \notin K$ $\forall s$ large enough : a contradiction. Hence $\alpha > 0$ as claimed. •

To need to apply our results on the existence of solutions to differential equations, we need to strengthen the property P1 to:

P1$\prime$: For each $j$, $Z_j(p)$ is bounded below and continuously differentiable function of $p$ for all $p \in R^n_{++}$.

We shall say that **Weak Axiom of Revealed Preference (WARP) holds at** $p$ if $p^*.Z(p) > 0$; otherwise, we shall say that WARP is **violated at** $p$**.**If WARP holds for all $p \in R^n_{++}$, then **WARP holds.** For discussions of WARP, see Hildenbrand and Jerison (1989). Given the above, note that

$$K = E \cup \{p \notin E : WARP \text{ is violated at } p\}.$$

To investigate this notion further, let $p^* \in E$ and $p \neq \theta p^*$; then it should be noted that by virtue of Walras Law, using the notation introduced above:

$$p.(X(p) - W) = p.Y(p);$$

further, by virtue of profit maximization, $p.Y(p) \geq p.Y(p^*)$; also by the definition of equilibrium, $Y(p^*) = X(p^*) - W$; consequently, putting all this together, we have $p.X(p) \geq p.X(p^*)$. Thus we have demonstrated the validity of:

**Claim 3.2.2** *At any p, the aggregate demand $X(p)$ costs no less than the aggregate demand $X(p^*)$ at equilibrium.*

**Remark 10** *Had the aggregate demand $X(p)$ originated from the maximization of a single utility function, it would have been possible to argue that at $p^*$, $X(p)$ should be more expensive than $X(p^*)$, i.e., $p^*.X(p^*) < p^*.X(p)$ or retracing the steps taken above, and using the fact that from profit maximization, $p^*.Y(p) \leq p^*.Y(p)$, it would follow that $p^*.Z(p) > 0$: which, of course, is WARP*[6]*.*

### 3.3 Tatonnement Processes

The so-called tatonnement processes have two major properties: first of all, in dis-equilibrium situations price adjustment in each market occurs in the direction of excess demand in that market and secondly, trades occur only at equilibrium prices. These two assumptions about price adjustment and trades figure in Walras' description of how the market figures out what the equilibrium prices are. The first treatment of the price adjustment equations as differential equations was provided by Samuelson (1946), where he wrote the price adjustment

[6]Thus aggregation across individuals has deprived us of these nice properties. It is, in this connection, that the results of Sonnenschein (1972) and Debreu (1974) are crucial. The only restrictions on excess demand functions are that they satisfy homogeneity of degree zero in the prices and obey Walras Law.

equations as

$$\dot{p}_j = F_j(p) \text{ for all j } \neq n \tag{3.1}$$

where $F_j(p)$ are required to satisfy the following restriction:

$$\text{Sign of } F_j(p) = \text{ Sign of } Z_j(p)$$

A special case of the above function $F_j(p)$ has been mostly used where $F_j(p) = k_j.Z_j(p)$ with $k_j > 0$ are some constants. The adjustment equations thus become

$$\dot{p}_j = k_j.Z_j(p) \text{ for all j } \neq n; k_j > 0 \forall j \tag{3.2}$$

It has also been argued that in the above case, one may so define the units of measurement of each commodity that $k_j = 1$ can be chosen without any loss of generality. Using this strategy we shall use the following equation:

$$\dot{p}_j = Z_j(p) \text{ for all j } \neq n;\ p_n = 1 \tag{3.3}$$

We shall have a chance to study the solutions to both ( 3.1 ) and ( 3.3 ) in our analysis below. The analysis of local stability of competitive equilibrium is restricted to situations when the initial point of the solution to the ( 3.1 ) is a point **close** to the equilibrium $p^*$. In such situations, the property of the functions $F_j$, viz., that they are of the same sign as $Z_j$, allows us some advantages. To see these, we have first of all[7],

**Claim 3.3.1** *If f(x) and g(x) are real valued linear functions,* $f, g : R^n \rightarrow R$*, such that* $f(x) \neq 0$ *for some* $x \in R^n$ *and* $f(x) = 0 \iff g(x) = 0$*, then* $f(x) = \alpha.g(x)$*,* $\forall x \in R^n$*, for some* $\alpha$*.*

[7]McKenzie (2002), p. .

Proof. Consider $\overline{x} \in R^n$ such that $f(\overline{x}) \neq 0$. Consider $y \in R^n$ and note that $f(y - \frac{f(y)}{f(\overline{x})}.\overline{x}) = 0$; consequently $g(y - \frac{f(y)}{f(\overline{x})}.\overline{x}) = 0$; hence $g(y) = \alpha.f(y)$ where $\alpha = \frac{g(\overline{x})}{f(\overline{x})}$. •

For any function $f(x)$, we write $\nabla f(x) = (\frac{\partial f(x)}{\partial x_j})$; by virtue of the above claim, if $\nabla F_j(p^*) \neq 0$ then it follows that $\nabla F_j(p^*) = \alpha_j \nabla Z_j(p^*)$ for some $\alpha_j > 0$. Consequently, the linear approximation to ( 3.1 ) may be taken to be

$$\dot{p} = D.A.p \tag{3.4}$$

where $A = (\frac{\partial Z_i(p^*)}{\partial p_j})$ and $D$ is a diagonal matrix with positive $\alpha_j$ down the diagonal. Now for example, redefining units of measurement, we may choose the system

$$\dot{p} = A.p \tag{3.5}$$

Consequently, there are two routes to arriving at a system such as ( 3.5 ); one, from a system such as ( 3.1 ) and the other, from a system such as ( 3.3 ). It would be more convenient to adopt the convention that we have arrived from ( 3.3 ). If the matrix $A$ has all its characteristic roots with real parts negative, we shall say ( 3.3 ) is **linear approximation stable.** This is a stronger requirement than ( 3.3 ) being locally asymptotically stable. The former implies the latter; but one can have the latter condition being met without the former being true.[8] Now the full Jacobian of $Z(p)$ at the equilibrium $p^*$ has some interesting

[8]Consider, for example, $x \in R, \dot{x} = -x^3$; and consider the linear approximation in a neighborhood of equilibrium ( $x = 0$ )given by $\dot{x} = 0$. The difference between the two, lies in the fact that even though linear approximation stability is violated, local asymptotic stability holds. This can happen only when the characteristic root of the matrix of partial derivatives evaluated at equilibrium ( the **Jacobian** of $Z(p)$ without the numeraire row and column ) has zero as a characteristic root.

properties which we note for future reference.

**Claim 3.3.2** *Consider the Jacobian $J(p^*)$ of the excess demand functions given by $(\dfrac{\partial Z_i(p^*)}{\partial p_j})$ where $i, j$ run over all the goods, including the numeraire and $p^*$ is an equilibrium. Then ${p^*}^T.J(p^*) = 0$ and $J(p^*).p^* = 0$ and further let $I = \{i : p_i^* > 0\}$; then $\forall\ i, j, r, s \in I$, $J_{ij}/p_i.p_j = J_{rs}/p_r.p_s$ and $J_{ij} = 0$ if either $i \notin I$ or $j \notin I$ where $J_{ij}$ denotes the cofactor of the i-jth element in $J(p^*)$.*

Proof. By virtue of Walras Law, we have $\sum_i p_i.Z_i(p) = 0$; hence differentiating, with respect to the $j$-th price $p_j$, we have $\sum_i p_i.Z_{ij}(p) + Z_j(p) = 0$ and evaluation at $p^*$ leads to ${p^*}^T.J(p^*) = 0$. Again noting the homogeneity of degree zero in the prices, we have from Euler's Theorem that $J(p^*).p^* = 0$. Next consider the matrix $B =$ adjoint $J(p^*) = (J_{ij})^T$. It is then known that $J(p^*).B = B.J(p^*) = \det J(p^*).I$ If $J_{ij} = 0\ \forall\ i, j$, then the claim follows trivially; hence suppose $J_{ij} \neq 0$ for some $i, j$; then note that rank of $J(p^*)$ is $n - 1$ and hence the solutions to $x^T.J(p^*) = 0$ and $J(p^*).x = 0$ constitute vector spaces of rank 1. Thus rows of $B$ must be scalar multiples of ${p^*}^T$ and columns of B must be scalar multiples of $p^*$; consequently, writing the $i$-th row of $B$ as $b_i$, $b_i = \theta_i.{p^*}^T$ for some scalar $\theta_i$ and the $j$-th column of $B$ as $b^j$, $b^j = \beta_j.p^*$ for some scalar $\beta_j$; it remains to note that $\theta_i = \beta_i$ and hence $J_{ij} = \theta_i p_j^* = \theta_j p_i^*$ and the claim follows. •

**Remark 11** *It should be noted that the full Jacobian $J(p)$ of the excess demand function $Z(p)$, can be symmetric only at the equilibrium $p^*$. This follows from the proof of the above claim; note that $p^T.J(p) = -Z(p)^T$ and $J(p).p = 0$ for any $p$. Consequently, $J(p)$ symmetric $\Rightarrow 0^T = p^T.J^T(p) = p^T.J(p) = -Z(p)^T$; hence $p$ must be an equilibrium.*

We consider next, whether with conditions such as the above, we can ensure that the equilibrium is locally stable, at least.

### 3.4 Local Stability of Tatonnement Processes

We shall consider the local stability of the system ( 3.3 ), i.e., the process

$$\dot{p}_j = Z_j(p) \text{ for all } j \neq n$$

and its linear approximation, around an equilibrium $p^*$, ( 3.5 ), i.e.,

$$\dot{p} = A.p$$

where $A = (\frac{\partial Z_i(p^*)}{\partial p_j})(i, j \neq n)$. It may be recalled also that the stability of ( 3.5 ) imples local asymptotic stability of ( 3.3 ); but not conversely. We shall say that the equilibrium $p^*$ is **linear approximation stable,** if the process ( 3.5 ) is stable i.e., if all the charcateristic roots of $A$ have negative real parts; we shall refer to $A$ being **stable** if all the characteristic roots of $A$ have their real parts negative.We shall say that the equilibrium $p^*$ is **locally stable** if the process ( 3.3 ) is locally asymptotically stable. The conditions for the stability of the matrix $A$ are contained in the Liapunov Theorem ( see Section ); to relate these conditions to excess demand functions, we have the following:

**Claim 3.4.1** *$A$ is stable $\Rightarrow \exists$ a positive definite matrix $B$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$ $\forall p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$, where*

$$\overline{B} = \begin{pmatrix} B & 0 \\ 0^T & 1 \end{pmatrix}.$$

Proof. Since $A$ is stable, by Liapunov's Theorem, there is a positive definite matrix $B$ such that $B.A + A^T.B$ is negative definite. Let $\overline{B}$ be as defined above and let $f(p) = (p^* - p)^T.\overline{B}.Z(p)$. Note that $f(p^*) = 0$; further, writing the elements of the matrix $\overline{B}$ as $(\bar{b}_{ij})$, the partial derivative of the function $f(p)$ as $f_k(p) = \dfrac{\partial f(p)}{p_k}$, $f_{kr}(p) = \dfrac{\partial^2 f(p)}{\partial p_k \partial p_r}$, we have the following:

$$f_k(p) = \sum_{i,j}(p_i^* - p_i).\bar{b}_{ij}.Z_{jk}(p) - \sum_j \bar{b}_{kj}.Z_j(p)$$

and

$$f_{kr}(p) = \sum_{i,j}(p_i^* - p_i).\bar{b}_{ij}.Z_{jkr}(p) - \sum_j \bar{b}_{rj}.Z_{jk}(p) - \sum_j \bar{b}_{kj}.Z_{jr}(p) \ ;$$

where $Z_{jkr}(p) = \dfrac{\partial^2 Z_j(p)}{\partial p_k \partial p_r}$. Now consider $k \neq n; r \neq n$ and evaluate all the above partial derivatives at $p^*$; we have then:

$$f_k(p^*) = 0 \forall k$$

and further,

$$\begin{aligned}(f_{kr}(p^*)) &= -(\sum_j \bar{b}_{rj}.Z_{jk}(p^*) + \sum_j \bar{b}_{kj}.Z_{jr}(p^*)) \\ &= -(\sum_j \bar{b}_{rj}.Z_{jk}(p) + \sum_j \bar{b}_{jk}.Z_{jr}(p))\end{aligned}$$

using the symmetry of the matrix $\overline{B}$; hence, using the fact $k, r \neq n$, we have:

$$(f_{kr}(p^*)) = -(B.A + A^T.B)$$

which ensures the fact that the matrix of second order partial derivatives of the function $f(p)$ at $p^*$, i.e., the matrix $(f_{kr}(p^*))$ is positive definite. Consequently, the function $f(p)$ attains a regular minimum at $p = p^*$ and hence there is a neighborhood $N_\delta(p^*)$ such that $p \in N_\delta(p^*), p \neq p^* \rightarrow f(p) > f(p^*) = 0$. •

**Remark 12** *If the matrix $A$ is stable with the relevant $B = I$, then a local version of the Weak Axiom of Revealed preference holds; since then there is a neighborhood $N_\delta(p^*)$, such that $p \in N_\delta(p^*) \to (p^* - p)^T.\overline{B}.Z(p) = {p^*}^T.Z(p) > 0$ when $B = I$.*

The Claim made above has the following converse:

**Claim 3.4.2** *Suppose there is a positive definite matrix $\overline{B}$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$ $\forall p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$ where*

$$\overline{B} = \begin{pmatrix} B & b \\ b^T & 1 \end{pmatrix}$$

*and the matrix $B.A + A^T.B$ has rank $(n - 1)$, then $A$ is stable.*

Proof. Define $f(p) = (p^* - p)^T.\overline{B}.Z(p)$; note that by the conditions specified, $f(p)$ attains a local minimum at $p = p^*$; hence it follows that the matrix $(f_{kr}(p^*)), k, r \neq n$ must be positive semi-definite; as shown in the proof of the above claim, $(f_{kr}(p^*)) = -(B.A + A^T.B)$; hence it follows that $(B.A + A^T.B)$ must be negative semi-definite; given the rank condition, we may conclude that there is a positive definite matrix $B$ such that $(B.A + A^T.B)$ is negative definite; hence by Liapunov's Theorem, A is stable. •

Without the rank condition, we have the following:

**Claim 3.4.3** *Suppose there is a positive definite matrix $\overline{B}$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$ $\forall p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$ where*

$$\overline{B} = \begin{pmatrix} B & b \\ b^T & \alpha \end{pmatrix}$$

*then $p^*$ is locally stable i.e., the process ( 3.3 ) is locally asymptotically stable.*

Proof. Define $V(p) = (p^* - p)^T.\overline{B}.(p^* - p)$and consider $\nu > 0$ such that $D_\nu(p^*) \subset N_\delta(p^*)$ where $D_\nu(p^*) = \{p : p_n = 1, (p^* - p)^T.\overline{B}.(p^* - p) < \nu\}$. Now consider $p^o \in D_\nu(p^*)$ and the solution to ( 3.3 ) with $p^o$ as initial point, $p(t, p^o)$. Note that

$$\dot{V}(p(t, p^o)) = -2(p^* - p(t, p^o))^T.\overline{B}.Z(p) < 0 \ , \ \forall p(t, p^o) \in D_\nu(p^*)$$

so long as $p(t, p^o) \neq p^*$; consequently $V(p(t, p^o)) \leq V(p^o)$ $\forall t$; moreover, the function $V(p)$ has all the properties of a Liapunov function in the region $D_\nu(p^*)$ and the claim follows. •

**Remark 13** *Supoose $A+A^T$ has rank $(n-1)$. Then the following conditions are equivalent:*

*(i) $p^*$ is an isolated point of the set $K$*

*(ii) ${p^*}^T.Z(p) > 0$ $\forall p \neq p^*$, $p \in N_\delta(p^*)$*

*(iii) $x^T.A.x < 0$ for all $x \neq 0$.*

It should be noted that the rank condition is only required for showing that Condition (ii) $\Rightarrow$ Condition (iii). Thus, the relationship between a local version of the Weak Axiom of Revealed Preference and stability of processes such as ( 3.3 ) or ( 3.5 ) are quite close. Kihlstrom et. al. (1976), contain related results. To put our results in this section in a clear perspective, let us define **Local Generalised WARP** near an equilibrium $p^*$ by: $\exists$ a positive definite matrix $\overline{B}$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$, $\forall p \neq p^*$, $p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$.

Returning to the the Claim , one may note that linear approximation stability of the process ( 3.3 ) implies Local Generalised WARP near the equilibrium $p^*$ where the matrix

$\overline{B}$ is given by:

$$\overline{B} = \begin{pmatrix} B & 0 \\ 0^T & 1 \end{pmatrix}$$

and $B$, a positive definite matrix, satisfies the equation:

$$A^T.B + B.A = -Q$$

for **any positive definite matrix** $Q$. For every such matrix $B$, one may construct $\overline{B}$ and for each such $\overline{B}$, we shall have $(p^* - p)^T.\overline{B}.Z(p) > 0 \ \forall p \neq p^*$, $p \in N_\delta(p^*)$ for some $\delta > 0$.

Thus, Local Generalised WARP is **necessary** for linear approximation stability of the process ( 3.3 ) and is **sufficient** for the local asymptotic stability of the same process; only when a rank condition is met is the condition sufficient for linear approximation stability too. These results indicate what kinds of restrictions have to be in place to ensure local stability.

The major conditions on excess demand functions which have been used to ensure local stability are one of the following:

i. Gross Substitution i.e., $\overline{A} = (\dfrac{\partial Z_i(p^*)}{\partial p_j})$, $\forall i, j$, has all its off-diagonal terms positive. Thus writing $Z_{ij} = \dfrac{\partial Z_i(p^*)}{\partial p_j}$, we must have $Z_{ij} > 0, i \neq j$[9].

ii. The matrix $A = (\dfrac{\partial Z_i(p^*)}{\partial p_j}), i, j \neq n$ has a dominant negative diagonal[10] i.e., the diagonal terms are negative $\exists$ positive numbers $c_1, c_2, \cdots, c_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n -$

[9] In the literature, the weaker sign, i.e., $Z_{ij} \geq 0, i \neq j$, has been referred to as the weak gross substitute case. Usually the matrix $\overline{A}$ is required to be indecomposable, then. We shall consider this case later.

[10] The fundamental property of dominant diagonal matrices is that they are non singular; if, in addition, the diagonal is negative, then all characteristic roots of the matrix have real parts negative: see, McKenzie (1959).

$1, c_j|Z_{jj}| > \sum_{i \neq j} c_i|Z_{ij}|$. An equivalent [11] way of defining diagonal dominance is the following: $\exists$ positive numbers $d_1, d_2, \cdots, d_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n-1, d_j|Z_{jj}| > \sum_{i \neq j} d_i|Z_{ji}|$. While the $c'$s ensure dominance of the diagonal terms over rows, the $d'$s ensure that the diagonal terms dominate across columns. Only in special cases (such as when the matrix is symmetric, for example), it is possible to conclude that the $c'$s and $d'$s match[12]. However, if the $c'$s and $d'$s do match then one may show:

**Claim 3.4.4** *If the matrix $A$ defined above has a dominant negative diagonal with $c_j = d_j \forall j$ then $A$ is quasi-negative definite*[13] *Further WARP is satisfied locally i.e., $\exists$ a neighborhood $N(p^*)$ such that $\forall p \in N(p^*), p \neq p^*, p^*.Z(p) > 0$.*[14]

Proof: Consider the matrix $B = A + A^T$; let a typical element of $B$ be written as $b_{ij}$; writing elements of $A$ as $a_{ij}$, $b_{ij} = a_{ij} + a_{ji}$; we are given that $\forall j$, $c_j|a_{jj}| > \sum_{i \neq j} c_i|a_{ij}|$ **and** $c_j|a_{jj}| > \sum_{i \neq j} c_i|a_{ij}|$ for some positive numbers $c_i$. It follows therefore that $2c_j|b_{jj}| > \sum_{i \neq j} 2c_i|b_{ij}|$, $\forall j$; consequently, the symmetric matrix $B$ has a dominant negative diagonal and is thus negative definite; this, in turn, establishes the fact that $A$ is quasi-negative definite. For the last part, we proceed as in the proof of Claim 3.4.1. Define $f(p) = (p^* - p)^T.Z(p)$; note that $f(p^*) = 0; \nabla f(p^*) = 0$[15] and that $(\frac{\partial^2 f(p^*)}{\partial p_j \partial p_k})_{j,k=1,2,\cdots,n-1} = -(A + A^T)$. These facts imply that the function $f(p)$ attain a **local minimum** at $p = p^*$ and the claim follows. •

[11] See, for example, Mukherji (1975).

[12] For example, the matrix $\begin{pmatrix} -1 & 0.5 \\ 1.5 & -1 \end{pmatrix}$ has a dominant negative diagonal; this may be checked by considering $c_1 = 9, c_2 = 15$ or by $d_1 = 15, d_2 = 9$; but choosing $d_1 = c_1, d_2 = c_2$, will not do.

[13] $A + A^T$ is negative definite.

[14] A global version of this result may be found in Fujimoto and Ranade (1988).

[15] Recall that $\nabla f(p)$ stands for the vector of partial derivatives of the function $f(p)$.

It is easy to check that, given Walras Law and homogeneity of degree zero in prices of the excess demand functions:

**Claim 3.4.5** *i.* $\Rightarrow$ *ii. with* $c_i = d_i = p_i^*$.

Consequently, by virtue of Claim 3.4.4, it follows that a local version of WARP is satisfied for the Gross Substitute case, as well. Thus the importance of WARP should be apparent, even for local stability of equilibrium. It should therefore come as no surprise that we need additional requirements to ensure this property. We turn next, to examples of instability which may be set right by appropriate changes in parameter values without really affecting the substantive nature of the example.

## 3.5 The Gale Example: The Choice of the Numeraire

One of the first examples of an unstable equilibrium is due to Gale (1963). We take that up first to analyze the possible causes of instability in this set up. It is of interest to note that this analysis will also reveal another rather crucial element in the construction of the tatonnement process.

It may be recalled that the ( 3.1 )involved choosing a numeraire; we chose good n to be the numeraire, the unit of account. Ideally of course, it should not matter which commodity is chosen; but before we can rest assured on this aspect, we need some further analysis on this question. One may argue, however, that in the real world, there is no choice of the numeraire available, since the medium of exchange is fixed and one should be interested in whether the economy is stable with respect to this choice of the numeraire; the fact that some other choice of numeraire would have rendered the system stable or unstable should be of no interest. As Arrow and Hahn (1971)[16], indicate, such a remark should be objected to at two levels: first, the argument about the medium of exchange being fixed is what has been termed to be 'casual empiricism' by Arrow and Hahn, since our general equilibrium model has no adequate theory of money. Second in actual situations, there **is** a choice of numeraire: consider, for example, the recent worries on the international markets: whether dollar retains its position as the pre-eminent currency with all transactions being in dollars or whether this position is taken over by the new currency, the euro. The question that will be addressed is whether this difference matters so far as stability questions are considered.

We shall use the Gale example to demonstrate that, unfortunately, the choice of the numeraire does matter. Note first of all that in a two-good economy, with only one relative price to worry about, this kind of problem does not appear. The example due to Gale which we shall consider now, was introduced to exhibit the possibility of the tatonnement process being unstable. There are three goods labelled 0, 1 and 2 and there are two individuals, C and P. C possesses $x^o$ units of good 0 and wishes to exchange it for goods 1 and 2 and has no desire for good 0. P, on the other hand has stocks of goods 1 and 2 given by $x^1, x^2$ which he wants to exchange for good 0 with P having no desire for good 0. Gale chooses good 0 to be the numeraire and analyzes the situation when C maximizes a concave utility function $U(x_1, x_2)$ subject to a budget constraint $p_1.x_1 + p_2.x_2 = x^o$, where $p_i$ denotes the price of good $i$ relative to good 0. The force of the paper was that if good 1 was a **giffen**

[16]See p.305.

good, then one may so choose speeds of adjustment $r_i > 0$ such that the system

$$\dot{p}_i = r_i.f_i(p) i = 1, 2 \tag{3.6}$$

is unstable; in the above, $f_i(p)$ denotes the excess demand for good $i$. In particular, Gale considers the linear approximation of the above system at equilibrium so that, as we have seen in the previous section, stability results depend entirely on the characteristic roots of the following matrix ( where $f_{ij}$ denotes the partial derivative of $f_i$ with respect to the price of good $j$ evaluated at equilibrium):

$$\begin{pmatrix} r_1 f_{11} & r_1 f_{12} \\ r_2 f_{21} & r_2 f_{22} \end{pmatrix}$$

By virtue of the giffen good assumption, $f_{11} > 0$; also, since C consumes good 1 and 2 only and no one else consumes goods 1 and 2, it follows that $f_{22} < 0$. However, $r_1$ can be so chosen that the trace of the above matrix $r_1 f_{11} + r_2 f_{22} > 0$, thus violating one of the necessary conditions for stability of ( 3.6 ). This was what Gale had shown.

Following Mukherji (1973), suppose we decide to choose good 1 to be the numeraire; clearly, this is possible, provided we assume that good 1 is not free. Note that no other giffen good can exist: we have already noted that good 2 cannot be giffen; and since P consumes good 0 only and no one else consumes good 0, good 0 cannot be giffen. With the change in the numeraire, C solves the problem

$$\text{Maximize } U(x_1, x_2)$$

$$\text{subject to } x_1 + p_2 x_2 = p_0 x^o$$

where $p_i$ for $i = 0, 2$ now represent the price of good $i$ relative to good 1. The demand functions are given by $x_i(p_0, p_2)$; consequently the excess demand functions are given by $f_i(p_0, p_2) = x_i(p_0, p_2) - x^i$ for $i = 1, 2$.So much for C. So far as P is concerned, P demands $x_0(p_0, p_2) = x^1 + p_2 x^2$; consequently, the excess demand for good 0 is given by $f_0(p_0, p_2) = x_0(p_0, p_2) - x^o$. The first point to note is that, writing $f_{ij}$ as the partial derivatives of $f_i$ with respect to $p_j$, $1 = 1, 2$,$j = 0, 2$ evaluated at equilibrium:

**Claim 3.5.1**

$$\det \begin{pmatrix} f_{10} & f_{12} \\ f_{20} & f_{22} \end{pmatrix} < 0$$

*provided that C's utility maximization exercise is solved at an interior point and that second order conditions are met.*

Proof. From the first order conditions at an interior maximum, we have the following:

$$\begin{pmatrix} U_{11} & U_{12} & -1 \\ U_{21} & U_{22} & -p_2 \\ -1 & -p_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ d\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ \lambda dp_2 \\ x_2 dp_2 - x^o dp_0 \end{pmatrix} \tag{3.7}$$

Let us write the determinant of the matrix on the left hand side as $\det A$ which may be taken to be positive assuming that the second order condition is satisfied; note further that the determinant in the claim is given by

$$f_{10}.f_{22} - f_{20}.f_{12} = x_{10}.x_{22} - x_{20}.x_{12}$$

from the definition of the the excess demand functions. Substituting the values of $x_{ij}$ from ( 3.7 ), we have that

$$\det \begin{pmatrix} f_{10} & f_{12} \\ f_{20} & f_{22} \end{pmatrix} = -\lambda.x^o / \det A < 0.$$

•

Next, notice that with the change in the numeraire, the system that we ought to be considering is

$$\dot{p}_i = r_i.f_i(p_0, p_2)\ ,\ i = 0{,}2 \tag{3.8}$$

and consequently the matrix which will decide the (local) stability of the above is given by

$$\begin{pmatrix} f_{00} & f_{02} \\ f_{20} & f_{22} \end{pmatrix}$$

We already have that the trace of this matrix is negative (recall: no other giffen goods). To establish that the determinant is positive, we need to use a result we had proved earlier.

By virtue of the above and given the result of Claim 3.3.2, it follows that

$$\det \begin{pmatrix} f_{00} & f_{02} \\ f_{20} & f_{22} \end{pmatrix} > 0$$

since the Claim 3.3.2 establishes that a cofactor of order 2 is positive ( the relevant cofactor is (-1) times the determinant in Claim 3.5.1 ). Consequently, the equilibrium is locally stable under the process ( 3.8 ).

Note that the example of instability provided by Gale has been converted to a stable case by changing the numeraire. There was only one good, in the Gale example, for which the price could move in the **wrong** direction; whereas Gale aggravated this movement, what has been done above is to eliminate the movement entirely. Note however, the two processes ( 3.6 ) and ( 3.8 ) are of course different. We can sum up the discussions here by means of the following:

**Proposition 3.1** *In general, stability properties of equilibrium depend upon the choice of the numeraire.*

We next provide a sufficient condition for stability of equilibrium to be insensitive to the choice of the numeraire. We shall focus entirely on the system ( 3.3 ):

$$\dot{p}_i = Z_i(p)\ \forall i \neq n\ ;$$

Let the equilibrium $p* > 0$. We shall say that the stability of equilibrium is **insensitive to the choice of the numeraire,** if the linear approximation to the above system is stable for all choice of n. The following points should be obvious: choosing another good , say good 1, as the numeraire, amounts to considering a new equilibrium price vector $q^* = \frac{1}{p_1^*}.p^*$; note that the system ( 3.3 ) also changes to

$$\dot{q}_i = Z_i(q)\ \forall i \neq 1\ ; \tag{3.9}$$

The stability of the linear approximation to ( 3.3 )depends on the matrix $J(p^*) = (Z_{ij}(p^*); i, j = 1 \cdots n-1)$; for the stability of the linear approximation to the system ( 3.9 ), on the other hand, we need to consider the matrix $J(q^*) = (Z_{ij}(q^*); i, j = 2 \cdots n)$. We need to connect the stability properties of these two matrices.

It would be more convenient, at this stage to consider the non-normalized price vector $P = (P_1, P_2 \cdots P_n)$ and the excess demand functions $Z_i(P)$. We shall denote the Jacobian, including the numeraire row and column by $A = (a_{ij}) = (\dfrac{\partial Z_i(P)}{\partial P_j}), i, j = 1, 2 \cdots n$; we write the cofactor of the $i-j$-th element in $A$ by $A_{ij}$. We have first of all,

**Claim 3.5.2** $A_{11} = \frac{1}{P_n^*}.J(p^*)$

Proof. By virtue of homogeneity of degree zero:

$$Z_i(P_1, P_2, \cdots P_{n-1}, P_n) = Z_i(p_1, p_2, \cdots, p_{n-1}, 1)$$

where each $p_i = P_i/P_n = g_i(P),$ say . Thus, for $i, j \neq n$

$$\frac{\partial Z_i(P)}{\partial P_j} = \sum_k \frac{\partial Z_i(p)}{\partial p_k} \cdot \frac{\partial g_k}{\partial P_j}$$

$$= \frac{1}{P_n} \cdot \frac{\partial Z_i(p)}{\partial p_j}$$

Evaluating at equilibrium $p^* = 1/P_n^*.P^*$, the claim follows. •

The principal minors $A_{ii}, A_{kk}$ of the matrix $A$ defined above have the following relationship:

**Claim 3.5.3** *Consider $i, k$ such that $P_i^*, P_k^* \neq 0$. Then $x^T.A_{ii}.x < 0$ for all $x \neq 0$ $\iff$ $x^T.A_{kk}.x < 0$ for all $x \neq 0$.*

Proof. Let $P_i^*, P_k^* > 0$ and let $x^T.A_{ii}.x < 0$ for all $x \neq 0$. Writing $B_{ii} = A_{ii} + A_{ii}^T$, note that then $B_{ii}$ is negative definite. Consider $B = A + A^T$, where $A$ is as defined above; we first claim that $A$ is negative semi-definite, i.e., $x^T.B.x \leq 0$ for all $x$; for suppose to the contrary, there is some $x^*$ such that ${x^*}^T.B.x^* > 0$; note that $x^* \neq 0$; in particular, $x_i^* \neq 0$; since otherwise, writing $\overline{x}^* = (x_1^*, \cdots x_{i-1}^*, x_{i+1}^*, \cdots x_n^*,$ ${x^*}^T.B.x^* = {\overline{x}^*}^T.B_{ii}.\overline{x}^* < 0$: a contradiction. Without any loss of generality, we can take $x_i^* > 0$. Now note that ${P^*}^T.B.P^* = 0$; define $t = x*_i/P*_i > 0$ and $v = x^* - t.P^*$; then $v^T.B.v = {x^*}^T.B.x^* > 0$; on the other hand, since $v_i = 0$, writing $\overline{v} = (v_1, \cdots v_{i-1}, v_{i+1}, \cdots v_n)$, we have $v^T.B.v = \overline{v}^T.B_{ii}.\overline{v} \leq 0$: a contradiction; hence no such $x^*$ can exist and $x^T.B.x \leq 0$ for all $x$.

Consequently $B_{kk} = A_{kk} + A_{kk}^T$, being a principal minor of $B$ is also negative semidefinite i.e., $x^T.B_{kk}.x \leq 0$ for all $x$. Next we note that in case $x^T.B_{kk}.x = 0$ for some $x \neq 0$, then we must have $B_{kk}.x = 0$; that is $B_{kk}$ must be singular; But since, ${P^*}^T.B = 0$ and $B.P^* = 0$, the results of Claim are applicable, and we may conclude that $B_{kk}$ must be non-singular, since $B_{ii}$ is given to be so; i.e., we must have $x^T.B_{kk}.x < 0$ for all $x \neq 0$; i.e., $x^T.A_{kk}.x < 0$ for all $x \neq 0$. Interchanging the roles of $A_{ii}$ and $A_{kk}$, the converse follows. •

By virtue of the above result, we may conclude as follows; suppose that we choose good 1 as numeraire; then we observe that the linear approximation to the process ( 3.3 ) is stable; i.e., the matrix $J(p^*)$ is stable. By virtue of Liapunov's Theorem, this means that there is some positive definite matrix $C$ such that $C.J(p^*) + J(p^*)^T.C$ is negative definite. Suppose further that this $C = I$; then note that $x^T.J(p^*).x < 0$ for all $x \neq 0$; further, by virtue of Claim , this means that $x^T.A_{11}.x < 0$ for all $x \neq 0$; consequently, by virtue of Claim , we have that $x^T.A_{kk}.x < 0$ for all $x \neq 0$ for all $k$ which are permitted to be chosen as numeraire (i.e., they are not free); again using Claim , it follows then that $x^T.J(q^*).x < 0$ for all $x \neq 0$ and this allows us to conclude that the linear approximation to the process ( 3.9 ) is stable. Thus, if the linear approximation to the tatonnement process for some choice of numeraire, is stable with the corresponding Jacobian to be quasi-negative definite[17], then stability of the linear approximation to the tatonnement with any other choice of the numeraire would follow.

The problem of sensitivity to numeraire choice appears therefore, because stability of

[17] A matrix $M$ is quasi-negative definite if $x^T.M.x < 0$ for all $x \neq 0$; in case, $M$ is symmetric, then $M$ is negative definite.

the linear approximation need not imply that the relevant Jacobian $J$ is quasi-negative definite; stability is equivalent to there being a positive definite $C$ such that $C.J + J^T.C$ is negative definite or that $C.J$ is quasi-negative definite for some positive definite $C$. Finally, it should be pointed out that for symmetric matrices, this gap is closed; i.e., a necessary and sufficient condition for all characteristic roots of a symmetric matrix to be negative (for symmetric matrices, all characteristic roots are real) is that the matrix be negative definite. Thus if the jacobian of the excess demand function happens to be symmetric, then stability of the linear approximation with one choice of numeraire implies that the linear approximation with any other choice of numeraire would also be stable.

### 3.6 The Scarf Example

We have already seen, in the last section, that the presence of giffen goods, for example, may destroy the stability properties of the tatonnement. In the present section we indicate that there may be other, seemingly more robust difficulties, for the stability of the tatonnement. We do this by considering an example due to Scarf (1960). Consider an exchange model where there are three individuals $h = 1, 2, 3$ and three goods $j = 1, 2, 3$. The utility functions and endowments are as under:

$$U^1(q_1, q_2, q_3) = min(q_1, q_2);\ w^1 = (1, 0, 0)$$

$$U^2(q_1, q_2, q_3) = min(q_2, q_3);\ w^2 = (0, 1, 0)$$

$$U^3(q_1, q_2, q_3) = min(q_1, q_3);\ w^3 = (0, 0, 1)$$

Routine calculations lead to the following excess demand functions, where good 3 is treated as numeraire (i.e., $p_3 = 1$):

$$Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}$$

$$Z_2(p_1, p_2) = \frac{p_2(p_1 - 1)}{(1 + p_2)(p_1 + p_2)}$$

$$Z_3(p_1, p_2) = \frac{p_2 - p_1}{(1 + p_1)(1 + p_2)}$$

and the tatonnement process, for this example is given by

$$\dot{p}_i = Z_i(p_1, p_2) \ i = 1, 2 \tag{3.10}$$

Notice that equilibrium for this exchange model (and for the process defined above) is given by $p_1 = 1, p_2 = 1$. It would be helpful to transform variables by setting $x_i = p_i - 1$ for $i = 1, 2$. With this change in variables, our process becomes

$$\dot{x}_1 = -\frac{x_2(1 + x_1)}{(x_1 + 2)(x_1 + x_2 + 2)} \ , \ \dot{x}_2 = \frac{x_1(1 + x_2)}{(x_2 + 2)(x_1 + x_2 + 2)} \tag{3.11}$$

In what follows, we shall analyse the answer to the following question: given an arbitrary $x^o = (x_1^o, x_2^o)$, how does the solution $x(t, x^o)$ to ( 3.11 ) behave as $t \to \infty$ ?

We introduce the function $v : R \to R$ by

$$v(x) = \frac{x^2}{2} + x - \ln(1 + x)$$

which is continuously differentiable for all $x$ such that $1 + x > 0$. One may show that

**Claim 3.6.1** *$v(x) > 0$ if $x > -1$, $x \neq 0$; $v(0) = 0$.*

Proof. Note that $v(0) = 0$, and for $x > -1$ we have $v'(x) = \dfrac{x(x+2)}{1+x}$ and $v''(x) = 1 + \dfrac{1}{1+x^2}$. Thus for $x > -1$, $v(x)$ is strictly convex with $v'(0) = 0$; hence $x = 0$ yields a global minimum for $v(x)$ for all $x > -1$. •

Next define $V(x_1, x_2) = v(x_1) + v(x_2)$. We have then:

**Claim 3.6.2** *Along the solution $x(t, x^o)$ to ( 3.11 ), $\dot{V} = 0$ provided $x_i(t, x^o) > -1$ for $i = 1, 2$.*

Proof. Note that

$$\dot{V} = v'(x_1).\dot{x}_1 + v'(x_2).\dot{x}_2 = 0$$

•

We may next claim

**Claim 3.6.3** *Given $x^o = (x_1^o, x_2^o), x_i^o > -1, i = 1, 2$ the solution $x(t, x^o)$ to ( 3.11 ) is such that $\exists a_i, b_i$ such that $-1 < a_i < b_i$ and $x(t, x^o) \in [a_1, b_1] \times [a_2, b_2]$ $\forall t > 0$.*

Proof. Follows from the last two claims. •

For local stability, it may be of some interest to note the following

**Claim 3.6.4** *For $x$ small, v(x) $\approx x^2$.*

Proof. This follows since for $x$ small one may use the following approximation:

$$\ln(1+x) \approx x - \frac{x^2}{2}$$

•

The above may be used to classify the the solution to ( 3.11 ) when the initial point $x^o$ is close to the equilibrium i.e., the origin. It is approximately a circle with the center origin and passing through $x^o$. In the general case, the nature of the orbit is provided by the Figure 5 below: the cyclical behavior of prices around equilibrium are revealed; however, since the figure cannot be taken as a demonstration, we provide such a demonstration, next.

FIGURE 5: The Orbits of the Scarf Example

First of all, note that since $V(t) = V(x_1(t), x_2(t)) = V(x_1^o, x_2^o)$ for all $t$, it follows that the solution or trajectory $x(t, x^o) = (x_1(t), x_2(t))$ is bounded and each $x_i(t)$ is bounded away from $-1$: since if either of these conditions is violated, $V(t)$ would tend to $+\infty$. Hence the $\omega$-limit set corresponding to $x^o$, $L_\omega(x^o)$, is non-empty and compact; also, $(0,0) \notin L_\omega(x^o)$ if $x^0 \neq (0,0)$ (remember, $(0,0)$ is the equilibrium for the system) hence by the Poincaré-Bendixson theorem[18] $L_\omega(x^o)$ must be a closed orbit. This means that either we have a *limit cycle* or the trajectory $x(t, x^o)$ itself is a closed orbit.

If there is a limit cycle $\mathcal{L}$, then by virtue of the Claim **??**, it follows that for any $y \in \mathcal{L}, V(y) = V(x^o)$; further, in such circumstances, there would be a neighborhood $\mathcal{N}$ of $x^o$ such that for any solution $x(t, y)$ originating from any $y \in \mathcal{N}$, $x(t, y) \to \mathcal{L}$[19]. Consequently, we must have $V(y) = V(x^o) \forall y \in \mathcal{N}$: this of course, is not possible, since the function $V$ cannot be constant on an open set. Hence no such limit cycle exists. And the solution $x(t, x^o)$ must be a closed orbit. Thus we have shown the following to be true:

[18] See, for instance, [10] p. 248.

[19] See, for instance, [10] p.251.

**Claim 3.6.5** *For any initial configuration $x^o$, the solution to ( 3.11 ), $x(t, x^o)$ is a closed orbit around the equilibrium* $(0, 0)$.

We turn, next to the set $K$ introduced in the Section 3.2. For the particular case under consideration, the set $K$ is given by:-

$$K = \{(p_1, p_2) : Z_1(p_1, p_2) + Z_2(p_1, p_2) + Z_3(p_1, p_2) \leq 0\}.$$

Using the expressions for $Z_i(p_1, p_2)$, we have that

$$K = \{(p_1, p_2) : (p_1 - p_2).(1 - p_1).(1 - p_2) \leq 0\}$$

FIGURE 6: The set K for the Scarf Example

In Figure 6, the shaded portions constitute the set $K$; the arrows indicate the direction of price movements on the boundary of the set K. In the unshaded portions, i.e., on $K^c$, $\sum_{i=1}^{3} Z_i(p) > 0$. Now consider $d(t) = \sum_{i=1}^{2}(p_i(t, p^o) - 1)^2$ where $p(t, p^o)$ denotes the solution to ( 3.10 ). Hence along this solution, we have

$$\dot{d} = 2\sum_{i=1}^{2}(p_i(t, p^o) - 1).\dot{p}_i = 2\sum_{i=1}^{3} Z_i(p(t, p^o))$$

Hence, it follows that

**Claim 3.6.6** *$\dot{d} < 0$ in $K^c$ while $\dot{d} \geq 0$ in $K$.*

Returning to the solution $p(t, p^o)$ of the system ( 3.10 ) through an arbitrary $p^o = (p_1^o, p_2^o) \neq (1, 1)$, by virtue of the Claims made above, one may conclude the following:

**Remark 14** *a.* $p_i(t, p^o) > 0$ *for all* $t$*;*

*b.* $W(p(t, p^o)) = \sum_i \{\frac{(p_i(t,p^o)-1)^2}{2} + (p_i(t, p^o) - 1) - \ln p_i(t, p^o)\}$ *is constant and* $= W(p^o)$ *for all* $t$*;*

*c.* $\dot{d} > 0$ *in the interior of* $K$*,* $\dot{d} < 0$ *in* $K^c$ *while* $\dot{d} = 0$ *on the boundary of* $K$ *( See Claim 3.6.6, above);*

*d. The solution* $p(t, p^o)$ *enters* $K$ **and** leaves $K$ repeatedly.

For the Scarf example, the solution is a closed curve given by the above Remark (see (b)); consequently, the set of limit points $L$ coincide with this curve; note that the Poincaré-Bendixson Theorem stated that so long as the set of limit points do not contain an equilibrium, again guaranteed by the point (b) noted above, a cycle is the only possible alternative. As we hope to show below, there are some more interesting features of the Scarf Example.

### 3.6.1 Hopf Bifurcation for the Scarf Example

We introduce next, a parameter say $b$, which stands for the amount of second good which individual 2 owns completely. Thus the value of $b = 1$ would revert back to the example considered above. We continue to treat good 3 as the numeraire and then compute excess demand functions for the non-numeraire commodities for the case at hand; it turns out that these are given, using the same notation as above, by the following expressions:

$$Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}$$

$$Z_2(p_1, p_2) = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)}$$

Consequently the system ( 3.10 ) now takes the form:

$$\dot{p_1} = \frac{p_1(1-p_2)}{(1+p_1)(p_1+p_2)} \text{ and } \dot{p_2} = \frac{p_2(p_1-b)+(1-b)p_1}{(1+p_2)(p_1+p_2)} \tag{3.12}$$

Once more standard computations ensure that the **unique equilibrium** is given by

$$p_1^* = \frac{b}{2-b} = \theta \text{ say}, p_2^* = 1$$

Thus it may be noted that our choice of the parameter places a restriction on its magnitude

$$0 < b < 2;$$

and we shall take it that this is met. Notice also that when $b = 1$, $\theta = 1$ too, and we have the earlier situation. That there have been some changes to the stability property of equilibrium is contained in the next claim:

**Claim 3.6.7** *For the process ( 3.12 ), $(\theta, 1)$ is a locally asymptotically stable equilibrium if and only if $b < 1$; for $b > 1$, the equilibrium is locally unstable.*

Proof: The characteristic roots of the Jacobian of the system ( 3.10 ) evaluated at the equilibrium are given by:

$$\frac{1}{8}(-b+b^2 \pm \sqrt{b}\sqrt{(-32+49b-26b^2+5b^3)})$$

and it should be noted that for $0 < b < 1.5$ approximately , the characteristic roots are imaginary; moreover, the real part, viz., $-b+b^2 < 0 \Leftrightarrow b < 1$ and the claim follows. •

We are now ready to show that:

**Claim 3.6.8** *For the system ( 3.12 ), the unique equilibrium* $(\theta, 1)$ *is globally stable whenever* $b < 1$. *When* $b > 1$ *any solution with an arbitrary non-equilibrium initial point is unbounded.*

Proof. Consider the function:

$$W(p_1, p_2) = 2(1-b)p_1 + (2-b)p_1^2/2 - b\log p_1 + p_2^2/2 - \log p_2$$

Then consider the derivative of the function $W(.,.)$ along any solution to the system ( 3.12 ), we have:

$$\begin{aligned}\dot{W} &= \{((2-b)p_1 - b)(1+p_1)\}\frac{\dot{p_1}}{p_1} + (p_2^2 - 1)\frac{\dot{p_2}}{p_2} \\ &= -(1-p_2)^2\frac{p_1(1-b)}{p_2(p_1+p_2)} < 0\end{aligned}$$

whenever $b < 1$ and $p_2 \neq 1$. We may now conclude that the function $W(p_1, p_2)$ is a Liapunov function for the system and the first part of the claim follows. For the remaining part note that whenever $b > 1$, $\dot{W} \geq 0$ along any solution. The main point of interest about the function $W(p_1, p_2)$ is that it is a **strictly convex function** with an absolute minimum at $(p_1^*, p_2^*)$. Suppose then that some solution remains bounded and hence, limit points exist; consequently, along such a solution, $W(t)$ will be monotonically non-decreasing and bounded too and hence convergent and thus $\dot{W}$ must converge to zero; one may conclude that any limit point for the bounded solution must be the equilibrium and consequently, $W(t)$ is **non-decreasing and converges** to its minimum value: thus $W(t)$ must be constant and the only possibility for a bounded solution is that it must begin from the equilibrium, as claimed. •

Thus an easy stability condition for the Scarf Example is that $b < 1$; just as, for a meaningful equilibrium to exist, we need to have $b < 2$ a more stringent requirement has to be placed on the magnitude of $b$ to ensure global stability[20]. More importantly, it is clearly demonstrated that income effects need not necessarily be the villain of the piece. In the Scarf example, there are no substitution effects, yet it is possible to have global convergence.

### 3.7 Global Stability of Tatonnement Processes

Recall the equations ( refeq:TG ):

$$\dot{p}_j = F_j(p) \text{ for all j } \neq n \tag{3.13}$$

where $F_j(.)$ has the same sign as the excess demand functions $Z_j(.)$. For price adjustment, which is triggered off from some arbitrary initial $p \in R^n_{++}$, the above form seems to be the best suited as a candidate. We have already seen that unless the excess demand functions are restricted in some manner (i.e., beyond the properties P1-P4 mentioned in Section 3.2 ), the convergence to equilibrium can not be assured, even when the initial price is **close** to the equilibrium. When the initial price is not subjected to this restriction, it is not

[20]In the context of the Scarf Example, several results of interest may be referred to: Hirota (1981) and (1985) and Anderson et. al. (2002). The first set of papers establishes the proposition that there are many distribution of the endowments which guarantee stability. Instead of the corner point chosen by Scarf (1960) and that we have followed, Hirota shows how redistribution of the totals will lead to global stability. In Anderson et. al., there is an example of an endowment distribution which ensures global stability; the endowment pattern is as follows: each individual possess the stock of the good that each is not interested in. A similar example in a two-good context is available in Gale(1963).

surprising that there have to be restrictions of some kind as well.

A principal condition under which convergence has been assured is that excess demand functions satisfy the assumption of **gross substitution (GS):**

$$\frac{\partial Z_i(p)}{\partial p_j} > 0 \text{ for } i \neq j, \text{ for all i, j, for all } p \in R^n_{++}$$

Note that the above definition is valid only over strictly positive prices. For, we have the following:

**Remark 15** *The extension of the above definition to include the entire non-negative orthant, i.e.,* $R^n_+$ *conflicts with the homogeneity property of excess demand functions. For suppose, we are to insist that the above holds over the boundary of the non-negative orthant as well; consider* $\overline{p}$ *with* $\overline{p_j} \geq o$ *for all* $j$*,* $\overline{p_1} = 0$*,* $\overline{p_k} > 0$ *for some* $k \neq 1$*. Now, consider* $\lambda > 1$ *and* $Z_1(\lambda\overline{p}) = Z_1(\overline{p})$*, by homogeneity of excess demand functions; whereas by gross substitution,* $Z_1(\lambda\overline{p}) > Z_1(\overline{p})$*.*

Sometimes gross substitution is defined with a weak inequality i.e.,

$$\frac{\partial Z_i(p)}{\partial p_j} \geq 0 \text{ for } i \neq j, \text{ for all i, j, for all } p \in R^n_{++}$$

we shall refer to this as the property of **weak gross substitution (WGS)** to distinguish it from the former. Let $I = [1, 2, \cdots, n]$; the economy is said to be **decomposable at** $p$ , if there is a **nonempty, proper subset** $J \subset I$ (i.e., $J \neq I$,) such that

$$\frac{\partial Z_i(p)}{\partial p_j} = 0 \text{ for } i \notin J, j \in J$$

If no such subset $J$ exists, the economy is said to be **indecomposable at** $p$**.** If the economy is indecomposable for every $p \in R^n_{++}$, then the economy is indecomposable. Weak gross substitution with indecomposability implies almost all the properties of gross substitution. First of all,

**Claim 3.7.1** *GS implies that the equilibrium is unique, upto scalar multiples.*

Proof: We note that by virtue of property P4, $p$ is an equilibrium implies that $p \in R^n_{++}$.Suppose then, that $p, q$ are equilibria with $p \neq \delta q$ for any $\delta > 0$. Define $\theta = \text{Max}_i p_i/q_i = p_k/q_k$, say. Note that $\theta q_i \geq p_i$ for all $i$; in particular note further that for $i = k$, equality holds (i.e., $\theta q_k = p_k$) while there must be some $i$ for which the strict inequality holds. By homogeneity property of excess demand functions, P3, it follows that $Z_k(\theta.q) = Z_k(q) = 0$; on the other hand, from GS and differentiability, we have:

$$Z_k(\theta.q) = Z_k(p) + \sum_j Z_{kj}(\overline{p}).(\theta q_j - p_j) > 0$$

where $\overline{p}$ is some point lying on the line segment joining $p$ to $\theta q$; this establishes a contradiction and hence equilibrium must be unique upto scalar multiples. •

**Remark 16** *We shall show later that uniqueness follows if we have WGS together with indecomposability.*

A second condition that has been used to generate global stability results is the condition of dominant negative diagonals, a condition which we saw also implied local stability in Section 3.4. We shall consider this restriction too, below. Finally, we shall consider motion on the plane and attempt to derive some global stability conditions.

### 3.7.1 Global Stability and WGS: The McKenzie Theorem

We first take up for consideration a result due to McKenzie(1960). The main interest lies in the fact that this result uses the most general form of the price adjustment process. Recall the equations ( 3.1 ):

$$\dot{p}_j = F_j(p) \text{ for all j } \neq n$$

where $F_j(.)$ has the same sign as the excess demand functions $Z_j(.)$. We assume that the excess demand functions satisfy the properties P1-P4; further, we assume that the assumption of **WGS** holds and that the economy is **indecomposable** . And finally, the functions $F_j(p)$,$j \neq n$ are assumed to be continuously differentiable for all $p \in R^n_{++}$. Under this assumption, for any $p^o \in R^n_{++}$, the solution $p(t, p^o)$ to the system ( 3.1 ) exists.

Let $I = \{1, 2, \cdots, n\}$: the set of all goods including the numeraire. We define the following:

$$P(p) = \{i \in I : Z_i(p) \geq 0\} \ ; \ N(p) = \{i \in I : i \notin P(p)\}$$

and

$$P'(p) = \{i \in I : \dot{p}_i \geq 0\} \ ; \ N'(p) = \{i \in I : i \notin P'(p)\}$$

Thus $P(p)$ denotes the set of goods at $p$ which have a non-negative excess demand; while $P'(p)$ refers to the set of goods whose price adjustment is non-negative at $p$. Notice that $n \in P'(p)$ since $\dot{p}_n = 0$ but $n$ may or may not be an element of $P(p)$. But $P(p) \subset P'(p)$. This distinction allows us to define:

$$V(p) = \sum_{i \in P(p)} p_i.Z_i(p) \text{ and } V'(p) = \sum_{i \in P'(p)} p_i.Z_i(p)$$

Further,

$$V'(p) = \sum_{i \neq n} Max(p_i.Z_i(p), 0) + Z_n(p)$$

Thus

$$V'(p) = \begin{cases} V(p) + Z_n(p) & \text{if } Z_n(p) < 0 \\ V(p) & \text{otherwise} \end{cases} \tag{3.14}$$

**Claim 3.7.2** *$Z(p) \neq 0 \Rightarrow V(p) > 0$; $p \in R^n_{++} \Rightarrow V'(p) \geq 0$.*

Proof. In case $Z(p) \neq 0$,$\exists i$ such that $Z_i(p) > 0$ and hence $V(p) > 0$. For the other part of the claim, note that by virtue of Walras Law, we have for all $p \in R^n_{++}$:

$$\sum_{i \in P'(p)} p_i.Z_i(p) + \sum_{i \in N'(p)} p_i.Z_i(p) = 0;$$

Note that the first term is $V'(p)$. In case $N'(p)$is empty, $V'(p) = 0$; in case $N'(p)$is non-empty, the second term above is negative; and hence the first term must be positive, i.e., $V'(p) > 0$. Since these are the only two possibilities, the claim follows. •

Let $E, F$ be two disjoint nonempty subsets of $I$ such that $E \cup F = I$. Since we have assumed that the economy is indecomposable, it follows directly that $\forall\, p \in R^n_{++}$, $\exists i \in E, j \in F$ such that $Z_{ij}(p) > 0$. For our convergence argument, we require something weaker, which does not require indecomposability but follows from WGS. This was what McKenzie had used and for the sake of completeness we provide the following:

**Claim 3.7.3** *Let $p \in R^n_{++}$ and $E, F$ be as defined above; further assume that $\sum_{i \in E} p_i.Z_i(p) \geq \epsilon > 0$ for some $\epsilon$; WGS $\Rightarrow \exists i \in E, j \in F$ such that $Z_{ij}(p) > 0$.*

Proof. Differentiating the Walras Law expression $\sum_i p_i.Z_i(p) = 0$ with respect to $p_j$, we have:

$$\sum_i p_i.Z_{ij}(p) = -Z_j(p).$$

Multiplying both sides by $p_j$ and summing over for $j \in E$, we have:

$$\sum_i \sum_{j \in E} p_i.Z_{ij}(p).p_j = -\sum_{j \in E} p_j.Z_j(p) \leq -\epsilon \text{ from hypothesis}$$

$$\text{Thus } \sum_{i \in E} \sum_{j \in E} p_i.Z_{ij}(p).p_j + \sum_{i \in F} \sum_{j \in E} p_i.Z_{ij}(p).p_j \leq -\epsilon$$

WGS implies that the second term is non-negative, given the fact that $E, F$ are nonempty and disjoint. Hence we may conclude

$$\sum_{i \in E} \sum_{j \in E} p_i.Z_{ij}(p).p_j \leq -\epsilon \tag{3.15}$$

Next from the property P3, that is the homogeneity property of excess demand functions, it follows that $\forall\ p \in R^n_{++}$, $\sum_j Z_{ij}(p).p_j = 0$. Hence $\forall\ p \in R^n_{++}$

$$\sum_{i \in E} \sum_j p_i.Z_{ij}(p).p_j = 0$$

$$\text{or } \sum_{i \in E} \sum_{j \in E} p_i.Z_{ij}(p).p_j + \sum_{i \in E} \sum_{j \in F} p_i.Z_{ij}(p).p_j = 0$$

or using ( 3.15 ), we have

$$\sum_{i \in E} \sum_{j \in F} p_i.Z_{ij}(p).p_j \geq \epsilon > 0$$

and the claim follows. •

Returning to the functions, $V(p), V'(p)$, note that these are both continuous functions of $p\ \forall\ p \in R^n_{++}$. However, given the involvement of the function $Max$ , the functions may lack derivatives at some p such that $Z_i(p) = 0$ for some $i$; in such situations, the right hand and left derivatives will always exist; these however may not be equal. Keeping this in mind, we can proceed as follows:

**Claim 3.7.4** *Let $V(t) = V(p(t,p^o))$ and $V'(t) = V'(p(t,p^o))$; whenever derivatives exist, $\dot{V}'(t) \leq 0$; if $V'(t) > 0$, the inequality is strict. $\dot{V}(t) \leq 0$ whenever derivatives exist.*

Proof. Writing $p(t)$ for $p(t,p^o)$, we note that

$$V'(t) = \sum_{i \in P'(p(t))} p_i(t).Z_i(p(t));$$

if derivatives exist, ( recall that $p_n(t) = 1 \forall t \Rightarrow \dot{p}_n = 0, \forall t$ ):

$$\dot{V}'(p(t)) = \sum_{i \in P'(p(t))} \{\dot{p}_i.Z_i(p(t)) + p_i(t).\sum_j Z_{ij}(p(t)).\dot{p}_j\}$$

$$= \sum_{i \in P'(p(t))} \{\dot{p}_i(-\sum_j p_j(t).Z_{ji}(p(t))) + p_i(t).\sum_j Z_{ij}(p(t)).\dot{p}_j\}$$

Splitting up the sum over $j$ into sum over $j \in P'(p(t))$and $j \notin P'(p(t))$, and cancelling terms, we have:

$$\dot{V}'(p(t)) = -\sum_{i \in P'(p(t))} \sum_{j \notin P'(p(t))} \dot{p}_i.Z_{ji}(p(t)).p_j(t)$$
$$+ \sum_{i \in P'(p(t))} \sum_{j \notin P'(p(t))} p_i(t).Z_{ij}(p(t).\dot{p}_j \leq 0$$

The last step follows because each term in the above is non-positive. Whenever, $p(t)$ is not an equilibrium price, $V'(p(t)) > 0$ and both $P'(p(t))$and its complement are non-empty; consequently, either directly from indecomposability or from the Claim ( 3.7.3 ), it follows that the second term is negative. This demonstrates the validity of the claim regarding $\dot{V}'(t)$. The claim for $\dot{V}(t)$ follows exactly as above, replacing $P'(p(t))$by $P(p(t)$. •

Next, we need to cover cases where derivatives of $V'(t)$ may not exist; say at $t = \bar{t}$; recalling the definition of $V'(p)$ from equation ( 3.14 ), it follows that at $t = \bar{t}$ the following must hold:

i. $\exists i \neq n$ such that $Z_i(p(\bar{t}) = 0$; and

ii. the right hand derivative, $\dot{V}'^{+}(\bar{t})$ and the left hand derivative, $\dot{V}'^{-}(\bar{t})$ exist but are unequal.

In view of the above, we define $Q(t) = \{i \neq n : Z_i(p(t)) \geq 0\}$; $Q_1(t) = \{i \in Q(t) : Z_i(p(t)) > 0\}$; $Q_2(t) = \{i \in Q(t) : Z_i(p(t)) = 0\}$. Thus $Q(t) = Q_1(t) \cup Q_2(t)$ and $P'(p(t))$= $Q(t) \cup \{n\}$. Note that $Q_1(t) \subset Q_1(t+h)$ for all $h > 0$ and small. The problems are created by virtue of the possibility that for $h > 0$ and small, there may be $i \in Q_1(t+h)$ but $i \notin Q_1(t)$. Clearly such an $i \in Q_2(t)$. We define $Q_3(t) = \{i \in Q_2(t) : i \in Q_1(t+h)\}$ for all $h > 0$ and small. With these definitions, we have:

$$V'(\bar{t}+h) - V'(\bar{t}) = Z_n(p(\bar{t}+h)) - Z_n(p(\bar{t})$$

$$+ \sum_{i \in Q(\bar{t}+h)} p_i(\bar{t}+h) Z_i(p(\bar{t}+h) - \sum_{i \in Q(\bar{t})} p_i(\bar{t}) Z_i(p(\bar{t})$$

The right hand side of the above may now be broken up further as follows, using the fact that for $h > 0$ and small, $i \in Q_1(\bar{t}+h) \Rightarrow$ either $i \in Q_1(\bar{t})$ or $i \in Q_3(\bar{t})$:

$$Z_n(p(\bar{t}+h)) - Z_n(p(\bar{t}) + \sum_{i \in Q_1(\bar{t})} \{p_i(\bar{t}+h) Z_i(p(\bar{t}+h) - p_i(\bar{t}) Z_i(p(\bar{t})\}$$

$$+ \sum_{i \in Q_3(\bar{t})} \{p_i(\bar{t}+h) Z_i(p(\bar{t}+h) - p_i(\bar{t}) Z_i(p(\bar{t})\}$$

$$- \sum_{i \in Q_2(\bar{t}) - Q_3(\bar{t})} p_i(\bar{t}).Z_i(p(\bar{t})$$

Note that the last term is by definition zero. Let $Q*(\bar{t}) = Q_1(\bar{t}) \cup Q_2(\bar{t}) \cup \{n\}$. Then we have:

$$V'(\bar{t}+h) - V'(\bar{t}) = \sum_{i \in Q*(\bar{t})} \{p_i(\bar{t}+h)Z_i(p(\bar{t}+h) - p_i(\bar{t})Z_i(p(\bar{t})\}$$

$$\text{Therefore } \dot{V}'^{+}(\bar{t}) = \sum_{i \in Q*(\bar{t})} \{\dot{p}_i.Z_i(p(\bar{t}) + p_i(\bar{t}).\sum_j Z_{ij}(p(\bar{t})).\dot{p}_j\}$$

We have already shown, by virtue of the Claim 3.7.4 , that the right hand side is non-positive in general, while it is negative at dis-equilibrium price configurations. Consequently, we may now claim:

**Claim 3.7.5** *$V'(t+h) \leq V'(t)$ for all $h > 0$ and small.*

Consequently, $V'(t)$ is monotonically non-increasing and hence, $0 \leq V'(p(t,p^o)) = V'(t) \leq V'(p^o)$ for all $t$. This allows us to make the following

**Claim 3.7.6** *$p(t,p^o)$ remains bounded for $\forall\ t$; further $p_i(t,p^o) \geq \delta_i > 0\ \forall\ t$ for some $\delta_i$ for all $i$.*

Proof. Note that $\|p(t,p^o)\| \to +\infty \Rightarrow \exists i \neq n$ such that $p_i(t,p^o) \to +\infty$ since $p_n(t,p^o) = 1$ $\forall\ t$. Define $q(t) = p(t,p^o)/p_i(t,p^o)$; then $q_i(t) = 1\ \forall\ t$ while $q_n(t) \to 0$. By Properties P3 and P4, $\sum_i Z_i(q(t)) = \sum_i Z_i(p(t,p^o)) \to +\infty$. Thus $V'(t) \to +\infty$ which is a contradiction. Hence the solution remains bounded beginning from any initial point. Finally, a similar argument, using property P4 establishes a contradiction if any price is not bounded away from 0. This establishes the claim. •

Considering the properties of the function $V'(p)$ established above, it follows that it satisfies all the requirements of a Liapunov Function; consequently, by virtue of Claim 3, we may conclude that every limit point of the solution $p(t, p^o)$ is an equilibrium of ( 3.1 ): quasi global stability.

We shall show below, that under WGS and indecomposability, equilibrium is unique. Putting these conclusions together we have demonstrated the validity of the following result due to McKenzie(1960):

**Proposition 3.2** *Given the properties P1-P4 of excess demand functions, WGS and indecomposability imply that the solution $p(t, p^o)$ to ( 3.1 ) from any initial point $p^0 \in R^n_{++}$ converges to the unique equilibrium.*

### 3.7.2 Global Stability and Dominant Diagonals

Let $Z : \Re^n_{++} \to \Re^n$ be excess demand functions satisfying assumptions P1 - P3 introduced in 3.2; for this section, we strengthen P4, introduced there, to the following:

P4*. Consider a sequence $P^s \in \Re^n_{++}$, $\forall s$, with $P^s_{i_o} = 1, \forall s$ for some fixed index $i_o$, such that[21] $||P^s|| \to +\infty$ as $s \to +\infty$; then $Z_{i_o}(P^s) \to +\infty$.

We shall consider the good $n$ as numeraire and consequently we represent prices by $(p_1, p_2, \cdots, p_{n-1}, 1)$ where we shall write $p = (p_1, p_2, \cdots, p_{n-1}) \in \Re^{n-1}_{++}$; thus the price configuration will be written as $(p, 1)$. Thus excess demand functions will be written as $Z(p, 1)$. We shall use the symbol $p_{\sim i}$ to denote all the components of the vector $p$ **except** the $i$-th. Unless stated to the contrary, all prices will be considered to be positive. The main advantage in the strengthening of P4 to P4* lies in the following:

[21] $||x||$ will be taken to be the Euclidean Norm, i.e., if $x \in \Re^n$, then $||x|| = +\sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)}$.

**Claim 3.7.7** *For each $i = 1, 2, \cdots, n-1$, there exists $\varepsilon_i > 0$ such that $Z_i(p_{\sim i}, p_i, 1) > 0$ if $p_i \leq \varepsilon_i$ for any $p_{\sim i}$.*

Proof: Suppose to the contrary, there is no such $\varepsilon_1$; i.e., for any sequence $\{p_1^s\}$, $p_1^s \to 0$ as $s \to +\infty$, one can find $p_{\sim 1}^s > 0$, such that $Z_1(p_1^s, p_{\sim 1}^s, 1) \leq 0$ for all $s$ large enough. Now consider the sequence $q^s = (1, 1/p_1^s.p_{\sim 1}^s, 1/p_1^s)$; notice that $||q^s|| \to +\infty$ as $s \to +\infty$; hence by P4*, $Z_1(q^s) \to +\infty$ or by homogeneity of excess demand functions, (P3), $Z_1(p_1^s, p_{\sim 1}^s, 1) \to +\infty$; thus for all $s$ large enough, $Z_1(p_1^s, p_{\sim 1}^s, 1) > 0$: a contradiction. This establishes the claim. •

We introduce, next the other main condition which has been imposed on excess demand functions to yield stability, is the condition of dominant negative diagonals, which we have encountered in Section 3.4. First of all, recall the definition of dominant diagonal condition. Consider the Jacobian of the excess demand functions $A(p, 1) = (Z_{ij}(p, 1)), i, j \neq n$.

For all $p \in \Re^{n-1}_{++}$ we have:

**DD i**. the diagonal terms must be negative ($Z_{jj}(p, 1) < 0, j \neq n$) and

**DD ii**. there exist positive numbers $d_1, d_2, \cdots, d_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n-1, d_j|Z_{jj}(p, 1)| > \sum_{i \neq j, n} d_i |Z_{ji}(p, 1)|$.

An equivalent [22] way of defining diagonal dominance is the following: $\exists$ positive numbers $c_1, c_2, \cdots, c_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n-1, c_j|Z_{jj}(p, 1)| > \sum_{i \neq j} c_i |Z_{ij}(p, 1)|$; the numbers $d$'s in **DD ii** provide row dominant diagonals while the $c$'s provide column dom-

[22] See, for example, Mukherji (1975).

inant diagonals. For the purpose at hand, we shall consider row dominant diagonals; also notice that the **same** constants $d$'s satisfy the condition **DD ii** $\forall p$.

It is known that under the condition of dominant diagonals, equilibrium is unique[23]. We denote this equilibrium by $(p^\star, 1)$. Recall the equations ( 3.1 ):

$$\dot{p}_j = F_j(p) \text{ for all j } \neq n$$

where $F_j(p)$ has the same sign as the excess demand functions $Z_j(p, 1)$. We shall assume, as before that the functions $F_j(p)$ are continuously differentiable on $\Re^{n-1}_{++}$. And we shall investigate the behavior of the solution $p(t, p^o)$ to (3.1) from any arbitrary initial point $p^o$ in $\Re^{n-1}_{++}$. One immediate consequence of P4* may now be noted:

**Claim 3.7.8** *For any adjustment process of the form (3.1), the solution $p(t, p^o)$ remains within a bounded region and is bounded away from the axes.*

Proof: Suppose that $||p(t, p^o)|| \to +\infty$; then P4* implies that $Z_n(p(t, p^o), 1) \to +\infty$; consequently, by Walras law, i.e., P2, we have, writing $p(t, p^o)$ as $(p_1(t), p_2(t), \cdots, p_{n-1}(t))$:

$$p_1(t)Z_1(p(t, p^o), 1) + \cdots + p_{n-1}(t)Z_{n-1}(p(t, p^o), 1) \to -\infty$$

Since excess demand functions are bounded below (P1), the above means that for some index $j \neq n$ $p_j(t).Z_j(p(t, p^o), 1) \to -\infty$; note this possible only if $p_j(t) \to +\infty$ **and** $Z_j(.) < 0$; thus for all $t$ large enough, say for $t > T$, $p_j(t).Z_j(p(t, p^o), 1) < 0$; which means that for all $t > T, \dot{p}_j < 0$; hence $p_j(t) \leq p_j(T) \forall t$: a contradiction. This establishes that $p(t, p^o)$ remains within some bounded region. Using the result of Claim 3.7.7, it follows that there

[23]See, for example, Arrow and Hahn (1971), p. 235.

is a rectangular region $R$ given by $0 < \varepsilon_i \leq p_i \leq M_i, i = 1, 2, \cdots, n-1$, such that $p^o \in R \Rightarrow p(t, p^o) \in R \forall t$. $\bullet$

Next, we shall deduce a consequence of assuming the dominant diagonal condition.

We begin by noting the following:

**Claim 3.7.9** *Given P1-P3 and P4*, $\exists \delta > 0$ such that for any $\lambda, 0 < \lambda < \delta$ and any $p \in \Re^{n-1}_{++}$, $p_i + \lambda Z_i(p, 1) > 0, i = 1, 2, \cdots, n-1$.*

Proof: By virtue of P1, we know that for all $p \in \Re^{n-1}_{++}$, there is $b_i > 0$, such that $Z_i(p, 1) \geq -b_i$, $i = 1, 2, \cdots, n-1$. Let $\bar{b} = \max_i b_i$.

Consider, next, the set $K_i = \{p \in \Re^{n-1}_{++} : Z_i(p, 1) \leq 0\}, i = 1, 2, \cdots, n-1$. The set $K_i$ is a closed subset of $\Re^{n-1}_{++}$ and we claim that $p \in K_i \Rightarrow p_i \geq \eta_i > 0$ for some $\eta_i$; if no such $\eta_i$ exists, then we can construct a sequence $p^s \in K_i \cap \Re^{n-1}_{++}$, $\forall s$, $p^s_i \to 0$ as $s \to +\infty$. Consider then the sequence $P^s = (p^s, 1)$ and $q^s = P^s.1/p^s_i$; note that $q^s_i = 1 \forall s$ and $||q^s|| \to +\infty$; hence by P4*, $Z_i(q^s) \to +\infty$; thus $Z_i(P^s) > 0$ for all $s$ large enough and hence $p^s \notin K_i$ for all $s$ large enough: a contradiction. Hence there is $\eta_i$ with the claimed property. Let $\bar{\eta} = \min_i \eta_i$.

Define $\delta = \bar{\eta}/\bar{b}$; now choose any positive $\lambda < \delta$. If possible, suppose for some $i$ and for some $p \in \Re^{n-1}_{++}$, $p_i + \lambda Z_i(p, 1) \leq 0 \Rightarrow p \in K_i \Rightarrow \lambda(-Z_i(p, 1)) \geq p_i \geq \bar{\eta}$; hence $\lambda \geq \bar{\eta}/(-Z_i(p, 1)) \geq \bar{\eta}/\bar{b} = \delta$: a contradiction. Hence no such $i$ and $p$ can exist and this proves the claim. $\bullet$

It would be appropriate to introduce in $\Re^{n-1}$ a norm which is somewhat different from the usual Euclidean norm which we have dealt with in most situations. We define the following norm[24] :

**d-norm**: For $x \in \Re^{n-1}$, $||x||_d = \max_{1 \le i \le n-1} |x_i/d_i|$ where the constants $d_i$'s are the same as the ones which appear in the definition of **DD ii**.

It is easy to check that this definition satisfies all the conditions required for being a norm. The reason for choosing such a norm will become clear with the next step:

**Claim 3.7.10** *$||x||_d$ is a norm on $\Re^{n-1}$; further if $A = (a_{ij}), i, j = 1, 2, \cdots, n-1$, $a_{ij}$ real numbers for all $i, j$, then $||A||_d = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} |a_{ij}| d_j / d_i$.*

Proof: That $||x||_d$ satisfies all the conditions is straightforward. Now for any square matrix $A$, $||A||_d$ is by definition $\sup_{||x||_d = 1} ||Ax||_d$[25]. From this definition, we observe that:

$$||Ax||_d = \max_i |\frac{\sum_j a_{ij}.x_j}{d_i}| = \max_i |\frac{1}{d_i}.\sum_j a_{ij}.d_j.x_j/d_j|$$

$$\le \max_i \frac{1}{d_i}.\sum_j |a_{ij} d_j|.|x_j/d_j| \le \max_i \frac{1}{d_i}.\sum_j |a_{ij}|.d_j.||x||_d$$

Now suppose that $\max_i \sum_j |a_{ij}| d_j / d_i = \sum_j |a_{kj}| d_j / d_k$; then define:

$$\bar{x}_j = \begin{cases} \dfrac{d_j a_{kj}}{|a_{kj}|} & \text{if } a_{kj} \neq 0 \\ d_j \text{ otherwise} \end{cases} \tag{3.16}$$

Notice that $||\bar{x}||_d = 1$; further for $i = k$, $\sum_j a_{kj}.\bar{x}_j = \sum_j |a_{kj}|.d_j$ and for any $i$, $|\sum_j a_{ij}.\bar{x}_j| \le \sum_j |a_{ij}|.d_j$; hence note that for this particular definition of $\bar{x}$, $||A\bar{x}||_d = \sum_j |a_{kj}|.d_j/d_k$, the bound we had obtained above, is attained. This proves the claim. •

We note next:

[24]The analysis follows the elegant treatment in Fujimoto and Ranade (1988).

[25]See, for example, Ortega and Rheinboldt (1970), p. 40-41.

**Claim 3.7.11** *Given any $\bar{p} \in \Re^{n-1}_{++} \neq p^\star$ there is some $\bar{\lambda} > 0$ such that $1 + \lambda Z_{ii}(p,1) > 0\ \forall \lambda, 0 < \lambda < \bar{\lambda}$ and for all $p \in [\bar{p}, p^\star]$, where $[\bar{p}, p^\star]$ consist of all points on the line segment connecting $\bar{p}$ to $p^\star$.*

Proof: By **DD i**, $Z_{ii}(p,1) < 0 \forall p \in [\bar{p}, p^\star]$; by virtue of P1,

$$\bar{\lambda} = \min_{p \in [\bar{p}, p^\star]} \frac{1}{-Z_{ii}(p,1)}$$

exists and is positive. It is straightforward to check that this definition of $\bar{\lambda}$ has the desired property. $\bullet$.

These preliminary steps allows us to prove the following crucial property of of dominant diagonal systems[26]

**Proposition 3.3** *Given **DD i-ii**, consider for any $p \in \Re^{n-1}_{++} \neq p^\star$, $\max_i \frac{|p_i - p_i^\star|}{d_i} = \frac{|p_k - p_k^\star|}{d_k}$, say. Then $p_k - p_k^\star > 0 \Rightarrow Z_k(p,1) < 0$; while $p_k - p_k^\star < 0 \Rightarrow Z_k(p,1) > 0$.*

Proof: Consider any $p \in \Re^{n-1}_{++} \neq p^\star$; next choose $\lambda < \min(\delta, \bar{\lambda})$, where $\delta$ and $\bar{\lambda}$ are as in Claims 3.7.9 and 3.7.11 respectively; for such a fixed choice of $\lambda$, define $\psi(p) = (p_i + \lambda Z_i(p))$; then by the Mean Value Theorem[27], we have the following:

$$||\psi(p) - \psi(p^\star||_d \leq \sup_{0 \leq t \leq 1} ||J_\psi(p^\star + t(p - p^\star))||_d . ||p - p^\star||_d$$

[26]This property was noted by Fujimoto and Ranade (1988).

[27]See Ortega and Rheinboldt (1970), p. 69.

where the matrix $J_\psi$ is given by the $(n-1)\times(n-1)$ matrix:

$$\begin{pmatrix} 1+\lambda Z_{11} & \cdots & \cdots & \lambda Z_{1n-1} \\ \lambda Z_{21} & 1+\lambda Z_{22} & \cdots & \lambda Z_{2n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda Z_{n-11} & \cdots & \cdots & 1+\lambda Z_{n-1n-1} \end{pmatrix}$$

Now by virtue of the Claim 3.7.10, we have:

$$||J_\psi||_d = \max_i [1 + \lambda \frac{d_i Z_{ii} + \sum_{j \neq i} |Z_{ij}| d_j}{d_i}]$$

using the fact that by our choice of $\lambda$, $|1+\lambda Z_{ii}| = 1 + \lambda Z_{ii}$; hence it follows from **DD ii** that $||J_\psi||_d < 1$ for all points on the line segment $[p, p^\star]$; consequently it follows that:

$$||\psi(p) - p^\star||_d < ||p - p^\star||_d$$

If the left hand side is $\frac{\psi_j(p) - p_j^\star}{d_j}$ and the right hand side is $\frac{p_k - p_k^\star}{d_k}$ then it follows that:

$$\frac{\psi_k(p) - p_k^\star}{d_k} \leq \frac{\psi_j(p) - p_j^\star}{d_j} < \frac{p_k - p_k^\star}{d_k}$$

which implies that $|p_k - p_k^\star + \lambda Z_k(p,1)| < |p_k - p_k^\star|$; since $\lambda > 0$ this is possible only if $(p_k - p_k^\star)$ and $Z_k(p,1)$ are of opposite signs. This establishes the claim. •

The above allows us to demonstrate the following:

**Proposition 3.4** *Given **DD i- ii**, the unique equilibrium is globally asymptotically stable under the adjustment process (3.1).*

Proof: Let $p(t, p^o)$ denote the solution to (3.1) from some arbitrary initial point $p^o \in \Re^{n-1}_{++}$. Define $V(t) \equiv V(p(t,p^o)) = ||p(t,p^o) - p^\star||_d$. Note that $V(p(t,p^o))$ is continuous in $p$; in case derivatives exist, $\dot{V} = Sgn(p_k - p_k^\star).\dot{p}_k$ where $||p(t,p^o) - p^\star||_d = |p_k - p_k^\star|/d_k$; hence by virtue of Proposition 3.3 and the properties of the system (3.1), it follows that when derivatives exist $\dot{V}(t) < 0$ if $p(t,p^o) \neq p^\star$. When derivatives do not exist, we need to investigate further.

Consider $S(t) = \{k : \dfrac{|p_k(t,p^o) - p_k^\star|}{d_k} \geq \dfrac{|p_i(t,p^o) - p_i^\star|}{d_i} \forall i\}$; thus $||p(t,p^o) - p^\star||_d = |p_k - p_k^\star|/d_k$ for $k \in S(t)$. Notice that if at $t$, $S(t)$ is a singleton, $\dot{V}(t)$ exists; when $S(t)$ has more than one member, derivatives may fail to exist. In this case, note that if $i \notin S(t)$ then $i \notin S(t+h)$ for $h$ positive and small. Thus $S(t+h) \subseteq S(t)$ for all $h$ positive and small enough. With these observations, consider some $k \in S(t+h)$, $h$ small enough; then $k \in S(t)$ and we have:

$$\lim_{h \to 0+} \frac{V(t+h) - V(t)}{h} = \lim_{h \to 0+} \frac{1}{h}\left(\frac{|p_k(t+h,p^o) - p_k^\star|}{d_k} - \frac{|p_k(t,p^o) - p_k^\star|}{d_k}\right)$$

thus we may observe that:

$$\lim_{h \to 0+} \frac{V(t+h) - V(t)}{h} = \frac{1}{d_k}\frac{d|p_k(t,p^o) - p_k^\star|}{dt} \text{ for some } k \in S(t) \ ;$$

note that the right hand side exists and is **negative** whenever $p(t,p^o) \neq p^\star$.

Consequently, $V(t+h) < V(t)$ for all $h$ sufficiently small and positive, so long as the equilibrium is not encountered. Thus $V(t)$ is a Liapunov function for the process (**??**); this also implies that $V(p(t,p^o)) < V(p^o)$ which implies that the solution is bounded. So all the ingredients required for our claim are in place. •

The above result is crucially dependant on the fact that **DD**holds or that the same constants provide the dominance condition **DD ii**, for all $p$. We present next an attempt to weaken this condition: but it will not be costless, since as we shall see the convergence result will have to be in terms of a somewhat special version of the tatonnement. Given this trade-off, it has been thought best to present both of these results.

We shall maintain the assumptions P1-P3 and P4* on excess demand functions; in addition, we shall require a modified version of the conditions **DD ii**:

**DD ii***: There exist continuously differentiable functions $h_j : \Re^{n-1}_{++} \to \Re^{n-1}_{++}$, $j = 1, 2, \cdots, n-1$ satisfying $h_j(p)|Z_{jj}(p,1)| > \sum_{i \neq j,n} |Z_{ji}(p,1)| h_i(p)$ for all $j = 1, 2, \cdots, n-1$.

We shall say that **DD*** holds when we have **DD i** and **DD ii***.

For **DD***, the adjustment process (3.1) needs to be specialized to the following[28]:

$$\dot{p}_j = h_j(p).Z_j(p,1) \text{ for all } j \neq n, p_n \equiv 1 \tag{3.17}$$

Notice that the weights which appeared in the definition of **DD ii*** also appear in the adjustment equations above; we shall return to an interpretation to these equations later. We note that by virtue of our assumptions, and due to Claims 3.7.7 and 3.7.8, we have :

**Claim 3.7.12** *Given any $p^o \in \Re^{n-1}_{++}$, the solution to (3.17) through $p^o$, $p(t, p^o)$ exists and is continuous with respect to the initial point; further the solution remains within a bounded region and is bounded away from the axes.*

We shall write the solution $p(t, p^o)$ in short as $p(t)$. Define $W(t) \equiv W(p(t)) = \max_{j \neq n} |Z_j(p(t), 1)| = |Z_k(p(t), 1)|$ say . Then, one may show:

**Claim 3.7.13** *Given **DD***, $W(t)$ is strictly decreasing along the trajectory, if $p(t, p^o) \neq p^\star$.*

Proof: Define $S(t) = \{j \neq n : |Z_j(p(t), 1)| \geq |Z_i(p(t), 1)| \forall i \neq n\}$; notice that if at $t$, $S(t)$ is a singleton, $\{k\}$, say, and if $p(t) \neq p^\star$, then $\dot{W}(t)$ exists and is given by $\dot{W}(t) =$

[28] The treatment presented here is a generalization of Mukherji (1974 a).

$(SgnZ_k(.)).\sum_{j\neq n} Z_{kj}(p(t),1).h_j(p(t)).Z_j(p(t),1)$. Suppose $Z_k(p(t),1) > 0$, we claim that the expression $\sum_{j\neq n} Z_{kj}(p(t),1).h_j(p(t)).Z_j(p(t),1) < 0$; if to the contrary, this expression is non-negative, we have, using the fact that **DD i** holds:

$$|Z_{kk}(.)h_k(.)Z_k(.)| \leq |\sum_{j\neq k,n} Z_{jk}(.).h_j(.)Z_j(.)| \leq \sum_{j\neq k,n} |Z_{kj}(.)|h_j(.)|Z_j(.)| \leq |Z_k(.)| \sum_{j\neq k,n} |Z_{kj}(.)|h_j(.)$$

where the first inequality follows from our contrary hypothesis and the fact that the diagonal term is negative; the second inequality follows on account of the absolute value of sum being less than or equal to the sum of absolute values and the last inequality follows because of the fact that $k \in S(t)$. We note then that since $Z_k(.)$ is not zero at dis-equilibrium, we have $|Z_{kk}(.)|h_k(.) \leq \sum_{j\neq k,n} |Z_{kj}(.)|h_j(.)$ which violates condition **DD ii*** for $j = k$. Hence $k \in S(t)$ and $Z_k(.) > 0$ implies that $\sum_{j\neq n} Z_{kj}(p(t),1).h_j(p(t)).Z_j(p(t),1) < 0$. In a similar fashion, in case $Z_k(.) < 0$, one may show that $\sum_{j\neq n} Z_{kj}(p(t),1).h_j(p(t)).Z_j(p(t),1) > 0$. Thus whenever, $S(t)$ is a singleton, $\dot{W}(t) < 0$ provided, $p(t) \neq p^\star$. If $S(t)$ is not a singleton, then derivatives may not exist. To cover such cases, notice that for $h > 0$ and small, $S(t+h) \subseteq S(t)$; so the situation is as in the proof of Proposition 3.4 and one may claim, exactly following those steps that $W(t+h) < W(t)$ for all $h$ positive and small, provided no equilibrium is encountered. Thus regardless of whether $S(t)$ is a singleton or not, so long as no equilibrium is encountered, $W(t)$ is strictly decreasing along the trajectory. •

We may now claim, the following to be true:

**Proposition 3.5** *Given **DD***, conditions P1-P3 and P4* on excess demand functions, the unique equilibrium* $(p^\star, 1)$ *is globally asymptotically stable under the process (3.17).*

Proof: First of all, we note that the process (3.17) satisfies the requirement of being a special case of the process (3.1) and hence, by virtue of Claims 3.7.7 and 3.7.8, any solution to (3.17) from an arbitrary $p^o$ will be trapped in some bounded region $R$ which is also bounded away from the axes. Consequently limit points exist and since $W(t)$ has been shown to satisfy the requirements for being a Liapunov function for the process (3.17), the claim is established. •

We have thus presented two results both of which assure global asymptotic stability for the unique equilibrium under appropriate adjustment processes. The first result, Proposition 3.4 considers the most general process but is more demanding in the restriction on excess demand functions, since the condition **DD ii** requires the **same** set of weights to provide row dominance for all prices. We used a condition derived by Fujimoto and Ranade (1988) to arrive at this conclusion; weakening the condition **DD ii** to **DD ii*** and allowing these weights to vary over prices we obtained the second result, Proposition 3.5, where the adjustment process (3.1) had to be specialized to (3.17). As we had remarked earlier, the generalization was not costless. The results are of interest since the adjustment processes, for both of these cases are more general than the ones in the existing literature. It may be possible to provide some other results for special cases of (3.1) but such exercises have not been reported.

Finally, the dominant diagonal condition has an inherent interest because what it ensures is that in any market, the excess demand is "most" responsive to its own price; the various formalizations provided, namely **DD** and **DD*** are alternative ways of capturing this simple idea. In contrast to the assumption of Gross Substitutes considered earlier, notice that dominant diagonals offer a scope for the presence of complementarities in the system subject to the provision that own price effects dominate. We should also mention that the gross substitute case satisfies the Weak Axiom of Revealed Preference in the Aggregate (WARP); it is of interest that so does the Dominant Diagonal Condition **DD** provided the same weights also provide column dominant diagonals; this has been shown by Fujimoto and Ranade (1988). Some what loosely speaking, thus, the condition WARP lies at the common intersection of the conditions which guarantee the stability of equilibrium.

As we have seen above, there are two issues involved. One is the nature of excess demand functions and the other is the form of the tatonnement process. If we specialize one, we are able to consider general forms of the other. We consider, next, general forms for the excess demand functions and, as is to be expected, we specialize the form of the tatonnement process.

### 3.8 The Structure of Limit Sets for the General Case

In this section, we consider excess demand functions satisfying the properties P1-P3 and P4* and the adjustment on prices given by the process ( 3.3 ):

$$\dot{p}_j = Z_j(p) \text{ for all j } \neq n;\ p_n = 1 \tag{3.18}$$

Then for any $p^o \in R^n_{++}$, the solution to ( 3.3 ), $p(t, p^o)$ exists. In addition, recalling the results of Claims 3.7.7 and 3.7.8, which remain applicable since the process (3.3) is a special case of the process (3.1), we may conclude:

**Claim 3.8.1** *There is a region $R$ given by $\{(x_1, x_2, \cdots, x_{n-1}) : 0 < \varepsilon_i \leq x_i \leq M_i\}$ for some*

$\varepsilon_i, M_i, i = 1, 2, \cdots, n-1$ *within which any solution to* $p(t, p^o)$ *lies provided* $p^o \in R$.

Thus any solution to (3.3) is bounded and bounded away from the axes. Let $E = \{p \in R^n_{++} : Z(p) = 0\}$: the set of equilibrium prices. To provide this set with some minimal structure, we shall assume that the economy is **regular** i.e.,

P5. For every $p \in E$, rank of

$$J(p) = (\frac{\partial Z_i(p)}{\partial p_j})$$

is $(n-1)$.

It is well known that P5 ensures that $E$ is a finite set. Recall that for any $p* \in E$ we had defined the set $K_{p*}$ by $K_{p*} = \{p \in R^n_{++} : p*^T .Z(p) \leq 0\}$: the set of prices where the Weak Axiom is **violated.** We shall use P1-P3, P4* and P5 to analyze the behavior of $p(t, p^o)$ as $t \to \infty$. By virtue of the Claims made above, we may conclude that the set

$$L = \{\bar{p} : \exists \text{ a susbsequence } p(t_s, p^o \text{ and } p(t_s, p^o) \to \bar{p} \text{ as } s \to \infty\}$$

is **non-empty** . In addition, given the properties of the solution to the system ( 3.3 ), one may also conclude that $L$ is a **compact and connected** subset of $R^n_{++}$. Further, we have:

**Claim 3.8.2** *$L \cap K_{p*} \neq \emptyset$ for any $p* \in E$.*

Proof. In case $L \cap K_{p*} = \emptyset$ for some $p* \in E$, then $p(t, p^o) \notin K_{p*}$ for all $t > T$ for some $T$ and consequently, $d(p(t, p^o), p*)$ is a Liapunov function for $t > T$ and $p(t, p^o) \to p*$ as $t \to \infty$, i.e.,$L = \{p*\} \subset K_{p*}$: a contradiction. Hence the claim. •

Next note that

$$d_*(p*) = \min_{p \in L} d(p, p*) \text{ and } d^*(p*) = \max_{p \in L} d(p, p*)$$

are well defined and further $d_*(p*)$ and $d^*(p*)$ are both attained in the set $L$. In particular, we have:

**Claim 3.8.3** *$d_*(p*)$ and $d^*(p*)$ are attained in $L \cap K_{p*}$.*

Proof. We shall write $K, d_*$ for $K(p*), d_*(p*)$ respectively. Suppose that $d_*$ is attained at $\overline{p} \notin K$; i.e., $\exists$ a subsequence $p(t_s, p^o) \to \overline{p}$ and $d(\overline{p}, p*) \leq d(p, p*)$, $\forall p \in L$. Thus $p * .Z(\overline{p}) = \delta > 0$ for some $\delta$. Also for some $\eta > 0$, $d(\overline{p}, K) > \eta$, where $d(\overline{p}, K) = \min_{p \in K} d(\overline{p}, p)$. Let $\epsilon > 0$ be such that $p \in N_\epsilon(\overline{p}) = \{p : d(p, \overline{p}) < \epsilon\} \Rightarrow d(p, K) > \eta$; further $\forall p \in N_\epsilon(\overline{p})$, $|p * .Z(p) - p * .Z(\overline{p})| < \delta/2$. Since $p(t_s, p^o) \to \overline{p}$, there is some number $S_\epsilon$ such that $\forall s > S_\epsilon$, $p(t_s, p^o) \in N_\epsilon(\overline{p})$ and there is a subsequence $p(t_s + \theta_s, p^o)$ such that $p(t_s + \theta_s, p^o) \notin N_\epsilon(\overline{p})$, while $p(t, p^o) \in N_\epsilon(\overline{p})$ for all $t$ such that $t_s \leq t < t_s + \theta_s$. Such a subsequence can always be constructed, since otherwise $p(t, p^o)$ has no other limit points and $L = \{\overline{p}\}$ and hence $L \cap K = \emptyset$: a contradiction. By definition, note that $d(p(t_s + \theta_s, p^o), K) \geq \eta$ and hence the subsequence $p(t_s + \theta_s, p^o)$ has all its limit points outside the set $K$. Now note that:

$$d(p(t_s + \theta_s, p^o), p*) = d(p(t_s, p^o), p*) + \theta_s \dot{d}(p(t_s + \lambda_s \theta_s, p^o), p*)$$

$$\text{for some } \lambda_s \text{ , } 0 \leq \lambda_s \leq 1$$

Recall that $\dot{d}(p(t, p^o), p*) = -p * .Z(p(t, p^o))$; thus, by hypothesis, $\forall s > S_\epsilon$, $\dot{d}(p(t_s + \lambda_s \theta_s, p^o), p*) = -p * .Z(p(t_s + \lambda_s \theta_s, p^o))$. Since $|p * .Z(p(t_s + \lambda_s \theta_s, p^o)) - p * .Z(\overline{p})| \leq \delta/2$ for all $s > S_\epsilon$, it follows that $\dot{d}(p(t_s + \lambda_s \theta_s, p^o), p*) = -p * .Z(p(t_s + \lambda_s \theta_s, p^o)) \leq \delta/2 - p * .Z(\overline{p}) = -\delta/2$. Since $p(t_s + \theta_s, p^o) \notin N_\epsilon(\overline{p})$ $\forall s > S_\epsilon$, $\theta_s$ cannot have 0 as a limit point. Hence

$$d(p(t_s + \theta_s, p^o), p*) \leq d(p(t_s, p^o) - \theta_s \delta/2$$

$$\leq d(p(t_s,p^o) - \liminf_{s\to\infty} \theta_s.\delta/2$$

Hence all limit points $\tilde{p}$ of $p(t_s+\theta_s,p^o)$ satisfy

$$d(\tilde{p},p*) \leq d_* - \liminf_{s\to\infty} \theta_s\delta/2 < d_*$$

which is a contradiction. Hence $\overline{p} \in K$ and the claim follows. An analogous argument establishes that $d^*$ is attained in $K$ too. • We now take into consideration the fact that $E$ may have other equilibria; but given P5, E is a finite set. To take this aspect into account, we make the following definitions:

$$K = \cup_{p*\in E} K(p*); d^* = \max_{p\in L} d(p,E); d_* = \min_{p\in L} d(p,E)$$

Hence note that $d_* = d(\hat{p},p*)$ for some $\hat{p} \in L, p* \in E$. By virtue of our last claim, $\hat{p} \in K(p*) \subset K$. In case this particular $p*$ happens to be an **isolated** point of $K(p*)$, we have the following:

**Claim 3.8.4** *Since $d_* = d(\hat{p},p*)$ for some $\hat{p} \in L, p* \in E$, $d_* = 0 \Rightarrow d^* = 0$ and $\lim_{t\to\infty} p(t,p^o) = p*$ provided $p*$ is an isolated point of $K(p*)$.*

Proof. Since $d_* = 0$, there is a subsequence $p(t_s,p^o) \to p*$ as $s \to \infty$. Further since $p*$ is an isolated point of $K(p*)$, there is a neighborhood $N(p*)$ of $p*$ such that for all $p \in N(p*), p \neq p* \Rightarrow p*.Z(p) > 0$. Moreover, $p \in N(p*) \Rightarrow \dot{d}(p,p*) < 0$. Let $\epsilon > 0$ be such that $d(p,p*) < \epsilon \Rightarrow p \in N(p*)$. Since $p(t_s,p^o) \to p*$, $d(p(t_s,p^o),p*) < \epsilon$ for all $s > S_\epsilon$, say. Also by our choice of $\epsilon$,$d(p(t,p^o),p*) < \epsilon \Rightarrow d(p(t+h,p^o),p*) < d(p(t,p^o),p*)$ for all $h > 0$. Thus $d(p(t,p^o),p*) < \epsilon$ for some $\overline{t}$ implies that $d(p(t,p^o),p*) < \epsilon$ for all $t > \overline{t}$ and the claim follows. •

We are now ready to consider the general nature of the set of limit points $L$ of the solution $p(t,p^o)$. By virtue of what we have shown above $L \cap K \neq \emptyset$. The question is whether $L \cap E$ is empty or not. In case it is empty, $d_* > 0$ and there are the following logically feasible possibilities:

(i) there is $\bar{t}$ such that for all $t > \bar{t}$, $p(t,p^o) \notin K$;

(ii) there is $\bar{t}$ such that for all $t > \bar{t}$, $p(t,p^o) \in K$;

(iii) there are subsequences $p(t_s,p^o), p(t_k,p^o)$ such that $p(t_s,p^o) \in K$ and $p(t_k,p^o) \notin K$ for all $s,k$ respectively.

Note that in case (i), $d(p(t,p^o),p*)$ is a Liapunov function for any $p* \in E$ and consequently $p(t,p^o) \to p*$ for any $p* \in E$: a contradiction. For the case (ii), $\dot{d}(p(t,p^o)) \geq 0$ for any $p* \in E$. Also since $p(t,p^o) \in K$ for all $t > \bar{t}$, which is a compact set, it follows that $d(p(t,p^o),p*)$ converges for any $p* \in E$ to some $D(p*)$; thus all points of $L$ are at the same distance $D(p*)$ from $p*$ and this is true for every $p* \in E$. In case (iii), the solution $p(t,p^o)$ fluctuates between a maximum distance $d^*$ and a minimum distance $d_*$ from the set of equilibria $E$, where the maximum and minimum are attained in the set $K$; recall this is what happened in the example due to Scarf considered above. When the set $L$ has this nature, we shall say that it has the **Scarf property.** . We note these possibilities in the form of

**Proposition 3.6** *The solution to the system ( 3.3 ) for any arbitrary initial point $p^o$ has a nonempty set of limit points $L$ which either contains an equilibrium or its points are equidistant from any equilibrium or $L$ has the Scarf property.*

Without putting more structure on the the sets $K(p*)$ or the set $K$, the set of limit points cannot be subjected to more restrictions. If the set of limit points $L$ does not contain an equilibrium, then it must be the case that either prices remain at some constant distance away from equilibria or fluctuate between a maximum and minimum distance from the set of equilibria. In some sense, therefore, **if the type of phenomenon exhibited during our analysis of the Scarf example is not present, convergence to equilibrium may be ensured.** We shall specifically consider these issues in smaller dimensions, next. In particular, we shall consider motion on the plane.

### 3.8.1 The Scarf Example Once More

By virtue of the comments made above, we shall consider the example due to Scarf as the point of departure. The model and the our analysis has been presented in Section 3.6. Consequently we shall not repeat them here except we provide an alternative proof of Claim 3.6.8. This will be instructive since it will reveal exactly what is required for convergence and to eliminate the type of problems envisaged by Scarf. For this purpose recall the system (3.12):

$$\dot{p_1} = \frac{p_1(1-p_2)}{(1+p_1)(p_1+p_2)} \text{ and } \dot{p_2} = \frac{p_2(p_1-b)+(1-b)p_1}{(1+p_2)(p_1+p_2)}$$

Recall that the **unique equilibrium** is given by

$$p_1^* = \frac{b}{2-b} = \theta \text{ say}, p_2^* = 1$$

Also recall that the characteristic roots of the relevant matrix at equilibrium are given by:

$$\frac{1}{8}(-b+b^2 \pm \sqrt{b}\sqrt{\{-32+49b-26b^2+5b^3\}}).$$

Consequently, for the the process ( 3.12 ), $(\theta, 1)$ is a locally asymptotically stable equilibrium if and only if $b < 1$ ; for $b > 1$, the equilibrium is locally unstable.

We had made the much stronger assertion: *For the system ( 3.12 ), the unique equilibrium* $(\theta, 1)$ *is globally asymptotically stable whenever* $b < 1$*; and any trajectory with* $(p_1^o, p_2^o) > (0,0)$ *as initial point remains within the positive orthant. When* $b > 1$*, any solution with an arbitrary non-equilibrium initial point is unbounded.* We offer below a somewhat different approach to the proving of this result, next.

We first note that for the system (3.12) **there can be no closed orbit in** $\Re^2_{++}$ so long as $b$ is different from unity. For this purpose we shall use, Dulac's Criterion[29]. Now consider the function:

$$f(p_1, p_2) = \frac{(p_1 + p_2)(1 + p_1)(1 + p_2)}{p_1 . p_2}$$

on $\Re^2_{++}$. Notice that:

$$\frac{\partial f(p_1, p_2) Z_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) Z_2(p_1, p_2)}{\partial p_2} = -(1 - b)/p_2^2$$

Thus $b \neq 1$ implies that Dulac's Criterion is satisfied by this choice of $f(p_1, p_2)$ and consequently there can be no closed orbits when $b \neq 1$. Applying next, the Poincaré-

[29]See, Andronov et. al. (1966), p. 305. This criterion looks for a function $f(p_1, p_2)$ which is continuously differentiable on some region $R$ and for which

$$\frac{\partial f(p_1, p_2) h_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) h_2(p_1, p_2)}{\partial p_2}$$

is of constant sign on $R$ (not identically zero), then there is no closed orbit for the system $\dot{p}_i = h_i(p_1, p_2), i = 1, 2$ on the region $R$.

Bendixson Theorem, it follows that for any initial $p^o \in \Re^2_{++}$, the unique equilibrium $p^\star = (\theta, 1) \in L_\omega(p^o)$ provided the $\omega$-limit set is non-empty.

Recall that for $b > 1$, the unique equilibrium is unstable; consequently no solution can enter a small enough neighborhood of $p^\star$; consequently, in this situation, $L_\omega(p^o)$ must be empty, if $p^o \neq p^\star$; thus the trajectories must be unbounded.

When $b < 1$, the unique equilibrium $p^\star$ is locally asymptotically stable; so if $L_\omega(p^o) \neq \emptyset$, $p^\star \in L_\omega(p^o) \Rightarrow p^\star = L_\omega(p^o)$; since once having entered a small enough neighborhood of the equilibrium, the trajectory cannot leave. Thus all that we need to guarantee convergence is that trajectories are bounded when $b < 1$.

This last step may be accomplished by considering the function

$$W(p_1, p_2) = 2(1-b)p_1 + (2-b)p_1^2/2 - b\log p_1 + p_2^2/2 - \log p_2$$

and noting that its time derivative, along any solution to the system ( 3.12 ):

$$\begin{aligned}\dot{W} &= \{((2-b)p_1 - b)(1+p_1)\}\frac{\dot{p_1}}{p_1} + (p_2^2 - 1)\frac{\dot{p_2}}{p_2} \\ &= -(1-p_2)^2\frac{p_1(1-b)}{p_2(p_1+p_2)} \leq 0\end{aligned}$$

whenever $b < 1$. Thus for $b < 1$, $W(p_1(t), p_2(t)) \leq W(p_1^o, p_2^o) \forall t$, where we write $(p_1(t), p_2(t))$ as the solution to (3.12). Note that if $p_i(t) \to +\infty$, for some $i$, $W(p_1(t), p_2(t)) \to +\infty$ and the boundedness **and** positivity of the solution are established[30]. This establishes the claim.

There are thus two things to be noted from the above demonstration: first that choosing a value of $b$ **different** from unity negates the existence of a closed orbit; and a value of $b$

[30]It may be recalled in our earlier approach, we had established that the function $W(.)$ was a Liapunov function for the problem at hand.

**less than** unity is required to ensure that trajectories remain bounded. In a sense to be made precise below, these are the two aspects we need to account for if we are interested in identifying global stability conditions. This is why the above analysis is revealing.

### 3.8.2 General Global Stability Conditions

If there are three goods and one of them is the numeraire, then the price adjustment equations of the type used for the Scarf example introduces dynamics on the plane.

For motion on the plane, we shall use the results introduced in Section 1.5[31]; we show next that it is possible to substantially weaken the conditions under which a global stability result may be deduced. This would allow us to conclude global stability for a competitive equilibrium as well as providing a general stability result which would be of some general interest, as well.

Consider the following systems of equations:

$$\dot{x} = f(x, y) \text{ and } \dot{y} = g(x, y) \tag{3.19}$$

where the functions $f, g$ are assumed to be of class $\mathcal{C}^1$ on the plane $\Re^2$. For any pair of functions $f(x, y), g(x, y)$ let $J(f, g)$ or simply $J$, if the context makes it clear, stand for the Jacobian[32]:

$$\begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

[31]These results are Propositions 1.4, 1.5 and 1.6.

[32]$f_x$ for any function $f$ will refer to the partial derivative of $f$ with respect to the variable x.

We shall use the setting of the tatonnement to investigate motion on the plane and for this purpose we introduce the notion of the excess demand functions $Z_i(p_1, p_2, p_3) : \Re^3_{++} \rightarrow \Re, i = 1, 2, 3$ which are required to satisfy the following:

**A** Conditions P1- P3 and P4* hold.

To study the dynamics on the plane, we shall investigate the solutions to a system of equations of the following type:

$$\dot{p_i} = h_i(p), i = 1, 2 \text{ with } p_3 \equiv 1 \tag{3.20}$$

where the functions $h_i(p)$ are assumed to satisfy the following: (we write $p = (p_1, p_2) \in \Re^2_{++}$)

**B** $h_i(p) = Z_i(p_1, p_2, 1), i = 1, 2.$

Thus the equation (3.20) defines motion on the positive quadrant of the plane and more importantly reduce to (3.3) for the plane.

A typical trajectory or solution to (3.20) from an initial $p^o \in \Re^2_{++}$ will be denoted by $\phi_t(p^o)$; the price configuration will be $(\phi_t(p^o), 1)$ for each instant $t$; this is just to signify that the numeraire (the third good) price is always kept fixed at unity. Also we note that any equilibrium for the dynamical system (3.20), say $\bar{p}$ where $h_i(\bar{p}) = 0, i = 1, 2$, implies that $(\bar{p}, 1)$ is an equilibrium for the economy, in the sense that $(\bar{p}, 1) \in E$ and conversely. We shall denote the equilibrium for (3.20) by $E_R$.

We are interested in the structure of the $\omega$-limit set $L_\omega(p^o)$ i.e., the limit points of the trajectory $\phi_t(p^o)$ as $t \rightarrow +\infty$[33].

First, we recall from Claims 3.7.7 and 3.7.8 *that there is a rectangular region* $R =$

[33]On the plane, the structure of ***non-empty*** $\omega$-limit sets was discussed by Proposition 1.5 in Section 1.5.

$\{(p_1, p_2) : \varepsilon_i \leq p_i \leq M_i\}$ *in the positive quadrant within which the solution gets trapped.*

Incidentally, this fact together with Poincaré's theory of indices for singular points[34], implies that $R$ contains equilibria; i.e., $E_R \neq \emptyset$; recall that, by virtue of our assumptions on excess demands, $(p_1, p_2, 1) \in E \Leftrightarrow (p_1, p_2) \in E_R$.

We shall assume now the following:

**C i**. Trace of the Jacobian $J(h_1, h_2)$ is not identically zero on $R$ nor does it change sign on $R$.

**C ii**. On the set $E_R$, the Jacobian $J(h_1, h_2)$ has a non-zero trace and a non-zero determinant.

Notice that while Olech (1963) demands an unique equilibrium, we do not. They demand a lot of other restrictions as well[35]. We have of course the properties of the excess demand function in A which have helped us to isolate a region such as $R$; **C i** and **C ii** appear weaker than the requirements demanded in Olech (1963) and Ito (1978). **C ii** ensures that the equilibria in $E_R$ have characteristic roots with real parts non-zero: this ensures that all equilibria or fixed points for the dynamic system (3.20) are hyperbolic or nondegenerate or simple. It follows that $E_R$ contains a finite odd number of equilibria since the sum of the indices of all must add up to $+1$[36].

**Proposition 3.7** *Under A, B and C, for any* $p^o \in R$, $L_\omega(p^o) = p^\star \in E_R$. *Thus all solutions converge to an equilibrium.*

Proof: Consider any $p^o \in R$ and the trajectory $\phi_t(p^o)$: the solution to (3.20); by virtue

[34]See, for instance, Section 1.5.

[35]See, for example conditions listed as O1-O4, in Section 1.5 and Proposition 1.6.

[36]See, for instance, Andronov et. al. (1966), p. 305.

of the Claim **??**, the $\omega$-limit set $L_\omega(p^o)$ is not empty. Again by the criterion of Bendixson [37], **C i** implies that there can be no closed orbits in the region $R$. Thus there can be no limit cycle and hence the Poincaré-Bendixson Theorem implies that $L_\omega(p^o) \cap E_R \neq \emptyset$. It follows therefore that $p^\star \in L_\omega(p^o)$ for some $p^\star \in E_R$; consequently there is a subsequence $\{t_s\}, t_s \to +\infty$ as $s \to +\infty$ such that $\phi_{t_s}(p^o) \to p^\star$ as $s \to +\infty$.

Since we know that the only types of equilibria are focii, nodes and saddle-points, the characteristic roots of the Jacobian $J(h_1, h_2)$ at $p^\star$ have real parts either both positive or negative, or they are real and of opposite signs, given **C ii**.

In the first case, there would be an open neighborhood $N(p^\star)$ which no trajectory or solution could enter; consequently since our trajectory $\phi_t(p^o)$ does enter *every* neighborhood of $p^\star$, it follows that at $p^\star$, the characteristic roots of the Jacobian, if complex, have real parts negative; and if real, then at least one must be negative. Thus the fixed point $p^\star$ is either a sink or at worst, saddle-point. If it is a sink, then any trajectory once having entered a small neighborhood of the equilibrium, can never leave. Consequently, the trajectory $\phi_t(p^o)$ has no other limit point. Thus $L_\omega(p^o) = p^\star$. In the case of a saddle-point, there is only a single trajectory which converges to the equilibrium; if $p^o$ happens to be on this trajectory, $L_\omega(p^o) = p^\star$ but otherwise it is not possible for a trajectory to have a saddle-point as a limit point. In any case therefore, the trajectory must **converge** to an equilibrium, as claimed.

•

We provide, next a set of remarks which highlight the implications of the above result.

[37] See, for example, Remark 7.

**Remark 17** *The above result provides a set of conditions under which an adjustment on prices on disequilibrium, in the direction of excess demand, will always lead to an equilibrium. Notice also that these conditions guarantee that there will always be at least one sink i.e., an equilibrium at which the Jacobian has characteristic roots with real parts negative. To see this note that if no such equilibrium existed, then the only equilibria are saddle-points and sources. Also in aggregate they are finite in number and moreover, as argued above, no trajectory can come close to sources; so the only possibility for a limit is a saddle-point; but each saddle-point has only one trajectory leading to it and there are an infinite number of possible trajectories. Thus there must be a sink.*

More importantly:

**Proposition 3.8** *Under **A**, **B** and **C**, if there is a unique equilibrium, it must be globally asymptotically stable.*

**Remark 18** *As mentioned above, there must be at least one equilibrium where the characteristic roots have real parts negative. Hence the trace of the Jacobian at that equilibrium must be negative; further, since the trace at that equilibrium will be negative and the trace cannot change sign nor can it be zero at equilibria, it follows that the trace of the Jacobian at every equilibrium must be negative.*

Consequently, we have:

**Proposition 3.9** *Under **A**, **B** and **C**, at every equilibrium, the sum of the characteristic roots of the Jacobian will be negative*[38].

[38] Thus, if roots are complex, the real parts must be negative.

**Remark 19** *If we consider $h_i(p_1, p_2)$ to have the same sign as $Z_i(p_1, p_2, 1), i = 1, 2$ then the assumptions in **C** are restrictions placed on the functions $h_i$. Of course these become difficult to interpret. One may show that the Jacobian of $(h_1, h_2)$ at equilibria is related to the Jacobian of $(Z_1, Z_2)$, where the partial derivatives are with respect to $(p_1, p_2)$, again at equilibria by means of the following:*

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix} \cdot \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

*where all partial derivatives are evaluated at an equilibrium and $d_i > 0, i = 1, 2$ are some positive numbers. This would provide some link between the equilibria for the dynamic process and equilibria for the economy.*

Let us reconsider the system (3.19); assume that the set of equilibria for this system $E = \{(x, y) : f(x, y) = 0, g(x, y) = 0\}$ is non-empty. The following general result follows from our analysis:

**Proposition 3.10** *If*

*i. There is a rectangular region $R = \{(x, y) : 0 \leq x \leq M, 0 \leq y \leq N\}$ such that any trajectory of (3.19) on the boundary of $R$ is either inward pointing or coincides with the boundary;*

*ii. Trace of $J(f, g)$ is not identically zero and does not change sign in the positive quadrant;*

*iii. On the set $E$, the trace and determinant of the Jacobian $J(f, g)$ do not vanish;*

*then any trajectory $\phi_t(x^o, y^o)$ where $(x^o, y^o) > (0, 0)$ converges to a point of $E$.*

A final remark considers the weakening of the assumption **C i**.

**Remark 20** *If we can find a function $\theta(p_1, p_2)$ which is continuously differentiable on the region $R$ and for which*

$$\frac{\partial\theta(p_1,p_2)h_1(p_1,p_2)}{\partial p_1} + \frac{\partial\theta(p_1,p_2)h_2(p_1,p_2)}{\partial p_2}$$

*is of constant sign on $R$, then there is no closed orbit for the system (3.20) on the region $R$*[39].

In some situations, the above may provide a weakening of the condition **C i**. It may be recalled that the sole purpose of **C i** was to rule out closed orbits in $R$. If, for example, $h_i(p_1, p_2) = Z_i(p_1, p_2, 1) = p_i.g_i(p_1, p_2), i = 1, 2$, then we may replace C i. by requiring that $p_1 g_{11}(p_1, p_2) + p_2 g_{22}(p_1, p_2)$ be of constant sign on $R$; note that we do not require the trace of $J(p_1 g_1, p_2 g_2)$ being constant on $R$. This follows by virtue of the fact that we may consider $\theta(p_1, p_2) = p_1^{-1} p_2^{-1}$ and then the condition in Remark 7 is satisfied for this choice of $\theta(p_1, p_2)$.

A more recent paper Anderson et. al. (2003), makes the important and significant contribution made that experiments conducted with agents with similar preferences and endowments, but engaging in double auctions would lead to price movements which are predicted by the tatonnement model. Thus the results provided by the tatonnement process, they argue, should be looked at with greater care because they seem to predict what price adjustments might actually occur.

As we showed in our analysis of the Scarf Example, the perturbation allowed us to get

[39] See Dulac's criterion, discussed in Section 1.5, Remark 7.

rid of closed orbits; for convergence, we needed to show that the solution was bounded. One of the reasons for our being able to obtain such a different result was due to the fact that at the original equilibrium, the relevant matrix had purely complex characteristic roots, with zero real parts. It is not surprising that in such a situation, a perturbation changed the real parts of the characteristic root from zero to positive or negative.

Notice that **C ii** rules out the Jacobian of the excess demand functions from having characteristic roots with zero real part or from being singular; both serve to ensure that the properties we observe are robust; non-singularity of the Jacobian, some times called regularity preserves static properties of the equilibria of the economic system for small changes in parameters; the trace being non-zero at equilibrium, preserves the dynamic properties from small changes in parameters. While **C i** rules out the trace of the Jacobian from changing signs on the positive quadrant which eliminates cycles. Thus **C i** rules out periodic behavior and **C ii** ensures **robustness**. These two together imply that the process will always lead to an equilibrium, provided trajectories are bounded; the particular equilibrium approached will depend on the initial configuration of prices, of course. It is also important to note that if there is a unique equilibrium, then that has to be globally asymptotically stable. Thus the feature of the original Scarf example, of a unique equilibrium which cannot be attained, is removed. However, these conclusions are for motion on the plane. Their interest lie in the fact that in many applications in economics, only such motions are considered.

In Hicks (1946), there is an enquiry relating to the following questions: if a market is stable by itself, can it be rendered unstable from the price adjustment in other markets

? Alternatively, if a market is unstable when taken by itself, can it be rendered stable by the price adjustment in the other markets ? To both an answer was provided in the negative. Notice that **C i** essentially ensured (together with **C ii**) that the trace of the relevant Jacobian remained negative; notice that this would be implied by assuming that $Z_{ii} < 0$ for each $i$, that is when each market when taken in isolation, was stable. This in turn has been seen to imply that the markets together must also be globally stable. Under certain conditions, **C i** may be weakened further; this involves the existence of a function $\theta()$ satisfying Dulac's criterion, as in the case of the perturbation of the Scarf example. But this is a matter of serendipity rather than design.

As we have seen, general results in this area are difficult to obtain. This is mainly due to **two** reasons: first of all, the excess demand functions are not expected, *a priori* to satisfy any other property apart from Homogeneity of degree zero in the prices and Walras Law; secondly, dynamics in dimensions greater than 2 may be quite difficult to pin down. Even on the plane, a variety of dynamic motions are possible. It is not surprising that in higher dimensions matters become a lot more complicated and Walras Law and Homogeneity do not help too much. We have just seen what these considerations will lead us to. And consequently, we must impose additional restrictions which may be called **global stability conditions** ; we have shown that an easy such condition for the Scarf example is $b < 1$. For the general case, on the plane, the conditions in **C** serve the same purpose.

## 3.9 Effective Tatonnement: An Example

### 3.9.1 The Problem

It is clear from what we have seen so far that a price adjustment on prices needs to be supported by additional restrictions on the excess demand functions to ensure convergence to equilibrium; without any such restrictions, which we have called stability conditions, dynamics of a very arbitrary character becomes possible. Even for the simplest version of the tatonnement (3.3), results were not clear. Strong assumptions such as gross substitutes or dominant diagonals have to be used to claim convergence.

Consequently any study to devise a theory for price adjustment behavior which successfully attains equilibria for any class of excess demand functions will be implausible. We have argued above, therefore, to identify stability conditions. The alternative may be to identify adjustment processes with strong convergence properties.

What then are the minimal ingredients for such a theory of adjustment processes? First of all, we have usually taken that price moves in the direction of the excess demand and secondly, that transactions occur when equilibrium is reached. These are the two cornerstones of the Walrasian tatonnement story. We report here some exercises where the first of these requirements are surrendered.

Historically, there have been the contributions of Smale (1976), Arrow and Hahn (1971) and Kamiya (1990). The main feature of these contributions are that they provide examples of adjustment processes which attain equilibria for **any family of excess demand functions**. The intuition behind the processes however is not always clear. Processes such as these, which converge regardless of the nature of excess demand functions are known as **effective processes**. The requirements for local processes to be effective were first studied by Saari and Simon (1978); they found that the dependence of price adjustment only on excess demand in the appropriate market was not sufficient for convergence; the entire jacobian of the excess demand function needs to be used to define the process. The later study by Jordan (1983) also proved to be quite revealing. It was shown there that for a local process to be effective a necessary condition was that it should violate a Walrasian requirement viz., the price of a commodity whose market was in equilibrium should not be adjusted even if there are disequilibria in other markets. Thus processes such as the ones we have considered so far do not have much chance of being able to attain equilibria. We provide an example of a process[40] which provides an economic rationale for such conclusions. As an interesting joint-product of this exercise, a condition for the uniqueness of the competitive equilibrium will be identified which has several interesting features.

### 3.9.2 Regular Economy

Consider an economy specified by means of excess demand function $Z : \mathcal{D} \to \Re^n$ where $\Re^n_{++} \subset \mathcal{D} \subset \Re^n_{+}$. We shall impose the restrictions P1-P3 and P4 which we reproduce here, for ease of reference:

P1. For all $p \in \mathcal{D}, Z(p)$ is **bounded below** and is continuously differentiable with continuous second order partial derivatives.

P2. $p^t.Z(p) = 0 \forall p \in \mathcal{D}$ (Walras Law).

[40] This is based on Mukherji (1995).

P3. $Z(\lambda p) = Z(p) \forall p \in \mathcal{D},$ for any $\lambda > 0$ (Homogeneity of degree zero in prices).

Let $S = \{p \in \Re^n_+ : \sum_i p_i^2 = 1\}$: the part of the unit n-sphere inside the non-negative orthant. Given P3, we may confine attention to the set $\mathcal{T} = S \cap \mathcal{D}$; it would be helpful to assume that in addition to the above, we have:

P4. If $p^s$ is a sequence in $\mathcal{T}$ for all $s$ and if $p^s \to p^o$ as $s \to \infty$, where $p^o \notin \mathcal{D}$ then $\sum_i Z_i(p^s) \to +\infty$.

Let $\mathcal{E} = \{p \in \mathcal{T} : Z(p) = 0\}$: the set of equilibria. We shall assume that there are **no boundary equilibria** [41]. Formally, we require that:

P5. Let $\partial S = \{p \in S : p_k = 0$ for some index $k\}$. Then $\mathcal{E} \cap \partial S = \emptyset$.

The Jacobian of the excess demand function is the matrix of all partial derivatives of the excess demand function:

$$J(p) = [\frac{\partial Z_i}{\partial p_j}, i, j = 1, 2, \cdots, n]$$

By virtue of P3, $J(p).p = 0 \forall p \in \mathcal{D}$; hence rank $J(p) \leq n - 1 \forall p \in \mathcal{D}$. An equilibrium $p \in \mathcal{E}$ is said to be **regular** if rank $J(p) = n - 1$; and the economy is said to be regular if every equilibrium is regular.

We shall write $\zeta(p)$, for any $p \in \mathcal{D}$, for the following square matrix of order $n + 1$:

$$\begin{pmatrix} J(p) & p \\ -p^t & 0 \end{pmatrix}$$

We note for future reference the following properties of the matrix $\zeta(p)$:

[41] It may be pointed out that our earlier restriction P4* ruled out boundary equilibria and so we had no reason to use this condition separately.

**Proposition 3.11** *For $p \in \mathcal{T}$, det $\zeta(p) \neq 0 \Leftrightarrow 0$ is NOT a repeated characteristic root of $J(p)$; if however $p \in \mathcal{E}$, det $\zeta(p) \neq 0 \Leftrightarrow 0$ is NOT a repeated characteristic root of $J(p) \Leftrightarrow$ rank $J(p)$ is $n-1$.*

Proof: Suppose then, that det $\zeta(p) = 0$; thus there must exist $(\bar{x}, \bar{\beta}) \neq (0,0)$ such that $\bar{x}^t.J(p) - \bar{\beta}p^t = 0$, and $\bar{x}^t.p = 0$; since $J(p).p = 0$ it follows that $\bar{\beta} = 0$; hence we have $\bar{x}^t.J(p) = 0$ and $\bar{x}^t.p = 0$, for some $\bar{x} \neq 0$. It therefore follows that 0 is a repeated characteristic root of $J(p)$; for suppose to the contrary that 0 is a simple characteristic root of the matrix $J(p)$; then rank of $J(p)$ is $n-1$ and moreover, the left and right characteristic vectors corresponding to 0 may be so chosen that their inner product is unity[42]; i.e., there are non-null vectors $u, v$ such that $u^t.v = 1, u^t.J(p) = 0, J(p).v = 0$; since the rank of $J(p) = n-1$; it follows that $p = \alpha v, \bar{x} = \gamma u$ for some scalars $\alpha, \gamma$; hence we have $0 = \bar{x}^t.p = \alpha.\gamma u^t.v = \alpha.\gamma$; since $\bar{x} \neq 0$; the scalar $\gamma$ cannot be zero; hence $\alpha = 0$ but then $p = 0$ but recall that too is not possible, since $p \in \mathcal{T}$. Thus 0 must be a repeated characteristic root for $J(p)$, as claimed. Conversely, if 0 is a repeated characteristic root of $J(p)$; then using the Jordan decomposition of the matrix $J(p)$ it follows that there exists $x \neq 0$ such that $x^t.J(p) = 0, x^t.p = 0$: notice that this implies that:

$$(x^t, 0).\begin{pmatrix} J(p) & p \\ -p^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}$$

which establishes the singularity of the matrix $\zeta(p)$.

If $p \in \mathcal{E}$, then we have in addition that $p^t.J(p) = 0$; notice now that if det $\zeta(p) = 0$;

[42] Consider for example, the Jordan decomposition of the matrix J(p); see, for example, Gantmacher (1959).

then it follows immediately that rank of $J(p) < n-1$; thus rank of $J(p) = n-1 \Rightarrow$ det $\zeta(p) \neq 0$. The converse argument is also straightforward. •

The property of regularity of equilibrium yields the following interesting consequences:

**Proposition 3.12** *Let $p^\star \in \mathcal{E}$ be regular. Then there exists a neighborhood $N(p^\star)$ of $p^\star$ satisfying the following:*

*i. $N(p^\star) \cap \mathcal{E} = p^\star$ (Local Uniqueness);*

*ii. $(p^\star)^t.[J(p)]^t.Z(p) \leq 0 \forall p \in N(p^\star) \cap \mathcal{T}$;*

*iii. $p \in N(p^\star) \cap \mathcal{T}, Z(p)^t.J(p) = 0 \Rightarrow p = p^\star$.*

Proof: For i., see, for example, Mas-Colell (1985), p. 185, Proposition 5.4.2.

For ii., define $f(p) = (p^\star)^t.[J(p)]^t.Z(p)$; note the following:

$$\nabla f(p) = (\frac{\partial f}{\partial p_k}) = (\sum_i \sum_j p_i^\star(\frac{\partial Z_{ji}(p)}{\partial p_k}.Z_j(p) + Z_{ji}(p).Z_{jk}(p)))$$

and $\nabla^2 f(p)$ is given by :

$$(\frac{\partial^2 f(p)}{\partial p_r \partial p_k}) = (\sum_i \sum_j p_i^\star(\frac{\partial Z_{ji}(p)}{\partial p_k}.Z_{jr}(p) + \frac{\partial^2 Z_{ji}(p)}{\partial p_r \partial p_k}Z_j(p) + \frac{\partial Z_{ji}(p)}{\partial p_r}.Z_{jk}(p) + Z_{ji}(p).\frac{\partial Z_{jk}(p)}{\partial p_r}))$$

Also note that $\nabla f(p^\star) = 0$; and the matrix $(\nabla^2 f(p)) = -2J^t(p^\star).J(p^\star)$ thus for $q \in N(p^\star) \cap \mathcal{T}$ we have $f(q) = 1/2(q-p^\star)^t.(\nabla^2 f(p^\star)).(q-p^\star) = -q^t.J^t(p^\star).J(p^\star).q \leq 0$; if $q \neq p^\star$, the inequality is strict and this proves the claim.

For iii., first recall that $p^\star$ being regular implies that $\zeta(p^\star)$ is non-singular; thus for $p$ 'close' to $p^\star$, $\zeta(p)$ must be non-singular too; note also that if $p \notin \mathcal{E}$, $Z(p)^t.J(p) = 0 \Rightarrow \zeta(p)$

is singular and hence the claim follows. •

With these properties at or near regular equilibria, we shall now set up an adjustment process which is locally effective.

### 3.9.3 A Locally Effective Process

Instead of the usual method of normalizing prices by choosing a numeraire and then considering all prices to be relative to this good, we shall adopt a some what different method; we consider prices to be normalized on the unit sphere $S$ introduced earlier; in particular we restrict attention to set $\mathcal{T}$ which is a subset of $S$ over which the excess demand functions are defined. Consider the following adjustment equation:

$$\dot{p_i} = -\sum_{k=1}^{n} Z_{ki}(p).Z_k(p), \ \ , i = 1, 2, \cdots, n \tag{3.21}$$

or alternatively, in matrix notation:

$$\dot{p} = -J^t(p).Z(p)$$

We shall refer to this process as the LS-process. Given any $p^o \in \mathcal{T}$, let $\phi_t(p^o)$ denote the solution to (3.21); such a solution exists, given our assumptions; we first ascertain that the solution always remains on the set $\mathcal{T}$ provided $p^o$ is 'close' to some regular equilibrium $p^\star$; we do this in two steps. First of all,

**Claim 3.9.1** $\phi_t(p^o) \in S$ *for all* $t \geq 0$.

Proof: Let us write the solution $\phi_t(p^o)$ as $(p_j(t))$; We note that $\sum_j p_j^2(t) = \sum_j (p_j^o)^2 \forall t$; this follows since $\sum_j p_j(t)\dot{p_j} = -p(t)^t.J(p(t))^t.Z(p(t)) = 0$; hence the claim follows. •

**Claim 3.9.2** *Let* $V(t) = \sum_j Z_j^2(p(t))$*; then* $\dot{V}(t) < 0$ *whenever* $p(t) \in N(p^\star) \cap \mathcal{T}, p(t) \neq p^\star$ *provided* $p^\star$ *is a regular equilibrium, and* $N(p^\star)$ *is as in the statement of Proposition 3.12.*

Proof: Note that $\dot{V}(t) = -2Z(p(t))^t.J(p(t))^t.J(p(t)).Z(p(t))$ ; this allows us to conclude that $\dot{V}(t) = 0$ only if $J(p(t)).Z(p(t)) = 0$; now recall and apply the result of the Proposition 3.12 iii, and the claim follows. •

Suppose $p^o \in N(p^\star) \cap \mathcal{T}$ where $N(p^\star)$ is as defined above; and consider $W(t) = 1/2 \sum_i (p_i(t) - p_i^\star)^2$; note that $\dot{W}(t) = -(p(t) - p^\star)^t.J(p(t))^t.Z(p(t)) = {p^\star}^t.J(p(t))^t.Z(p(t)) \leq 0$ by virtue of Proposition 3.12 ii. Thus $\phi_t(p^o) \in N(p^\star) \forall t$.

Now suppose that $\phi_t(p^o)$ approaches $\bar{p} \notin \mathcal{D}$; then by virtue of the assumption P4., $V(t) \to +\infty$ which contradicts Claim 3.9.2. So $\phi_t(p^o) \in \mathcal{D} \forall t$. Hence by virtue of the above, we may claim:

**Claim 3.9.3** $\phi_t(p^o) \in N(p^\star) \cap \mathcal{T} \forall t$ *provided* $p^o \in N(p^\star) \cap \mathcal{T}$ *and* $p^\star$ *is regular.*

Note then, that by virtue of the result in Claim 3.9.2, $V(t)$ is a local Liapunov Function and hence using the conclusion of Claim 3.9.3, one may conclude that

**Proposition 3.13** *Every regular equilibrium is locally asymptotically stable under the LS-process.*

### 3.9.4 The Significance of the LS-Process

In our studies, till we encountered the LS-process, we considered the Walrasian tatonnement. This is based on the intuition that price adjustment in a market is determined by the level of excess demand in that market: we shall refer to this as the Walrasian Adjustment

Hypothesis. The naive expectation that this will work is based on an argument which may be constructed as follows: if the price of a good is raised (lowered), we may expect its excess demand to be lowered (raised, respectively); so if the market is in excess demand (excess supply), one should raise (lower, respectively) the price to move towards equilibrium.

To understand the rationale behind the LS-process it will be convenient to write the equation (3.21) as follows:

$$\dot{p}_i = -Z_{ii}(.)Z_i(.) - \sum_{k \neq i} Z_{ki}(.)Z_k(.)$$

thus the price adjustment in the $i$-th market may be decomposed into two terms: the first relates to its own market excess demand and the other relates to the excess demands in all other markets. In case $Z_{ii}(.) < 0$, the first term on the right thus satisfies the Walrasian Adjustment Hypothesis; the second term on the right has no counterpart in the Walrasian paradigm since the second term indicates how price adjustment in market $i$ should depend on the state of excess demand in the $k-$th market $k \neq i$: if for instance, $Z_k(.) > 0$, then $p_i$ should be revised upwards (downwards) depending on whether $Z_{ki}(.) < 0(> 0$ respectively ). Thus the LS-process acknowledges the interconnections between markets when formulating the price adjustment hypothesis; price adjustment in any market should not only help curb level of excess demand in its own market, it should also help curb excess demand in other markets as well; if the terms $Z_{ki}(.) = 0$, for example, then this latter requirement is not an additional burden on price adjustment in the $i$-th market and the second set of terms disappear. When this is not so, the requirements placed on what $\dot{p}_i$ should be is an aggregation of many terms and consequently the if the first terms dominates, then we shall have the Walrasian Adjustment Hypothesis satisfied; when $Z_i(.) = 0$ in a disequilibrium configuration ($Z_k(.) \neq 0$ for some $k$), the first term cannot dominate and it is easy to understand why Jordan (1983) reached the conclusion that an effective process will necessarily violate the requirement $\dot{p}_i = 0$ whenever $Z_i(.) = 0$.

The naive belief that price adjustment in a market is governed by excess demand in only that market, as in the Walrasian Adjustment Hypothesis takes into account only what may be called the direct effects of disequilibrium. In a slightly different context, Hicks (1946) indicated that there are indirect effects too: those which work through other markets. For example suppose that in market $j$, $Z_j(.) > 0$; $j \neq i$; buyers in the market for $j$ would bid the price up; this would affect the $i$-th market too through the excess demand $Z_i$ depending on the sign of $Z_{ij}(.)$. It appears that while stressing the interconnections between markets while discussing issues relating to the existence of equilibrium, Walras did not take into account of precisely these interconnections while setting up the dynamic considerations. In other words, processes such as (3.1), where price adjustment in any market is governed entirely by excess demand in that market is at best a partial equilibrium approach to price adjustment. To be able to reduce disequilibria in **all** markets, we need to take into account not only excess demands in other markets but also the links between markets provided by the partial derivatives $Z_{ij}(.)$. This provides a justification for why the entire Jacobian of the excess demand function must be involved.

As a final point to clinch the matter, consider the situation when $n = 2$; then the process

(3.21) for the $i$-th market reduces to:

$$\dot{p}_i = -Z_{ii}(.)Z_i(.) - Z_{ji}(.)Z_j(.)$$

It is easy to check then that if $Z_{ii}(.) < 0$ then $\dot{p}_i.Z_i(.) > 0$ whenever $Z_i(.) \neq 0$: this follows from properties P2 and P3. Notice that $n = 2$ means that effectively, there is only one market and hence interconnections through other markets need not be considered. In such situations, the adjustment process (3.21) satisfies the Walrasian Hypothesis.

There is another aspect of the above that should be pointed out. Given $Z(p)$, define $f_i(p) = -\sum_k Z_{ki}(p).Z_k(p)$ and $f(p) = (f_i(p))$. Note that $p^t.f(p) = 0$; also $f(\lambda p) = f(p)/\lambda$; so that if we define

$$\theta(p) = ||p||.f(p) \tag{3.22}$$

then $\theta(p)$ satisfies $p.\theta(p) = 0$ and $\theta(\lambda p) = \theta(p)$: thus $\theta(p)$ may be called a **pseudo-excess demand function**. More importantly, by virtue of Proposition 3.12, it follows that for any regular equilibrium $p^\star$, for $p \in N(p^\star)$, $\theta(p) = 0$ if and only if $p = p^\star$ and $(p^\star)^t.\theta(p) > 0$. Thus in a neighborhood of regular equilibria, regardless of the nature $Z(p)$, $\theta(p)$ satisfies a local Weak Axiom of Revealed Preference. Thus one way of justifying a process such as (3.21) is to claim that it is a local modification of a process such as (3.3) where $\dot{p}_j = Z_j(p)$ and the excess demands have been replaced by the pseudo-excess demands.

In the literature there are two other processes which may be called effective; these are due to Smale (1976) and Kamiya (1990). Both of these processes involve the choice of a numeraire; by choosing the $n$-th good as numeraire; since, in this section, we have used a different normalization, viz., on the sphere, we shall use $q$ to denote the relative prices, i.e

$q = (q_1, q_2, \cdots, q_{n-1})$; also we shall use the notation $\bar{Z}(q), \bar{J}(q)$ to denote respectively, the excess demand functions of the non-numeraire goods and their Jacobian. With this notation, the process due to Smale (1976), the Global Newton Method (N) may be presented as:

$$\bar{J}(q).\dot{q} = -\lambda \bar{Z}(q) \tag{3.23}$$

where $\lambda$ is a real number such that sign $\lambda$ = sign det $-\bar{J}(q)$. The result due to Kamiya (1990) behaves very much like the Global Newton Method near equilibria and since we shall be comparing local behavior only, we confine attention to (3.23). To make comparisons, we need to consider a normalized version of (3.21) where commodity $n$ is a numeraire; such a process may be written as:

$$\dot{q} = -\bar{J}(q)^t.\bar{Z}(q) \tag{3.24}$$

Notice this normalization preserves the local stability of (3.21) since the linearized version of (3.24) around an equilibrium $q^\star$ is given by:

$$\dot{q} = -\bar{J}(q^\star)^t.\bar{J}(q^\star).q$$

and since the matrix $-\bar{J}(q^\star)^t.\bar{J}(q^\star)$ is negative definite at all regular equilibrium, it follows that every regular equilibrium is locally asymptotically stable under (3.24). Consider the case when $n = 3$ then under the process (N) near a regular equilibrium, the matrix on the left of (3.23), is invertible and we have:

$$\dot{q} = -\frac{\lambda}{\Delta} \begin{pmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{pmatrix} .\bar{Z}(q)$$

where $\Delta$ is the determinant of $\bar{J}(q)$ and hence $\lambda/\Delta$ is positive. For the process (3.24) on the other hand, we have:

$$\dot{q} = -\begin{pmatrix} Z_{11} & Z_{21} \\ \\ Z_{12} & Z_{22} \end{pmatrix} .\bar{Z}(q)$$

Thus for the former, we have on the right hand side $\bar{J}(q)^{-1}$; whereas for the latter, we have $\bar{J}(q)^t$. To see the differences even more starkly, assume that the off diagonal terms are zero; then under (3.23), the Global Newton Method, we have:

$$\dot{q}_i = -\lambda Z_i(q)/Z_{ii}(q)$$

and for the process (3.24), we have:

$$\dot{q}_i = -Z_{ii}(q).Z_i(q)$$

the processes begin to differ when the sign of $\lambda$ happens to be negative; say $Z_{22}.Z_{11} < 0$ and say $Z_{11} < 0$. In such a situation, the process N will adjust $q_1$ in a direction opposite to excess demand while moving $q_2$ in the same direction as its excess demand: given the sign pattern of the $Z_{ii}$ this is not what we would have expected. What we would have expected to move the first price in the same direction as excess demand and the second opposite to its excess demand and this is exactly what happens under the process (3.24).

Finally, neglecting excess demands in other markets, when would $\dot{q}_j$ have the same sign as $Z_j(q)$: for the process (3.24), this is so whenever $Z_{ii}(.) < 0$; for the process (N), however, this is so whenever

$$\lambda \frac{\text{cofactor of } Z_{jj}(q) \text{ in } \bar{J}(q)}{\det\ \bar{J}(q)} < 0$$

It is a bit difficult to explain the necessity of this condition for convergence. Thus, our discussion should convince people of the necessity of including information about all excess demands while formulating price adjustment mechanisms. But it is also important to remember that there has to be some rationale for explaining how the inclusion of other excess demands guarantee convergence.

### 3.9.5 Uniqueness of Competitive Equilibrium

As we hope to demonstrate, the exercise reported in the sections above, allows us to identify a set of conditions which imply the uniqueness of competitive equilibrium. We ask the following question: we know that for global uniqueness, we should certainly have local uniqueness of equilibrium; are there some conditions which together with local uniqueness add up to global uniqueness ? As we shall see, it is possible to answer this question in the affirmative.

In particular we shall show that the conditions identified in the Proposition 3.12 play major role. Notice that these conditions all hold in some neighborhood of a regular equilibrium. Consider the first two of them, which we restate here, for ease of reference:

P6. (**Near Equilibria)** Let $p^\star \in \mathcal{E}$. There exists a neighborhood $N(p^\star)$ of $p^\star$ satisfying the following:

i. $N(p^\star) \cap \mathcal{T} = p^\star$ (Local Uniqueness);

ii. $(p^\star)^t.[J(p)]^t.Z(p) \leq 0 \forall p \in N(p^\star) \cap \mathcal{T}$.

As we had shown in Proposition 3.12, P6 is implied at all **regular** equilibria; demanding that P6 holds is however weaker than insisting on regularity of equilibrium. To show this, we need to construct an example of an excess demand function where regularity is violated but P6 holds; so consider the following example:

**Exercise 4** *Consider the function*

$$Z_1(p_1,p_2) = \begin{cases} (3-\frac{p_1}{p_2})^3 & \textit{provided } p_1 < 4p_2 \\ = 48\frac{p_1}{p_2} - 13 & \textit{otherwise} \end{cases}$$

Notice that the function $Z_1(.,.)$ is a differentiable continuous function, homogeneous of degree zero and bounded below for all $p_1, p_2 > 0$; by virtue of Proposition 10 of Shafer and Sonnenschein (1982), p. 683, $Z_1(.,.)$ and $Z_2(p_1,p_2)$ defined by $= -p_1.Z_1(p_1,p_2)/p_2$ constitute excess demand functions for an appropriate economy. The unique equilibrium in $S$ may be seen to be $p^\star = (\sqrt{\frac{9}{10}}, \sqrt{\frac{1}{10}})$; at this price $J(p^\star)$ is the null matrix: thus the economy is not regular. In any neighborhood $N(p^\star)$ small enough, it must be the case that $p_1 < 4p_2$; hence we note that:

$$p^{\star^t} J^t(p).Z(p) = \sqrt{(\frac{1}{10})}.\frac{1}{p_2}(3-\frac{p_1}{p_2})^5[4(\frac{p_1}{p_2})^3 - 15(\frac{p_1}{p_2})^2 + 12(\frac{p_1}{p_2}) - 9]$$

$$= \sqrt{(\frac{1}{10})}.\frac{1}{p_2}(3-\frac{p_1}{p_2})^5[(-3+\frac{p_1}{p_2}).(4(\frac{p_1}{p_2})^2 - 3(\frac{p_1}{p_2}) + 3]$$

Check that when $p_1 = 3p_2$, the rhs is zero; when $p_1 > 3p_2$, the first term in parenthesis is negative; while the term in square brackets is positive so the product is negative. When $p_1 < 3p_2$; the first term in parenthesis is positive while the second term in square brackets is negative and the product is once again, negative. **Thus the requirements of P6 are met, while the economy is not regular**.

Recollect that the Proposition 3.12 had a third part which holds near regular equilibria: $p \in N(p^\star) \cap \mathcal{T}, Z(p)^t.J(p) = 0 \Rightarrow p = p^\star$. We shall consider the strengthening this to:

P7. $p \in \mathcal{T}, Z(p)^t.J(p) = 0 \Rightarrow Z(p) = 0$ or $p \in \mathcal{E}$.

In contrast to P6, P7 is a restriction which is supposed to hold for all of $\mathcal{T}$. We know of course that near a regular equilibrium, P6 and P7 hold; if P6 and P7 hold then we shall show that equilibrium must be unique. This would thus ensure uniqueness of equilibrium based on conditions which necessarily hold near regular equilibria P6 and if a condition which holds near regular equilibria holds elsewhere, as well (P7).

We first examine the plausibility of meeting the requirement in P7; towards this end, observe, first of all:

**Remark 21** *The exercise presented above satisfies P7.*

To see this note that when $p_1 \geq 4p_2$

$$Z_1 = 48\frac{p_2}{p_1} - 13 \ ; \ Z_2 = -48 + 13\frac{p_1}{p_2};$$

consequently the Jacobian $J(p)$ is given by:

$$\begin{pmatrix} -48\frac{p_2}{p_1^2} & 48/p_1 \\ 13/p_2 & -13\frac{p_1}{p_2^2} \end{pmatrix}$$

Thus $Z(p)^t.J(p) = 0 \Rightarrow$

$$(48\frac{p_2}{p_1} - 13)(-48\frac{p_2}{p_1^2} - 13\frac{p_1}{p_2^2}) = 0 \Rightarrow p_1/p_2 = 48/13 \Rightarrow$$

$p_1 < 4p_2$: a contradiction. Similarly, when $p_1 < 4p_2$ it is easy to check that to satisfy the first component of $Z(p)^t.J(p) = 0$, requires $p_1 = 3p_2$ or $Z(p) = 0$; so that $Z(p)^t.J(p) = 0$ is possible only at equilibria. Hence P7 holds.

That the above example is not a stray case satisfying P7 will be established next by considering some conditions which imply P7.

**Claim 3.9.4** *If $\zeta(p)$ is non-singular for all $p \in \mathcal{T}$, P7 is satisfied for all $p \in \mathcal{T}$.*

Proof: Recall from the proof of Proposition 3.11 that if $\zeta(p)$ is non-singular then P7 follows: in fact, we had used regularity to guarantee non-singularity near regular equilibria. Hence the claim follows. •

We next show that under the assumption of indecomposable weak gross substitutes, P7 is also satisfied. We have:

**Claim 3.9.5** *Let $\mathcal{D} = \Re^n_{++}$ and let $Z_{ij}(p) \geq 0 \forall p \in \mathcal{T}$; further, for all such $p$, the index set $I = \{1, 2, \cdots, n\}$ has no non-empty proper subset $K$ such that $Z_{ij}(p) = 0$ for all $i \notin K, j \in K$. Then P7 holds.*

Proof: Suppose $p \notin \mathcal{E}$ and $Z(p)^t.J(p) = 0$. Then we have, $\sum_j Z_i(p).Z_{ij}(p) = 0$ for all $j$. Thus, we have:

$$-Z_j(p).Z_{jj}(p) = \sum_{i \neq j} Z_i(p) Z_{ij}(p) \forall j$$

or, we have:

$$|-Z_j(p).Z_{jj}(p)| = |\sum_{i \neq j} Z_i(p) Z_{ij}(p)| \leq \sum_{i \neq j} |Z_i(p) Z_{ij}(p)| \forall j$$

and hence:

$$|Z_j(p)|(-Z_{jj}(p)) \leq \sum_{i \neq j} |Z_i(p)|.Z_{ij}(p) \forall j$$

the last step follows by virtue of the sign restriction on the $Z_{ij}$; writing $v_j = |Z_j(p)|$ and $v = (v_j)$, we have $v^t.J(p) \geq 0$; since $J(p).p = 0$ and $p \in \Re^n_{++}$ it follows that we must have $v^t.J(p) = 0$; this means that $(v + Z(p))^t.J(p) = 0$; since $p \notin \mathcal{E}$, the set $K = \{i : Z_i(p) \leq 0\} \neq \emptyset$ and there is $i \notin K$; thus $K$ is a non-empty proper subset of $I$. Consider the equation

$(v + Z(p))^t.J(p) = 0$ that is $\sum_i (v_i + Z_i).Z_{ij} = 0 \forall j$; in particular, for $j \in K$, $v_j + Z_j = 0$; thus the above equation implies that $Z_{ij} = 0$ when $i \notin K$: thus we arrive at a contradiction and our claim is established. •

Since we had demonstrated the local asymptotic stability of every regular equilibrium under a process such as (3.21) in Proposition 3.13 by using the properties such as P6 and P7 near regular equilibria (Proposition 3.12), we can dispense with regularity and claim:

**Claim 3.9.6** *Under P1-P7, every equilibrium $p^\star \in \mathcal{E}$ is locally asymptotically stable under (3.21).*

But now since P7 holds on all of $\mathcal{T}$, we can claim the following:

**Proposition 3.14** *Let $L(p^o)$ denote the set of limit points to the solution $p(t, p^o)$ of (3.21) for any $p^o \in \mathcal{T}$; then under P1 - P7, $L(p^o) = p^\star$ for some $p^\star \in \mathcal{E}$.*

Proof: First of all, consider $W(t) \equiv W(p(t, p^o)) = \sum_j Z_j^2(p(t, p^o))$ and note that $\dot{W}(t) = -2Z(p(t, p^o))^t.J(p(t, p^o)).J^t(p(t, p^o)).Z(p(t, p^o)) < 0$ unless $p(t, p^o) \in \mathcal{E}$ by virtue of P7. We know already from Claim 3.9.1 that $p(t, p^o) \in S$; since $W(t) \leq W(p^o) \forall t$, it follows, by virtue of P4 that $p(t, p^o) \in \mathcal{T} \forall t$ and hence that $L(p^o) \neq \emptyset$ and $L(p^o) \subseteq \mathcal{T}$. Notice that we have thus established that $W(.)$ is a Liapunov function for the process (3.21) and every limit point of the trajectory $p(t, p^o)$ is in $\mathcal{E}$; we know by virtue of P6 that equilibria are locally unique and locally asymptotically stable, as we have just noted above. it follows therefore that once having entered a small enough neighborhood of any equilibrium, the trajectory cannot leave; hence $L(p^o)$ can have only one limit point and this proves the result. •

This allows us to apply a result due to Arrow and Hahn (1971)[43] and claim the following:

**Proposition 3.15** *Under P1-P7, the set $\mathcal{E}$ is a singleton: equilibrium is unique.*

Proof: Under P1-P7, we know that $\mathcal{E}$ is compact and made up of locally unique equilibrium: hence $\mathcal{E}$ is a finite set. Thus let the elements of $\mathcal{E}$ be written as $p^i, i = 1, 2, \cdots, m$; our claim is that $m = 1$. Suppose this is not the case and choose any two $p^1, p^2 \in \mathcal{E}$ and consider the line segment $C = [p^1, p^2]$ i.e., $C = \{p : p = tp^1 + (1-t)p^2 \text{ for some } t, 0 \leq t \leq 1\}$; let $C(p^i) = \{p^o \in C : p(t, p^o) \to p^i\}$; we know that since each $p^i$ is locally asymptotically stable, $C(p^i) \neq \emptyset, i = 1, 2$; indeed, due to the Proposition 3.14, we know that every $p \in C$ belongs to some $C(p^j)$ and hence one may write $C = \cup_{j=1}^m C(p^j)$, where some of the sets $C(p^j)$ are non-empty; further we also know that $C(p^j) \cap C(p^k) = \emptyset$, if $j \neq k$ and we also note that since $C$ is connected it cannot be decomposed into the union of two or more disjoint non-empty closed sets; consequently, it must be the case that for some $j$, $C(p^j) \neq \emptyset$ and is not closed; i.e., there is a sequence $p(s) \in C(p^j)$ with $\bar{p}$ as a limit point and $\bar{p} \notin C(p^j)$; since $p(s) \in C$ and $C$ is a closed set, $\bar{p} \in C$ and $\bar{p} \notin C(p^j)$ implies that $\bar{p} \in C(p^k)$ for some index $k$ different from $j$. Thus $p(t, p(s)) \to p^j$ and $p(t, \bar{p}) \to p^k$; from the property of continuity of the solution, we know that there is a neighborhood $N$ of $\bar{p}$ such that $p^o \in N$ implies $p(t, p^o) \to p^k$ (recall, equilibria are locally unique); now since $p(s) \to \bar{p}$, for $s$ large enough, $p(s) \in N$ and hence $p(t, p(s)) \to p^k$: a contradiction. Thus we cannot construct a set such as $C$ and hence $\mathcal{E}$ must be a singleton as claimed. •

To sum up: it is to be noted that we used P6 and P7 to be main conditions for guar-

[43]See, Theorem 5, p. 280

anteeing uniqueness of equilibrium. The main interest in these conditions lie in the fact that these conditions will **always be satisfied near regular equilibria**. Note that since we dispensed with regularity, P6 was imposed near equilibria while P7 was needed over $\mathcal{T}$. If one were to ask the question that in addition to local uniqueness, what else would ensure global uniqueness, then our analysis provides an answer. There is another point of interest about these conditions. Recall the pseudo-excess demand functions $\theta(p)$ (3.22); we had remarked that these functions satisfy a local weak axiom of revealed preference. Now consider the following: given an excess demand function system $Z(p)$ satisfying P1-P5. Consider a fictitious economy made up of the the pseudo excess demand functions, $\theta(p)$ given the $Z(p)$; notice P7 ensures that the equilibria of the fictitious economy coincide with those of the original economy; now if the original economy was regular then P6 would hold and the equilibrium for the original economy must be unique. Notice that what P6 achieves is to ensure that the fictitious economy satisfies local uniqueness as well as a local weak axiom of revealed preference. Thus the close links between WARP, uniqueness and stability are exhibited once more. In fact, it may be recalled that it is this link which we have investigated in our analysis throughout this chapter.

## 3.10 Discrete Time Tatonnement

### 3.10.1 Preliminary Difficulties

Among economists, the problem that discreteness, *per se* introduces, it seems to me, are not well appreciated. To exhibit the problem, we consider first of all, a discrete version of the tatonnement process studied earlier:

$$p_j(t+1) = \max\{0, p_j(t) + \gamma Z_j(p(t))\} \ \forall j \neq n \ , p_n(t) = 1 \forall t \tag{3.25}$$

where $\gamma > 0$ is interpreted as a speed of adjustment, assumed for the sake of simplicity, to be constant across commodities/markets. Further to ease our demonstration, we assume that the excess demand functions $Z_j(.)$ exhibit the property of Gross substitution (GS) for all $p \in R^n_{++}$i.e.,

$$\frac{\partial Z_j(p)}{\partial p_k} > 0 \ \forall j \neq k$$

and for all $p \in R^n_{++}$. Under the property of GS, it is well known that

i. There is an unique $p* = (p*_1, \cdots, p*_{n-1}, 1) \in R^n_{++}$ such that $Z(p*) = 0$.

ii. For every $p \in R^n_{++}$,$p \neq p*$, $p * .Z(p) > 0$ (the Weak Axiom of Revealed Preference holds).

The continuous version of the process ( 3.25 ) converges to the unique equilibrium and the demonstration is relatively straightforward, using the fact that the function $V(p) = \|p - p*\|^2 = \sum_j (p_j - p*_j)^2$ is a Liapunov function. It would be instructive to try to see what we can say about the iterates of the process ( 3.25 ).

Using a method suggested by Uzawa(1958), one may show the following:

**Claim 3.10.1** *If $\gamma$ is sufficiently small, then for any $\epsilon > 0$, there is an integer $t^o$ such that $V(p(t+1)) \leq V(p(t))$ for $0 \leq t < t^o$ and $V(p(t) \leq \epsilon$ for $t \geq t^o$.*

.

Proof: Now consider the following:

$$\left.\begin{array}{c} \sum_{j\neq n} p_j^2(t+1) \leq \sum_{j\neq n} p_j^2(t) + 2\gamma \sum_{j\neq n} p_j(t) Z_j(p(t)) + \gamma^2 \sum_{j\neq n} Z_j^2(p(t)) \\ -2\sum_{j\neq n} p*_j \, p_j(t+1) \leq -2\sum_{j\neq n} p*_j \, p_j(t) - 2\gamma \sum_{j\neq n} p*_j \, Z_j(p(t) \end{array}\right\} \tag{3.26}$$

Also note that by definition, we have

$$V(p(t+1)) = \sum_{j\neq n} p_j^2(t+1) - 2\sum_{j\neq n} p_j(t+1) p*_j + \sum_{j\neq n} p*_j^2$$

so that using the equation ( 3.26 ) and using Walras Law, we have:

$$V(p(t+1)) \leq V(p(t)) + \gamma[\gamma \sum_{j\neq n} Z_j^2(p(t) - \sum_{j=1}^{n} p*_j \, Z_j(p(t))$$

Next let, for any $\epsilon > 0$, $K = \max[\epsilon, V(p^o)]$ where $p^o$ denotes the initial price; and further choose:

$$\gamma < \min\{\min_{p\in B_\epsilon} \frac{p*.Z(p)}{\sum_{j\neq n} Z_j(p)^2}, \min_{p\in C_\epsilon} \sqrt{\frac{\epsilon/2}{\sum_{j\neq n} |Z_j(p)|}}\}$$

where $B_\epsilon = \{p \in R^n_{++} : K \geq V(p) \geq \epsilon/2\}$ and $C_\epsilon = \{p \in R^n_{++} : V(p) \leq \epsilon/2\}$. Note first of all that such a choice of $\gamma$ can be made and taken to be positive. This follows, by the continuity of the functions and the sets being compact over which positive minimum may be shown to exist. Next note that for such a choice of $\gamma$, we have:

a. if $p(t) \in B_\epsilon$ then $V(p(t+1)) < V(p(t))$

b. if $p(t) \in C_\epsilon$ then $V(p(t+1)) \leq \epsilon$.

Since $V(p^o) \leq K$, it follows that $V(p(t)) \leq K, \forall t$ and hence the sequence of prices $p(t)$ is bounded and limit points exist: let $\hat{p}$ be a limit point such that $V(\hat{p}) \leq V(\overline{p})$ for any other limit point $\overline{p}$. It may be shown that $V(\hat{p}) \leq \epsilon/2$; to see this, let $p(t_s) \to \hat{p}$

while $p(t_s + 1) \to \overline{p}$ say; then it follows that $\overline{p}_j = \max\{0, \hat{p}_j + \gamma Z_j(\hat{p})\}$ for all $j \neq n$; thus $V(\hat{p}) > \epsilon/2 \Rightarrow V(\overline{p}) < V(\hat{p})$; hence we must have $V(\hat{p}) \leq \epsilon/2$; hence the claim follows. •

It should be easy to see that choosing the appropriate $\gamma$ is a difficult task indeed; for that we need to have pretty exhaustive information about the price space; in fact, it would appear that we must know the equilibrium it self. And in that case, we need not worry about processes such as the one constructed above. If knowledge about the equilibrium and the price space is less than perfect, and if we may only be able to guess what the speed of adjustment should be, what happens to the convergence question? To see this and related questions, we turn to the next section.

There is another point which is perhaps noteworthy about the example considered below. The fact that non-linear discrete processes may give rise to problems has not been properly appreciated by economists, as we mentioned above, has to do with the fact that the examples considered were mainly the logistic equation mentioned in Section 3.1. What we show below is that similar situations can be found within models which are very familiar to economists.

### 3.10.2 Bifurcation and Complex Dynamics in a Discrete Tatonnement

Consider an exchange model involving two individuals A and B, and two goods x and y. A's preferences are given by $x^{\alpha}y^{1-\alpha}$ where $0 < \alpha < 1$ and B's preferences are given by $x^{\beta}y^{1-\beta}$ where $0 < \beta < 1$; further A has the endowment $(x^{\circ}, 0)$ and B has the endowment $(0, y^{\circ})$ where $x^{\circ}, y^{\circ}$ are both assumed to be positive. To continue with our analysis, standard calculations lead to the following excess demand function for good x, $Z(p)$, where $p$ is the price of x relative to y

$$Z(p) = \frac{\beta y^\circ}{p} - (1-\alpha)x^\circ \tag{3.27}$$

See Figure 7; a better behaved excess demand function may be difficult to locate.

FIGURE 7: The Excess Demand Function

Equilibrium $p^\star$ is then uniquely determined by

$$p^\star = \frac{\beta y^\circ}{(1-\alpha)x^\circ}. \tag{3.28}$$

Consider the standard adjustment on prices in disequilibrium, the tatonnement, for the model described above:

$$p(t+1) = p(t) + \gamma Z(p(t)) \tag{3.29}$$

where $\gamma > 0$ is some speed of adjustment, assumed constant; we can rewrite the equation ( 3.29) as $p(t+1) = f(p(t))$ where

$$f(p) = p + \gamma Z(p) = p + \gamma(\frac{\beta y^\circ}{p} - (1-\alpha)x^\circ);$$

note that at $p = \overline{p} = \sqrt{\gamma\beta y^\circ}$, $f(.)$ attains a minimum value which is given by

$$f(\overline{p}) = 2\sqrt{\gamma\beta y^\circ} - \gamma(1-\alpha)x^\circ$$

and for the adjustment process to be well defined for all values of $p$, we shall require that the parameter values are so defined that $f(p) > 0$ for all $p$; this is ensured by requiring that the minimum value, $f(\overline{p})$, is positive i.e.,

$$4 > \frac{\gamma((1-\alpha)x^\circ)^2}{\beta y^\circ} \tag{3.30}$$

Let us, for future reference, define

$$K = \frac{\gamma((1-\alpha)x^\circ)^2}{\beta y^\circ}$$

It follows that

$$K = \frac{(\overline{p})^2}{(p^\star)^2}$$

Consequently, $\overline{p} \leq p^\star \Leftrightarrow K \leq 1$. Figure 8 depicts the nature of the function $f(p)$ when $K > 1$.

FIGURE 8: THE f-MAP WITH K > 1

We first note that

**Claim 3.10.2** *There is an interval $I$ = [a,b] such that $f : I \to I$.*

Proof: Consider the case when $K > 1$; i.e., $\overline{p} > p^\star$. Let $b = Max[f(f(\overline{p})), \overline{p}]$; $a = f(\overline{p})$

Since $\overline{p} > p^\star$, and $f'(p) < 0 \ \ \forall p < \overline{p}$ , note that

$$a = f(\overline{p}) < f(p^\star) = p^\star < \overline{p} \leq b$$

Consider I = [a,b]; clearly, $p^\star \in$ I; consider $p \in I$ and suppose, if possible, that $f(p) \notin I$.

$$\text{Since} f(p) \geq f(\overline{p}) = a; f(p) \notin I \Rightarrow f(p) > b.$$

That is, we must have $Z(p) > 0$ or $f(p) > p$ or $a = f(\overline{p}) \leq p < p^\star < \overline{p}$; hence $b \geq f(f(\overline{p})) \geq f(p) > b$: a contradiction. Hence no such $p$ can exist and the claim is established. A similar construction for I may be provided when $K \leq 1$. •

Since $f'(p^\star) = 1 - K$, we have by virtue of, say, Proposition 1 in Saari (1991), (see our Proposition **??**) the following:

**Claim 3.10.3** *$K < 2 \Rightarrow p^\star$ is locally stable for the process ( 3.29 ); $p^\star$ is locally stable for the process ( 3.29 ) $\Rightarrow K \leq 2$.*

Given above, we shall investigate what happens when $2 < K < 4$.

It should be next pointed out that

**Claim 3.10.4** *For $2 < K < 2.5$ there exists a stable 2-cycle.*

Proof: For a two cycle to exist we need to show that there are $q, s$ such that $q \neq s$ satisfying $q = f(s)$ and $s = f(q)$, i.e.,

$$q = s + \gamma(\frac{\beta y^\circ}{s} - (1-\alpha)x^\circ) \text{ and } s = q + \gamma(\frac{\beta y^\circ}{q} - (1-\alpha)x^\circ)$$

Thus $2sq = \gamma\beta y^\circ$ and $s + q = \gamma(1-\alpha)x^\circ$ or s, q are roots of the quadratic equation

$$z^2 - \gamma(1-\alpha)x^\circ z + \frac{\gamma\beta y^\circ}{2} = 0$$

Real roots of the quadratic exist whenever

$$(\gamma(1-\alpha)x^\circ)^2 - 2(\gamma\beta y^\circ) > 0 \text{ or } K > 2.$$

Thus 2-cycles exist for all $K > 2$ ; to examine the stability of these cycles (See Lauwerier (1986), p. 40, for example), we need to examine

$$\mid f'(q)f'(s) \mid \ = \ \mid 5 - \gamma\beta y^\circ(\frac{1}{s^2} + \frac{1}{q^2}) \mid \ = \ \mid 9 - 4K \mid;$$

$$\text{now } \mid 9 - 4K \mid < 1 \text{ if and only if } 2 < K < 2.5$$

and the claim follows. •

Notice that just as $K$ crosses the value 2, the unique equilibrium $p^\star$ loses stability and a stable 2-cycle is born; the 2-cycle loses stability just as $K$ crosses the value 2.5 and a stable 4-cycle may be shown to exist. To analyze the behavior of the attractors for different values of $K$, we fix the values of all the parameters except $\gamma$:

$$\beta y^\circ = 1 \text{ and } (1-\alpha)x^\circ = 6;$$

so that $K = 36\gamma$ and our difference equation ( 3.29 ) takes the particular form

$$p(t+1) = p(t) + (\frac{1}{p(t)} - 6)K/36 \tag{3.31}$$

Let $a_n$ denote the critical value of $K$ where a $2^n$ cycle is born; then $a_1 = 2$ and $a_2 = 2.5$. It is known that the sequence $a_n$ converges to $a_\infty$ as n becomes large and further, for any one dimensional iterate, with a single parameter, there is a constant, F (the Feigenbaum constant = 4.669202...) which allows us to approximate the value of $a_\infty$ by means of the following (See, Lauwerier (1986), p.44-45.):

$$a_\infty \approx \frac{Fa_{n+1} - a_n}{F-1}$$

Using the above formula, one may see that for the example we have been discussing,

$$a_\infty \approx 2.6362694$$

To properly appreciate the nature of the dynamics, please see Figure 9: the so called bifurcation diagram.

FIGURE 9: THE BIFURCATION DIAGRAM

In this diagram, on the x-axis we have values of the parameter K; on the vertical axis, we have plotted the iterates $f^n(x)$, say for $n = 200$ to $n = 300$ for a point chosen at random from the corresponding [a,b]. This experimental method generates the period doubling phenomenon mentioned above; beginning with a two period cycle, bifurcating into 4 period cycles and so on, till $a_\infty$; beyond this value, apart from small gaps the picture is diffused; within the gaps, we have stable cycles. Outside these regions of stable cycles, we may conclude that either there is no stable cycle or there are cycles with very long periods: the reason we are not able to capture these may be attributed to the fact that we have looked at only a few iterates. This is only to be expected from running such experiments.

### 3.10.3 Evidence of Chaos

To proceed analytically in the matter, we need to be in a position to apply results such as Proposition 2; however, these results cannot be directly applied. We recall ( 3.31 ):

$$p(t+1) = p(t) + (\frac{1}{p(t)} - 6)K/36$$

now introduce $q = 1/p$; then

$$\begin{aligned} q(t+1) = \frac{1}{p(t+1)} &= \frac{1}{p(t) + (\frac{1}{p(t)} - 6)K/36} \\ &= \frac{36q(t)}{36 + K(q(t) - 6)q(t)} \\ &= g(q(t)), say; \end{aligned}$$

We consider the map $g(q)$ and note its properties. See Figure 10, in this connection.

FIGURE 10: THE g-MAP

Note first, that there is a single parameter $K$ and since we shall be considering variations in the value of the parameter, we shall write the map as

$$g_K(q) = \frac{36q}{36 + Kq^2 - 6Kq} \tag{3.32}$$

and restrict attention to $K \in \Delta = (2, 4)$. Any value of $K$ in $\Delta$ will be called an admissible value of the parameter. The following properties are immediate:

P1. $g_K(q)$ is well defined for all non-negative values of $q$, whenever $K < 4$; also $g_K(q)$ is of class $C^3$ for all $q \geq 0$. For each admissible value of $K$, there exists a unique $\overline{q_K}$ such that $g'_K(\overline{q_K}) = 0; \overline{q_K} = 6/\sqrt{K}$; further, $q < \overline{q_K} \Rightarrow g'_K(\overline{q_K}) > 0$; while $q > \overline{q_K} \Rightarrow g'_K(\overline{q_K}) < 0$. Note also that in $I_K$, the unique fixed point for each admissible value of $K$ is $q = 6 > \overline{q_K}$.

P2. Define $b(K) = g_K(\overline{q_K})$; $a(K) = Min[g_K(b(K)), \overline{q_K}]$; then $a(K) \leq \overline{q_K} < b(K)$; further, writing $I_K = [a(K), b(K)]$, $g_K : I_K \to I_K$. Also note that $a(K), b(K)$ vary continuously with $K$ in $\Delta$.

Thus

$$b(K) = \frac{6}{\sqrt{K}(2 - \sqrt{K})}$$
$$a(K) = \text{ Min } [\frac{6(2 - \sqrt{K})}{\sqrt{K}(5 + 2K - 6\sqrt{K})}, \frac{6}{\sqrt{K}}]$$

It may be noted that for $K < 2.25, a(K) \ = 6/\sqrt{K}$ while for $K > 2.25$,

$$a(K) = \frac{6(2 - \sqrt{K})}{\sqrt{K}(5 + 2K - 6\sqrt{K})};$$

thus for $K = 2$,

$$g_K(b(K)) = \frac{6(2 - \sqrt{K})}{\sqrt{K}(5 + 2K - 6\sqrt{K})} > \frac{6}{\sqrt{K}};$$

and further, for the same value of $K$,

$$g_K^2(b(K)) =$$

$$\frac{6(2-\sqrt{K})(5+2K-6\sqrt{K})}{\sqrt{K}[6K^2-34K^{3/2}+74K-74\sqrt{K}+29]} > \frac{6}{\sqrt{K}};$$

while for $K = 3.6$, $g_K^2(\overline{q_K}) = g_K(b(K)) < 6/\sqrt{K}$; moreover, $g_K^3(\overline{q_K}) = g_K^2(b(K)) < 6/\sqrt{K}$.

Thus the family of maps $g_K$ for $K \in \Delta$, constitute a **full family of single peaked** maps. Further $g(q) > q$ for all q $\in (a(K), 6)$. In addition, the Schwartzian derivative is given by:

$$Sg_K(q) = \frac{-216K}{(Kq^2-36)^2} < 0$$

for all $q \in I_K, q \neq \overline{q_K}$.

The conditions A1 and A2 are thus verified; consequently, Proposition 2.8 holds; in addition, the results for the map ( 5.3 ) should apply to our earlier excess demand map, since these are topologically conjugate to one another. Consequently, our preliminary comments about bifurcation (Figure 9) have been provided with analytical support by virtue of Proposition 2.8. Further, to specify the nature of chaotic behavior we use the notions of topological chaos. We may claim:

**Claim 3.10.5** *For $K \in \Delta_1 = (25/9, 4)$ the map $g_K(.)$ exhibits topological chaos.*

Proof: Consider any $K \in \Delta_1$;

$$b(K) = g_K(\overline{q_K})\ ;\ a(K) = g_K(b(K)),\ \text{since } K > 2.25.$$

Consider the expression for $g_K^2(b(K)) = g_K^3(\overline{q_K})$; by virtue of Proposition 10, a sufficient condition for topological chaos to be exhibited is that

$$g_K^2(\overline{q_k}) < \overline{q_K}$$

and

$$g_K^3(\overline{q_K}) < 6.$$

The first inequality reduces to

$$2 - \sqrt{K} < 5 + 2K - 6\sqrt{K}$$

which holds for all $K > 2.25$. The second inequality is written as:

$$\frac{6(2-\sqrt{K})(5+2K-6\sqrt{K})}{\sqrt{K}[6K^2 - 34K^{3/2} + 74K - 74\sqrt{K} + 29]} - 6 < 0.$$

As Figure 11 would show, the above inequality is true for all $K \in \Delta_1$. Thus the claim is true by virtue of Proposition 10. •

FIGURE 11: THE RANGE FOR TOPOLOGICAL CHAOS

We can show the existence of ergodic chaos too.

**Claim 3.10.6** *For $K = 25/9$, the map $g_K$ exhibits ergodic chaos; in addition, there exists $K \in \Delta_1$ such that $g_K$ exhibits ergodic chaos.*

Proof: Consider the expression for $g_K^2(\overline{q_K})$; note that for $K = 25/9$, $g_K^2(\overline{q_K}) = 6/K$ Consequently, $g_K^3(\overline{q_K}) = 6$. And since $g_K'(6) = 1 - K = -16/9$, Proposition 2.9 applies, and one may conclude that for $K = 25/9$, the map $g_K$ exhibits ergodic chaos. To follow the above method, in order to identify other $K$ values for which the corresponding $g_K$ exhibits ergodic chaos, we need to solve the following equation for $K$,

$$g_K^n(\overline{q_K}) = \frac{6}{K}$$

for some integer n; then the corresponding $g_K$-map exhibits ergodic chaos by virtue of Proposition 2.9. In fact, we have solved this equation for n = 2. If we try to solve this equation for n = 3, recalling the expression for $g_K^3(\overline{q_K}) = g_K^2(b(K))$, we have to solve:

$$\frac{6(2-\sqrt{K})(5+2K-6\sqrt{K})}{\sqrt{K}[6K^2 - 34K^{3/2} + 74K - 74\sqrt{K} + 29]} = \frac{6}{K}$$

or, say

$$\sqrt{K}X - Y = 0$$

Figure 12 plots right hand side for various values of $K \in \Delta_1$ It is straightforward to check that the equation has a solution for $K$ near 3.2 •

FIGURE 12: EXISTENCE OF ERGODIC CHAOS

## 3.11 A Summing Up

We have investigated over the past pages, the notion of stability of equilibrium. The notion of stability, goes back to Walras and his idea of a groping process, the tatonnement, which searches for and attains equilibria. The main idea behind the process, which we have called the Walrasian Adjustment Hypothesis, is that in disequilibrium situations, prices are revised upwards in the direction of excess demand. There is another aspect, which we did not perhaps specify and that is that trades take place only at equilibrium. We have kept this latter condition throughout; the former was relaxed only in the section on Effective processes.

We began with an analysis of local stability and it was found straightaway that we needed to impose some additional restrictions on excess demand functions; a necessary condition was identified which was the first inkling that one had of the importance of the Weak Axiom of Revealed Preference; without any such condition cyclical behavior around equilibrium was possible too. We saw how such behavior could be tackled and this paved the way for the central point of this chapter that stability conditions need to be investigated and imposed. Without such conditions, the notion of excess demand functions themselves would not play any meaningful roles.

While theorists have been inclined to impose restrictions to imply existence of equilibrium, the imposition of conditions to imply stability have not been popular. We saw the derivation of some stability conditions. So far as the adjustment process was considered, we studied two variations: one where the price adjustment had the same sign as excess demand and the other, the more specific one where price adjustment was proportional to excess demand; the more general form of the adjustment process was seen to require the stricter restrictions on excess demand viz., gross substitution, dominant diagonals were analyzed. Both had close links with WARP. The weaker version of the price adjustment process could handle some other types of restrictions on excess demands: we saw what could happen in the general n-goods case as well as in the 3-goods case. In the latter case, with the choice of the numeraire, the dynamics moved to studying motion on the plane and hence once could identify global stability conditions. In the n-goods case, there was no easy way of obtaining general results but we did identify the structure of the limit sets in this case too.

Given the problems identified above, we tried to give up on the Walrasian hypothesis altogether; this led us to a study of effective processes. Two results flowed from this source: one was to identify a process which worked always to reach equilibria locally and the other was to provide an interesting set of conditions for the uniqueness of equilibrium. Again, the crucial role of the Weak Axiom was revealed.

In fact, there appears to be a trade off between the nature of processes and the conditions required for convergence to equilibrium. The more general processes require more restrictions on excess demand functions; specific forms of the processes may not need too many restrictions on the nature of excess demands for convergence.

Finally, we turned to discrete price adjustment processes; here the dynamic results turned out to be quite complex indeed. And convergence, is the exception. Possibilities of variety of periodic behavior in very simple contexts were identified.

# 4 Non-Tatonnement Processes

## 4.1 Introduction

One of the major restrictions employed in the analysis of the tatonnemnt processes has been the requirement that **trading if at all, takes place at equilibrium price configurations**. One may have wondered exactly how this assumption figured in our analysis in the previous chapter: this was implicit when we represented excess demand functions $Z_j(.)$, we denote them as being functions of prices alone. In the current chapter, we shall attempt to weaken this restriction, by allowing for transactions or trades to take place at dis-equilibrium prices or at non-market clearing prices. There was no need in the previous chapter, thus, to distinguish between desired transactions and actual transactions, since there was no difference between the two: actual transactions being confined to take place at equilibrium match desired transactions. But when we allow for transactions to take place at non-market clearing prices, we need to investigate what these actual transactions could be.

To follow the development of the literature, we consider first certain properties of such trades and then, at a later stage, we indicate what transactions can take place when prices fail to clear markets. There are two distinct sets of properties of dis-equilibrium transactions which have been analyzed: the first due to Hahn and Negishi (1962) and the second due to Uzawa (1962). Before considering them in turn, we need to clearly identify the model in the context of which the entire analysis will proceed.

We consider the standard exchange model with $n$-goods, numbered 1 to $n$; good $n$ is the numeraire; there are $M$ individuals $i = 1, \cdots, M$ each with an utility function $U^i(.)$ and an endowment $\overline{w}^i \in R^n_+$. Any vector of prices $p$ will be taken to be $p = (p_1, p_2, ..., p_{n-1}, 1) \in R^n_{++}$; and let $R^n_+$ denote the consumption possibility set of individual $i$. We shall assume the following:

**A 1** For each $i, U^i(.) : R^n_+ \to R$, strictly quasi-concave, increasing and continuously differentiable on $R^n_{++}$, the strictly positive orthant. Further, for each $i$,$U^i(\overline{x}) = 0$ if $\overline{x}_k = 0$ for some $k$ and $x \in R^n_{++} \Rightarrow U^i(x) > 0$.

Consequently, as before, the **desired trades** or **individual excess demand** of each individual $z^i(p, \overline{w}^i)$, is formulated on the basis of the **demand functions** , $x^i(p, \overline{w}^i)$ of each individual $i$ obtained from the following maximization problem ( **U-Max** say):

max $U^i(x)$

subject to $p.x \leq p.\overline{w}^i,\ x \in C^i$.

Now, $z^i(p, \overline{w}^i) = x^i(p, \overline{w}^i) - \overline{w}^i$. Further, the excess demand function $Z(p, \{\overline{w}^i\}) = \sum_i z^i(p, \overline{w}^i)$. The effect of trades at dis-equilibrium or non-market clearing prices mean that transactions take places at $p$ where $Z(p, \{\overline{w}^i\}) \neq 0$; consequently actual trades $t^i$ will necessarily differ from desired trades $z^i(p, \overline{w}^i)$, since a feature of the situation is that the desired trades can not be made. What properties should such trades have ?

We note that first of all,

$\sum_i t^i = 0$

since every purchase ( an addition to the endowment for some) must also indicate a sale ( a subtraction from endowment for some other) and hence, for each good $j$, purchases and sales must balance. This, of course, is an **aggregate feasibility condition** for trades; an **individual feasibility condition**, which too must hold is that

$$p.t^i = 0$$

which is just the budget equation and guarantees that trading at constant prices, by itself, cannot add or subtract to or from wealth.

Next note that after such transactions or trades are complete, endowments, post-trade, are $w^{i'} = \overline{w}^i + t^i$; the point to note is that the result of the change in endowments do not affect the demands, i.e.,$x^i(p, \overline{w}^i) = x^i(p, w^{i'})$ and consequently, the desired trades post-trade, $z^i(p, w^{i'}) = z^i(p, \overline{w}^i) - t^i$ so that the market excess demand, $Z(p, \{w^{i'}\})$ remains the same as before the trade $Z(p, \{\overline{w}^i\})$. With these preliminary remarks, we need to investigate the consequences of assuming some special features of the trades $t^i$.

## 4.2 The Hahn-Negishi Process

To characterize the disequilibrium trades $t^i$, Hahn and Negishi (1962) assumed that the trades reflected the following:

**A 2** In a market with over all excess demand $Z_j(.) > 0$(excess supply, $Z_j(.) < 0$), after trades are complete, no individual agent is left with an individual excess supply (respectively, individual excess demand) i.e.,

$$z^i_j(p, w^{i'}).Z_j(p, \{w^{i'}\}) \geq 0 \; \forall i, j.$$

Hahn and Negishi went on to assume that once the trades are completed prices changed according to the familiar equation:

$$\dot{p}_j = Z_j(p(t), \{w^i(t)\})\ \forall j \neq n \tag{4.1}$$

It is assumed that $p_n = 1$ so that $\dot{p}_n = 0$; in addition, the Hahn-Negishi assumption on trades hold, namely:

$$\dot{w}^i(t) = \phi^i(p(t), \{w^i(t)\})\ \forall i \tag{4.2}$$

for some continuous functions $\phi^i(p(t), \{w^i(t)\})$ so that $w^i(0) = \overline{w}^i \forall i$ and at each instant, we have:

$$\forall i,\ z^i_j(p(t), w^i(t)).Z_j(p(t), \{w^i(t)\}) \geq 0 \forall t \geq 0 \text{ and } \forall j \neq n$$

and further, the trades satisfy

$$p(t).\dot{w}^i(t) = 0 \text{ for all } i \text{ \& } t \text{ and } \sum_i w^i(t) = \sum_i \overline{w}^i\ \forall t$$

We may now make the following claim for the price adjustment and trading rule, called the Han-Negishi process:

**Claim 4.2.1** *Along any solution to the Hahn-Negishi process,(4.1),(4.2),* $\forall i, \dot{U}^i(x^i(p(t), w^i(t)) \leq 0$. *The inequality is strict so long as* $\dot{p} \neq 0$.

Proof: We begin by noting that first of all at each $t$, $x^i(p(t), w^i(t))$ solves the maximum problem U-Max; hence it follows that each instant $t$, for each $i$, there is some $\lambda^i(t) > 0$ such that $\nabla U^i(x^i(p(t), w^i(t))) = \lambda^i(t)p(t)$. Consequently, $\dot{U}^i(x^i(p(t), w^i(t))) = \nabla U^i(x^i(p(t)), w^i(t))).\dot{x}^i = \lambda^i(t)p(t)\dot{x}^i(t)$.

Again from the budget constraint, which must be exactly met, given our assumptions, we have:

$$p(t)\dot{x}^i(.) + \dot{p}.x^i(.) = p(t)\dot{w}^i(t) + \dot{p}.w^i(t)$$

Since by the trading rule, the first term on the right is zero, we have,

$$p(t).\dot{x}^i(.) = -\dot{p}.z^i(p(t), w^i(t)) = -\sum_{j=1}^{n-1} Z_j(p(t)\{w^i\}) z^i_j(p(t), w^i(t)) \leq 0$$

The inequality will be strict whenever $\dot{p} \neq 0$. The claim now follows, since, $\lambda^i(t) > 0$ for all $t$. •

Thus along the solution to any Hahn-Negishi Process, the target utilities for each $i$ or the indirect utility for each $i$ is decreasing unless price adjustment stops. We shall use this information next. Suppose that along the solution to the Hahn Negishi process, either $\|p(t)\| \to \infty$ or $p_j(t) \to 0$ for some $j$. These two situations are similar since in the first case, it is the relative price of the numeraire which is going to zero. In either of these cases, the budget set for some individual will tend to become unbounded and hence the target utility, for that individual may be increased as much as possible: which contradicts the fact that target utilities for all individuals must never exceed the initial level. Since endowments always sum up to the initial levels, the boundedness of the solution to (4.2) is more straightforward. Consequently, we may claim that the following holds:

**Claim 4.2.2** *Along the solution of Hahn-Negishi process, (4.1) and (4.2), beginning with any arbitrary initial $p(o), w^i(0) = \overline{w}^i$, $p(t, \{w^i(t)\})$ remains bounded and $p(t)$ remains bounded away from boundary of $R^n_+$.*

Thus limit points exist for the Hahn-Negishi process and we may claim that all limit points are competitive equilibria, since we have just shown that the sum of target utilities behave as a Liapunov function. Note that given the two sets of equations, equilibrium would mean that no price adjustment can take place and no further trades can occur: i.e., a no-trade competitive equilibrium. We note this conclusion as:

**Proposition 4.1** *(Hahn-Negishi) Given any initial configuration of prices and endowments,* $(p^0, \{\overline{w}^i\})$ *the Hahn-Negishi process described by (4.1) and (4.2) generates solutions which converges to a no-trade competitive equilibrium for the economy.*

One of the deficiencies of the process such as the one we have described is the fact that there does not appear to be any incentive for individuals to make these kinds of transactions; in other words, why should some one trade so that the Hahn Negishi post-trade assumptions are met ? The view adopted has been that these trades reflect the following type of transactions: people queuing up to trade in each market and then given the imbalances, if more persons wish to buy, then all those who wish to sell get to carry out their plans; conversely if more people wish to sell, then those who wish to buy get to carry out their plans. While these may seem reasonable at first glance, a moment's reflection will indicate a problem. Consider an individual who wishes to buy in the market $j$ and wants to sell in the market $k$: that is, in market $j$ the person wants to surrender the required amount of the numeraire to acquire units of commodity $j$; while in market for $k$, the **same** person wants to get rid of some amount of commodity $k$ and receive units of the numeraire in exchange. Now assume further, that in the market $j$ where the individual wants to purchase, most people want to sell: then according to the Hahn Negishi assumption, this person should sell. However now suppose that in market for $k$ where the person wants to sell, most people want to sell too and once more, Hahn Negishi assumption indicates that the person may be unable to sell. Now given this constraint, it is quite possible that the person is worse off after making this transaction and hence, in such a situation, the person would not have participated in the transaction. Thus to accept a set of feasible transactions, we need to ensure that they be **voluntary** ; the easiest such consideration comes from demanding that no one should lose by making these transactions. This provides the contribution of Uzawa (1962) with his point of departure.

## 4.3 Uzawa's Edgeworth Process

### 4.3.1 Edgeworth Barter and Voluntary Trades

As we had indicated in the last section, the Hahn-Negishi assumption on trades may be difficult to justify in the context of multiple markets. The set-up is similar to the one considered in the previous section, except in some details; first, we need to strengthen the assumption on individual tastes:

**A 1*** In addition to the requirements in **A 1**, each utility function $U^i(.)$, for each $i = 1, 2, \cdots, m$ is assumed to be strictly increasing and strictly concave in the strictly positive orthant $R^n_{++}$.

Thus for any two positive bundles $x, y \in R^n_{++}, x \neq y, U^i(\lambda x + (1 - \lambda y)) > \lambda U^i(x) + (1 - \lambda)U^i(y)$ , for all $\lambda, 0 < \lambda < 1$, for each $i$. Moreover, the marginal utilities $U^i_j(x) =$

$\dfrac{\partial U^i(x)}{\partial x_j} > 0$ for all $x \in R^n_{++}$ and for every $i$ and $j$.

Thus, given endowments $w^i \in R^n_{++}, \forall i$, $p \in R^n_{++}$, the solution to the U-Max problem leads to demand functions for each $i$, $x^i(p, w^i)$ with the following properties:

- For each $i$, $x^i(p, w^i) \in R^n_{++}$ and is a continuous function of $p, w^i$.
- $p.x^i(p, w^i) = p.w^i$;
- $U^i(x^i(p, w^i)) \geq U^i(y)$ for all $y$ such that $p.y \leq p.w^i$.

Note that excess demands are then given by $Z(p, \{w^i\}) = \sum_i x^i(p, w^i) - \sum_i w^i$. The usual restrictions on excess demand functions will be taken to hold here as well[1]:

**P** $Z : R^n_{++} \times R^{mn}_{++} \to R^n$ is continuously differentiable function of its arguments, is bounded below and satisfies Walras Law ($\forall (p, \{w^i\}) \in R^n_{++} \times R^{mn}_{++}, p.Z(p, \{w^i\}) = 0$); in addition, $Z(\ \lambda p, \{w^i\}) = Z(p, \{w^i\})$ for any $\lambda > 0$ (Homogeneity of degree zero in prices); further, if $\{p^s\}$ is a sequence in $R^n_{++}$ for all $s$ and if $p^s_j \to 0$ for some $j$ then $Z_j(p^s, \{w^i\}) \to +\infty$ for any $\{w^i\} \in R^{mn}_{++}$.

Next, we shall define the transaction rule which formalizes the notion of voluntary trades or Edgeworth Barter according to Uzawa (1962). Given $W = \sum_i w^i$, the total stock of goods available, consider the set $\Omega = \{Y = (y^i) \in R^{mn}_{+} : \sum_i y^i = W\}$: each element of $\Omega$ constitute a feasible allocation of the stock of goods $W$ among the $m$ individuals with the $i$-th individual receiving $y^i$. The transaction rule at each configuration $p, \{w^i\}$ chooses a $Y$

[1] These are the same as the ones employed in the last chapter, see for example P1 - P4 and P4*. We need to take into account the fact that endowments will change along the process.

from the set $\Omega$ and we shall write $Y = G(p, \{w^i\})$: $G$ is the transaction rule[2] and we shall require the following:

**A 2***$G : R^n_{++} \times \Omega \to \Omega$ written $G(p, \{w^i\}) = (g_i(p, \{w^i\}) = (y^i), i = 1, 2, \cdots, m$ satisfies for each $i$, $p.y^i = p.w^i$ and $U^i(y^i) \geq U^i(w^i)$. In case $U^i(y^i) = U^i(w^i) \forall i$ then $y^i = w^i$ for all $i$; and further $g^i(p, \{w^i\}) = w^i$ , $\forall i \Leftrightarrow \{w^i\}$ is the only element of $\{Y = (y^i) : Y \in \Omega, p.y^i = p.w^i, U^i(y^i) \geq U^i(w^i) \ , \forall i\}$ . Finally, if $\sum_i x^i(p, w^i) = W$ then $y^i = x^i(p, w^i)$.

The interpretation is that prior to transactions, each individual has $w^i$; after transactions, each individual has $y^i$. Thus notice that the transaction does not create or destroy goods since we are restricting the definition to set $\Omega$; further trades will take place according to the budget constraint for each individual and no one loses through these transactions. Thus we are not allowing any consumption or production to take place during the process.

Also, for any transaction to occur, at least one person must be better off, after the transactions. And finally, if the demands are achievable, then the transactions will be such as to achieve them. With these requirements, a natural question arises whether all these conditions can in fact be met.

To answer this query consider some positive numbers $c_1, c_2, \cdots, c_m$, one for each individual and consider the Social Welfare Function $\Psi_c(\{U^i(y^i)\}) = \sum_i c_i U^i(y^i)$ and the following maximum problem:

$$\max \Psi_c(\{U^i(y^i)\}) = \textstyle\sum_i c_i U^i(y^i)$$

$$\text{subject to } py^i = p.w^i$$

[2]Note that we have stuck to the notation and terminology of Uzawa (1962); thus, transactions should have been defined by $G(p, \{w^i\}) - \{w^i\} = \{y^i - w^i\}$.

$$U^i(y^i) \geq U^i(w^i)$$

$$Y = \{y^i\} \in \Omega$$

Notice that the objective function is strictly concave (**A 1***) and it is being maximized over a non-empty, convex, compact set; as a result, the problem has a unique solution $Y = \{y^i\}$ satisfying all the requirements of **A 2***.

Given the initial distribution $\{w^i\}$, we define $Y* = \{y^{i*}\} \in \Omega$ to be **Pareto Optimal** if there is no $Y = \{y^i\} \in \Omega$ such that $U^i(y^i) \geq U^i(y^{i*})\forall i$ with inequality strict for some $i$. Thus if the distribution $\{w^i\}$ happens to be Pareto Optimal then for any transaction rule $G(.)$ satisfying **A 2*** and any $p$, we must have $G(p, \{w^i\}) = (w^i)\forall i$. This follows immediately since if, to the contrary, there exists $Y = \{y^i\} \in \Omega$ satisfying $py^i = p.w^i \forall i$ and $U^i(y^i) \geq U^i(w^i)$ for all $i$ as required by **A 2***, then consider $Z = \{z^i\}, z^i = \dfrac{y^i + w^i}{2}$; note that $Z \in \Omega$ and $y^i \neq w^i$ for some $i$ implies that, for that particular $i$, $U^i(z^i) > U^i(w^i)$ while for the others we have $U^i(z^i) \geq U^i(w^i)$ so that the Pareto Optimality of $\{w^i\}$ is violated.

### 4.3.2 A Process of Price Adjustment and Trades

With these preliminary matters in place, we can now proceed to define the following process given an initial $(p^o, \{w^{io}\})$, $p^o \in R^n_{++}, w^{io} \in R^n_{++}, \forall i$. Now define $\Gamma = R^n_{++} \times \Omega$, where $\Omega = \{Y = (y^i) \in R^{mn}_{+} : \sum_i y^i = W = \sum_i w^{io}\}$:

$$\dot{p}_j = k_j Z_j(p, \{w^i\}) \forall j \neq n \; p_n \equiv 1 \text{ and } \dot{w}^i = g^i(p, \{w^i\}) - w^i \forall i \tag{4.3}$$

where the constants $k_j > 0$ for all $j$. Thus, at each instant, while price adjustment, for all non-numeraire commodities, takes place in the direction of excess demand, transactions too occur according to the transaction rule specified by the function $G = (g^i)$ in **A 2***.

An equilibrium for the process is a configuration $(p^\star, \{w^{i\star}\})$ at which both price adjustment and trades stop that is: $Z_j(p^\star, \{w^{i\star}\}) = 0$ and $g^i(p^\star, \{w^{i\star}\}) = w^{i\star}$ for all $i, j$.

To enable us to proceed with the analysis of the solution to (4.3), we shall also require:

**A 3***: The function $G : \Gamma \to \Omega$ is continuously differentiable function of its arguments.

Given **P, A 1* - A 3***, there is a solution $\phi_t(p^o, \{w^{io}\}) = (p(t), \{w^i(t)\})$ to (4.3) and we need to guarantee that it stays within the set defining the entire process:

**Claim 4.3.1** *The solution $\phi_t(p^o, \{w^{io}\}) = (p(t), \{w^i(t)\})$ to (4.3) remains within $\Gamma^o$, a compact subset of $\Gamma$ for all $t$.*

Proof: Recall the Claims 3.7.7 and 3.7.8. Since price adjustment under the process (4.3) is a special case of the process studied earlier (3.1), these claims continue to apply. Hence we may conclude that $p(t)$ remains within a compact set and remains bounded away from the axes: thus $p_j(t) > 0 \forall j, \forall t$.

Next note that under our assumptions, since $\{w^{io}\} \in \Omega$, and $\sum_i \dot{w}^i(t) = \sum_i g^i(.) - \sum_i w^i = 0$ since $(g^i(.)) \in \Omega$ consequently, $\sum_i w^i(t) = \sum_i w^{io} = W$. Also, we have that along the solution to the process, for each $i$, $\dot{U}^i(w^i(t)) = \nabla U^i(w^i(t)).\dot{w}^i(t) = \nabla U^i(w^i(t)).(g^i(p(t), \{w^i(t)\}) - w^i(t))$[3] In addition, from the definition of the transaction rule, we have that $U^i(g^i(p(t), \{w^i(t)\})) \geq U^i(w^i(t))$; hence using the concavity of the utility functions, **A 1***, we have $0 \leq U^i(g^i(.)) - U^i(w^i(.)) \leq \nabla U^i(w^i(.)).(g^i(.) - w^i(.))$; thus $\dot{U}^i(w^i(t)) \geq 0 \forall t$. Thus $U^i(w^i(t)) \geq U^i(w^{io}) > 0$, where the strict inequality follows by virtue of **A 1**[4]; this also implies that for all $t$, $w^i(t) \in \Omega$.

[3] $\nabla f(x)$ denotes the vector of partial derivatives of the function $f(x)$ i.e., $\nabla f(x) = (\frac{\partial f(x)}{\partial x_j})$.

[4] Recall that **A 1** is included in **A 1***.

Hence the solution lies in some $\Gamma^o$, a bounded subset of $\Gamma$ for all $t$, as claimed. $\bullet$

Note that along the solution, we have shown that $U^i(w^i(t))$ are **non-decreasing**; we next show that if the initial configuration $\{w^{io}\}$ is not Pareto Optimal, these utilities cannot remain constant for all $i$. That is some trades will actually take place, provided the initial configuration is not Pareto Optimal. This is established through the following two steps. First of all, define, at any configuration $(p, \{w^i\}) \in \Gamma$, for each $i, j$ (recall $p_n \equiv 1$:

$$\Psi^i_j(p, w^i) = \frac{U^i_j(w^i)/p_j}{U^i_n(w^i)} - 1$$

It should be apparent that in case $\Psi^i_j(p, w^i) > 0$, then if individual $i$ gives up (*pays*) a small amount of good $n$ and receives whatever the equivalent value of good $j$ (*purchases*)in return, then individual $i$ is made better off and conversely, whenever the quantity $\Psi^i_j(p, w^i) < 0$ an exchange of good $j$ for good $n$ will benefit the individual.

### 4.3.3 When Trades Do Not Occur

It should also be clear that $\Psi^i_j(p, w^i) = 0 \forall j \Leftrightarrow w^i$ solves the problem $\max U^i(x)$ subject to $p.x \leq p.w^i, x \geq 0$. On an aggregate level, however, individuals may be stuck with their endowments, for another reason and this is in case, all individuals wish to make the same transactions. More specifically, define $B(p, \{w^i\}) = \{Y \in \Omega : Y = (y^i), U^i(y^i) \geq w^i \text{ , } p.y^i \leq p.w^i \; \forall i$. Then one may show:

**Claim 4.3.2** *$B(p, \{w^i\}) = \{w^i\} \Leftrightarrow \exists$ non-negative numbers $c_i, i = 1, 2, \cdots, m$ and a vector $b \in R^n$ such that for all i, $\Psi^i(p, w^i) = c_i.b$ provided $w^i \in R^n_{++}$.*

Proof: Consider $I = \{i : \Psi^i(p, w^i) \neq 0\}$; if $I$ is empty, then we may just choose $b$ to be any vector and the numbers $c_i = 0$; so say $I$ is non-empty and $\#I \geq 2$; even if $I$ has only one member, say, $i_o$, there is nothing to prove, since we just choose $b = \Psi^{i_o}(p, w^{i_o}); c_{i_o} = 1, c_i = 0, i \neq i_o$. Notice then $B(p, \{w^i\}) = \{w^i\}$ if and only if for each $i \in I$ $w^i$ solves the following problem $\mathcal{P}_i$:

$$\max U^i(y^i)$$

$$\text{subject to } U^h(y^h) \geq U^h(w^h) \forall h \neq i, h \in I$$

$$\textstyle\sum_{h \in I} y^h \leq \sum_{h \in I} w^h$$

$$p.y^h \leq p.w^h \ , \forall h \in I$$

$$y^h \geq 0 \ , \forall h \in I$$

It is clear that if there are non-negative numbers $c_h$ and a vector $b$ satisfying the conditions of the claim, no trades can take place. For the converse, suppose that no trades are possible; then, for each $i \in I$, $w^i$ solves the problem $\mathcal{P}_i$ which is a concave programming problem with only one set of constraints non-linear. Note that we can find $y^h, h \in I, h \neq i$, $U^h(y^h) > U^h(w^h)$, $p.y^h = p.w^h$ by using the fact that for all such $h$ $\Psi^h(p, w^h) \neq 0$ and making small transactions, which we adjust with $w^i$ to ensure $\sum_{h \in I} y^h = \sum_{h \in I} w^h$; the fact that $w^i$ is positive for all $i$ is crucial at this stage. This implies that the above maximum problem is equivalent to the saddle-value problem[5] and hence the fact that each $w^i$ solves $\mathcal{P}_i$ implies the existence of non-negative numbers $\lambda_h, \mu_j, \theta^h$ satisfying the following:

[5]See, for example, Arrow, Hurwicz and Uzawa (1961), Corollary 3 to Theorem 3 which shows the validity of the Kuhn-Tucker theorem on Non-Linear programming under this kind of Slater's condition applied to the only the nonlinear constraints.

$\lambda_h U^h_j(w^h) - \mu_j - \theta_h p_j = 0 \ \forall h \neq i, h \in I$

$U^i_j(w^i) - \mu_j - \theta_i p_j = 0$

where we have equality on account of the fact that $w^h_j > 0$ for all $j$ and for all $h \in I$. Thus eliminating $\mu_j$, we have for all $h \in I$:

$\lambda_h U^h_j(w^h)/p_j - \theta_h = U^i_j(w^i)/p_j - \theta_i \forall j$

and hence subtracting the equation for $j = n$ from both sides we have:

$\lambda_h (U^h_j(w^h)/p_j - U^h_n(w^h)/p_n) = U^i_j(w^i)/p_j - U^i_n(w^i)/p_n$

which implies that $\Psi^i_j(w^i) = \rho_h \Psi^h_j(w^h)$ for some positive numbers $\rho_h$. The numbers $\rho_h$ are positive since by definition $i \in I$. This proves the claim, since we may now define $b = \sum_{h \in I} \Psi_h(p, w^h)$ so that $\Psi_h(p, w^h) = c_h b$, $c_h > 0 \forall h \in I, c_h = 0, h \notin I$. $\bullet$.

The second step is to prove the following:

**Proposition 4.2** *Consider the process (4.3) and an initial configuration* $(p^o, \{w^{io}\})$, $p^o \in R^n_{++}, w^{io} \in R^n_{++}, \forall i$. *If along the solution* $\phi_t(p^o, \{w^{io}\}) = (p(t), \{w^i(t)\})$, $\{w^i(t)\} = \{w^{io}\}$ , $\forall t$, *then* $\{w^{io}\}$ *is Pareto Optimal and* $p(t)$ *converges to* $p^o$ *which is such that* $Z(p^o, \{w^{io}\}) = 0$.

Proof: Since $\{w^i(t)\} = \{w^{io}\} \forall t$ it must be the case, given the property of the transaction rule (see **A 2***, above) that $B(p(t), \{w^i(t)\}) = \{w^i(t)\} = \{w^{io}\} \forall t$; also since $\{w^{io}\} \in R^n_{++}$, the above result is applicable and we have that there is some vector $b(t)$ and $c_i(t) \geq 0$ such that $\Psi^i(p(t), \{w^i(t)\}) = c_i(t) b(t)$ for all $t$, where we may take $b(t) = \sum_h \Psi^h(p(t), w^{ho})$ and $\Psi_h(p(t), w^{ho}) = c_h(t) b(t)$, $c_h \geq 0, \forall h$.

We assume that $b(t)$ does not vanish for any $t$; since if $b(t)$ is zero, then all the $\Psi_h(.)$ vanish too but then, as we have seen, the proposition holds since demands coincide with $\{w^{ho}\}$. With this, one may define $c_h(t) = \Psi_h(t).b(t)/b(t).b(t)$ and hence $\sum_h c_h(t) = 1$ with $c_h \geq 0 \forall h$.

Next, note that we may rewrite (recall that $p_n = 1$) $\Psi_j^h(t) = r_j^h/p_j(t) - 1$ where $r_j^h = U_j^h(w^{ho})/U_n^h(w^{ho})$ If $r_j^h = r_j^i = \bar{r}_j$ for all $h, i$ and $j = 1, 2, \cdots, n$ then notice that $\{w^{ho}\}$ constitutes a Pareto Optimum and the vector $(\bar{r}_1, \bar{r}_2, \cdots, 1)$ constitute an equilibrium price vector corresponding to $\{w^{ho}\}$. Hence we take the case when $r_j^h \neq r_j^k$ for some $h, k$ and some $j$. We have then:

$r_j^h - r_j^k = p_j(t)(\Psi_j^h(t) - \Psi_j^k(t)) = (c_h(t) - c_k(t))q_j(t)$ where $q_j(t) = p_j(t).b_j(t)$.

Note that $c_h(t) \neq c_k(t)$, $\forall t$; since the left hand side is independent of $t$, $c_h(t) = c_k(t)$ for some $t$ will imply that $r_j^h = r_j^k \; \forall j$: a contradiction.

Since $\forall i$, $\Psi^i(t) = c_i(t).b(t)$ and $\Psi_j^i(t).p_j(t) = r_j^i - p_j(t)$, we have $-\dot{p}_j(t) = \dot{c}_i.q_j(t) + \dot{q}_j(t).c_i(t)$ for all $i$ and $j$. Hence summing over all $i$ and noting that $\sum_i c_i(t) = 1 \forall t$ and hence, $\sum_i \dot{c}_i(t) = 0$, we have $-m\dot{p}_j(t) = \dot{q}_j(t)$. Thus using the earlier expression[6], we have that:

$$\dot{p}_j(t) = \frac{1}{m} \frac{\dot{c}_h(t) - \dot{c}_k(t)}{c^h(t) - c^k(t)} q_j(t) = \frac{r_j^h - r_j^k}{m}.\omega(t)$$

where $\omega(t) = \dfrac{\dot{c}_h(t) - \dot{c}_k(t)}{(c_h(t) - c_k(t))^2}$. Since we have assumed that $b(t) \neq 0, \forall t$, it follows that $\omega(t) \neq 0$[7]; since otherwise if $\omega(t_o) = 0$ for some $t_o$, then $\dot{p}_j = 0$ for all $j$ and consequently, it

[6] $c_i q_j(t) = r_j^i - p_j(t)$ for $i = h, k$ and subtracting, we have $q_j(t) = \dfrac{r_j^h - r_j^k}{c_h(t) - c_k(t)}$. In addition we have that $0 = (\dot{c}_h(t) - \dot{c}_k(t))q_j(t) + (c_h(t) - c_k(t))\dot{q}_j(t)$.

[7] Notice that $\omega(t)$ is well defined for all $t$ since the denominator does not vanish.

must be the case that $p(t_o)$ is an equilibrium price vector which in turn would imply that $b(t_o) = 0$: a contradiction. Consequently given this conclusion, it follows that $\omega(t)$ is either always positive or always negative: thus prices behave monotonically. Now since we have already shown that prices are bounded this implies that $p(t)$ converges say to $p^\star$ and that $(p^\star, 1)$ must be an equilibrium and hence it must follow that $\{w^{io}\}$ is Pareto Optimal, as was claimed. •

### 4.3.4 Convergence

It may be recalled that the equilibrium for the process (4.3) has two properties: first that there can be no further trades and secondly that there cannot ba any price adjustment as well. The convergence to equilibrium will take up these properties one by one.

We know that any solution $(p(t), w^i(t)\})$, possesses limit points (since the solution lies in a bounded region); let $p^\star, \{w^{i\star}\})$ be one such. Note that this configuration must also be positive i.e., $p^\star_j > 0, w^{i\star}_j > 0$ for all $i, j$. Now since the utilities $U^i(w^i(t))$ are nondecreasing and bounded above, it follows that $U^i(w^i(t))$ converges say to $U^{i\star}$ for each $i$; thus for any limit point $(\bar{p}, \{\bar{w}^i\})$ for the process, it must be the case that $U^i(\bar{w}^i) = U^{i\star}$ for each $i$. Consider next, the solution originating from the limit point $(p^\star, \{w^{i\star}\})$, denoted by $(p*(t), \{w^{i\star}(t)\})$: since every point of this solution is a limit point of the original solution with $(p^o, \{w^{io}\}))$ as initial point, it follows that $U^i(w^{i\star}(t)) = U^{i\star}$ for all $t$ and hence no trades can take place. This implies, that $\{w^{i\star}\}$ is Pareto Optimal. Notice also that for every limit point $\bar{w}^i = w^{i\star}$. Thus $w^i(t) \to w^{i\star}$ and we have proved the following:

**Proposition 4.3** *The solution $\phi_t(p^o, \{w^{io}\}) = (p(t), \{w^i(t)\})$ to (4.3) for any $p^0 \in R^n_{++}, w^{io} \in R^n_{++}$ for all $i$ is such that $\{w^i(t)\}$ converges to a unique Pareto Optimal configuration $\{w^{i\star}\}$.*

Notice that the above result is silent on the convergence of prices; except we may use the proof of the Proposition 4.2 to conclude that along the limit path, i.e., the solution originating from the limit point $(p^\star, \{w^{i\star}\})$, denoted by $(p*(t), \{w^{i\star}(t)\})$, $p*(t)$ converges. But this clearly is not good enough: since as matters stand now, there could be many limit points for the prices. But, fortunately, it is possible to close this gap and we do so next[8].

Before we proceed we need some elementary facts which we shall note for future use. First of all,

**Claim 4.3.3** *Given $W^\star = \{w^{i\star}\}, w^{i\star} \in R^n{++}, \forall i$, Pareto Optimal, there is a unique price configuration $(p^\star, 1) = (p_1^{star}, \cdots, p^\star_{n-1}, 1)$ such that $Z((p^\star, 1), W^\star) = 0$.*

Proof: Suppose that $(p^\star, 1)$ and $(\bar{p}, 1)$ both satisfy $Z((p, 1), W^\star) = 0$; where $W^\star$ is Pareto Optimal. That is for all individuals $i$, the demands $x^i(p, 1, w^{i\star}) = w^{i\star}$ when $p = \bar{p}$ and $p = p^\star$. This implies that for each $i$, $w^{i\star}$ solves the problem:

$\max U^i(x)$ subject to $\sum_j^{n-1} p_j x_j + x_n \leq \sum_j^{n-1} p_j w^{i\star}_j + w^{i\star}_n$

when $p = \bar{p}$ and when $p = p^\star$. Since $w^{i\star} \in R^n_{++}$, we have the following conditions satisfied for all $j$:

$$\frac{\partial U^i(w^{i\star})}{\partial w_j} / \frac{\partial U^i(w^{i\star})}{\partial w_n} = \bar{p}_j = p^\star_j$$

which proves the claim. •

[8] The following argument, is based on Mukherji (1974).

Next, it should be pointed out that, for any $p^\star$ whose uniqueness was established in Claim 4.3.3, and excess demands $Z(p, W^\star)$ we have the following:

**Claim 4.3.4** *Given* $W^\star = \{w^{i\star}\}, w^{i\star} \in R^n{++}, \forall i$, *Pareto Optimal, we have* $\sum_j^{n-1} p_j^\star . Z_j(p, W^\star) + Z_n(p, W^\star) > 0$ *for any* $p > 0$, $p \neq p^\star$. *Thus WARP holds.*

Proof: Consider any $p \neq p^\star$ and note that at from the property of demand functions, $x^i(p, 1, w^{i\star})$, for every individual $i$, we have $\sum_{j \neq n} p_j x_j^i(.) + x_n^i(.) = \sum_{j \neq n} p_j w_j^{i\star} + w_n^{i\star}$; thus we may conclude $U^i(x^i(.)) > U^i(w^{i\star})$; hence at $p*$, when $x^i(p^\star, 1, w^{i\star}) = w^{i\star}$, we must have $\sum_{j \neq n} p_j^\star x_j^i(p, 1, w^{i\star}) + x_n^i(.) > \sum_{j \neq n} p_j^\star w_j^{i\star} + w_n^{i\star}$: this must be true for every $i$: summing over all $i$ provides us with the desired inequality. •

Now define $V(t) = 1/2 \sum_{j \neq n} (p_j(t) - p_j^\star)^2 / k_j$ where $(p(t), W(t))$ denotes the solution to the process (4.3); we know that $W(t) \to W^\star$, a Pareto Optimal configuration such that $p^\star$ is the unique corresponding equilibrium configuration. We have:

**Claim 4.3.5** $V(t) \to 0$ *as* $t \to +\infty$.

Proof: Suppose first of all that

$$V(t) \geq \epsilon > 0 \text{ for all } t \tag{4.4}$$

. We know from the Claim 4.3.1, that $p(t)$ lies within a bounded region, say $B$ in $R_{++}^{n-1}$; let $B_\epsilon = \{p \in B : 1/2 \sum_j^{n-1} (p_j - p_j^\star)^2 < \epsilon\}$; notice that from the Claim 4.3.4, $f(p, W^\star) \equiv \sum_j^{n-1} p_j^\star Z_j(p, W^\star) + Z_n(p, W^\star) > 0$ on the set $B - B_\epsilon$: which is a compact set; thus on this set, $f(p, W^\star)$, a continuous function, attains a *minimum* $\delta > 0$.

Notice also that the function $f(p,W) = \sum_j^{n-1} p_j^\star Z_j(p,1,W) + Z_n(p,1,W)$ is a continuous function on $\Gamma^o$ a compact set (see, Claim 4.3.1) and hence is uniformly continuous on this set. Thus for any $\lambda > 0$ there is a $\mu > 0$ such that:

$$|f(p,W) - f(p,W^\star)| < \lambda \text{ if } |W - W^\star| < \mu$$

Notice that along the solution to (4.3), $\dot{V}(t) = \sum_j^{n-1}(p_j(t) - p_j^\star).\dot{p}_j = -f(p(t),W(t))$, using Walras Law. Now if $V(t) \geq \epsilon$ as we have assumed, then $p(t) \in B - B_\epsilon$ , $\forall t$. Hence $f(p(t),W^\star) > \delta > 0$. Now choose $\lambda = \delta/2$; then there is $\mu(\delta)$ such that[9] :

$$|f(p(t),W) - f(p(t),W^\star)| < \delta/2 \text{ whenever } |W - W^\star| < \mu(\delta)$$

Thus $f(p(t),W) > f(p(t),W^\star) - \delta/2 > \delta/2$ whenever $|W - W^\star| < \mu(\delta)$. Now since $W(t) = \{w^i(t)\} \rightarrow W^\star$ by virtue of Proposition 4.3, it follows that there is some $T(\mu(\delta))$ such that $|W(t) - W^\star| < \mu(\delta)$ for all $t > T(\mu(\delta))$; hence for all such $t$, we have $f(p(t),W(t)) > \delta/2$ i.e., for all $t > T(\mu(\delta))$, $\dot{V}(t) = -f(p(t),W(t)) < -\delta/2 < 0$ which contradicts (**??**).

Thus for any $\epsilon > 0$ and for any $N$, say $N > T(\mu(\delta))$, there is a $T(\epsilon) > N$, such that $V(T(\epsilon)) < \epsilon$; if possible let there exist $h > 0$ such that $V(t) < \epsilon$ for all $t \in [T(\epsilon), T(\epsilon)+h)$ and $V(T(\epsilon)+h) = \epsilon$; that is $\dot{V}(T(\epsilon)+h) \geq 0$: this contradicts the fact that $p(T(\epsilon)+h) \in B - B_\epsilon$ and consequently, since $T(\epsilon) + h > N > T(\mu(\delta))$, $\dot{V}(T(\epsilon) + h) < -\delta/2 < 0$. Thus no such $h$ exists and $V(t) < \epsilon$ for all $t > T(\epsilon)$. This proves the claim. •

Collecting together the above claims, we have demonstrated the validity for the following:

[9]We use $|x - y|$ to denote the Euclidean distance: $\surd(\sum_j(x_j - y_j)^2)$; and when $W = \{w^i)\}, W^\star = \{w^{i\star}\}$ $|W - W^\star|$ will denote $\sum_i |W^i - w^{i\star}|$.

**Proposition 4.4** *Consider the process (4.3) and an initial configuration* $(p^o, \{w^{io}\})$, $p^o \in R^n_{++}, w^{io} \in R^n_{++}, \forall i$. *The solution* $(p(t), \{w^i(t)\})$ *converges to* $(p^\star, \{w^{i\star}\})$ *where* $W^\star = \{w^{i\star}\}$ *is Pareto Optimal and* $Z(p^\star, 1, \{w^{i\star}\}) = 0$.

This is thus a further step from the Hahn-Negishi Process which we have considered earlier. Notice that in this case, we have a better description of trades at disequilibrium prices since these have been provided with some rationality properties. However the price adjustment mechanism is really borrowed from the tatonnment analysis undertaken earlier. Besides, the instantaneous nature of transactions and price adjustment makes the entire process difficult. We proceed in the next section to re-examine these issues in a discrete time environment. This allows us to describe the entire process in a more satisfactory way. However, as we shall see the steps taken by Hahn-Negishi and Uzawa will provide important building blocks towards such an explanation.

## 4.4 A Discrete Non-Tatonnement Process

### 4.4.1 Motivation

The most appealing story that one can tell about disequilibrium behavior is the following: if excess demands are positive, then prices are bid up; whereas, if excess demands are negative, prices are bid down. We shall attempt here a model for the above story; the story would be partly modified, but hopefully, its basic appeal would be maintained.

Consider agents who encounter a price - they know their own demands and supplies but they do not know whether the price is an equilibrium one (i.e., market clearing). We shall assume that each agent visits each market in succession - there are $n$ markets if there are $n+1$ commodities; thus if commodity $n+1$ is the numeraire, first $n+1$ is traded against 1; then $n+1$ against 2 and so on till $n+1$ is traded against $n$. This completes **one round of trading.** At the end of one round of trading, either no agent experienced difficulty in making desired transactions or some such problems are encountered.

There are two type of problems that one may encounter: (a) some agent wanted to buy commodity $j$, some one wishes to sell $j$, but the agent who wanted to purchase $j$ has run out of the numeraire: the **no-cash** problem; (b) some agent wants to buy $j$ and has the numeraire (cash) but there is no agent who wants to sell $j$ at these prices; alternatively, there is a seller of $j$ but no purchaser of $j$ even though there is no-cash problem. When agents encounter problems of the type (b), they correctly infer that the price is not an equilibrium one. We shall assume that *unsuccessful buyers bid the price up and unsuccessful sellers bid the price down;* these bids are subjected to some bounds which will be made explicit below. It is also clear that the new price lies somewhere between the largest and smallest bid and we leave the matter at this stage. For our purpose, we do not need to tie down things further. Once new prices are formed, another round of trading occurs. The succession of trades and price adjustments are then analyzed and we provide a theorem on the convergence of endowments and prices.

Thus the price adjustment story we began with has been modified at two levels: first of all, trades occur at disequilibrium prices and it is through such trades that agents find out about the nature of the market; and secondly, it is when a buyer (seller) wants to buy (sell) more but is unable to do so that prices are bid up (down). To make some of these ideas more precise, we shall consider first of all, a situation involving two goods only and use that to prepare the ground for the more general consideration.

### 4.4.2 A First Step: The Two-good Case

We try to provide a first introduction to the story outlined above. Consider the case of two goods; the advantage then is that there is only a single market and we are interested in capturing what happens in that market.

The individuals $i$ have stocks of the goods $w^i = (w_1^i, w_2^i)$ with $w_j^i > 0$ for all $i, j$; consider a price $p$ which is the price of good 1 relative to good 2, the numeraire. At such a price, each individual would like to consume $x^i = (x_1^i, x_2^i)$ where each $x^i$ solves the problem:

$$\max U^i(y_1, y_2) \text{ subject to } p.y_1 + y_2 \leq p.w_1^i + w_2^i$$

and thus would like to make the following transactions $z^i = x^i - w^i$; we shall refer to these as the **desired transactions**. Summing over $z^i$ for all $i$, we get the excess demands $Z(p, \{w^i\})$, of course. Notice that unless $p$ was market clearing, i.e., $Z = 0$, the desired transactions $z^i$ are not mutually compatible in the sense not all individuals can make their desired transactions. This then is the the most important feature of a disequilibrium configuration. More importantly, it is only when some individual fails to make their desired transactions that it is revealed that the $p$ is not market clearing.

So suppose then that $p$ is not market clearing $Z(.) \neq 0$. What actual transactions can take place ? We shall use the notation $\{t^i\}$ to denote **actual transactions**. First of all, actual transactions must be *feasible* i.e., $\sum_i t^i = 0$. Thus every purchase by individual $i$ should also be a sale by some person $h$. Next, *no one should lose* from making such transactions i.e., $U^i(w^i + t^i) \geq U^i(w^i)$; and transactions should respect *the budget constraint* i.e., $pt_1^i + t_2^i = 0$ for all $i$. Notice that had the price being market clearing $Z = 0$, then $t^i = z^i$ is possible and desired and actual transactions match. When $p$ is not market clearing, the feasibility requirement rules out the matching of actual transactions with desired transactions. In such situations, are there transactions which satisfy the three conditions outlined above ?

To answer the above consider the following set of individuals at any $p$: $S(p, \{w^i\}) = \{i : z_1^i(p, w^i).Z_1(p, \{w^i\}) \leq 0\}$ which we may call the *short side of the market*; we define $L(p, \{w^i\}) = \{i : i \notin S(p, \{w^i\})\}$ as the *long side of the market.* It is easy to see that whenever $p$ is not market clearing, the long side is not empty; but the short side of the market may be empty. This is so when even though the price is not market clearing, every individual wants to buy the same commodity. Clearly in such situations, there is no one who is willing to sell them this commodity and no voluntary transaction can take place.

It would thus appear reasonable to add two other facets to our definition of voluntary transactions. First of all that $t_1^i$ should have the same sign as $z_1^i$ (actual transactions must be in the same direction as desired transactions) and $|t_1^i| \leq |z_1^i|$ the fact that no one should be coerced into buying (selling) more than the desired quantities. An alternative specification may be provided as follows: no individual should be able to increase utility by curbing transactions. Notice that these are all in terms of the non-numeraire commodity only; what happens with the numeraire will now follow from the budget constraint. It turns out in case the short side of the market is non-empty, it is possible to define a set of actual transaction meeting all the requirements:

$$t^i = \begin{cases} z^i(p, w^i) & \forall i \in S(p, \{w^i\}) \\ \delta^i z^i(p, w^i) & \text{otherwise} \end{cases} \tag{4.5}$$

where $0 \leq \delta^i \leq 1$ are so defined that $\sum_i t^i = 0$. thus individuals on the short side achieve their desired transactions; those on the long side try to move up as close as possible to their desired transactions. Notice that such a set of transactions meets our requirements but these transactions require that there are individuals on the short side of the market.

After such transactions are complete, notice that the individuals on the short side, having achieved their desired transactions have no way of figuring out that the price $p$ was not a market clearing one; it is those on the long side who have recognized that the $p$ is not market clearing and they have figured out the nature of the imbalance too. Now if nothing happens to the price then notice that those who were on the short side have no desire to participate while the only individuals who wish to transact are the unsatisfied persons on the long side of the market and all of them wish to buy the same commodity: no further transactions can take place. Thus these transactions are really very much like the Hahn-Negishi transactions that we have analyzed earlier.

As we had indicated above, the individuals on the long side have been able to identify the nature of the market imbalance and therefore would like to bid the price up or down in an attempt to achieve their desired transaction. So such bids are made and the price gets revised somewhere in between the highest and smallest bids made and once gain individuals transact to find out what the nature of the market imbalance is. This is the story that we shall analyze formally in the multi-good setting. Notice that with many markets, we need to work a bit more at identifying transactions since it is impossible to define the short side of THE market. Fortunately, the developments in Non-Walrasian Equilibria will help us in identifying such transactions.

### 4.4.3 The Model and Preliminary Definitions

We consider the standard exchange model with $n$-goods numbered 1 to $n$; good $n$ is the numeraire; there are $M$ individuals $i = 1, \cdots, M$ each with an utility function $U^i(.)$ and an endowment $\overline{w}^i$, with $\overline{w}^i_n > 0 \forall i$. It would be convenient to define $X = \{x \in R^n_+ : x_n > 0\}$; thus $\forall i, \overline{w}^i \in X$. We shall assume the following:

A. For each i, $U^i(.)$ is a strictly quasi-concave, increasing and continuously differentiable on $R^n_+$.

Define

$$s^i_j(x) = \frac{\partial U^i(x)}{\partial x_j} / \frac{\partial U^i(x)}{\partial x_n}$$

B. $s^i_j(x) > 0$ for all $i, j$ and for all $x \in X$; further $s^i_j(x^t) \to 0$ for some subsequence $\{x^t\} \in X$ $\Rightarrow$ either $x^t_j \to \infty$ or $x^t_n \to 0$.

C. For each $i$, $U^i(x) \geq U^i(\overline{w}^i) \Rightarrow x \in X$.

The assumptions of this section are thus quite similar to the ones made earlier. Now given a price vector, $p$,$p_j > 0$ for all $j$, demand functions $x^i(p, \overline{w}^i)$ and excess demand

$z^i(p, \overline{w}^i) = x^i(p, \overline{w}^i) - \overline{w}^i$ may be defined for every individual $i$. The market excess demand is then $Z(p, \{\overline{w}^i\}) = \sum_i z^i(p, \overline{w}^i)$. Given a price $p$ such that $Z(.) \neq 0$, $p$ is a disequilibrium price vector. To define transactions which might take place at such a disequilibrium configuration, we take the help of constructs of a Non-Walrasian Equilibrium. The essential nature of such an equilibrium lies in developing rations or constraints on transactions such that the constrained transactions are mutually compatible. We begin, as in the two-good case with a list of requirements for feasible transactions.

Consider the notion of **a Younes Equilibrium at** $\{p, \{w^i\}\}$ . We shall say that the array $\{\overline{t}^i\}$ constitute a Younes Equilibrium at $\{p, \{w^i\}\}$ provided the following conditions hold:

a. $\sum_i \overline{t}^i = 0$

b. For each $i$, $\overline{t}^i$ solves the problem:

$$\max_t U^i(w^i + t)$$

$$\text{subject to } p.t \leq 0$$

$$\min(0, \overline{t}^i_j) \leq t_j \leq \max(0, \overline{t}^i_j) \text{ for all } j \neq n$$

$$w^i + t \geq 0$$

and

c. There does **not** exist a pair of consumers $(i, h)$, a commodity $j$ and a real number $\epsilon > 0$ such that

$$U^i(w^i + \overline{t}^i) < U^i(w^i + \overline{t}^i + \epsilon a^j)$$

$$U^h(w^h + \overline{t}^h) < U^h(w^h + \overline{t}^h - \epsilon a^j)$$

with $w^i + \overline{t}^i + \epsilon a^j \geq 0$ and $w^h + \overline{t}^h - \epsilon a^j \geq 0$, where $a^j = (a^j_k)$, $a^j_k = 0$, $k \neq j, n$, $a^j_n = -p_j$, $a^j_j = 1$. The vector $a^j$, thus, denotes the transaction involved in the purchase of one unit of the commodity $j$.

Notice that the transactions defined above satisfy the requirements of feasibility (a), voluntariness (b) and exhaust all possible trading opportunities (c). Notice that the constraints in the maximization problem in (b), ensure, first of all, that no one loses; secondly, transactions respect the budget constraint and finally, no one can increase utility by trading less. These three combine to make the transactions voluntary. The conditions (c) ensure that all trading opportunities have been exploited so that after transactions have been completed, there is no scope for further utility enhancing transactions.

An alternative notion to the above is that of **transactions** $\{\overline{t}^i\}$ **associated with a Dreze Equilibrium at** $\{p, \{w^i\}\}$ : transactions $\{\overline{t}^i\}$ are said to be associated with a Dreze Equilibrium at $\{p, \{w^i\}\}$ provided there exist $c^i_j, C^i_j$, $j = 1, 2, ..., n-1$, $i = 1, 2, \cdots, M$ satisfying the following three conditions:

i. $c^i_j \leq 0 \leq C^i_j \forall i, \forall j \neq n$

ii. $\sum_i \overline{t}^i = 0$ and for each $i$, $\overline{t}^i$ solves the following problem (**DE**):

$$\max_t U^i(w^i + t)$$

$$\text{subject to } p.t \leq 0$$

$$c^i_j \leq t_j \leq C^i_j \text{ for all } j \neq n$$

$$w^i + t \geq 0$$

and

iii. If for some $j$ there is $i_1$ such that $c_j^{i_1} = \overline{t}_j^{i_1}$ then $C_j^h > \overline{t}_j^h$ $\forall h$; similarly if there is some $j$ and $i_1$ such that $C_j^{i_1} = \overline{t}_j^{i_1}$ then $c_j^h < \overline{t}_j^h$ $\forall h$.

Notice now that the Dreze Equilibrium introduces explicitly the notion of constraints on transactions $c_j^i, C_j^i$: the first lost are constraints on sales while the second lot are constraints on purchases; these constraints are such that the constrained plans are mutually compatible (ii) and are voluntary in that they satisfy the budget constraint as well as the fact that by reducing the volume of trades, no one would be able to increase utility. Condition iii ensures that the non-trivial nature of the constraints in that the constraints may bind only on one side of the market. Markets with such constraints have been termed **orderly** in the literature[10]. As it is easy to see, this requirement is similar to requiring that all possible transactions have been carried out. It turns out that these two forms of transactions are related[11]:

**Proposition 4.5** *Given any Younes equilibrium* $\{\overline{t}^i\}$ *at* $(p, \{w^i\})$ *with* $w_n^i + \overline{t}_n^i > 0$, $\forall i$, *one may construct* $\{c^i, C^i\}$ *such that* $\{\overline{t}^i\}$ *constitute transactions associated with a Dreze Equilibrium at* $(p, \{w^i\})$*. Conversely, any transactions* $\{\overline{t}^i\}$ *associated with a Dreze Equilibrium at* $(p, \{w^i\})$ *constitute a Younes equilibrium.*

As we shall see, crucial use will be made of this result in the analysis provided below. Some implications of transactions associated with a Dreze Equilibrium are noted below for future reference. Consider the fact that the transactions $\{\overline{t}^i\}$ solve the problem **DE**, where each utility function $U^i$ is assumed to be strictly quasi-concave and where each constraint

[10] See, for instance, Hahn (1978).

[11] Silvestre (1982)

is linear (affine); then for each $i$, there exist non-negative numbers $\lambda^{i\star}, \mu^{\star}_{ij}, \delta^{\star}_{ij}, \theta^{\star}_{ij}$ satisfying the following conditions[12]:

4.i. $\overline{U}^i_j - \lambda^{i\star} p_j + \mu^{\star}_{ij} - \delta^{\star}_{ij} + \theta^{\star}_{ij} = 0 \ \forall j \neq n$

4.ii. $\overline{U}^i_n - \lambda^{i\star} + \theta^{\star}_{in} = 0$

4.iii. $\lambda^{i\star} p.\overline{t}^i = 0$

4.iv. $\theta^{\star}_{ij}(w^i_j + \overline{t}^i_j) = 0 \forall j$

4.v. $\mu^{\star}_{ij}(\overline{t}^i_j - c^i_j) = 0 \forall j \neq n$

4.vi. $\delta^{\star}_{ij}(C^i_j - \overline{t}^i_j) = 0 \forall j \neq n$

In case, $w^i_n + \overline{t}^i_n > 0$, it follows from 4.iv that $\theta^{\star}_{in} = 0$ and hence from 4.ii that $\overline{U}^i_n = \lambda^{i\star} > 0$ where the inequality follows from the strictly increasing nature of the utility functions. In this situation, we may define:

$$\overline{s}^i_j \equiv s^i_j(w^i + \overline{t}^i) = \frac{\overline{U}^i_j}{\overline{U}^i_n} \ \forall j \neq n$$

Then from 4.i, we have

4.vii. $\overline{s}^i_j = p_j + \{\delta^{\star}_{ij} - \mu^{\star}_{ij} - \theta^{\star}_{ij}\}/\lambda^{i\star} \forall j \neq n$

By virtue of condition (iii) of a Dreze Equilibrium, we also have the following:

$$\delta^{\star}_{ij} > 0 \text{ for some } i, j \Rightarrow \mu^{\star}_{ij} = 0 \ \forall i \ ;$$

[12] We shall write $\overline{U}^i_j$ to designate the partial derivative of $U^i$ with respect to $j$ and evaluated at $w^i + \overline{t}^i$.

and

$$\mu^{\star}_{ij} > 0 \text{ for some } i, j \Rightarrow \delta^{\star}_{ij} = 0 \; \forall i.$$

Thus at any Dreze Equilibrium and for any market $j \neq n$ the following possibilities exist:

There is some $i$ such that $c^i_j = \overline{t}^i_j$: the market is *sales-constrained*; or

There is some $i$ such that $C^i_j = \overline{t}^i_j$: the market is *purchase-constrained*; or

For all $i$, $c^i_j < \overline{t}^i_j < C^i_j$: the market is *unconstrained.*

When a market is unconstrained, we have, by virtue of 4.v and 4.vi, $\mu^{\star}_{ij} = \delta^{\star}_{ij} = 0$ and hence from 4.vii, $\overline{s}^i_j \leq p_j$ for all $i$ and the equality holds if $i$ is such that $w^i_j + \overline{t}^i_j > 0$.

Thus when a market is sales constrained, we have that $\delta^{\star}_{ij} = 0$ for all $i$ and hence $\overline{s}^i_j \leq p_j - \mu^{\star}_{ij}/\lambda^{i\star}$ with equality in case $i$ satisfies $w^i_j + \overline{t}^i_j > 0$; and similarly, it follows that when a market is purchase constrained, we have that $\mu^{\star}_{ij} = 0$ for all $i$ and hence $\overline{s}^i_j \leq p_j + \delta^{\star}_{ij}/\lambda^{i\star}$ with equality in case $i$ satisfies $w^i_j + \overline{t}^i_j > 0$. We shall summarize the above conclusions in the form of the following:

**Claim 4.4.1** *At transactions $\{\overline{t}^i\}$ associated with a Dreze Equilibrium, for any $j \neq n$, either the market is sales constrained and $\overline{s}^i_j \leq p_j \forall i$ or the market is purchase constrained and $\overline{s}^i_j \geq p_j \forall i$ such that $w^i_j + \overline{t}^i_j > 0$ .*

.

We shall refer to the above as Condition A. Now notice that it is not necessary that the transactions associated with a Dreze equilibrium be non-zero. Condition A however makes it possible for us to identify the following condition for some transactions to take place: if there exists individuals $i, h$ and some commodity $j$ such that

$$s_j^{i_1}(w^{i_1}) > p_j > s_j^{i_2}(w_j^{i_2}) \text{ and } w_j^{i_2} > 0$$

then transactions associated with a Dreze Equilibrium will be non-zero. We shall refer to this as Condition T.

With the above set-up, it should be clear that transactions at a Younes Equilibrium or those associated with a Dreze Equilibrium have rather nice properties and if we are to identify transactions which might take place at a disequilibrium configuration, then we may consider these transactions. There is one drawback however and that is there does not appear to be any way of guaranteeing that these transactions will take place quickly. This aspect makes us continue our search for what kinds of transactions may take place. The ideal candidate appears to be to have only one round of transactions (to be made specific below). On the basis of these transactions, individuals form some idea of the market imbalances or the nature of binding constraints. And on the basis of this the bids are put in place. The new price will be somewhere in between the largest and smallest bid in each market. And then there would be another round of transactions and we shall study the resulting process below.

### 4.4.4 One Round of Trading

To overcome the problems mentioned above, we assume that first agents exchange good 1 against $n$; then 2 against $n$ and so on till they exchange $n-1$ against $n$. This would be one round of trading. At the end of this one round of trading, individual $i$ would have traded and $i$'s endowment would be altered $\overline{w}^i + e^i$, if at the beginning $i$ held $w^i$; $e^i$ denotes the transactions made by $i$. We demand that the transactions $\{e^i\}$ satisfy the following[13]:

1. $\sum_i e^i = 0$

2. For each $i$, $e^i$ solves the problem **YE-1**

$$\max_t U^i(w^i + t)$$

$$\text{subject to } p.t \leq 0$$

$$\min(0, e^i_j) \leq t_j \leq \max(0, e^i_j) \text{ for all } j \neq n$$

$$-w^i_n + \textstyle\sum_{k \leq j} p_k.t_k \leq 0 \text{ for all } j \neq n$$

$$w^i + t \geq 0$$

and

3. There does **not** exist a pair of individuals $i_1$ and $i_2$, a commodity $h \neq n$ and a number $\epsilon > 0$ such that

$$U^{i_1}(w^{i_1} + e^{i_1} + a_h) > U^{i_1}(w^{i_1} + e^{i_1})$$

$$U^{i_2}(w^{i_2} + e^{i_2} - a_h) > U^{i_2}(w^{i_2} + e^{i_2})$$

$$w^{i_1}_n - \textstyle\sum_{k \leq h} p_k.e^{i_1}_k - \epsilon p_h \geq 0$$

$$w^{i_2} + e^{i_2} - a_h \geq 0 \text{ and } w^{i_1} + e^{i_1} + a_h \geq 0$$

where $a_h$, as defined before, denotes the effect on the endowment when $\epsilon$ units of good $h$ is purchased at the unit price of $p_h$. If transactions $\{e^i\}$ satisfy the above conditions, we shall say that $\{e^i\}$ constitute a **Y1-Equilibrium** at $(p, \{w^i\})$.

[13]This notion is based on Younes (1975).

We need to point out some differences ( between conditions 1 to 3 and the conditions for a Younes Equilibrium. First of all notice the problem **YE-1** has an extra constraint that was missing from the earlier constructs of a Younes equilibrium or Dreze Equilibrium : since the individual visits the markets sequentially, there is a sequential budget constraint: when the individual visits the first market then the stock of numeraire is $w_n^i$ which gets adjusted by the amount of transaction that is made and this adjusted amount serves as the constraint when the individual visits the 2-nd market and so on. Similarly, condition 3 has the same type of constraint: trading possibilities are exhausted subject to the fact that when the agents visited the markets for j, those who have the numeraire and those who have the good do not want to undertake further transactions. This sequential nature of the budget constraint is natural in the present context. To relate the transactions after one round $\{e^i\}$ with our earlier definitions, we have the following claim:

**Claim 4.4.2** *If* $\{e^i\}$ *constitute a* ***Y1-Equilibrium*** *at* $(p, \{w^i\})$ *and further if for every i we have:*

$$-w_n^i + \sum_{j \leq k} p_j \bar{e}_j^i < 0 \forall k \neq n \tag{4.6}$$

*then* $\{e^i\}$ *are transactions associated with a Dreze Equilibrium at* $(p, \{w^i\})$.

Proof: If (4.6) holds at $\{e^i\}$ then since conditions a and b of a Younes Equilibrium hold, it remains to check that condition c is also met. Suppose to the contrary that condition c of a Younes equilibrium is violated. That is there exist individuals $(i, h)$, a commodity $j_1$ and a real number $\epsilon > 0$ such that

$$U^i(w^i + e^i) < U^i(w^i + e^i + \epsilon a^{j_1})$$

$$U^h(w^h + e^h) < U^h(w^h + e^h - \epsilon a^{j_1})$$

where $a_j = (0, \cdots, 0, \epsilon, 0, \cdots, -\epsilon p_j)$; this implies that $w^h_{j_1} > 0$ and at the market for $j_1$, since condition (4.6) holds for $i$ and for $k = j_1$, this implies that condition 3 for a **Y1-Equilibrium** is violated. Hence condition c of a **Younes Equilibrium** must hold as well and thus, $\{e^i\}$ constitute a **Younes Equilibrium** at $(p, \{w^i\})$. By virtue of Proposition 4.5, the claim follows. •

The crucial nature of the condition (4.6) is now revealed. As we have indicated above at a **Y1-Equilibrium**, after transactions are complete, markets may not be orderly, in general. However (4.6) ensures that this property is guaranteed; alternatively, (4.6) ensures that a **Younes Equilibrium** is attained in one round of transactions.

### 4.4.5 The Process and the Main Result

Thus if individuals began with $w^i$, then at the end of a round of trading, they have

$$w^{i'} = w^i + \overline{e}^i \tag{4.7}$$

where the array $\{\overline{e}^i\}$ satisfies conditions 1-3 of a **Y1-Equilibrium**. And more importantly, individuals would have noted which are the binding constraints each faced; for example, an individual could have been unsuccessful in buying or selling in some market, $j$, say where good $j$ was exchanged against the numeraire $n$.

We shall assume that an unsuccessful buyer (seller) bids the price up (down, respectively); the highest (lowest) bid, we shall assume would be the person's marginal rate of substitution, $s^i_j(w^i + e^i)$. Given these bids, the new price in the $j-$th market would be somewhere between the lowest and the largest bid and hence, between the lowest and largest $s_j^i$. It would be convenient to represent the new price as :

$$p_j' = \alpha_j \max_i \ s_j^i(w^i + e^i) + (1 - \alpha_j) \min_{i \ni w_j^i + e_j^i > 0} \ s_j^i(w^i + e^i) \tag{4.8}$$

where $1 > \alpha_j > 0$. The minimum is being taken only over those who have positive amounts of the commodity; only then can one ensure that some persons would be willing to sell the commodity. The numbers $\alpha_j$ play no role in the analysis below except that we require them to be bounded away from zero and 1. After the new prices are formed as described above, a further round of trading occurs and subsequently, prices are adjusted once more.

Thus, beginning with $\{w^i(k-1)\}$, at the end of round $k$ of trading, in accordance with the equation ( 4.7 ), each agent $i$ has

$$w^i(k) = w^i(k-1) + \overline{e}^i(k-1) \tag{4.9}$$

where the array $\{\overline{e}^i(k-1)\}$ satisfies conditions 1-3 of a **Y1-Equilibrium** at $p(k-1), \{w^i(k-1)\}$.

Subsequently, we have price adjustment provided by the following (writing $s_j^i(k)$ for $s_j^i(w^i(k))$ and $I(j,k) = \{i : w_j^i(k) > 0\}$):

$$p_j(k) = \alpha_j(k) \max_i \ s_j^i(k) + (1 - \alpha_j(k)) \min_{i \in I(j,k)} \ s_j^i(k) \text{ for all } j \neq n \tag{4.10}$$

where $0 < \delta_j < \alpha_j(k) < 1 - \eta_j$ for some positive numbers $\delta_j, \eta_j$. Initially, we assume that there is some arbitrary $p^o = (p_1^o, p_2^o, \cdots, p_{n-1}^o, 1)$ where $p_j^o > o \forall j$ and $\{w^{io}\}$ where for each $i$, $w_n^{io} > 0$. Thus $\overline{e}^i(0)\}$ denotes the transactions at a **Y1-Equilibrium** at $(p^o, \{w^{io}\})$ which defines through (4.9), $\{w^i(1)\}$ and hence $p(1)$ through (4.10) and so on.

As explained above, a sequence of $p(k), \{w^i(k)\}$ is generated after $k$ rounds of trading. We shall analyze the behavior of this sequence as $k \to \infty$. The main result in this connection is the following:

**Proposition 4.6** *$p(k), \{w^i(k)\}$ converges to a no-trade Walrasian configuration $p*, \{w^i*\}$.*

It would be convenient to introduce the following notation for future use:

$$a_j(k) = \max_i s^i_j(k)\ ;\ b_j(k) = \min_{i \in I(j,k)} s^i_j(k)\ ;\ D_j(k) = a_j(k) - b_j(k) \tag{4.11}$$

Notice then that from (4.10) we have :

$$b_j(k) \leq p_j(k) \leq a_j(k)$$

and consequently,

$$a_j(k) - p_j(k) = \alpha_j(k) D_j(k) \geq \delta_j D_j(k) \tag{4.12}$$

and also

$$p_j(k) - b_j(k) = (1 - \alpha_j(k)) D_j(k) \geq \eta_j D_j(k) \tag{4.13}$$

First of all, we to restate Claim 4.4.2 to meet current requirements:

**Claim 4.4.3** *If at some round $k$, transactions $\overline{e}^i(k)$ are such that for all $i$*

$$-w^i_n(k-1) + \sum_{j \leq h} p_j(k-1)\overline{e}^i_j(k) < 0 \forall h \neq n \tag{4.14}$$

*then $\{\overline{e}^i(k)\}$ are transactions associated with a Dreze equilibrium at $\{p(k-1), \{w^i(k-1\}\}$.*

The significance of the above rests on the crucial property iii of transactions at a **Dreze Equilibrium**; this says that only one side of the market can be rationed /constrained. One may now note, trivially, that:

**Claim 4.4.4** *If at some round $k$ no trades occur and $w_n^i(k-1) > 0\ \forall i$ then condition ( 4.6 ) holds.*

The proof is immediate since $\{\overline{e}^i\} = 0\ \forall i$.

Consider then the sequence $\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\}\}$ generated by the equations ( 4.9 ) and ( 4.10 ). Consider $\Omega \in R_+^{Mn}$ which is such that $\Omega = \{\{x^i\} : i = 1 \cdots M, U^i(x^i) \geq U^i(\overline{w}^i), \sum_i x^i = \sum_i w^{io}\}$ where the array $\{\{w^{io}\}$ denotes the initial endowment, i.e., $\{w^i(0)\}$.

Note that by virtue of our assumption C., $\{x^i\} \in \Omega \Rightarrow x^i \in X \forall i$. Note also that the set $\Omega$ is a compact subset of $R^{Mn}$ and by construction, $\{w^i(k)\} \in \Omega \forall k$; thus all limit points $\{\tilde{w}^i\} \in \Omega$; consequently, for each $i$, $s_j^i$(k) has a maximum and minimum in $\Omega$, say $\beta_j^i$ and $\theta_j^i$, respectively. In other words, we have

$$\max_i \beta_j^i \geq p_j(k) \geq \min_i \theta_j^i > 0$$

where the strict inequality follows from assumption B. Thus one may conclude:

**Claim 4.4.5** *$\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\}\}$ is a bounded sequence and hence has limit points $\{\hat{p}, \{\hat{w}^i\}, \{\overline{e}^i\}\}$ such that $\hat{p} > 0$ and $\hat{w}^i \in X \forall i$.*

In addition, we have for $k > 1$, if no transactions occur, then:

**Claim 4.4.6** *$\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\}\}$ is a no-trade Walrasian configuration if $\overline{e}^i(k) = 0 \forall i$.*

Proof: First of all by Claim ( 4.4.4 ), condition ( 4.6 ) holds; hence, by virtue of our Claim ( 4.4.3 ), $\{\bar{e}^i(k)\} = \{0\}$ are transactions associated with a **Dreze Equilibrium** at $\{p(k), \{w^i(k)\}\}$. Thus the conditions ii. and iii. of the Dreze equilibrium imply one of the following holds at any $j \neq n$[14]:

$$\text{Either } s_j^i(k) \leq p_j(k) \forall i$$

$$\text{Or } s_j^i(k) \geq p_j(k) \forall i \in I(j,k)$$

on account of the fact that no transactions have taken place, $\bar{e}^i(k) = 0 \forall i$; but on account of price adjustment we know that $a_j(k) - p_j(k) \geq \delta_j.D_j(k)$ and $p_j(k) - b_j(k) \geq \eta_j.D_j(k)$; thus on either count we must have $D_j(k) = 0$. This establishes the claim. •

*Thus unless a Walrasian configuration is encountered at the end of some round of trading, trading and price adjustment would continue.* Let $\{\hat{p}, \{\hat{w}^i\}, \{\hat{e}^i\}\}$ be an arbitrary limit point of the sequence $\{p(k), \{w^i(k)\}, \{\bar{e}^i(k)\}\}$. We have:

**Claim 4.4.7** *The array $\{\hat{e}^i\}$ satisfies conditions 1, 2 and 3 of **Y1-Equilibrium** at $\{\hat{p}, \{\hat{w}^i\}\}$.*

Proof: Note that, from Claim 4.4.5, $\hat{p} > 0$ and $\{\hat{w}^i\} \in X \ \forall i$. Suppose that the subsequence $\{p(s), \{w^i(s)\}, \{\bar{e}^i(s)\}\}$ converges to $\{\hat{p}, \{\hat{w}^i\}, \{\hat{e}^i\}\}$. By definition, $\bar{e}^i(s)$ solves the following problem:

$$\max_t U^i(w^i(s) + t)$$

$$\text{subject to } p(s).t \leq 0$$

[14] Recall Claim 4.4.1.

$$\min(0, \overline{e}^i_j(s)) \leq t_j \leq \max(0, \overline{e}^i_j(s)) \text{ for all } j \neq n$$

$$-w^i_n(s) + \textstyle\sum_{k \leq j} p_k(s).t_k \leq 0 \text{ for all } j \neq n$$

$$w^i(s) + t \geq 0$$

Let $\gamma^i\{p(s), w^i(s), \overline{e}^i(s)\}$ denote the constraint set of the above problem. We shall show below that $\hat{e}^i$ solves the following:

$$\max_t U^i(\hat{w}^i + t)$$

$$\text{subject to } t \in \gamma_i(\hat{p}, \hat{w}^i, \hat{e}^i)$$

Thus, the array $\{\hat{e}^i\}$ satisfies conditions 1 and 2. Now suppose contrary to condition 3., there exist individuals $i_1, i_2$, a commodity $h$ and a number $\epsilon > 0$ such that

$$\left.\begin{array}{ll} U^{i_1}(w^{i_1} + \hat{e}^{i_1} + a_h) > U^{i_1}(w^{i_1} + \hat{e}^{i_1}) & \\ U^{i_2}(w^{i_2} + \hat{e}^{i_2} - a_h) > U^{i_2}(w^{i_2} + \hat{e}^{i_2}) & \\ w^{i_1} + \hat{e}^{i_1} + a_h \geq 0 & w^{i_2} + \hat{e}^{i_2} - a_h \geq 0 \\ -w^{i_1}_n + \sum_{j \leq h} \hat{p}_j \hat{e}^{i_1}_j + \epsilon p_h \leq 0 & \end{array}\right\} \tag{4.15}$$

Consequently, for $s$ large enough, we must have the following:

$$\left.\begin{array}{l} U^{i_1}(w^{i_1}(s) + \overline{e}^{i_1}(s) + a_h) > U^{i_1}(w^{i_1}(s) + \hat{e}^{i_1}(s)) \\ U^{i_2}(w^{i_2}(s) + \hat{e}^{i_2}(s) - a_h) > U^{i_2}(w^{i_2}(s) + \hat{e}^{i_2}(s)) \end{array}\right\} \tag{4.16}$$

But since the array $\{\overline{e}^i(s)\}$ satisfies condition 3 by definition, the equation ( 5.2 ) implies that we must have for each $s$ large enough:

$$\text{Either } -w^{i_1}_n(s) + \sum_{j \leq h} p_j(s)\overline{e}^{i_1}_j(s) + \epsilon p_h > 0$$

$$\text{Or } w^{i_2}_h + \overline{e}^{i_2}_h(s) - \epsilon < 0$$

thus, equation ( 4.15 ) cannot hold. Hence the array $\{\hat{e}^i\}$ must also satisfy condition 3 at $\{\hat{p}, \{\hat{w}^i\}\}$ and the claim is true. •

To complete the proof, recall that we need to clinch a fact assumed in the proof above: viz., $\hat{e}^i$ solves the following:

$$\max_t U^i(\hat{w}^i + t)$$

$$\text{subject to } t \in \gamma_i(\hat{p}, \hat{w}^i, \hat{e}^i)$$

We do so now through the following two steps:

**Claim 4.4.8** *Consider* $(c_j, C_j), c_j \leq 0 \leq C_j, j = 1, 2, ..., (n-1)$ *and* $w \in R^n_+$. *Define* $\sigma(c, C, w) = \{t \in R^n : c_j \leq t_j \leq C_j, j \neq n, w + t \geq 0\}$. *For any sequence* $(c^s, C^s, w^s) \rightarrow (c^o, C^o, w^o)$ *where* $c^o_j \leq 0 \leq C^o_j, j \neq n$ *and* $w^o \geq 0$ *and any* $t^o \in \sigma(c^o, C^o, w^o)$ *there is a sequence* $t^s$ *such that* $t^s \in \sigma(c^s, C^s, w^s)$ *for all* $s$ *and* $t^s \rightarrow t^o$.

Proof: Notice first of all that for any $(c, C, w)$ satisfying $c_j \leq 0 \leq C_j, j \neq n$ and $w \geq 0$, $\sigma(c, C, w)$ is non-empty. Next since $t^o \in \sigma(c^o, C^o, w^o)$ we can define the following: $J_1 = \{j : C^o = t^o_j = c^o_j\}$ , $J_2 = \{j : C^o_j > t^o_j = c^o_j\}$ , $J_3 = \{j : C^o_j = t^o_j > c^o_j\}$ , $J_4 = \{j \notin J_1 \cup J_2 \cup J_3 : t^o_j + w^o_j > 0\}$ , $J_5 = \{j \notin J_1 \cup J_2 \cup J_3 : t^o_j + w^o_j = 0\}$. Now define $t^s$ as follows:

$$t^s_j = t^o_j \text{ for } j \in J_1 \cup J_4 \text{ ; } t^s_j = C^s_j \text{ for } j \in J_3$$

for $j \in J_2$, define $t^s_j = \lambda^s c^s_j + (1 - \lambda^s) C^s_j$ where $\lambda^s = \max_{[0,1]} \lambda$ such that $w^s_j + t^s_j \geq 0$; finally for $j \in J_5$ define $t^s_j = -w^s_j$.

Notice that by definition, $t^s_j + w^s_j \geq 0 \forall j$ and for all $s$ large enough and for all $j \notin J_2$ $t^s_j \rightarrow t^o_j$; for $j \in J_2$, notice that since $\lambda^s$ is a bounded sequence there is a limit point $\bar{\lambda}$ and consequently for such $t_j^s$, there is a limit point $\hat{t}_j = c_j^o(= t_j^o) + (1-\bar{\lambda})(C_j^o - c_j^o) > t_j^o$ if $\bar{\lambda} \neq 1$; thus in this case, $w_j^o + \hat{t}_j > 0$ and hence, there exists a subsequence $t_j^{s_r}$ such that for all $r$ large, $w_j^{s_r} + t_j^{s_r} > 0$ but then $\lambda^{s_r}$ cannot be maximal; hence for every limit point we must have $\bar{\lambda} = 1$ and hence $t_j^s \to t_j^o$ for $j \in J_2$ as well. This proves the claim. •

We write $c_j^i = min(0, e_j^i)$ , $C_j^i = max(0, e_j^i)$ and recall the definition of $\gamma^i(p, w^i, e^i) = \{t \in R^n : p.t \leq 0, c_j^i \leq t_j \leq C_j^i \text{ for } j \neq n, -w_n^i + \sum_{k \leq j} p_k.t_k \leq 0 \text{ for all } j \neq n, w^i + t \geq 0\}$. We may now show that

**Claim 4.4.9** *Consider* $(p^o, w^{i_o}, e^{i_o}), p_j^o > 0 \forall j, w_n^{i_o} > 0$, $t^o \in \gamma(p^o, w^{i_o}, e^{i_o})$ *and a sequence* $(p^s, w^{i_s}, e^{i_s}) \to (p^o, w^{i_o}, e^{i_o})$. *Then there is a sequence* $t^s$ *such that* $t^s \in \gamma(p^s, w^{i_s}, e^{i_s})$ *and* $t^s \to t^o$.

Proof: Since $w_n^{i_o} > 0 \; \exists \bar{t} \in \gamma(p^o, w^{i_o}, e^{i_o})$ such that $p^o.\bar{t} < 0$ and $-w_n^{i_o} + \sum_{k \leq j} p_k^o.\bar{t}_k < 0 \forall j \neq n$: for example $\bar{t}_j = 0 \forall j \neq n, \bar{t}_n = -w_n^{i_o}/2$. Next notice[15] that $t^o \in \sigma(c^{i_o}, C^{i_o}, w^{i_o})$ and hence from Claim 4.4.8, there is a sequence $t^s$, such that $t^s \in \sigma(c^{i_s}, C^{i_s}, w^{i_s})$ such that $t^s \to t^o$. Define next $u^s = \lambda^s t^s + (1-\lambda^s)\bar{t}$ where $\lambda^s$ is $\max_{[0,1]} \lambda$ such that $p^s.u^s \leq 0$; it is to be noted that $\lambda^s < 1 \Rightarrow p^s.u^s = 0$; since $t^s, \bar{t} \in \sigma(c^{i_s}, C^{i_s}, w^{i_s})$ which is convex, we note that $u^s \in \sigma(c^{i_s}, C^{i_s}, w^{i_s}) \cap \{t \in R^n : p^s.t \leq 0\}$; since $\lambda^s$ forms a bounded sequence, it will have a limit point $\bar{\lambda}$; if possible let this be $< 1$; then there is a subsequence $u^{s_r} \to \bar{u} = \bar{\lambda} t^o + (1-\bar{\lambda})\bar{t}$ and since $\lambda^{s_r} < 1$, $p^{s_r}.u^{s_r} = 0$ along this subsequence so that $p^o.\bar{u} = 0$; since $p^o.\bar{t} < 0$ this means that $p^o.t^o > 0$: a contradiction. Thus $\bar{\lambda} = 1$ and $\bar{u} = t^o$. Thus there is a sequence $u^s$ such that $u^s \to t^o$ and $u^s \in \sigma(c^{i_s}, C^{i_s}, w^{i_s}) \cap \{t \in R^n : p^s.t \leq 0\}$.

[15]We shall write $c^{i_o} = min(0, e^{i_o}), C^{i_o} = max(0, e^{i_o})$ and $c^{i_s} = min(0, e^{i_s}), C^{i_s} = max(0, e^{i_s})$.

Next consider the inequalities $\varsigma(w^{i-o}, p^o) = \{t \in R^n : -w_n^{io} + \sum_{k \leq j} p_k^o.t_k^o \leq 0$ for all j $\neq n\}$.

Consider first of all the inequality for $j = 1$, i.e., $-w_n^{i_o} + p_1^o t_1^o \leq 0$; if this inequality is satisfied with a strict inequality, then for all $s$ large enough, we must have $-w_n^{i_s} + p_1^s.u_1^s < 0$ and consequently, $u^s \in \sigma(c^{i_s}, C^{i_s}, w^{i_s}) \cap \{t \in R^n : p^s.t \leq 0\} \cap \{t \in R^n : -w_n^{i_s} + p_1^s u_1^s \leq 0\}$.

On the other hand, if the inequality is satisfied with an equality then $t_1^o > 0$ and $-w_n^{i_o} + p_1^o.t_1^o = 0$; now define $u_j^s(1) = u_j^s, j \neq 1$ and for $j = 1$, define $u_1^s(1) = \lambda^s u_1^s$ where $\lambda^s$ is the $\max_{[0,1]} \lambda$ such that $-w_n^{i_s} + p^s.u_1^s(1) \leq 0$. As before, notice that $\lambda^s < 1 \Rightarrow -w_n^{i_s} + p^s.u_1^s(1) = 0$. If $\lambda^s \to \bar{\lambda} < 1$, then $u_1^s(1) \to \bar{\lambda} t_1^o$ and consequently, from hypothesis, $-w_n^{i_o} + p_1^o \bar{\lambda} t_1^o < 1$ but then, if $\lambda^{s_r} \to \bar{\lambda} < 1$ then $\lambda^{s_r} = 1$ for all $r$ large and we arrive at a contradiction and hence $\bar{\lambda} = 1$ and $u_1^s(1) \to t_1^o$ and we have a sequence $u^s(1) \to t^o$ where $u^s(1) \in \sigma(c^{i_s}, C^{i_s}, w^{i_s}) \cap \{t \in R^n : p^s.t \leq 0\} \cap \{t \in R^n : -w_n^i + p_1.t_1 \leq 0\}$.

We consider the next inequality in $\varsigma(w^{i_o}, p^o)$ and proceed exactly as above and continue this till we have exhausted all the inequalities in $\varsigma(w^{i_o}, p^o)$ to conclude that there is a sequence $u^s \in \gamma(p^s, w^{i_s}, e^{i_s})$ and $u^s \to t^o$ as claimed. •

Finally, suppose that $(\hat{p}, \hat{w}^i, \hat{e}^i)$ is the limit of a sequence of $(p^s, w^{i_s}, e^{i_s})$ where $\hat{w}_n^i > 0$ for all $i$, then $\hat{e}^i$ solves the problem $\max_t U^i(w^i + t)$ subject to $t \in \gamma(\hat{p}, \hat{w}^i, \hat{e}^i)$; since $\hat{e}^i \in \gamma(\hat{p}, \hat{w}^i, \hat{e}^i)$, $\hat{e}^i$ does not solve the problem implies that there is $t \in \gamma(\hat{p}, \hat{w}^i, \hat{e}^i)$ such that $U^i(\hat{w}^i + t) > U^i(\hat{w}^i + \hat{e}^i)$; by virtue of the Claim 4.4.9, it follows that there is a sequence $t^s \in \gamma(p^s, w^{i_s}, e^{i_s})$ and $t^s \to t$; thus for $s$ large enough it follows that $U^(w^{i_s} + t^s) > U^i(w^{i_s} + e^{i_s})$: a contradiction. This proves the claim.

The above allows us to demonstrate the validity of:

**Claim 4.4.10** $\{\hat{e}^i\} = 0 \; \forall i.$

Proof: Note first of all that the sequence $U^i(w^i(k))$ is nondecreasing and bounded above by virtue of the Claim ( 4.4.5 ); thus for each $i$, $U^i(w^i(k))$ converges to $\hat{U}^i$ say as $k \to \infty$; note further, that $U^i(w^i(k+1)) \to \hat{U}^i$ too, as $k \to \infty$. Thus $U^i(\hat{w}^i) = U^i(\hat{w}^i + \hat{e}^i) = \hat{U}^i$ for all $i$. By virtue of the Claim ( 4.4.7 ), it follows that since $\{\hat{e}^i\}$ satisfies conditions 1,2 and 3 at $\{\hat{p}, \{\hat{w}^i\}\}$, and $U^i(.)$ is strictly quasi-concave, we must have $\hat{e}^i = 0$ for all $i$ and our claim follows. •

We are now ready to show that:

**Claim 4.4.11** *At any limit point $\{\hat{p}, \{\hat{w}^i\}, \{\hat{e}^i\}\}$ the condition ( 4.6 ) holds: i.e.,*

$$\forall i : -w_n^i + \sum_{j \leq h} \hat{p}_j \hat{e}_j^i < 0 \forall h \neq n$$

Proof: Follows by virtue of the Claims ( 4.4.10 ), ( 4.4.7 ) and ( 4.4.4 ). •

This allows us to conclude that:

**Claim 4.4.12** *For all $k$ large enough, the sequence $\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\}\}$, is such that $\{\overline{e}^i(k)\}$ are transactions associated with a Dreze equilibrium at $\{p(k), \{w^i(k)\}\}$.*

Proof: By virtue of the Claim ( 4.4.11 ), it follows that for all $k$ large enough, the condition ( 4.6 ) must hold; consequently, the claim follows by virtue of Claim ( 4.4.3 ). •.

The above shows that even though, the initial steps of the sequence $\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\}\}$ may not be a Dreze equilibrium, for $k$ large enough, they do constitute such an Equilibrium.

To complete the proof, we shall consider a sequence $\{p(k), \{w^i(k)\}, \{\overline{e}^i(k)\})$ such that $\{\overline{e}^i(k)\}$ are transactions associated with a Dreze equilibrium at $\{p(k), \{w^i(k)\}\}$[16].

### 4.4.6 The Proof of the Main Result

The final steps in the proof will now be provided. We have demonstrated above that for the sequence $(p(k), \{w^i(k)\}, \{\overline{e}^i(k)\})$, $\overline{e}^i(k) \to 0$ for each $i$ (Claim 4.4.10). From the definition of the process (4.9), we have:

**Claim 4.4.13** *For each $i$, $w^i(k+1) - w^i(k) \to 0$; in addition for any fixed $h$, $w^i(k+h) - w^i(k) \to 0$.*

Note that there would be some associated sequence of $\{c^i(k), C^i(k)\}$ such that $\overline{e}^i(k)\}$ are the associated transactions satisfying conditions i - iii of a **Dreze Equilibrium**. Hence we have at each stage $k$ and any market $j$ is either constrained in sales or constrained in purchases or may not be constrained at all. First, we consider what happens if a particular market $j$ is sales constrained along some subsequence $k_s$. The notations and relationships introduced in (4.11), (4.12) and (4.13) will be used in the analysis.

**Claim 4.4.14** *If for some commodity $j$, $\exists k_s$, a subsequence, such that for each $s$ there is some $i$ satisfying $c^i_j(k_s) = \overline{e}^i_j(k_s) = w^i_j(k_s+1) - w^i_j(k_s) < C^i_j(k_s)$ then $D_j(k_s) \to 0$ as*

[16] In other words, we neglect the first steps and concentrate on those stages where the Claim 4.4.12 is valid.

$s \to +\infty$*; moreover,* $D_j(k_s+1) \to 0$.

Proof: From the condition of sales constrained, we know that (see Condition A above) that:

$$s_j^h(k_s+1) \le p_j(k_s) \text{ for all } h$$

and from the price adjustment equations (4.10) that

$$p_j(k_s) \le a_j(k_s)(= \max_h s_j^h(k_s))$$

Thus we have $a_j(k_s+1) \le p_j(k_s) \le a_j(k_s)$; now from Claim 4.4.13, we know that $a_j(k_s+1) - a_j(k_s) \to 0$; hence $a_j(k_s) - p_j(k_s) \to 0$ and consequently it is immediate that $D_j(k_s) \to 0$.

For the remaining part of the claim, assume to the contrary that $D_j(k_{s_r}+1) \ge \epsilon > 0$ for some subsequence $k_{s_r}$. Then we have :

$$\epsilon \le a_j(k_{s_r}+1) - b_j(k_{s_r}+1)$$

$$\le [a_j(k_{s_r}+1) - a_j(k_{s_r}) + a_j(k_{s_r}) - b_j(k_{s_r})] + [b_j(k_{s_r}) - b_j(k_{s_r}+1)]$$

The first term within square brackets approaches zero, by virtue of Claim 4.4.13. Consequently for all $r$ large enough, we must have $[b_j(k_{s_r}) - b_j(k_{s_r}+1)] \ge \epsilon/2$.

Let $i_1 \in I(j, k_{s_r}+1)$ such that $s_j^{i_1}(k_{s_r}+1) = b_j(k_{s_r}+1)$; note that from the definition of price adjustment (4.10), it follows that $p_j(k_{s_r}) \ge b_j(k_{s_r})$. Then $i_1 \in I(j, k_{s_r})$ as well. Since other wise, $w_j^{i_1}(k_{s_r}) = 0 < w_j^{i_1}(k_{s_r}+1)$ and hence $c_j^{i_1}(k_{s_r}) < w_j^{i_1}(k_{s_r}+1) - w_j^{i_1}(k_{s_r})$ Since individual $i_1$ is not constrained in the $j$-th market at stage $k_{s_r}$, it follows that $b_j(k_{s_r}+1) = s_j^{i_1}(k_{s_r}+1) \ge p_j(k_{s_r}) \ge b_j(k_{s_r})$: a contradiction. Hence we have : $D_j(k_{s_r}) = a_j(k_{s_r}) - b_j(k_{s_r}) \ge s_j^{i_1}(k_{s_r}) - b_j(k_{s_r}) \ge 0$. Thus we have:

$$\epsilon/2 < |b_j(k_{s_r}) - b_j(k_{s_r}+1)| \le |b_j(k_{s_r}) - s_j^{i_1}(k_{s_r})| + |s_j^{i_1}(k_{s_r}) - s_j^{i_1}(k_{s_r}+1)|$$

But the two terms on the right are both tending to zero: the first because $D_j(k_{s_r})$ is approaching zero, as just demonstrated and the second on account of Claim 4.4.13. We arrive at a contradiction and thus the claim is established. •

Next consider any limiting configuration for the process (4.9) and (4.10): $[\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$; we know by virtue of our claims that $\bar{e}^i = 0$ ( see Claim 4.4.10 ) for **any** limiting configuration $[\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$. Define $\bar{a}_j = \max_i s^i_j(\bar{w}^i), \bar{b}_j = \min_{i \ni \bar{w}^i_j > 0} s^i_j(\bar{w}^i), \bar{D}_j = \bar{a}_j - \bar{b}_j$. First of all note that:

**Claim 4.4.15** *For any commodity $j \neq n$ we have only the following possibilities:*

*4.i.* $\bar{a}_j \leq \bar{p}_j$ *or*

*4.ii.* $\bar{b}_j \geq \bar{p}_j$ *or*

*4.iii.* $\bar{a}_j = \bar{b}_j = \bar{p}_j$

Proof: Since $[\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$ is a limiting configuration, there is a subsequence $(k_r)$ such that $[p(k_r), \{w^i(k_r)\}, \{e^i(k_r\}] \rightarrow [\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$ as $r \rightarrow +\infty$; also the transactions $\{e^i(k_r)\}$ are transactions associated with a Dreze Equilibrium at $[p(k_r), \{w^i(k_r)\}]$ for all $r$ large enough ( see Claim 4.4.3 ). Thus for all $r$ large enough, we must have Condition A (Claim 4.4.1): for any $j \neq n$, either the market is sales constrained and $s^i_j(w^i(k_r+1)) \leq p_j(k_r) \forall i$ or the market is purchase constrained and $s^i_j(w^i(k_r+1)) \geq p_j(k_r) \forall i$ such that $w^i_j(k_r+1) = w^i_j(k_r) + e^i_j(k_r) > 0$ . Also recall that by virtue of Claim 4.4.13, $w^i(k_r+1) \rightarrow \bar{w}^i$ for all $i$. With these facts in place, assume to the contrary that there is some $j$ and individuals $i_1, i_2$ such that $\bar{w}^{i_2}_j > 0$

and

$$s_j^{i_1}(\bar{w}^{i_1}) > \bar{p}_j > s_j^{i_2}(\bar{w}^{i_2})$$

then notice that Condition A will be violated for large values of $r$. This establishes the claim. •

We now claim that

**Claim 4.4.16** *At any limiting configuration, $[\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$, for any commodity $j \neq n$, we must have $\bar{D}_j = 0$.*

Proof: There is a subsequence $(k_r)$ such that $[p(k_r), \{w^i(k_r)\}, \{e^i(k_r\}] \to [\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$ as $r \to +\infty$; consider for any $j$, the situation (4.i); since along the subsequence $a_j(k_r) - p_j(k_r) \geq \eta_j D_j(k_r)$, we must have $\bar{a}_j - \bar{p}_j \geq \eta_j \bar{D}_j$ thus (4.i) $\Rightarrow \bar{D}_j = 0$.

Next suppose we have at $j$ the situation (4.ii) with $\bar{D}_j = \epsilon > 0$ so that $\bar{a}_j > \bar{b}_j \geq \bar{p}_j$. Thus there must exist $i_1$ such that $s_j^{i_1}(\bar{w}^{i_1}) > \bar{p}_j$. Since for all $i$, $w_j^i(k_r+1) \to \bar{w}^i$, for all $r$ large enough we must have $s_j^{i_1}(k_r+1) > p_j(k_r)$ and hence by Condition A, for all large $r$, $b_j(k_r+1) \geq p_j(k_r)$. This means that we must have, for all large $r$, $p_j(k_r+1) - p_j(k_r) \geq p_j(k_r+1) - b_j(k_r+1) \geq \delta_j D_j(k_r+1)$. Thus one may conclude that $\liminf p_j(k_r+1) \geq \bar{p}_j + \delta_j \bar{D}_j$; repeating this argument, for $h$ suitably chosen, we must have:

$$\liminf p_j(k_r+h) \geq \liminf p_j(k_r+h-1) + \delta_j \bar{D}_j \geq \bar{p}_j + h.\delta_j \bar{D}_j \geq \bar{a}_j$$

Thus, the sequence $[p_j(k_r+h), \{w^i(k_r+h\}, \{e^i(k_r+h\}]$ has, as a limit point $[\hat{p}, \{\bar{w}^i\}, \{0\}]$ where we have $\hat{p}_j \geq \bar{a}_j$ or that condition (4.i) holds, but then, $\bar{D}_j = 0$: a contradiction. Hence in condition (4.ii), as well, we must have $\bar{D}_j = 0$. This proves the claim. •

This allows us to claim:

**Claim 4.4.17** $\{w^i(k)\} \to \{\bar{w}^i\}$ *as* $k \to +\infty$*, a unique Pareto Optimal configuration.*

Proof: At any limiting configuration $[\bar{p}, \{\bar{w}^i\}, \{\bar{e}^i\}]$, since $\bar{D}_j = 0$ for all $j \neq n$ it follows that $\{\bar{w}^i\}$ is Pareto Optimal. Now suppose that $\{w^i(k)\}$ has another limit point $\{\hat{w}^i\}$ where $\hat{w}^i \neq \bar{w}^i$ for some $i$. The we have that $\{\hat{w}^i\}$ is Pareto Optimal too. Recall that $U^i(w^i(k))$ being monotonic non-decreasing and bounded must converge to some $\bar{U}^i$ as $k \to +\infty$; hence it must be the case that $U^i(\bar{w}^i) = U^i(\hat{w}^i) = \bar{U}^i$ for all $i$; the strict quasi-concavity of the utility functions may now be exploited to conclude that the allocation $\{(\hat{w}^i + \bar{w}^i)/2\}$ is feasible and for all those $i$ such that $\hat{w}^i \neq \bar{w}^i$, $U^i(\hat{w}^i + \bar{w}^i)/2) > \bar{U}^i$: which contradicts the Pareto Optimality of $\{\bar{w}^i\}$. Hence there cannot be any other limit point and hence the claim follows. •

Finally, the convergence of prices is established in:

**Claim 4.4.18** $p(k) \to \bar{p}$ *such that* $[\bar{p}, \{\bar{w}^i\}]$ *constitute a no-trade Walrasian Equilibrium.*

Proof: We have already established that $\{w^i(k)\} \to \{\bar{w}^i\}$, a unique Pareto Optimal configuration. Define $\bar{p_j} = \bar{a}_j = \bar{b}_j$. We notice that due to the price adjustment equation, we have for all $k$, $p_j(k) \leq a_j(k)$; hence it follows that $\limsup p_j(k) \leq \bar{a}_j$. We first show that this must be satisfied with an equality.

Notice that for all $k$ large enough, we know that $\{\bar{e}^i(k)\}$ are transactions associated with some Dreze Equilibrium at $[p(k), \{w^i(k)\}]$; thus there are some constrains of the form $[c^i_j(k), C^i_j(k)]$ for every $i$ and all $j \neq n$. Suppose there is a subsequence $k_s$ such that for each $s$ along this subsequence, there is some $i$ such that $c^i_j(k_s) = \bar{e}^i_j(k_s) = w^i_j(k_s + 1) - w^i_j(k_s)$

that is, there is some subsequence $k_s$ along which the $j$-th market is sales rationed. Thus $a_j(k_s+1) \leq p_j(k_s) \leq a_j(k_s)$. Thus it follows $\lim p_j(k_s) = \bar{a}_j$.

Suppose there is no such subsequence, then $p_j(k+1) \geq p_j(k)$ for all $k$ large enough and hence is monotonic non-decreasing and bounded above must converge i.e., $\lim p_j(k) = p_j*$ for some $p_j*$. Also we know that $p_j(k+1) - p_j(k) \geq p_j(k+1) - b_j(k+1)$, since the market is purchase constrained, if at all, we have that $p_j(k) \leq b_j(k+1)$ and hence, $p_j(k+1) - p_j(k) \geq \eta_j D_j(k+1) \geq 0$ and since the left hand side goes to zero (given that the price sequence converges), it follows that $D_j(k) \to 0$; since $a_j(k) - p_j(k) = \eta_j D_j(k)$ it follows that $p_j* = \bar{a}_j$.

Thus $\limsup p_j(k) = \bar{a}_j = \bar{b}_j$. If possible, let $\liminf p_j(k) = \tilde{p}_j < \bar{a}_j$; then for $\epsilon > 0$ there is some subsequence $k_r$ such that $p_j(k_r) \in (\bar{a}_j - \epsilon, \bar{a}_j + \epsilon)$ while $p_j(k_r+1) \notin (\bar{a}_j - \epsilon, \bar{a}_j + \epsilon)$; since $\limsup p_j(k) = \bar{a}_j$, for $r$ large enough we must have $p_j(k_r+1) < \bar{a}_j - \epsilon < p_j(k_r)$: notice that this means that along the subsequence $(k_r)$, in the $j$-th market, the only possibility of a binding constraint is a sales constraint and hence by virtue of Claim 4.4.14, $D_j(k_r) \to 0$ and $D_j(k_r+1) \to 0$: thus $p_j(k_r+1) \to \bar{a}_j$: thus no such subsequence can exist and hence $\limsup p_j(k) = \liminf p_j(k) = \bar{a}_j = \bar{b}_j$ for all $j$. •

This concludes the proof of the Proposition 4.6.

## 4.5 An Appraisal of the Non-Tatonnement Processes

We began by considering the entire question of allowing transactions at disequilibrium prices. Several processes were considered. in the case of multiple markets, the problem of devising rules of transactions, when prices are not market clearing have been considered in detail. we began with the short-side rule which is the Hahn-Negishi restriction. We subsequently replaced that by the rationality requirement that no one should lose from taking part in disequilibrium transactions which was what Uzawa considered. For both of these cases, the convergence of prices, when price adjustment was proportional to excess demand could be guaranteed.

From the point of view of convergence alone, non-tatonnement processes seem to perform much better. What is significant is that we do not need to impose conditions such as gross substitutes or dominant diagonals any longer. The really surprising result has been the one in connection with a discrete process of trading and and price adjustment: we were able to guarantee convergence without imposing too many restrictions on the utility functions. If only one recalls our experience with a discrete tatonnement with a single market, one should appreciate the result in this connection.

The restriction on transactions were mainly that transactions should be **feasible, voluntary and exhaust all possibilities**. In particular, our demand for one round of transactions were even weaker since we lifted the last restriction. However this meant that we needed to restrict the transaction involving the numeraire: the fact that no one gives up the total stock of numeraire at any stage. Thus the problem of not having "money" at any stage was resolved, artificially by means of the consideration of the set $X$ and the condition C. Thus to take into account all of these problems, our convergence result will run into problems if people are able to unload their total stock of money.

There is another aspect of non-tatonnement processes which we need to consider. This is best discussed within the two-good example we have provided above. We reproduce the description of trades in the context a single market when the short-side of the market is non-empty:

$$t^i = \begin{cases} z^i(p, w^i) & \forall i \in S(p, \{w^i\}) \\ \delta^i z^i(p, w^i) & \text{otherwise} \end{cases}$$

where $\delta_i$ are non-negative and such that $\sum_i t^i = 0$; notice that these trades are not unique. To elaborate, suppose that in this market there is excess demand of 100 units at the current price; suppose also that in this situation there are some people who wish to sell this commodity and the total amount that some persons would like to sell is 50 units; this means that the remaining would want to purchase 150 units, since over all the excess demand is of 100 units. Now the rule would say that those who want to sell, do indeed manage to sell. But then for individuals on the long side, who get to buy and how much is not specified. The numbers $\delta_i$ could be redefined in a variety of way and each would be an admissible set of transactions. Thus given that there are many ways of defining transactions, the resulting equilibrium which will be approached will also vary. Thus the equilibrium approached depends on the actual transactions carried out.

Originally, the investigation of whether an equilibrium was approached or not was supposed to answer another line of query. This was the method of comparative statics: if equilibrium was stable, then changes in equilibrium values due to changes in parameters were meaningful. Since then the new values would in fact be attained. But if we do not really know what equilibrium would be attained then comparing equilibrium values may not make too much sense.

Notice also the aspect of production: in fact, we have exclusively considered exchange models and assumed that there is no production and consumption along the process of adjustment. It is as if, producers first decide on how much of the various goods to produce based on expectations of what prices will be and then bring their goods to the market. Consumers may have received an advance on their sale of labor and other factors and decide in turn on what to consume and during this stage, decide to on what to purchase. Basically it is this stage that we have tried to describe.

When we bring in the treatment of actual time, a time for production, a time for transaction and so on, we need to handle questions of a somewhat different nature. If one were to allow for people to take decisions during one period which will affect outcomes in another, we need to also study the notion of equilibrium over time. We shall turn to the resultant dynamics of such considerations, next.

# 5 Growth Processes

## 5.1 Introduction

We consider in the present chapter, equilibrium over time; the idea being that whatever decisions we take today should influence what decisions we can take tomorrow so that there is another dimension of the competition for resources which we have not considered so far: do we use up the resources today and leave nothing for the future or do we follow a more prudent path. These types of issues are at the core of macroeconomics and naturally, as should be clear, are essentially dynamic in nature. There are three basic types of growth models that have been considered in the literature. I refer here exclusively to what has been called the 'neoclassical' literature. The first type has been called descriptive and owes its origin to the contribution of Solow (1956) and Swan (1956). Although it has been pointed out that there are features of such a construction which tend to be regarded as unsatisfactory[1], we shall begin with such models. Among the remaining two sets of models, derives savings behavior from maximizing behavior over the consumer's lifetime. The optimal economic growth model owes its origin to Ramsey (1928) and subsequently refined by Cass (1965) and Koopmans (1965) and derives savings behavior on the basis of the assumption that households live forever[2]; the third type of model, the overlapping generations model of growth originates from the Samuelson (1958) and Diamond (1965)

[1]See for instance, Azariadis (1993), p.4, " In present terminology, Solow made an ad-hoc assumption - and there are few sins as grave as this for a self-respecting economist."

[2]This is found eminently acceptable and a great improvement over the assumption of constant savings behavior.

contributions and the simplest version assumes that each decision maker lives for exactly two periods. We shall take them up for consideration one by one. Some times, in addition to this classification, growth models are classified according to whether there are many goods or not. Thus one talks of one sector (where there is a single good which can be stored and consumed), two sectors and multi-sector growth models. We begin with the simplest, one sector, descriptive growth model.

There is a single good which can be produced, consumed and invested; its production requires, inputs of Labor employed ($N_t$) and Capital ($K_t$); the latter is just the good itself from previous periods. At time $t$, the output of this good $Y_t$ may be written as $Y_t = F(K_t, N_t)$, where the function $F(,)$ is assumed to be continuously differentiable satisfying constant returns to scale, with positive marginal products and diminishing returns to each factor. Thus $F(\lambda K, \lambda N) = \lambda F(K, N); F_i > 0; F_{ii} < 0, i = K, N$. In such circumstances, it is more convenient to work with the per-capita terms defined as follows:

$$y_t = Y_t/N_t \ ; \ k_t = K_t/N_t$$

for then we have that $y_t = F(1, k_t) \equiv f(k_t)$ and hence it is straightforward to check that $F_N = f(k) - kf'(k) \ ; \ F_K = f'(k)$. To avoid problems, we shall also make use of the Inada conditions[3]:

$$\lim_{k \to 0} f'(k) = +\infty \text{ and } \lim_{k \to +\infty} f'(k) = 0$$

Investment $I_t$ is by definition, $\dot{K_t}$[4]; in contrast to employment $N_t$, the supply of labor

[3]These may be identified as boundary conditions which we have encountered in the earlier chapters.

[4]We assume that there is no depreciation in capital.

$L_t$ is assumed to grow at an exogenously specified exponential rate:

$$L_t = L_o e^{nt} \tag{5.1}$$

Further there are two types of agents, those who earn wages by selling their labor and those who earn profits; these two types of agents have differing behavior (utility functions) resulting in different consumption patterns. We capture this aspect by postulating that each type consumes a fixed portion of their income $1 - s_w, 1 - s_p$ being respectively the consumption propensities out of wages and those out of profits, respectively. In other words, $s_w, s_p$ denote the savings propensities, assumed constant. Thus if $w_t$ denotes the wages (in terms of the good being produced), then consumption by wage earners amount to $(1 - s_w)w_t N_t$ while the consumption by profit earners amount to $(1 - s_p)(Y_t - w_t N_t)$ and thus aggregate consumption is given by

$$X_t = (1 - s_w)w_t N_t + (1 - s_p)(Y_t - w_t N_t)$$

Thus we have

$$X_t = (1 - s_p)Y_t + (s_p - s_w)w_t N_t \tag{5.2}$$

Notice that in case the saving propensities match, i.e., $s_w = s_p = s$, aggregate consumption reduces to $(1 - s)Y_t$. In any case, we shall assume that $s_p \geq s_w$.

Also in the single produced good market, demand consists of consumption needs $X_t$ and investment requirements $I_t$; thus equilibrium in each period $t$ requires

$$Y_t = X_t + I_t$$

which in turn, taking into account the facts mentioned above, implies the following:

$$\dot{K}_t = s_p Y_t - (s_p - s_w) w_t N_t \tag{5.3}$$

This adjustment of capital stock, ensuring that demand and supply match over all periods will be the focus of our enquiry in the current chapter.

## 5.2 Full Employment

With full employment of labor ensured, $N_t = L_t$, we can replace $N_t$ by $L_t$ in all of the above equations; thus, for instance, we have $k = K/L$ in this section; in addition, we know also that given $K_t$, $w_t$ must be such that labor demand should be $L_t$ or that

$$w_t = F_L(K_t, L_t) = f(k_t) - k_t f'(k_t) \tag{5.4}$$

Notice that by virtue of our assumptions, it is clear that $f(k) - kf'(k)$ is monotonically decreasing function. Given $K_t$ and $L_t$, $k_t$ is determined and this determines $w_t$ uniquely. With this clarification, note that the equilibrium over time equation (or the equation adjusting capital stock) may be re-written in per-capita terms as follows:

$$\frac{\dot{K}_t}{L_t} = s_p f(k_t) - (s_p - s_w)(f(k_t) - k_t f'(k_t)) = s_w f(k_t) + (s_p - s_w)(k_t f'(k_t)$$

Since we have $k = K/L$, $\dot{k} = \dot{K}/L - k\dot{L}/L = \dot{K}/L - kn$ and hence we have:

$$\dot{k}_t = s_w f(k_t) + (s_p - s_w) f'(k_t) - nk_t \tag{5.5}$$

or we may rewrite it as

$$\frac{\dot{k}_t}{k_t} = \frac{s_w f(k_t)}{k_t} + (s_p - s_w) f'(k_t) - n \text{ (say } = \theta(k_t) - n) \tag{5.6}$$

We shall analyze the behavior of the solution to this equation next.

The equilibrium for (5.6) is provided[5] by $k^\star$ such that $\theta(k^\star) = n$; we are sure that there is a unique equilibrium since, notice that $\theta'(k) = s_w \dfrac{kf'(.) - f(.)}{k^2} + (s_p - s_w) f''(.) < 0$ and by virtue of the Inada conditions, $\lim_{k\to 0} \theta(k) > n$ whereas $\lim_{k\to\infty} \theta(k) < n$. Thus, we have also that $k < k^\star \Rightarrow \theta(k) > n$ while for $k > k^\star \Rightarrow \theta(k) < n$. Not only does this imply that we have such a $k^\star$, but we may show that, beginning from any arbitrary $k_o$, the solution $\phi_t(k_o)$ to (5.6) converges to $k^\star$.

Consider, for example $V(t) = (\phi_t(k_o) - k^\star)^2$; we have just shown that[6] $\dot{V}(t) = 2(\phi_t(k_o) - k^\star)\dot{k} < 0$ unless $\phi_t(k_o) = k^\star$: thus $V(t)$ is a Liapunov function for system (5.6). In fact, plotting $\theta(k)$ will give us the above result as well:

Figure 1: Convergence to Steady State

Thus we have the following:

**Proposition 5.1** *Given full employment of labor, there is an unique $k^\star$ at which $\dot{k}$ is zero and moreover, beginning from any initial configuration, the solution will approach $k^\star$ over time.*

$k^\star$ is called the balanced growth configuration since when $K/L = k^\star$, $K$ must be growing at the same rate as $L$ to maintain the constancy of the capital-labor ratio. This is an optimistic result; since, it seems to suggest that initial conditions do not matter as over time, $k^\star$ will be approached, regardless of initial conditions. Consequently, regardless of the past, differences

[5]When $s_p = s_w = s$, $k^\star$ is determined by the equation $f(k) = nk$: this is the Solow (1956) case.

[6]Recall that since $\phi_t(k_o)$ is the solution to (5.6), $\dot{\phi}_t(k_o) = \dot{k}$.

will get eroded and ultimately, an economy would be able to achieve balanced growth which will be tied down by $K, Y$ growing at the rate of growth in population[7].

## 5.3 Unemployment

The story in the last section although optimistic may not have been very convincing and among the many restrictions imposed, we need to weaken some of them to get at a more realistic description of the growth path. The first and foremost that comes to mind is the consideration of unemployment. Recall that when we introduced the basic single good model, we had kept the notion of employed labor $N_t$ separate from the available labor $L_t$ and we need to restore that; clearly then, the wage rate $w_t$ should determine the level of employment $N_t$; but what determines the wage rate then ? Writing the employment ratio $N/L$ as $v$, a standard way of approaching this problem is to require that[8]

$$\frac{\dot{w}}{w} = \psi(v) \tag{5.7}$$

where $\psi'(v) > 0;$ further $\psi(0) < 0, \psi(1) > 0$. We shall of course take $\psi(.)$ to be a **continuously differentiable function**; hence there is a unique $v^\star < 1$ such that $\psi(v^\star) = 0$[9].

---

[7] Usually, in the context of there being a depreciation in the capital stock over time, say at a constant rate $\delta > 0$, $I_t = \dot{K}_t - \delta K_t$ and hence at the steady state $k^\star$, we should have $\theta(k^\star) = n + \delta$. Thus $K, Y$ will have to grow at the rate of $n + \delta$ in the steady state.

[8] See, for instance, Goodwin (1967), Akerlof and Stiglitz (1969), Rose (1967). Some version of such an adjustment, the Phillips curve, has been often employed.

[9] $1 - v^\star$ has some times been referred to as the natural rate of unemployment; see, for instance Akerlof and Siglitz (1969).

### 5.3.1 Capital-output Ratio and Employment-Capital Ratio: Constant

Consider the situation when $N_t/K_t = b$ and $Y_t/K_t = C$ where $b, C$ are constants[10]. First of all note that $v = bk$ where $k = K/L$, if there is unemployment; otherwise $v = 1$. We may thus write $v = \min\{bk, 1\}$. Recall the equation (5.3):

$$\dot{K} = s_p Y_t - (s_p - s_w) w_t N_t$$

Note that

$$\frac{\dot{K}}{L_t} = s_p \frac{Y_t}{L_t} - (s_p - s_w) w_t \frac{N_t}{L_t} = s_p C k - (s_p - s_w) w_t v_t$$

Since we have $\dot{k} = \frac{\dot{K}}{L} - k\frac{\dot{L}}{L}$, we have:

$$\dot{k} = s_p C k - (s_p - s_w) w v - nk \tag{5.8}$$

Thus the system of equations, we need to analyze are given by the equations (5.7) and (5.8). We shall write the system as follows:

$$\dot{k} = k\{(s_p C - n) - (s_p - s_w) w \min(b, 1/k)\} \text{ and } \dot{w} = w\psi(\min(bk, 1))$$

The only possible non-trivial equilibrium for the system is given by $k^\star = v^\star/b, w^\star = \frac{s_p C - n}{b(s_p - s_w)}$. It is to be noted that the nature of the terms $\min\{b, 1/k\}$ would complicate matters[11] Accordingly, we rewrite the system in the region $k < 1/b$ first:

$$\begin{pmatrix} \dot{k} \\ \\ \dot{w} \end{pmatrix} = \begin{pmatrix} k\{\alpha - \beta w\} \\ \\ w\{\theta(k) - c\} \end{pmatrix} \tag{5.9}$$

[10] We are using the assumptions employed by Akerlof and Stiglitz (1969).

[11] This is on account of the fact that these terms lack differentiability at the switch points; we shall argue that these switch points are not encountered. In contrast to the local considerations in Akerlof and Siglitz (1969), we shall attempt a global analysis below.

where $\alpha = (s_p C - n), \beta = (s_p - s_w)b$ are both taken to be positive; in addition, we define $\theta(k) \equiv \psi(bk) + c$ where $c$ is a positive constant. In particular, we shall choose $c = -\psi(0) > 0$; with such a choice, we ensure that $\theta(0) = 0$ and that $\theta'(0) = \lim_{x\to 0} \frac{\theta(x)}{x} > 0$ and is well defined.

It should be noted then that $\theta'(k) > 0$ and the system (5.9) is a Lotka Volterra type equation[12] except for the equation defining $\dot{w}$ where the function $\theta(k)$ need not be of the form $\gamma k + \delta$ with the coefficients constant. However, we shall show that even for this form, the conclusions of the Lotka-Volterra equations remain valid[13].

The next thing that we need to ensure is that the trajectory or solutions remain bounded. We take up these considerations one by one. We consider, first of all, the question of determining the nature of the trajectories of the system (5.9). For this purpose, we introduce the function:

$$\Phi(x) = \int_0^x \frac{\theta(s)}{s} ds$$

where $\theta(.)$ is as above. Some properties of the function $\Phi(.)$ are immediate:

**Claim 5.3.1** $\Phi(x)$ *is well defined for all* $x \geq 0$*;* $\Phi(0) = 0; \Phi'(x) = \frac{\theta(x)}{x}, x \neq 0.$

We note next

**Claim 5.3.2** *Let* $x^\star$ *be such that* $\theta(x^\star) = c$*. Then the function* $\Psi(x) \equiv \Phi(x) - c \log x$ *attains a global minimum at* $x^\star$*; further,* $\Psi(x) \to +\infty$ *as* $x \to 0$*.*

[12] See, Section 1.5.1.

[13] It is in this sense that our results provided below are stronger than the original Akerlof-Stigliz (1969) formulation; while their conclusion were based on the linear approximation to (5.9), we shall work with the equations themselves.

Proof. Notice that $\Psi'(x) = \dfrac{\theta(x) - c}{x}$ and $\Psi''(x) = \dfrac{x\theta'(x) - \theta(x)}{x^2} + \dfrac{c}{x^2}$; consequently it follows that at $x^\star$, $\Psi'(x^\star) = 0; \Psi''(x^\star) > 0$: thus $x^\star$ is a point of regular local minimum for the function $\Psi(x)$. Suppose, if possible there is $\bar{x} \neq x^\star$ such that $\Psi(\bar{x}) < \Psi(x^\star)$. For the sake of definiteness, let $\bar{x} > x^\star$; since $x^\star$ is a point of regular local minimum it follows that for some $\delta > 0$, $\Psi(x^\star + \delta) > \Psi(x^\star) > \Psi(\bar{x})$ hence for some $\hat{x} \in [x^\star + \delta, \bar{x}], \Psi(\hat{x}) = \Psi(x^\star)$; hence for some $x \in [x^\star, \hat{x}], \Psi'(x) = 0$ or $\theta(x) = c$ with $x > x^\star$, which contradicts the strictly increasing nature of the function $\theta(.)$. From the definition of the function, it is immediate that as $x \to 0$, $\Psi(x) \to +\infty$. $\bullet$.

Now consider the function $V(k, w) = \Psi(k) + \{\beta w - \alpha \log w\}$; by virtue of the above, we should note the following features of this function:

- Along any solution to (5.9), $\dot{V}(k, w) = \Psi'(k).\dot{k} + (\beta w - \alpha).\dot{w}/w = (\theta(k) - c)\dot{k}/k + (\beta w - \alpha)\dot{w}/w = 0$.

- Thus if the solution has $(k^o, w^o)$ as initial point, and writing the solution to ( 5.9) as $(k_t, w_t)$, we have, $V(k_t, w_t) = V(k^o, w^o)$ for all $t \geq 0$.

- The function $V(k, w)$ attains a unique minimum at $k^\star, w^\star$. This follows by virtue of the property of the function $\Psi(x)$ noted above and by virtue of the fact that $\beta w - \alpha \log w$ is a strictly concave function.

- Consequently, if the initial point differs from $(k^\star, w^\star)$ no solution can approach the equilibrium.

- Along any solution, $(k_t, w_t) > (0, 0)$ if $(k^o, w^o) > (0, 0)$. This follows since the axes

$k = 0, w = 0$ are trajectories for the system (5.9) and two distinct trajectories cannot cross.

- Along any solution $(k_t, w_t)$ $w_t$ remains bounded above: since if $w_t \to +\infty$, $\beta w - \alpha \log w \to +\infty$: for if not, then there is some $M$ such that $\beta w^\star - \alpha \log w^\star \leq \beta w_t - \alpha \log w_t \leq M$ or we have $0 < \beta \leq M/w_t + \alpha(\log w_t)/w_t$; notice that the term on the right tends to zero as $w_t \to +\infty$: which is a contradiction; hence no such number $M$ exists. Thus if $w_t \to +\infty$, $V(k_t, w_t) \to +\infty$ but we know that $V(k_t, w_t)$ remains constant and hence $w_t$ must remain bounded.

- Recall that by definition $k_t = \dfrac{K_t}{L_t} = \dfrac{K_t}{N_t} . \dfrac{N_t}{L_t} \leq 1/b$.

Thus, any solution to the system (5.9),$(k_t, w_t)$ remains bounded and bounded away from the axes and further, $k_t \leq 1/b$ for all $t$; thus the $\omega$-limit set is non-empty; notice too that from the properties of the function, $V(k, w)$, that no equilibrium can belong to this set and consequently, by the Poincaré-Bendixson Theorem[14], the $\omega$-limit set set must be a closed orbit. Hence there are two possibilities: either we have a limit cycle or the entire trajectory it self is a closed orbit. Thus the situation is exactly similar to the one encountered in the proof of Claim 1.5.5 and hence by following the same line of argument, one may conclude that there cannot be a limit cycle. Thus the entire discussion may be summarized thus[15]:

**Proposition 5.2** *For the system (5.9), any trajectory beginning from $k^o, w^o$, both positive,*

[14]See Proposition 1.4.

[15]This is the general form of the result in Section III of Akerlof and Stiglitz (1969), where "there is no choice of technique" and in which, "wages adjust to their equilibrium value with a lag".

*with $k^o < 1/b$ is a closed orbit around the equilibrium $(k^\star, w^\star)$, with $k_t < 1/b$ for all $t$.*

The cyclical behavior around equilibrium was explained in terms of the the relationship (5.7) where high levels of employment are seen to lead to high wages; but this also implies that average savings rate for the economy must have declined if wage-earners save less than profit-earners; this lowering of the savings rate leads to fewer jobs in the future and hence an increase in the rate of unemployment. This sets up a movement in the reverse direction with lower wages and so on. Consequently the predator-prey relationship between wages and the capital-labor ratio was used to explain the fluctuation of rates of unemployment around some equilibrium value.

### 5.3.2 The Goodwin Growth Cycle

The Goodwin (1967) contribution was perhaps one of the more influential studies on growth theory. The model was close to the model we analyzed in the last section. First we relax the assumption of "no choice of technique" which had frozen the ratio $N_t/K_t$; we maintain the assumption of a fixed capital-output ratio $Y_t/K_t = C$; the new feature is the presence of technical progress through growth of labor productivity $a_t = Y_t/N_t$ given by $a_t = a_o e^{\alpha t}$; further it is assumed that $s_w = 0, s_p = 1$: thus all wages are consumed and all profits are invested. As before, the labor force $L_t$ grows a steady rate of $n$ per period so that $L_t = L_o e^{nt}$. Instead of our earlier variables, Goodwin conducts the entire analysis in terms of the variables $u, v$ where $u$ is the share of workers in total product $u = \frac{wN}{Y} = \frac{w}{a}$ and $v$, as before is the employment ratio $v = \frac{N}{L}$. Finally, the wage rates are governed by a Phillips curve type relation, we have used before, viz., (5.7).

First of all note that profits in this set-up amount to $Y_t - w_t L_t$ account for the total amount of investment given our assumption on savings propensities. Thus we have:

$$\dot{K}_t = \{Y_t - w_t L_t\} = \{1 - \frac{w_t}{a_t}\}Y_t \tag{5.10}$$

Given the fact that the capital-output ratio is constant, we must have $\dot{Y}/Y = \dot{K}/K$ and from the definition of labor productivity, we must have $\dot{Y}/Y - \dot{N}/N = \alpha$ so that using the equation (5.10), we have $\dot{N}/N = \dot{Y}/Y - \alpha = \dot{K}/K - \alpha = \{1 - w_t/a_t\}C - \alpha = \{1 - u_t\}C - \alpha$. Now since by definition, once again, $\dot{v}/v = \dot{N}/N - \dot{L}/L$, we have the following:

$$\dot{v}/v = C(1 - u_t) - (\alpha + n) \tag{5.11}$$

and finally using the the Phillips curve relation (5.7), in conjunction with the definition of share of workers in total product, we have

$$\dot{u}/u = \dot{w}/w - \dot{a}/a = \psi(v_t) - \alpha \tag{5.12}$$

The system which Goodwin analyzes is made up of equations (5.11) and (5.12); in particular, he considers a linear approximation for the function $\psi(v)$. However, there is no need to consider the linear approximation to the function since the methods adopted to analyze the system made up of (5.9) may be used once more to arrive at the same conclusion[16]:

[16] A comparison of the two systems will show that they are exactly identical and the same method of proof may be used; consequently, we do not provide the proof in this case.

**Proposition 5.3** *The solution to the system made up of (5.11) and (5.12),* $(u_t, v_t)$ *from some arbitrary initial point* $(u^o, v^o) > (0, 0)$ *is a closed orbit around the equilibrium* $(u^\star, v^\star)$*, where* $u^\star = \dfrac{C - (\alpha + n)}{C}$ *and* $v^\star$ *satisfies* $\psi(v^\star) = \alpha$[17].

The above model generated great interest since it describes a scenario where two ratios $u, v$ (respectively, the share of workers and the employment ratio) are chasing one another along a closed orbit. To see what this has to say about Goodwin's model, we need to note a few things first:

$$u = \frac{w}{a} : \text{share of workers; } 1 - u\text{: share of capitalists}$$

Thus the rate of profit is given by $\frac{1-u}{\sigma}$ which, given the assumptions made above is also the rate of growth (i.e., $\frac{\dot{Y}}{Y}$). It should be clear that along any solution to the Goodwin equations, the variables $u, v$ will fluctuate between limits say $u_{min} \leq u \leq u_{max}$ and $v_{min} \leq v \leq v_{max}$. In the words of Goodwin (1967), one may note the following: ".. when profit is at its greatest ($u = u_{min}$), employment is average ($v_{min} < v < v_{max}$) and the high growth rate pushes employment up to its maximum ($v = v_{max}$), which squeezes the profits to its average level ($u_{min} < u < u_{max}$). The deceleration of growth lowers employment.....The improved profitability carries the seed of its own destruction be engendering a too vigorous expansion of output and employment.."[18].

In the last section, the same cyclical behavior was modelled in terms of $w, k$ chasing one

[17] Recall that $\psi(0) < 0 < \psi(1)$; consequently for such a solution to exist, we need to ensure that $\alpha < \psi(1)$; further for $u^\star$ we require $C > \alpha + n$.

[18] See Goodwin (1967), pp. 57-8. According to Goodwin, this is what Marx's idea of "contradictions of capitalism" was all about.

another; this was made possible by virtue of the assumption of fixed methods of production; this meant that the share of workers $u = wN/Y = wN/K.K/Y = wb/C$ so that $u$ was a constant multiple of $w$; similarly, $v = N/L = N/K.K/L = bk$. Thus the approach, in the last two sections have been quite similar. In contrast to the earlier section, the fixed $N/K$ ratio has been discarded in the present context, of course but the conclusions have not really differed. The type of cyclical behavior studied in the last two sections will be referred to as Goodwin Cycles. We shall examine, next, how robust this explanation has been.

### 5.3.3 Robustness of Goodwin Cycles

To enable us to relate to results of the Section 1.6.1, let us as first step consider the function $\psi(v)$ to be specified by $\psi(v) = \rho v - \gamma$ where $\rho, \gamma$ are positive constants. With this specification, we revert back to the original set of Goodwin equations:

$$\begin{pmatrix} \dot{v} \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} v\{C(1-u) - (\alpha + n)\} \\ \\ u\{\rho v - (\gamma + \alpha)\} \end{pmatrix} \tag{5.13}$$

This of course is the standard form of the Lotka-Volterra equations and as we had demonstrated in Section 1.6.1, a slight perturbation to the model destroys the periodic behavior. For example, if instead of the standard Phillips curve in (5.7), we were to consider that $\dot{w}/w = \psi(v) + \tau u$ where $\tau$ is a constant and is **small in magnitude** then the resulting equations would be a perturbation of the Goodwin model[19]. Consider next the system

[19]Such formulations ar not implausible and have been considered in the literature; see, for instance, Lorenz (1993), p. 70. This says that the rate of change in real wages depends not only on the level of employment ($v$) but also on the share of workers ($u$).

(5.13) in the light of this adjustment:

$$\begin{pmatrix} \dot{v} \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} v\{C(1-u) - (\alpha + n)\} \\ \\ u\{\rho v + \tau u - (\gamma + \alpha)\} \end{pmatrix} \tag{5.14}$$

Using the terminology of Section 1.6.1, the NTE (Non-Trivial Equilibrium) for the above system is given by[20]

$$u^\star = 1 - \frac{\alpha + n}{C},\ v^\star = \frac{\gamma + \alpha - \tau u^\star}{\rho}$$

At the NTE, the Jacobian of the system is given by:

$$\begin{pmatrix} 0 & -Cv^\star \\ \\ \rho u^\star & u^\star \tau \end{pmatrix}$$

Consequently, we have:

**Claim 5.3.3** *The NTE is locally asymptotically stable (a sink) when $\tau < 0$; for $\tau > 0$, the NTE is unstable (a source). Further, the real part of the characteristic root of the Jacobian of (5.14) is given by $u^\star\tau/2$; and $\tau = 0$ is a point of Hopf Bifurcation for the system.*

And using methods similar to the ones employed in Section 1.6.1, one may show:

**Proposition 5.4** *Any solution $(u_t, v_t)$ to the system (5.14) with $(u_o, v_o) > (0, 0)$ as an initial point converges to the NTE whenever $\tau < 0$; if $\tau > 0$ the only solution which is bounded is the one originating from the NTE.*

Thus for cyclical behavior in a system such as (5.14), $\tau = 0$ is necessary and sufficient. We next examine whether the approximation of the function $\psi(v)$ is essential to reach the

[20] We are of course assuming that such a solution exists in the positive quadrant with $0 < u^\star, v^\star < 1$; in addition, we are assuming that the coefficient $\tau$ is small in magnitude.

above conclusions. In other words, if we maintain that original formulation of the Phillips curve in (5.7), viz., $\psi(v)$ is an increasing function with $\psi(0) < 0 < \psi(1)$; we shall thus write, as before $\psi(v) = \theta(v) - \gamma$ where $\gamma = -\psi(0) > 0$ so that $\theta(v) = \psi(v) - \psi(0)$ and hence $\theta(0) = 0$; with these modifications, the system (5.14) becomes:

$$\begin{pmatrix} \dot{v} \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} v\{C(1-u) - (\alpha + n)\} \\ \\ u\{\theta(v) + \tau u - (\gamma + \alpha)\} \end{pmatrix}$$

or writing $C - (\alpha + n) = A$, we have:

$$\begin{pmatrix} \dot{v} \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} v\{A - Cu\} \\ \\ u\{\theta(v) + \tau u - (\gamma + \alpha)\} \end{pmatrix} \tag{5.15}$$

Now consider the function:

$$W(u, v) = \{Cu - A \log u\} + \{\int_0^v \frac{\theta(s)}{s} ds - \sigma \log v\}$$

where $\sigma = \gamma + \alpha - \frac{\tau A}{C}$ and is a constant. Then along any solution to (5.15), we have:

$$\dot{W} = \{Cu - A\}\frac{\dot{u}}{u} + \{\theta(v) - \sigma\}\frac{\dot{v}}{v}$$

$$= \{Cu - A\}[\theta(v) + \tau u - (\gamma + \alpha) - \theta(v) + \sigma] = \tau\{Cu - A\}^2/C$$

Consequently, it follows that whenever $\tau < 0$, $\dot{W} < 0$ along any solution to (5.15) unless $Cu = A$; we may therefore conclude, as in the proof of Claim 3.6.8, that $W(u, v)$ is a Liapunov function for (5.15) and the solution converges to the non-trivial equilibrium $u^\star = A/C; v^\star = \theta^{-1}(\gamma + \alpha - \tau u^\star)$[21]. We may thus summarize our findings in the following:

[21]The boundedness of the solution follows from the definition of the variables $u, v$; these are both fractions and cannot exceed 1; in addition any solution originating from an initial positive configuration cannot approach the axes since the axes $u = 0, v = 0$ are trajectories or solutions. Finally, it should also be pointed out that the function $W(.)$ is a strictly concave function of $(u, v)$ and attains a global minimum at $(u^\star, v^\star)$.

**Proposition 5.5** *Any solution* $(u_t, v_t)$ *to the system (5.15) with* $(u_o, v_o) > (0, 0)$ *as an initial point converges to the NTE whenever* $\tau < 0$*; if* $\tau > 0$ *the only solution which is bounded is the one originating from the NTE.*

Thus the conclusions of Proposition 5.4 are independent of the approximation of the function $\psi(v)$; with a more general form too, identical conclusions may be obtained without imposing any additional restrictions[22].

Before passing on to related considerations, it should be pointed out that possibilities of convergence in Goodwin type models have been noted in the literature. An exercise due to Flaschel (1984) may be reported to indicate differences from the above approach. Flaschel considers the following variation to the Goodwin basic model:

$$\frac{\dot{w}}{w} + \eta\pi = \psi(v) \ , \ \psi' > 0$$

where $\pi = g((1+r)u - 1)$ with $g' > 0, g(0) = 0$; all other aspects are as above, except for the fact that:

$$\frac{\dot{K}}{K} = s(u)Y/K = s(u)C \ , \ s' < 0$$

Thus, in contrast to what we have described above, the Phillips curve (5.7) has been adjusted for "money illusion" so that when $\eta > 0$ ($< 0$) workers receive a lower (respectively, higher) real wage than they bargained for together with an equation which describes how the rate of inflation $\pi$ is formed via a mark-up process. Finally the investment or accumulation equation is a straightforward generalization from the constant savings rate assumption.

[22] For an alternative approach, via the Dulac's Criterion, see below.

Combining these, we have the following system:

$$\begin{pmatrix} \dot{v} \\ \\ \dot{u} \end{pmatrix} = \begin{pmatrix} v\{Cs(u) - (\alpha + n)\} \\ \\ u\{\psi(v) - \alpha - \eta g((1+r)u - 1)\} \end{pmatrix} \tag{5.16}$$

The difference between (5.15) and the above is easy to spot: first of all notice that (5.15) is a particular case of (5.16) where the functions $s(.), g(.)$ have been taken to be linear. For the system (5.16), the following result is claimed: *Assume that the Jacobian $J = (J_{ik})$ of the system (5.16) satisfies*

- *trace $J < 0$,*
- *det $J > 0$ and*
- $J_{12}.J_{21} \neq 0$

*everywhere on $\Re^2_+$, then the equilibrium $(u^\star, v^\star)$ is asymptotically stable in the large.* Now $J$ is given by the following matrix:

$$\begin{pmatrix} Cs(u) - (\alpha + n) & vCs'(u) \\ \\ u\psi'(v) & \{\psi(v) - \alpha - \eta g((1+r)u - 1)\} - u\eta g'(.)(1+r) \end{pmatrix}$$

It should be easy to see that at the equilibrium all the conditions mentioned above are satisfied; notice too that the only condition which is easily seen to hold all over the non-negative orthant is the requirement on the off-diagonal terms, given the sign restrictions on the derivatives of the functions $\psi, s$; it is not at all clear how the other two requirements on the trace and the determinant are going to be met on the entire non-negative orthant.

However **without imposing any of the above requirements**, notice that one may claim :

**Claim 5.3.4** *For the system (5.16), there can be no cyclical orbits in the positive orthant, given the sign restrictions on the derivatives of the functions* $g(.)$.

Proof: Consider the function $\theta(v, u) = \frac{1}{uv}$ and then consider:

$$\frac{\partial\{\theta(v,u)v[Cs(u) - (\alpha + n)]\}}{\partial v} + \frac{\partial\{\theta(v,u)u[\psi(v) - \alpha - \eta g((1+r)u - 1)]\}}{\partial u}$$

$$= 0 - \frac{\eta g'(.)(1+r)}{v} < 0$$

on the positive quadrant: hence by Dulac's Criterion, there can be no cycles in the positive orthant. •

Thus for convergence to equilibrium from any initial positive configuration, one may note that additionally we need to show that the solution is bounded and one may use the Poincaré-Bendixson Theorem to complete the demonstration. It may be recalled that the variables $u, v$ are, by definition, fractions and cannot exceed unity; consequently the bounded nature of the solution should follow from the model itself.

## 5.4 A Digression: A General Lotka-Volterra Model

We have seen that a Lotka-Volterra Model has made its appearance from time to time in various contexts; the original form of the Lotka Volterra Model studied in Sections 1.5.1 and 1.6.1 used specific forms for the pair of differential equations; we saw in the last section, that replacing these forms by more general forms often did not change the nature of results. We attempt here to provide a very general treatment of Lotka-Volterra Models and identify conditions for periodic behavior and convergence. Hopefully, apart from being of independent interest, they might be of specific interest to the situations we have been analyzing at present.

The basic feature of the predator-prey model or the Lotka-Volterra models is the following pair of functions: let $M, N$ be two functions $M, N : \Re_+ \times \Re_+ \to \Re$, $M, N \in \mathcal{C}^1$ satisfying the following:

**P1** $M(0,0) > 0 \geq N(0,0)$

**P2** $M_x(x,y) \leq 0, M_y(x,y) < 0; N_x(x,y) > 0, N_y(x,y) \leq 0$ , $\forall (x,y) \in \Re_+ \times \Re_+$.[23]

A pair of functions $M, N$ satisfying the above two conditions define a Generalized Lotka-Volterra System (GLVS) given by:

$$\dot{x} = xM(x,y) \text{ and } \dot{y} = yN(x,y) \tag{5.17}$$

The interpretation of the above system is as follows: there are two species in a particular environment, one of which preys on the other (the predator and the prey). The predator requires prey in order to subsist while the prey can live off the environment; denoting the population of the prey be denoted by $x$ while that of the predator by $y$; thus the rates of growth of the population of the species are related as follows: for the prey, the greater the population of the predators, the lower is the prey's rate of growth other things being equal ($M_y < 0$); and since the environment too is limited in some way, the rate of growth of the prey, given any fixed level of the population of predators, is decreasing, if at all, in its own population level ($M_x \leq 0$). For the predator, on the other hand, the rate of growth of its

[23]A subscript indicates a partial derivative with respect to the variable in the subscript. Thus $M_x(x,y)$ is the partial derivative of the function $M$ with respect to the variable $x$.

population increases with the population of the prey and decreases with its own population, other things remaining the same. This is the rationale for **P2**. The rationale for **P1**, follows from the above. We call the system (5.17)[24]. We begin my noting that for the system (5.17), given **P1 & P2**, the following are the types of equilibria[25]:

- No Species $E_1(0,0)$.
- No Predator $E_2(\hat{x},0), \hat{x} > 0$.
- Both Species $E_3(\hat{x},\hat{y}) > (0,0)$.

Consider for example the following specifications of the functions $M, N$:

$$M(x,y) = a - \alpha x - by \text{ and } N(x,y) = dx - \beta y - c \tag{5.18}$$

where $a, b, c, d > 0, \alpha, \beta \geq 0$. It is immediate to note that

**Claim 5.4.5** *For the above specification of the functions $M, N$, $E_2, E_3$ both exist if and only if $\alpha > 0; \alpha c < ad$. If $\alpha = 0$, $E_3$ exists but $E_2$ does not; while if $\alpha > 0$ but $\alpha c > ad$ $E_2$ exists but $E_3$ does not.*

Consider, next, the following forms of the functions $M, N$

$$M(x,y) = \frac{1+ax}{1+x} - by \text{ and } N(x,y) = \frac{1+dx}{1+x} - \beta y - c \tag{5.19}$$

[24]Since the standard form of the Lotka-Volterra system or the Predator-Prey Model is given by the following: $M(x,y) = a - by$ and $N(x,y) = dx - c$ with $a, b, c, d > 0$: see, for instance Section 1.5.1.

[25]A fourth possibility with only predators $(0,\hat{y})$ cannot be an equilibrium since that would require $N(0,\hat{y}) = 0$ and this would contradict our assumptions.

where $0 < a < 1, b > 0, d > c > 1, \beta > 0$. Note that $M(0,0) = 1; N(0,0) = 1 - c < 0; M_x < 0, M_y < 0, N_x > 0, N_y < 0$.

**Claim 5.4.6** *For $M, N$ given by (5.19) there is no $E_2$ and if $\frac{a}{b} > \dfrac{d-c}{\beta}$ there is no $E_3$.*

Proof: That there is no $E_2$ is immediate since $M(x,0) = 0$ has no positive solution. Notice that along $M(x,y) = 0$, $by = \dfrac{1+ax}{1+x}$ so that along the curve, we have $y > \frac{a}{b}$. Similar considerations along $N(x,y) = 0$ lead us to conclude that along this curve, $y < \dfrac{d-c}{\beta}$. Consequently if the parameters are so chosen that $\frac{a}{b} > \dfrac{d-c}{\beta}$ then the curves $M(x,y) = 0$ and $N(x,y) = 0$ have no intersection and there is no $E_3$, as claimed. •

The above discussion goes to show that the existence of equilibria $E_2 \& E_3$ are independent of one another and we need to strengthen our assumptions in order to specify existing equilibria. We do so below:

**P3** (i) $M_x(x,y) < 0, N_y(x,y) < 0 \forall (x,y) \in \Re_+ \times \Re_+$.

(ii) There is some positive number $K$ such that $M(x,y) \leq 0$, if either $x \geq K$ or $y \geq K$.

(iii) For any $y \geq 0$ there is a positive number $x(y)$ such that $N(x(y), y) > 0$; further for each $x \geq 0$, $\exists K(x)$ such that $N(x,y) \leq 0$ if $y \geq K(x)$.

To maintain our analogy with $x$ being the population of prey and $y$ being the population of predators, the above specifies that first of all, the rates of growth of population of preys and predators, are decreasing functions of their own population, other things remaining the same; **P2** had merely required that these be non-increasing functions. Secondly, if either the population of preys or that of the predators are large enough, the rate of growth of population of preys cannot be positive; and finally, for the growth rate of the population of predators to be positive, given any current level of its population, requires an adequate population of preys; and given any population of preys, if the population of predators is large enough, the growth rate of the predator population will be non-positive. That these are not too restrictive a set, may be seen by referring to the specification given by (5.18) when all the parameters $a, b, c, d, \alpha, \beta$ are positive: this system satisfies **P1, P2 and P3**. We note that:

**Claim 5.4.7** *Under **P1, P2 and P3**, there are at most three equilibria; $E_1$&$E_2$ always exist; $E_3$ may also exist under some conditions*[26].

Proof. We have already seen that there are three types of equilibria possible; **P3** ensures that there cannot be multiple equilibria of type $E_2$ and $E_3$ since intersections between the curves $M(x, y) = 0$ and the $y = 0$ line is unique if at all; since the former has a negative slope[27]; similarly, the intersection between $M(x, y) = 0$ and $N(x, y) = 0$ , if any, has to be unique too since while the former has a negative slope the latter has a positive slope. Hence there can be at most three equilibria.

Since $M(0, 0) > 0 \geq M(K, 0)$ there is some $\hat{x} \in (0, K]$ such that $M(\hat{x}, 0) = 0$; $(\hat{x}, 0)$ is the $E_2$ equilibrium with no predators.

Next by virtue of the restriction placed on the function $N(x, y)$, we note that there is $x(0)$ such that $N(x(0), 0) > 0 \geq N(0, 0)$ and hence there is $\bar{x} \in [0, x(0))$ such that

[26] Essentially, for $E_3$ to exist, the curves $M(x, y) = 0$ and $N(x, y) = 0$ have to intersect in the positive quadrant; the conditions we shall identify, guarantee that this happens.

[27] The slope of $M(x, y) = 0$ is given by $-\frac{M_x}{M_y} < 0$; and the slope of the curve $N(x, y) = 0$ is given by $-\frac{N_x}{N_y} > 0$.

$N(\bar{x}, 0) = 0$. If $\bar{x} < \hat{x}$ then an $E_3$ equilibrium exists; to see this note that under this condition $M(\bar{x}, 0) > M(\hat{x}, 0) = 0 = N(\bar{x}, 0)$ while $N(\hat{x}, 0) > N(\bar{x}, 0) = 0 = M(\hat{x}, 0)$; thus from continuity, there has to be some $\tilde{x}, \bar{x} < \tilde{x} < \hat{x}$ and corresponding $\tilde{y} > 0$ such that $M(\tilde{x}, \tilde{y}) = N(\tilde{x}, \tilde{y})$: the $E_3$ equilibrium. •

### 5.4.1 Local Stability Properties of Equilibria

We examine the local stability of the equilibria $E_1, E_2$ and $E_3$ in this section. First of all, we need to compute the Jacobian of the system (5.17) given **P1, P2 & P3**. This is given by the following matrix:

$$\begin{pmatrix} M(x,y) + xM_x(x,y) & xM_y(x,y) \\ yN_x(x,y) & yN_y(x,y) + N(x,y) \end{pmatrix}.$$

Thus at $E_1(0,0)$ we have from **P1**:

$$\begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix};$$

this establishes that at $E_1$, the Jacobian has one positive and one negative characteristic root and hence the equilibrium is a saddle-point. Next at $E_2(\hat{x}, 0)$ the Jacobian is given by:

$$\begin{pmatrix} - & - \\ 0 & N(\hat{x}, 0) \end{pmatrix};$$

where the sign of the term $N(\hat{x}, 0)$ depends on whether $\hat{x} > \bar{x}$: the condition for the existence of $E_3$, as we had shown in the proof of Claim 5.4.7. Since by definition, $N(\bar{x}, 0) = 0$, if $E_3$ exists, $E_2$ is a saddle point, since then one characteristic root is positive and the other negative; whereas if there is no $E_3$ equilibrium then $E_2$ is locally asymptotically stable since both characteristic roots are then negative.

Whenever $E_3$ exists, the Jacobian evaluated at $E_3$ has the following sign pattern:

$$\begin{pmatrix} - & - \\ + & - \end{pmatrix};$$

consequently, whenever $E_3$ exists, the Jacobian has both roots with real parts negative and the equilibrium is locally asymptotically stable.

Finally, consider the system 1.3; we may consider this to be a special case of 5.18 where $\alpha = \beta = 0$. It is easy to check that under this specification, the system 1.3 has two equilibria: $E_1(0,0)$ and $E_3(\frac{c}{d}, \frac{a}{b})$ and at $E_3$, the Jacobian has the following sign pattern:

$$\begin{pmatrix} 0 & - \\ + & 0 \end{pmatrix};$$

thus under, 1.3, the characteristic roots are pure complex, the real parts being zero. Thus $E_3$ is a center.

### 5.4.2 Global Stability Properties

**Invariant Set** For the system (5.17), we shall refer to a solution originating from some point $(x^o, y^o)$ by the notation $\phi_t(x^o, y^o)$ and our objective here is to tie down what happens to this solution as $t \to \infty$, for an arbitrary $(x^o, y^o) \in \Re_+ \times \Re_+$. We show that under our assumptions **P1, P2 and P3**, there is an *invariant set* $Q \subset \Re_+ \times \Re_+$. That is the solution $\phi_t(x, y)$ is defined for all $(x, y) \in Q$ and *remains within the set $Q$ for all $t$.* We do this constructively in the following steps.

- From **P1, P3**, we have, $M(0,0) > 0 \geq M(0,K)$; hence, $\exists \bar{y} \in (0,K]$ such that $M(0,\bar{y}) = 0$.

- Under **P1, P2 and P3**, $E_2$ exists; denote this by $(\hat{x}, 0)$.

- Given $\bar{y}$ noted above, **P3** guarantees that there is $x(\bar{y})$ such that $N(x(\bar{y}), \bar{y}) = 0$.

- Define $x^\star = \max[\hat{x}, x(\bar{y})]$. Then define $y^\star$ by $N(x^\star, y^\star) = 0$. That this is well defined may be seen as follows. If $x^\star = x(\bar{y})$ then $y^\star = \bar{y}$. On the other hand, if $x^\star = \hat{x} > x(\bar{y})$, then $N(\hat{x}, \bar{y}) > 0$[28]. Thus by **P3**, there is $K(\hat{x})$ such that $N(\hat{x}, y) \leq 0 \forall y \geq K(\hat{x})$; hence there is $y^\star$ such that $N(\hat{x}, y^\star) = 0$ as claimed.

Now consider the rectangle $Q$ made up with the points $E_1 = (0,0), (0, y^\star), (x^\star, y^\star), (0, x^\star)$. We may now show:

**Claim 5.4.8** *The set $Q$ defined above is invariant.*

Proof: Consider any $(x, y) \in Q_B$ where $Q_B \subset Q$ is the boundary of $Q$ and consider the solution (trajectory) $\phi_t(x, y)$. We shall show that the trajectory either coincides with the boundary or enters $Q$.

$E_1$ is an equilibrium so any trajectory originating there stays put; in case $x^\star = \hat{x}$, the point $(x^\star, 0)$ is another equilibrium, $E_2$ and once again, any trajectory originating there stays put; in case $x^\star = x(\bar{y}) > \hat{x}$, any trajectory originating from $(x^\star, 0)$ has $\dot{y} = 0, \dot{x} < 0$ since $M(x^\star, 0) < M(\hat{x}, 0) = 0$ and so the trajectory coincides with the $x-$ axis, and is directed inside $Q$ along the boundary. Notice that any trajectory originating from a point on the $x-$axis is directed along the $x-$axis towards the equilibrium $E_2$ and hence stays within $Q$.

[28]Since $N(,)$ is increasing in $x$ and $N(x(\bar{y}), \bar{y}) = 0$.

Consider, next, any point $(0, y)$ with $0 < y \leq y^\star$; for the trajectory originating from such a point has $\dot{x} = 0, \dot{y} < 0$ since $N(0, y) < 0$ and hence the trajectory coincides with the $y-$axis and is directed towards $E_1$.

Any trajectory originating from a point of the type $(x, y^\star), 0 < x \leq x^\star$ has $\dot{x} < 0, \dot{y} < 0$ since $M(x, y^\star) < 0, N(x, y^\star) < 0$ and consequently, the trajectory is directed inside $Q$.

Similarly, any trajectory originating from a point of the type $(x^\star, y), 0 < y \leq y^\star$ is also directed inside $Q$. This completes the demonstration of our claim. •

Figures 2A and 2B would clarify the claims made above[29].

FIGURES 2A AND 2B: THE INVARIANT SET

#### 5.4.3 General Conclusions

By appealing to the Poincaré-Bendixson Theorem, the following claim can be made:

**Remark 22** *For any $(x, y) \in \Re_+ \times \Re_+$, the trajectory $\phi_t(x, y)$ must enter $Q$ and either approaches an equilibrium or there is a limit cycle in $Q$. Any limit cycle, if there is one, must surround an equilibrium.*

**Remark 23** *Further, when there is no $E_3$, as in Figure 2A, the triangular region bounded by the axes and the curve $M(x, y) = 0$ is an invariant subset of $Q$: once entered, it cannot be left. In such a situation, there can be no limit cycle and hence any trajectory must approach the equilibrium $E_2$. Consequently, ultimately, all predators disappear.*

[29]Although the lines $M(x, y) = 0, N(x, y) = 0$ have been drawn as straight lines, they need not be so, of course. It is their slopes which are of interest; see footnote 27 above.

**Remark 24** *On the other hand, when we have an equilibrium such as $E_3$ as in Figure 2B, there* appear to be only *two possibilities for any trajectory: beginning from a strictly positive configuration, either we have convergence to $E_3$ or a limit cycle around $E_3$. Neither $E_1$ nor $E_2$ can be approached.*

However, given **P3**, we can go further. Recall that under this restriction, we have first of all that $M_x < 0; N_y < 0$ over the domain of discussion. Then consider the function $\theta(x, y) = \frac{1}{x.y}$ in the positive quadrant and consider the expression:

$$\frac{\partial\{xM(x,y)\theta(x,y)\}}{\partial x} + \frac{\partial\{yN(x,y)\theta(x,y)\}}{\partial y} = \frac{1}{y}M_x(x,y) + \frac{1}{x}N_y(x,y) < 0$$

over the entire positive quadrant: hence by Dulac's criterion[30], there can be **no cycles in the positive quadrant**.

Thus when there is no equilibrium with Both Species present, all trajectories converge to the No Predator equilibrium; if an equilibrium with Both Species present exist, we will have convergence to this equilibrium. To apply Dulac's criterion, it may be recalled that we do not need the strict signs of the partial derivatives $M_x, N_y$; the weak signs admitted under the restriction **P1** are sufficient together with a proviso that they are not identically zero. Consequently, if we are interested in exhibiting cyclical behavior of any kind, we must have some variation in the sign of $M_x, N_y$[31].

[30]See Remark 7.

[31]See, for example, the example in Section 1.6.2.

### 5.4.4 Robust Unemployment Cycles

We consider in this section, the possibility of having robust unemployment cycles in set-ups similar to the ones we have been discussed above. The model[32] which we shall consider is somewhat reminiscent of the Generalized Lotka-Volterra models we have considered in the last section. As we shall see, to obtain a robust example of periodic behavior, we need to follow the formal structure of the example in Section 1.6.2. In terms of our construct of unemployment cycles, the Phillips curve will play a major role too. And we shall identify conditions which are required to ensure that we have such a configuration.

Consider an economy with imperfectly competitive firms; a representative firm[33]indexed by the subscript $r$ is assumed to satisfy the following:

5.i. The output-capital ratio for the entire economy $Y/K$ and its employment capital ratio $N/K$ are respectively equal to the ratios $Y_r/K_r, N_r/K_r$ chosen by the firm. Let $x \equiv N/K \equiv N_r/K_r$. Then

$$Y/K \equiv Y_r/K_r = f(x) \tag{5.20}$$

where $f(.)$ is well defined for all $x > 0$ and $f'(x) > 0$[34].

5.ii. $K_r = \lambda K$ , $0 < \lambda < 1$ and $\lambda$ is a constant.

5.iii. Current production is planned to to equal expected demand at the current price; given the current price, the growth of expected demand is an increasing function of the ex-

[32]This is based on the model of Rose (1967).

[33]See, Rose (1967), p.154 on such firms. By the action of the representative firm, average firm behavior is being conveyed.

[34]Thus production function for the representative firm is homogeneous of degree one in $N_r, K_r$.

cess of actual demand over expected demand in the past; the relationship between aggregate demand and aggregate production determines the relationship between actual demand and expected demand. Thus aggregate demand greater (or less than equal) to aggregate production implies that actual demand is greater (or less than equal to, respectively) to expected demand. If actual demand and expected demand match, then the current growth of excess demand is equal to the growth of aggregate capital.

5.iv. The planned growth rate $I/K$ where $I$ is planned investment of the whole economy equals that for the representative firm.

Consider the expected demand function to be given by $p = Ad(Y_r/B)$ where $p$ is the expected demand price of output $Y_r$; and $A, B$ are time dependent functions of time, both positive; thus, $Y_r/B = d^{-1}(p/A) = \phi(p/A)$; the price elasticity of demand $\eta$ is given by

$$\eta = -\phi'(p/A)1/A\frac{p}{Y_r/B} = -\frac{d(Y_r/B)}{d'(Y_r/B)}\frac{1}{Y_r/B} > 1$$

where the inequality is **by assumption** . At constant prices, expected demand varies according to:

$$\dot{Y}_r/Y_r = \eta\dot{A}/A + \dot{B}/B$$

where $\dot{A}/A$ will be related to the difference between **aggregate** demand and **aggregate** production; $\dot{A}/A$ will be zero if the difference is zero: this is in line with the specifications mentioned above. The specifications also imply that when aggregate production and demand match, $\dot{Y}_r/Y_r = \dot{K}/K$; hence we must have $\dot{B}/B = \dot{K}/K$; we may thus take $B = K_r$;

thus expected demand reduces to:

$$p = Ad(Y_r/K_r) = Ad(f(x)) \tag{5.21}$$

Expected profit maximization given $K$ and hence $K_r$, and $w/A$ leads to the maximization of :

$$\pi = [A\ d(f(x)).f(x) - wx]K = [d(f(x)).f(x) - (w/A)x]A\ K$$

by choosing an appropriate $x$[35]. The first order condition for this maximization problem leads to

$$(1 - 1/\eta)d(f(x)).f'(x) = w/A. \tag{5.22}$$

Using this expression, the expected rate of profit $r$ is given by:

$$r = \pi/pK = f(x) - (1 - 1/\eta)f'(x)$$

Consider the aggregates next; planned savings $S$ depends on expected real income (production) or on expected real profits and wages, the stock of capital and the rate of interest; we may express this as the following, suppressing the rate of interest:

$$S/K = g(x) \tag{5.23}$$

where $g'(x) > 0$ so that the marginal propensity to save is positive. Next, regarding planned investment $I$ and given the assumption that that the planned growth rate $I/K$ is the same as the representative firm's, one may note that this rate for the representative firm should be a decreasing function of the rate of interest and an increasing function of the state of

[35] Actually choosing an employment level $N_r$.

" long-term expectations". It is further assumed that the long-term is dominated by what happens in the short run and thus, the firm's planned growth rate is an increasing function of the rate of profit and as before suppressing the rate of interest, we obtain:

$$I/K = k(x) \tag{5.24}$$

where $k'(x) > 0$ thus the marginal propensity to invest is positive.

In each short period, the market for goods is in temporary equilibrium with any gap between savings and planned investment being met by movements of stocks; some times the latter is taken to be unplanned (dis)investment[36]. Thus buffer stocks should enter in the the stock of capital which is considered in the production function, since its role some times add to costs in terms of carrying charges and some times saves costs by reducing the risk of depletion according to Rose (1967). Consequently

$$S = \dot{K}. \tag{5.25}$$

We are now ready to tie up the term $\dot{A}/A$ in the following manner:

$$\dot{A}/A = E(k(x) - g(x)) = H(x) \tag{5.26}$$

where $E(0) = 0$ and $E'(x) > 0$. Notice that $H'(x) = E'(.)\{k'(x) - g'(x)\}$ and consequently the sign of $H'(.)$ depends upon the sign of $k'(x) - g'(x)$: or the difference between the marginal propensities to save and invest.

Finally, the behavior of wages $w$ will be taken to be of the type we have investigated

[36]See Rose (1967), p. 157, paragraph (f) for details.

above, viz., (5.7), namely:

$$\frac{\dot{w}}{w} = \psi(v) \tag{5.27}$$

where $v = N/L$, $L$ is the supply of labor so that $v$ is the employment ratio as before. Further the function $\psi(.)$ is restricted by the following on an interval $[a, b]$ where $0 < a < b \leq 1$:

$$\psi'(v) > 0 \ , \ \lim_{v \to a} \psi(v) = -\infty \ , \ \lim_{v \to b} \psi(v) = +\infty$$

Labor supply $L$ is assumed to grow at a constant rate $n$; the labor capital ratio $L/K$ will be denoted by $z$ and is the counterpart on the supply side to $x$, the employment capital ratio $N/K$. It follows therefore that

$$\dot{z}/z = \dot{L}/L - \dot{K}/K = n - g(x). \tag{5.28}$$

It should be remembered that by definition, we have

$$v \equiv \frac{x}{z}. \tag{5.29}$$

From the first order condition, (5.22), we have:

$$\frac{\dot{x}}{x} = \gamma(x)(\frac{\dot{A}}{A} - \frac{\dot{w}}{w})$$

where

$$\frac{1}{\gamma(x)} = -\frac{d \log\{(1 - 1/\eta)d(f(x))f'(x)\}}{d \log x}$$

and $\gamma(x) > 0$ in the relevant region. Substituting from equations (5.26) and (5.27) and using (5.29), we have

$$\frac{\dot{x}}{x} = \gamma(x)\{H(x) - \psi(x/z)\} \tag{5.30}$$

The system of equations that we shall analyze are now in place and is made up of equations (5.28) and (5.30):

$$\dot{x}/x = \gamma(x)\{H(x) - \psi(x/z)\} \text{ and } \dot{z}/z = n - g(x). \tag{5.31}$$

**Trajectories and Equilibrium for the system (5.31)** Since we are assuming that the various functions appearing in the system are differentiable and well defined in the subset $\mathcal{R}$ of the positive quadrant, where $\mathcal{R} = \{(z,x) : z, x > 0, 0 < a < x/z < b \leq 1\}$; through any point$(z^o, x^o)$ in $\mathcal{R}$ there is a trajectory (solution) to the system (5.31) which we shall denote by $\Phi_t(z^o, x^o)$ and which in turn denotes the time path of all relevant variables in the model. The equilibrium for the system in $\mathcal{R}$, is given by solving for $\dot{x}/x = 0, \dot{z}/z = 0$ and noticing the fact that the function $\gamma(x) > 0$ in the region, to obtain $x^\star = g^{-1}(n)$ and $z^\star = x^\star/\psi^{-1}(H(x^\star))$[37]. Thus there is a unique equilibrium for the system in $\mathcal{R}$.

Notice that at this equilibrium, there is a steady growth of capital and income at the 'natural rate' $n$ since with $g(x^\star) = n$, hence $\dot{K}/K = S/K = g(x^\star) = n$ and since $Y/K = f(x^\star)$, a constant, $\dot{Y}/Y = \dot{K}/K = n$; further at this equilibrium, since $H(x^\star) = \psi(x^\star/z^\star)$, prices and wages rise at the rate $H(x^\star)$ and the employment rate $v^\star = x^\star/z^\star$ is just appropriate and self-sustaining. To analyze the local stability of the equilibrium, we need to look at the Jacobian of the system (5.31) evaluated at equilibrium:

$$\begin{pmatrix} \gamma(x^\star)\{x^\star H'(x^\star) - v^\star\psi'(v^\star)\} & \gamma(x^\star){v^\star}^2\psi'(v^\star) \\ -z^\star g'(x^\star) & 0 \end{pmatrix}$$

Notice that the determinant is positive; while the trace is given by $\gamma(x^\star)\{x^\star H'(x^\star) -$

[37]Notice that the inverse functions exist since the functions involved in the inversion are monotonic.

$v^\star\psi'(v^\star)\}$ whose sign is determined by the term $\{x^\star H'(x^\star) - v^\star\psi'(v^\star)\}$; notice here that the second term is negative; hence if the sign of $H'(x^\star) = E'(.)(k'(x^\star) - g'(x^\star))$ or that of $k'(x^\star) - g'(x^\star)$ is negative, the trace is negative and the equilibrium is locally stable for the system (5.31). In any case, the equilibrium cannot be a saddle-point. We thus have the following:

**Claim 5.4.9** *The unique equilibrium $(z^\star, x^\star)$ is locally stable if $g'(x^\star) > k'(x^\star)$.*

We may also note the following:

**Proposition 5.6** *If $g'(x) \geq k'(x)$ for all $x > 0$ then there can be no cyclical trajectories in $\mathcal{R}$.*

Proof: We shall demonstrate the validity of this proposition by the application of Dulac's Criterion. Consider the function $\theta(z,x) = \dfrac{1}{x\gamma(x)z}$ and note that

$$\frac{\partial\{x\gamma(x)(H(x) - \psi(x/z))\theta(z,x)\}}{\partial x} + \frac{\partial\{z(n - g(x))\theta(z,x)\}}{\partial z}$$

$$= \{H'(x) - \psi'(x/z).1/z\}1/z$$

and under the given condition, $H'(x) = E'(.)(k'(x) - g'(x)) \leq 0$ on $\mathcal{R}$ so that the entire expression is always negative on $\mathcal{R}$: thus there can be no cyclical trajectory in $\mathcal{R}$. This completes the demonstration. •

To proceed with a further characterization of possibilities, it would be best to consider the phase-plane for the system (5.31).

Figure 2C: Phase Plane for the Rose Model

Notice that the region $\mathcal{R}$ has been divided up into four subregions named *I, II, III and IV* by means of the line $\dot{x} = 0$ and $\dot{z} = 0$; these lines intersect at the equilibrium $E$ ; and we have drawn the the curve $\dot{x} = 0$ to have a positive slope, for reasons which may become clear soon. The slope of the curve $\dot{x} = 0$ may be determined thus: first of all, along this curve, we must have $H(x) = \psi(x/z)$; hence

$$\frac{dx}{dz}_{\dot{x}=0} = -\frac{(x/z^2)\psi'(.)}{H'(.) - \psi'(.)};$$

hence the slope of the curve depends on the sign of the term $\psi' - H'$; the first term is positive but the second term could be of any sign depending on the relative magnitudes of marginal propensities to save and invest. For the moment, we need not really worry about this sign.

Consider an initial configuration $(z^o, x^o) \in \mathcal{R}$; the question we need to analyze is whether the solution $\Phi_t(z^o, x^o)$ can leave the region $\mathcal{R}$: notice leaving the region from subregions I and III is not possible; this is so, because the arrows on the rays *a, b* respectively, indicate that the trajectory will be directed back into the region $\mathcal{R}$; from the remaining two subregions, *II, IV* , notice that $\dot{x}$ is $+\infty$ in the former, and $-\infty$ in the latter and these direct the trajectory towards $\mathcal{R}$; the only movement away comes from the movement on the variable $z$ but on both instances, $\dot{z}$ is finite consequently the rays *a, b* cannot be reached; thus the trajectory $\Phi_t(z^o, x^o)$ remains within $\mathcal{R}$; also as the arrows will indicate, the variables cannot both go to $+\infty$ nor can they both go to 0 since in regions *III and IV* one variable is always increasing and in regions *I, II* one variable is always increasing. This establishes the following:

**Claim 5.4.10** *For the system, (5.31), for any arbitrary point $(z^o, x^o) \in \mathcal{R}$, the solution $\Phi_t(z^o, x^o)$ lies within a bounded subset of $\mathcal{R}$.*

Claim 5.4.10 simplifies matters enormously since now we have a set of immediate implications. First of all,

**Proposition 5.7** *If $g'(x) \geq k'(x)$ for all $x > 0$ and the strict inequality holds at $x^\star$, then for any initial configuration, $(z^o, x^o) \in \mathcal{R}$, $\Phi_t(z^o, x^o)$ converges to the unique equilibrium.*

Proof. Since the Claim 5.4.10 implies that the $\omega$-limit set is non-empty; the Proposition 5.6 implies that under the given condition, there can be no cycles. Hence the Poincaré-Bendixson Theorem implies that the equilibrium must be a limit point for every trajectory. But by Claim 5.4.9, the equilibrium is locally stable and hence if any trajectory enters a neighborhood of the equilibrium, it cannot leave. This concludes the demonstration. •

Finally, we have:

**Proposition 5.8** *A necessary condition for the existence of cyclical orbits is that for some $x > 0$, $k'(x) > g'(x)$; if $x^\star H'(x^\star) - v^\star \psi'(v^\star) > 0$, then there will be a limit cycle: i.e., for any arbitrary $(z^o, x^o) \in \mathcal{R}$, the $\omega$-limit set will be a closed orbit.*

Proof: The necessity is immediate from Proposition 5.6. By virtue of Claim 5.4.9, it follows that if $k'(x^\star) > g'(x^{star})$, the unique equilibrium is locally unstable: the relevant Jacobian has a positive trace and the unique equilibrium is unstable[38]. Consequently for any initial

[38]The Figure 2-C depicts such a situation. Notice too that at such an equilibrium, we must have $k'(x^\star) > g'(x^\star)$.

configuration $(z^o, x^o) \in \mathcal{R}$, the non-empty $\omega$-limit set cannot contain a equilibrium and hence by virtue of the Poincaré-Bendixson Theorem, the $\omega$-limit set must be a closed orbit.

•

It should be pointed out that there may be more than one periodic orbits: with the outermost and innermost being attracting orbits. What happens along such a periodic orbit ? Note that this cyclic orbit surrounds the unstable equilibrium at $E$ . When $x/z$ or $v$, the employment ratio is at a minimum, $x < x^\star$, so that $\dot{z} > 0$ but the rate of profit and the employment-capital ratio and $x$ are rising; employment is raised since $x$ is now rising faster than $z$. The relatively flat section of the Phillips curve (contained in the specification of the function $\psi(v)$) does not allow $x, z$ to rise to their equilibrium values; as $x$ crosses $x^\star$, $z$ starts to fall and $v$ rises and this causes wage inflation to overtake demand-price inflation consequently causing the rate of profit to fall, once employment is high enough and subsequently, $x$ starts to fall. The rate of profit and $x$ continues to fall; a falling $x$ reduces the growth of capital and ultimately reduces the fall in $z$ and the employment rate $v$ reaches a maximum and declines since $z$ continues to fall less rapidly than $x$. The inflation in wages is sustained over the flat part of the Phillips curve once more and the rate of profit is continues its fall as well. Once more $x$ crosses $x^\star$ (this time falls below) and so $z$ starts to rise and the employment rate $v$ declines; once this has declined far enough, wages first rise less rapidly and then fall and causes the rate of profit and $x$ to rise and ultimately $x$ and $v$ both rise and the minimum $v$ is attained once more. Basically, what has been described is a movement along the orbit beginning from sub-region $II$ of the phase plane and continuing through region *III* , *IV, I* and back to subregion *II* .

Notice that such a limit cycle is robust in the sense that small changes in initial conditions will lead to the same cyclical orbit; small changes in parameters are unlikely to alter the unstable character of the equilibrium too. Consequently cyclical behavior should persist. Rose (1967) attributes the possibility of there being an unstable equilibrium to government policies directed towards the avoidance of both excessive inflation and excessive unemployment [39]. Finally, it should be pointed out that it has been possible to isolate the aspects which are crucial for such a configuration to exist: the relative magnitudes of the marginal propensities to save and invest.

## 5.5 Discrete Growth Processes

### 5.5.1 Classical Growth Processes

Till relatively recently, one of the fundamental presumptions about processes of economic growth has been the approach over time to a steady state; that the dynamics, on its own, is capable of showing highly complex behavior without any exogenous shocks was established only around the 1980's through the works of persons such as Day ( ), etc. We consider these aspects below. As always, it is best to begin with the simplest possible constructs to understand the interplay of various aspects. Classical growth theory, analyzes very familiar terrain for most economists and so when the authors named above came up with their conclusions, the results made an impact to say the least.

[39]See, Rose (1967), p. 162.

### 5.5.2 Population Dynamics

Consider a single good being produced by means of labor alone according to a production function of the type $f(\ell)$; we consider first, a completely egalitarian agrarian society where the output is completely distributed among the population according to the average product $f(\ell)/\ell$ which is the wage rate $w$. Malthus argued that when the necessities were plentiful, population tended to grow at its maximum biological rate or natural rate, say $n$; when the situation was adverse, Malthus assumed that the net birth rate to be the maximum that could be attained under some exogenously determined subsistence level $\sigma$. Finally it was also being assumed that the net birth-rate was being determined by the income or level of output. We are considering a formalization of all of these ingredients which are due to Day (1983).

Thus it is being assumed that population growth rate $\frac{\triangle\ell}{\ell}$ would be governed by an equation of the form:

$$\frac{\triangle\ell}{\ell} = \min\{n, \frac{w-\sigma}{\sigma}\} \tag{5.32}$$

We note for future reference the fact that the population and labor employed are being treated as one and the same variable $\ell$. Now writing $\triangle\ell = \ell_{t+1} - \ell_t$, the above equation reduces to

$$\ell_{t+1} = \min\{(1+n)\ell_t, w_t\ell_t/\sigma\} \tag{5.33}$$

The next ingredient, in this simple economy is the production function $Y = f(\ell)$ which determines output as a function of labor employed and a distribution rule which ensures that the total product is distributed according to the average product so that we have

$$w_t = f(\ell_t)/\ell_t \tag{5.34}$$

The production function is assumed to satisfy the following:

5.i. $f(0) = 0$

5.ii. $f(.)$ is continuous and single-humped

Then the equation (5.33) reduces to

$$\ell_{t+1} = \theta(\ell_t) = \min\{(1+n)\ell_t, f(\ell_t)/\sigma\} \tag{5.35}$$

Notice that the function $\theta(.)$ is well behaved and continuous, single-humped although there are two segments to it.

To make matters somewhat more definite we assume that, in addition to the ones listed above

- $f'(0)$ exists and further, it is greater than $\sigma(1+n)$

On the basis of this restriction, we can say that for low values of $\ell$, $(1+n)\ell < f(\ell)$ and there is a switch-point $\ell_s > 0$ beyond which the inequality is reversed. Thus for values of $\ell$ low enough, $0 < \ell < \ell_s$, $\theta(\ell) = (1+n)\ell$: this is the region when the rate of growth is determined purely by biological conditions, and the natural rate $n$ prevails. Beyond the range $\ell_s$, however, $\theta(\ell) = f(\ell)/\sigma$ the population growth rate is determined by subsistence considerations. Thus the former range (less than $\ell_s$ ) is referred to as the B-phase and the latter range (beyond $\ell_s$) is referred to as the S-phase.

The equilibrium or steady state for the equation (5.35), say $\ell^\star$, is determined by $\ell^\star = \theta(\ell^\star)$. To proceed, we need to tie matters down a bit more. For example, recall single-humped has not been provided with a formal definition; we do so by requiring[40]the following:

- $f''(.)$ exists and is non-positive for all $\ell \geq 0$.
- There is $\bar{\ell} > 0$ ($\bar{\ell}$ may be $+\infty$) such that $f'(\ell) > 0$ for all $\ell \in [0, \bar{\ell})$.
- There is $\hat{\ell} > 0$ such that $f(\hat{\ell})/\sigma < \hat{\ell}$.

We note that under these conditions, see, for example, Figure 3 below. The diagram is drawn under the assumption that $\bar{\ell}$ is finite. In such a situation the map consists of two segments: the first for $\ell \in [0, \ell_s]$,the operative portion is the straight line (the dashed line in the diagram) AD; and the next segment is for $\ell \in [\ell_s, \ell_m]$, the dotted parabola D onwards. The first is what has been termed the B-phase where rates of growth is the natural biological rate; the latter, is the subsistence phase, S-phase where rates of growth are constrained by the subsistence levels. Note first of all that under the conditions stated above, we have:

**Claim 5.5.11** *$\ell^\star$ exists and $\ell^\star > \ell_s$.*

Figure 3 here

This may be seen as follows: note that $\ell_s$ solves $(1 + n)\ell = f(\ell)/\sigma$; such a solution exists since for small $\ell$, it follows from the boundary condition that $(1 + n)\ell < f(\ell)/\sigma$ and moreover, we have $f(\hat{\ell})/\sigma < \hat{\ell} < (1 + n)\hat{\ell}$ thus the two curves must cross somewhere

[40]This need not be the most general set of conditions for our analysis. But these are certainly very convenient.

in between. Hence $f(\ell_s)/\sigma = (1+n)\ell_s > \ell_s$; thus there must be $\ell^\star \in (\ell_s, \hat{\ell})$ such that $f(\ell^\star)/\sigma = \ell^\star$.

Thus notice that the steady state equilibrium $\ell^\star$ must necessarily be in the S-phase. We consider what happens when $\bar{\ell}$ is finite and the strictly increasing nature of the production function ceases at $\bar{\ell}$. It follows that $f(.)$ reaches a global maximum at $\bar{\ell}$ and there can be no other humps on account of the restriction on the sign of the second order derivative of the function $f(.)$. The situation then is exactly as depicted in Figure Day. Thus $f(\bar{\ell})/\sigma = \ell_m$, say; note that $\ell_m = \max_{\ell \geq 0} \min\{(1+n)\ell, f(\ell)/\sigma\}$.

Next note that at $\bar{\ell}$, $f(\bar{\ell})/\sigma > \bar{\ell}$ and hence $\bar{\ell} < \ell^\star \leq \ell_m = f(\bar{\ell})/\sigma = \theta(\bar{\ell})$. As a reference to the section of Discrete processes will make clear it is of crucial importance to study the relative magnitudes of $\theta(\ell_m) = \theta^2(\bar{\ell}) = f(\ell_m)/\sigma$ and $\bar{\ell}$. For complicated dynamics, as we know, it is necessary that $f(\ell_m)/\sigma < \bar{\ell}$; let $\ell_c$ be defined by the lower of the two values generated by the equation $\theta(\ell_c) = \bar{\ell}$. Thus $\ell_c < \bar{\ell}$; further $\ell_c < \bar{\ell} < \ell_m$; now if $f(\ell_m) < \ell_c$, the conditions for Proposition **??** are satisfied and the map $\theta(.)$ is chaotic. Thus very complicated dynamics would follow in such situations and the growth path, stemming from the simple dynamics of (5.35) is capable of generating cycles of any order. The source of this complex set of results is clear: it has to do with the function $f(.)$ being such that $\bar{\ell}$ is finite.

To consider what happens when $\bar{\ell}$ is infinite so that $f'(\ell) > 0 \forall \ell > 0$ together with $f''(\ell) < 0$ and given the existence of $\hat{\ell}$ such that $f(\hat{\ell})/\sigma < \hat{\ell}$, it follows that$\ell^\star$ exists and is in the S-phase, as before. But given the geometry of the situation, specified by the other conditions, it follows that at $\ell^\star$, $0 < f'(\ell^\star)/\sigma < 1$ and consequently, the steady state is immediately known to be locally asymptotically stable, since at the steady state, we are in the S-phase, so that the map $\theta(.)$ is given by the function $f(.)/\sigma$. For global stability, we note that the map $\theta(.)$ is monotonic increasing; consequently, $\ell_1 < \ell^\star \Leftrightarrow \theta(\ell_1) < \theta(\ell^\star) = \ell^\star$; this shows that the iterates form a monotonic sequence: increasing, bounded above by $\ell^\star$, if the initial point $\ell_o < \ell^\star$; the iterates form a monotonic decreasing sequence bounded below by $\ell^\star$ if the initial point $\ell_o > \ell^\star$. Thus the sequence of iterates $\{\ell_t\}$ converge and they must converge to $\ell^\star$. Thus the situation is as expected.

### 5.5.3 Distribution Dynamics

There may be other sources for complicated dynamics in even rather simple classical model. We consider in this section the following Ricardian model, analyzed by Bhaduri and Harris (1987) where the essential dynamics of the Ricardian system is governed solely by distributional changes and we shall follow the Bhaduri and Harris treatment closely. The simple model that they consider is one where some homogeneous output, made from labor and land, is distributed as wages, profits and rents. There is an abstraction from the kind of Malthusian dynamics considered in the last section and it is assumed that with real wages fixed, the labor supply is perfectly elastic. The profits are reinvested by extending the margin of cultivation and the paper studies the interaction between distribution and accumulation with unlimited labor supply at constant real wages.

Thus labor ($\ell$) is applied in fixed proportions to to less and less fertile land as the margin of cultivation is extended and there is a falling marginal product ($MP_\ell$) curve for labor taken to be:

$$MP_{\ell_t} \equiv \frac{dY_t}{d\ell_t} = a - b\ell_t \tag{5.36}$$

where $a, b > 0$. This leads to to the total product curve for labor

$$Y_t = a\ell_t - b\ell_t^2 = \ell_t(a - b\ell_t) \tag{5.37}$$

by specifying that labor is essential for production, so that $\ell = 0 \Rightarrow Y = 0$. Thus the average product of labor ($AP_\ell$) is given by

$$AP_{\ell_t} = a - b\ell_t/2 \tag{5.38}$$

and this allows us to consider at stage $t$, the rents $R_t$ to land as the difference between the average and marginal products of Labor:

$$R_t = AP_{\ell_t} - MP_{\ell_t} = bN_t^2 \tag{5.39}$$

The final category Profits $P_t$, is then the residual after the payment of rents and replacement of the wage fund advanced to hire labor. Thus we have

$$P_t = Y_t - W_t - R_t \tag{5.40}$$

further since as we have indicated above the real wage is fixed at some $w$, $W_t = w\ell_t$; accumulation of additional wage funds to increase employment arises from the reinvestment of profits (assuming that there is no consumption out of such accruals):

$$W_{t+1} - W_t = P_t \tag{5.41}$$

so that using the above, we have:

$$\ell_{t+1} = \frac{a}{w}\ell_t - \frac{b}{w}\ell_t^2 \tag{5.42}$$

If we rewrite the above using the transformation $x_t = b\ell_t/a$, then notice that (5.42) reduces to

$$x_{t+1} = Ax_t(1 - x_t) \tag{5.43}$$

where $A = a/w$. Now this is the logistic map which we have studied in some detail earlier and for the present, we note that even here complicated dynamics may result for values of the constant $A$ close to 4. It would thus appear that simple classical models contain within them scope for generating very complicated trajectories.

### 5.5.4 A Re-examination of Source of Complicated Dynamics

In both of the cases considered above, we need to examine one particular aspect which is common to both of the models. The common feature of both of these models is the existence of a production function which is single-humped. See, for example, Figure 3 above.

The point to note is that at $\bar{\ell}$ the marginal product of labor is zero and beyond this level of employment, marginal product becomes negative. Of course for positive output, we must have $0 < \ell < OC$.

Any rational producer should never employ in the range $\bar{\ell} < \ell < OC$ since by cutting down on employment, profits can be made to increase on two counts: first in the cutting back in wages and secondly due to the negative marginal product of labor. Consequently we need to take this matter into account when we consider production functions of this type. We investigate in this section the effect of this step on the conclusions arrived at in the last two sections.

Consider first of all the Malthusian population dynamics analyzed through (5.35) and consider the case when the production function is of the type depicted above[41] in Figure . For the egalitarian society being considered, it would make little sense to employ beyond $\bar{\ell}$: notice that $f(\bar{\ell}+h) < f(\bar{\ell}) \forall h > 0 \Rightarrow f(\bar{\ell}+h)/(\bar{\ell}+h) < f(\bar{\ell})/(\bar{\ell}+h)$ so it is best to keep the extra $h$ unemployed and share out the maximum output among every one. The result of this would make the basic dynamics to be governed by the following equation:

$$\ell_{t+1} = \phi(\ell_t) \equiv \min(\theta(\ell_t), \bar{\ell}) \tag{5.44}$$

First of all notice that this would mean that there are two possibilities for equilibria:

- $\ell^\star = \theta(\ell^\star) < \bar{\ell}$
- $\ell^\star = \bar{\ell} < \theta(\bar{\ell})$

In the first case, notice that first of all, $\ell^\star$ occurs where $0 < f'(.)/\sigma < 1$ so that the steady state is locally asymptotically stable; next since the only region for consideration, is the interval $[0, \bar{\ell}]$; either the initial point lies in $[0, \ell^\star]$ or in $(\ell^\star, \bar{\ell}]$; in the former case, the iterates are monotonic increasing and bounded above; while in the latter case, they are monotonic decreasing bounded below and convergence in both cases to the steady state is guaranteed.

In the second case, notice that the situation is even simpler: since now the only region for consideration is the interval $[0, \bar{\ell}]$ and the iterates are monotonic increasing and bounded above; hence they converge and the only possible point of convergence is the steady state.

[41] We analyzed this by referring to the case when $\bar{\ell}$ is finite.

Thus complicated Malthusian population dynamics, which we saw was plausible, is no longer feasible once we make provisions for rational employment rules. To obtain the conclusions that were obtained in Day (1983), it would have to be ensured that the population dynamics was governed by the earlier equation (5.35): it would thus have to be said that the function $f(.)$ is not really a production function but a relationship between output and population denying the notion of employment.

The Bhaduri-Harris (1987), conclusion, similarly may appear to be completely dependent on the single humped production with an unique maximum at $\bar{\ell}$. In this paper, there is a restriction $0 < x_t = b\ell_t/a < 1$ which guarantees that marginal products are non-negative (see (5.36): thus always $\ell < \bar{\ell}$.) However regardless of this, the logistic equation appears and so does the possibility for complicated dynamics. It seems that the difference between the two (Day (1983) and Bhaduri and Harris (1987)) lie in the fact that whereas the production function itself appeared as part of the map in the former this is not so in the latter study. In fact, it may be useful to see a somewhat more general setup in the Bhaduri-Harris (1983) context. To investigate the nature of the map, consider a general production function $Y = f(\ell), f'(.) > 0, f''(.) < 0$; then notice that the crucial map (5.42) reduces to:

$$\ell_{t+1} = \frac{f'(\ell_t)}{w}.\ell_t \tag{5.45}$$

Put this way, it is easy to see what is happening along the process: if the current level of marginal products exceed the fixed real wage, then employment is increased; whereas if the marginal products happen to be less than the real wages, then employment is curtailed. Now given the specification of marginal products in (5.36), the function $f'(\ell_t).\ell_t/w$ yields the logistic curve even after constraining employment levels to yield non-negative marginal products.

However, this need not necessarily be so. For example, consider the function $f(\ell) = A.\ell^{\gamma}$ where $0 < \gamma < 1, A > 0$. Such a specification meets all the requirements; however, it is easy to check that $f'(\ell).\ell = \gamma A\ell^{\gamma}$ and consequently the map is monotonic increasing, strictly concave, ensuring the existence of $\ell^{\star}$ which is locally asymptotically stable; the last assertion follows since first of all $\ell^{\star} = \{\frac{A\gamma}{w}\}^{\frac{1}{1-\gamma}}$ and evaluating the *derivative* of $\gamma A\ell^{\gamma}/w$ at this equilibrium, yields $\gamma$ which is less than unity. Having established this and the monotonic increasing nature of the map, it is straightforward to establish that iterates form a monotonic bounded and hence convergent, sequence as before. Thus there are no possibilities for any complicated dynamics. Thus the source of the complicated dynamics may be traced to the nature of the function $\phi(\ell) = f'(\ell).\ell$. It is usual to take $\phi'(\ell) > 0$ for small values of $\ell$[42]. Thus unless this derivative changes sign, there can be no chance of complicated dynamics. For example, for the Bhaduri and Harris (1987) set-up, this derivative is $\frac{a - 2b\ell}{w}$ which changes sign at $\ell = a/2b$[43].

We summarize this discussion in the form of the following:

**Claim 5.5.12** *In the Ricardian Model a necessary condition for the existence of complicated dynamics is that the function $\phi(\ell)$ defined above is non-monotonic.*

[42] This follows from the fact that $\phi'(\ell) = \ell f''(\ell) + f'(\ell)$ and the second term is usually assumed to become large as $\ell \to 0$ while the first term vanishes if one were to take the second derivative to be bounded, for example.

[43] Recall that non-negative marginal products make us confine attention the interval $[0, a/b]$.

### 5.5.5 Growing Through Cycles

Matsuyama (1999)(and (2001)) presented a model which appears to have rather interesting implications; the standard neoclassical model of growth focuses on capital accumulation to be the sole process through which an economy grows and consequently, if there is diminishing returns to capital, such a process of growth cannot be sustained. The more recent endogenous theories of growth suggest a way around this problem of diminishing returns through a consideration of innovations of new products which then sustain growth indefinitely.

That neoclassical models of growth may be capable of non-convergence and generating complicated behavior has been known for some time. Matsuyama shows that there is a trade-off between growth and innovation; while innovation is necessary for sustaining growth, during the process of innovation, growth rate is low. It is during the period when no innovation occurs, that growth rate picks up and these two states alternate to achieve a high rate of growth. We shall reexamine these conclusions below. We present the model next.

There is one final good which is produced under competitive conditions with the help of labor ($L$) supplied inelastically and several intermediate goods; the single final good is both consumed and invested; say that at the end of period $t-1$, $K_{t-1}$ units of the final good is unused and is thus available for production in period $t$ and is to play the role of capital.

Additionally in period $t$, there are several types of intermediate goods available denoted by $z$, $z \in [0, N_t]$; prior to period $t$, types of intermediate goods up to $N_{t-1}$ have been introduced ($N_0 > 0$); in period $t$, production of type $z$ up to $N_{t-1}$ (the old intermediate goods) require just $a$ units of capital per each unit of any of the old intermediate goods. New intermediates in the range $[N_{t-1}, N_t]$ may also be introduced and sold exclusively by those who choose to innovate; for each of the new intermediate, in addition to the $a$ units of capital per unit, a fixed cost of $F$ units of capital has to be incurred before production can be carried out. The old intermediate goods are sold competitively and hence, price must reflect marginal costs i.e.,

$$p_t(z) = ar_t, z \in [0, N_{t-1}]$$

where $r_t$ is the price of a unit of capital in period $t$; the new intermediates are sold under conditions of monopoly by the investor who innovates and to compute sales we need to find out about the demand for the new intermediates.

Production of the final good in period $t$ takes place according to the production function:

$$Y_t = AL^{1/\sigma}\{\int_0^{N_t} [x_t(z)]^{1-1/\sigma} dz\} \tag{5.46}$$

where $x_t(z), z \in [0, N_t]$ is the intermediate good of type $z$ in period $t$; $\sigma \in [1, \infty]$ is the constant direct partial elasticity of substitution between any two pairs of intermediates; the following features should be noted:

•given $N_t$, production in period $t$ satisfies constant returns to scale.

•the price elasticity of demand for each intermediate is $\sigma$.

•the share of labor is $1/\sigma$.

Given the above demand condition, the price of the new intermediates must satisfy

$$p_t^m(z) = \frac{\sigma a r_t}{\sigma - 1}, z \in (N_{t-1}, N_t]$$

Since all intermediate goods enter the production function symmetrically, we may take

$$x_t(z) = x_t^c \forall z \in [0, N_{t-1}]; x_t(z) = x_t^m \forall z \in (N_{t-1}, N_t]$$

and given the demand conditions specified above,

$$\frac{x_t^c}{x_t^m} = [\frac{p_t^c}{p_t^m}]^{-\sigma} = [1 - \frac{1}{\sigma}]^{-\sigma} \tag{5.47}$$

Note that the above has been obtained on the assumption that both the new and the old intermediate goods are produced. The new intermediates are produced by monopolists whose one period profit $\pi_t^m$ is the sole incentive for production. Note that $\pi^m(t) = p_t^m.x_t^m - r_t(ax_t^m + F) = r_t(ax_t^m \frac{1}{\sigma-1} - F)$. Free entry guarantees that profits are non-positive so that we have $ax_t^m \leq (\sigma - 1)F$ with the proviso that: $N_t \geq N_{t-1}$ and

$$(N_t - N_{t-1})[a.x_t^m - (\sigma - 1)F] = 0 \tag{5.48}$$

Thus when $N_t > N_{t-1}$, innovation occurs and new products are introduced; the innovator earns no extra profits and operates at its break even point; while if this is not possible, potential sales do not break even, then $N_t = N_{t-1}$ and no innovation takes place. In this situation, of course there is a constraint which needs to be looked into. Recall that available capital is $K_{t-1}$ and the production of both types of intermediates requires the use of this resource; hence we must have

$$K_{t-1} = N_{t-1}a.x_t^c + (N_t - N_{t-1})(a.x_t^m + F)$$

Recall that if production takes place for both the sets of intermediate products (the new and the old), we have, $N_t > N_{t-1}$ and from (5.47), $a.x_t^c = a.x_t^m\{\frac{\sigma-1}{\sigma}\}^{-\sigma}$ and from (5.48), $ax_t^m = (\sigma-1)F$, so that $ax_t^c = \theta F\sigma$ where $\theta = \{\frac{\sigma-1}{\sigma}\}^{1-\sigma}$. It should be pointed out that as $\sigma \in [1, \infty]$, $\theta \in [1, e]$. If on the other hand, $N_t = N_{t-1}$, $ax_t^c = \frac{K_{t-1}}{N_{t-1}}$. Thus, we have:

$$ax_t^c = \min[\frac{K_{t-1}}{N_{t-1}}, \theta\sigma F] \tag{5.49}$$

And further,

$$N_t = N_{t-1} + \max[0, \frac{K_{t-1}}{\sigma F} - \theta N_{t-1}] \tag{5.50}$$

Substituting the above into (5.46), we have:

$$Y(t) = \begin{cases} AL^{\frac{1}{\sigma}} N_{t-1}(\frac{K_{t-1}}{aN_{t-1}})^{1-\frac{1}{\sigma}} & \text{for } \sigma F\theta N_{t-1} \geq K_{t-1} \\ AL^{\frac{1}{\sigma}}[N_{t-1}(\frac{\theta\sigma F}{a})^{1-\frac{1}{\sigma}} + (\frac{K_{t-1}}{\sigma F} - \theta N_{t-1})(\frac{(\sigma-1)F}{a})^{1-\frac{1}{\sigma}}] & \text{otherwise} \end{cases} \tag{5.51}$$

And finally to close the model, we have that a constant fraction of the output is left unused so that it may be used as capital in the next period,[44] i.e.,

$$K_t = \mu Y_t \tag{5.52}$$

[44]It is this which provides the point of departure for the contribution of Matsuyama (2001) where capital accumulation is derived from intertemporal optimization of the infinitely lived agent. This does lead to a complication in the dynamics.

To simplify matters, we introduce the following notation:

$$\alpha = \frac{A(aL)^{\frac{1}{\sigma}}}{a(\theta\sigma F)^{\frac{1}{\sigma}}},\ k_t = \frac{K_t}{N_t\sigma F\theta}$$

The dynamics of the entire system is captured through the equations (5.50), (5.51) and (5.52). And we may combine them in to a single equation, using the variable $k_t$ introduced above as follows:

$$k_t = \begin{cases} \mu\alpha k_{t-1}^{1-\frac{1}{\sigma}} & \text{if } k_{t-1} \leq 1 \\ \dfrac{\mu\alpha k_{t-1}}{1+\theta(k_{t-1}-1)} & \text{otherwise} \end{cases} \tag{5.53}$$

In short, we shall write

$$k_t = \phi(k_{t-1}) \tag{5.54}$$

and given any initial $k^0$, we shall study the iterates $\phi^t(k^0)$, for $t = 1, 2, ...$

It should be noted $\phi(1) = G = \mu\alpha$; further that $\phi'(k) = (1 - \frac{1}{\sigma})Gk^{-\frac{1}{\sigma}} > 0$ if $k < 1$ and $\phi'(k) = \dfrac{G(1-\theta)}{(1+\theta(k-1))^2} < 0$ if $k > 1$. Thus the map $\phi(.)$ is of the standard uni-modal variety with a maximum value at 1.

FIGURES 4a and 4b HERE

### 5.5.6 Possible Equilibria

It may be easily seen by referring to (5.53) that a crucial parameter for the system is $G = \phi(1)$. Note also that as $k \to \infty$, $\phi(k) \to \dfrac{G}{\theta}$, which is finite, so that for all $k$ large enough, $\phi(k) < k$; also since $\phi(k) > k$ for all $k$ small enough (since $\phi(0) = 0$ and $\phi'(k) > 1$

for all $k$ small enough), it follows that there exists $k > 0$ such that $k = \phi(k)$. Further such a positive $k$ is unique too.

There are two possibilities depending on the magnitude of $G$; first of all notice that if $G < 1$, $k^*$ is an equilibrium $\Rightarrow k^* < 1$; consequently $k^* \leq 1 \Rightarrow k^* = G^\sigma$; and if $G > 1$ then $k^* > 1 \Rightarrow k^* = \frac{G-1}{\theta} + 1$. Consider then these two cases one by one.

Consider first, $G \leq 1$; as indicated above, in this case, $k^* = G^\sigma$. In this equilibrium, notice that $N$ remains fixed and hence no innovation occurs; since $N$ is constant and given the definition of the variable $k$, $K$ too remains fixed and hence $Y$, so that no growth takes place. Beginning from any arbitrary $k^0$, the iterates settle to the regime $k < 1$ (Matsuyama calls this the Solow regime) and converges to $k^*$ since, $|\phi'(k^*)| = |(1 - \frac{1}{\sigma})| < 1$; thus $k^*$ is locally asymptotically stable.

Consider next $G > 1$; $k^* = \frac{G-1}{\theta} + 1$ and $|\phi'(k^*)| = |\frac{\theta - 1}{G}|$; thus it is seen that local asymptotic stability requires that $\theta - 1 < G$. If on the other hand, $\theta - 1 > G$, the equilibrium is unstable. At this equilibrium notice that $N$ is increasing continuously and keeping pace with it is $K$ and as we may see $Y$ too is increasing all at the same rate: the case of balanced growth. Along the balanced growth path, where $k_t = k^* > 1$, it follows from (5.50) that $\frac{N_{t+1}}{N_t} = G$ and consequently, $\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = G$ as well; thus $G - 1$ is the growth rate per period along the balanced growth path.

### 5.5.7 2-cycles and Their Stability

When $G > 1$ and $\theta - 1 > G$, the equilibrium is unstable as we have shown above; but there is a 2-cycle, i.e., $\exists k_1 > 1 > k_2$ such that $k_1 = \phi(k_2)$ and $k_2 = \phi(k_1)$.

Thus it is asserted that there exist $k_1 > 1 > k_2$ satisfying the following:

$$k_2 = \frac{Gk_1}{1 + \theta(k_1 - 1)}$$

and

$$k_1 = Gk_2^{1-\frac{1}{\sigma}}$$

Using the second relation to eliminate $k_1$, we have the following claim: there is $0 < k_2 < 1$ such that

$$k_2^{1-\frac{1}{\sigma}}[(1-\theta)k_2^{\frac{1}{\sigma}} + G\theta k_2 - G^2] = 0$$

Note that writing $f(k_2) = [(1-\theta)k_2^{\frac{1}{\sigma}} + G\theta k_2 - G^2]$, $f(0) = -G^2 < 0$ while $f(1) = (G-1)(\theta - 1 - G) > 0$ and hence the claim follows. Also note that $f'(0) < 0$ and $f'(k) = 0 \Rightarrow k = \overline{k} = \{\frac{\theta - 1}{\sigma\theta G}\}^{\frac{\sigma}{\sigma-1}}$ and $f(.)$ is decreasing for $k < \overline{k}$ and increasing for $k > \overline{k}$; consequently, there is a unique $k_2$ satisfying the above claim.

It is easy to see that there cannot be any two period cycle $(k_1, k_2)$, $0 < k_1 < k_2 < 1$. For then $k_1 = Gk_2^{1-\frac{1}{\sigma}}$ and $k_2 = Gk_1^{1-\frac{1}{\sigma}}$; this implies that $k_1 = G^\sigma > 1$ which is a contradiction.

Nor can there be a cycle $(k_1, k_2)$, $1 < k_2 < k_1$; for other wise $k_2 + k_2\theta(k_1 - 1) = Gk_1$ and $k_1 + k_1\theta(k_2 - 1) = Gk_2$ so that $(k_2 - k_1)(1 - \theta + G) = 0$ which too is a contradiction.

Thus:

**Claim 5.5.13** *There is a unique 2-period cycle* $(k_1, k_2)$ *with* $0 < k_2 < 1 < k_1$ *when* $\theta - 1 > G > 1$.

The stability of this cycle depends on whether the following is greater or less than unity:

$$|\phi'(k_1).\phi'(k_2)| = |\frac{G^2(1-\frac{1}{\sigma})(\theta-1)}{k_2^{\frac{1}{\sigma}}\{1+\theta(k_1-1)\}^2}| = [\frac{G^2(\theta-1)}{k_2^{\frac{1}{\sigma}}}].\frac{(1-\frac{1}{\sigma})}{\{1+\theta(k_1-1)\}^2}$$

$$= \frac{(\theta-1)(1-\frac{1}{\sigma}).G^2.k_1^2}{\{1+\theta(k_1-1)\}^2.k_1^2.k_2^{\frac{1}{\sigma}}}$$

$$= \frac{\theta-1}{G}.\frac{k_2^{\frac{1}{\sigma}}(1-\frac{1}{\sigma})}{G}$$

$$= A.B \text{ say}$$

Note that $A > 1$ while $B < 1$ so that the product could go either way. If the above happens to be less than unity then, the 2-period cycle is attracting while if the quantity is greater than unity the cycle is unstable. It should be clear that the smaller is the ratio $A$, the greater is the chance of the cycle being stable. As we shall see, this point is of crucial importance for locating a robust range of parameter values for which the cycles will be stable. There can be nothing more definite about the aspect of stability since as is clear from the above expression, both possibilities exist.

For instance, consider the following situations for $\sigma = 5$ and hence, $\theta = 2.44$; but for alternative similar looking values of $G$, we have:

**Example 1** *Let* $G = 1.075 < \theta - 1 = 1.44$. *Then the two period cycle is given by* $k_1 = 1.06487 > 1 > k_2 = 0.988226$; $|\phi'(k_1).\phi'(k_2)| = 0.995503$ *hence the cycle is stable.*

**Example 2** *Let* $G = 1.070 < \theta - 1 = 1.44$. *Then the two period cycle is given by* $k_1 = 1.06041 > 1 > k_2 = 0.988805$; $|\phi'(k_1).\phi'(k_2)| = 1.00491$ *hence the cycle is unstable.*

Since both kinds of cycles are possible, consider then the merit of the following proposition (Proposition 2, Matsuyama (1999)):

*Let $g_x$ be the gross growth rate of the variable $x$. Along the period-2 cycles*

*(a)* $g_N = 1 < G < G(k_2)^{-\frac{1}{\sigma}} = g_K = g_Y$ *in the Solow Regime.*

*(b)* $g_N = 1 + \theta(k_1 - 1) > G = g_k = g_Y$ *in the Romer Regime.*

*(c)* $g_N = g_K = g_Y = \{1 + \theta(k_1 - 1\}^{\frac{1}{2}} = Gk_2^{\frac{1}{2\sigma}} > G$ *over the cycles.*

The claims made above were meant to show that output $(Y)$ and investment $(K)$ grow faster in the Solow Regime: i.e., (when $k < 1$ and no innovation takes place); in other words, even though innovation is essential to sustain growth, during the process of innovation (the Romer regime when $k > 1$) the economy registers lower growth. After innovation stops, markets become competitive; only then, higher growth is possible. And it is along the cycles that the economy grows faster than along the unstable balanced growth path.

As should be obvious, this conclusion is of some interest only if the cycle is stable, since otherwise the fact, that along the cycle a higher growth rate is possible, is of little interest. It is because of this that the calculations about the stability of the cycles provided above assume significance.

### 5.5.8 Are Robust Stable 2-Cycles Plausible?

What kind of parameter values are sure to provide us with stable cycles ? The Examples (1) and (2) show that with roughly similar looking values of $G$ one may get different results. But there is some thing more that one may say in addition to the above.

Consider the situation when $G = \theta - 1$; it is immediate, by referring back to (5.53), that in this case, $\phi(1) = G$ and $\phi(G) = 1$; further $|\phi'(1).\phi'(G)| = (1 - \frac{1}{\sigma})/G < 1$ so that this cycle is stable (here $\phi'(1)$ is to be interpreted as the derivative on the left); this assertion follows by considering the expression $A.B$ derived earlier; now $A = 1$ and hence the assertion follows. But in this particular situation there is an embarrassment of cycles; since for any $k_1 > 1, k_1 < G$ there is $k_2 > 1$ such that $\phi(k_1) = k_2$ and $\phi(k_2) = k_1$. But these cycles disappear as soon as $G < \theta - 1$, as we have seen above but, it should be clear that the cycle close to $(1, G)$ remains and that cycle retains stability.

Thus if $G$ is close but less than $\theta - 1$ then the resultant cycle is stable. Note that for the situation described in Examples 1 and 2, $\theta - 1$ is about 1.44; thus some where between the values for $G$ given by 1.07 and 1.075, the two-period cycles change their stability property and we have a point of bifurcation; for lower values of G there may be other points of bifurcation. Indeed, Mitra (2001) has shown that this model is capable of exhibiting topological chaos too (i.e., there may be a cycle whose period is not a power of 2)[45] for certain configuration of parameters. The fluctuation in the interval $[\phi(G), G]$ which we noted earlier, may in fact be quite complicated.

In other words, very complicated dynamics may be possible in this framework and consequently, the relevance of the Proposition such as the one mentioned above may be ascertained only if we are assured that parameters have certain values which guarantee the robust stability of the two-period cycles i.e., $G$ is close to $\theta - 1$. This is robust because

[45] Consequently, for every integer $n$, there is a cycle with period $n$ and there is an uncountable set $X$ such that for every $k^o \in X$ the iterates $\phi^t(k^o)$ do not exhibit any periodicity. See the Example 3 below for one such configuration.

there would be an open set $O = (\overline{G}, \theta - 1)$ and any $G \in O$ would ensure stability of the two period cycles. The lower bound on this interval amounts to where the product $A.B$ becomes unity. Matsuyama seems to be aware of this problem although he does not appear to quite attribute either the stability or the robustness aspect the importance each deserves[46].

However before leaving the question of stability of the two period cycles, we present an estimate of the of the range of $G$-values for which the two cycles retain stability. To obtain an estimate for $\overline{G}$, we may note that the expression in $A.B$ may be written as

$$\frac{(\theta - 1)(1 - \frac{1}{\sigma})}{G^2} . k_2^{\frac{1}{\sigma}} = C.D \text{ say}$$

where we know that $D < 1$ so now if $C < 1$ then we are sure that $A.B < 1$ and the cycle is stable. Thus a sufficient condition for the stability of the two-cycle is:

**Claim 5.5.14** *If $G > \hat{G} = \sqrt{((\theta - 1)(1 - \frac{1}{\sigma}))}$ then the 2-cycle is stable.*

The claim follows by noticing that whenever the condition is satisfied, the term $C < 1$. In fact for $\sigma = 5$, for example, we have, $\hat{G} = 1.07384$ so that, whenever $G > 1.07384$ the two-cycle is stable[47]. This allows us to obtain an estimate of the open interval $O$, viz., $(\hat{G}, \theta - 1)$

[46]See, for example, Matsuyama (1999), p. 344.; it is clear from this that there may be ranges over which the 2-cycles may be unstable; also over this range, the attracting cycles may have periods which are powers of 2, as we see below. It appears that the author is more concerned about the empirical plausibility of the existence of 2-cycles, as his discussion on plausibility considers only the condition for the existence of 2-cycles, viz., $\theta - 1 > G > 1$, with $G$ being 'close' to $\theta - 1$. The entire plausibility exercise considers $G = \theta - 1$.

[47]It should be clear that $\hat{G} > \overline{G}$. Thus for $\sigma = 5$, $G \in [1.07384, 1.44)$ implies stability of the two period cycle. In Matsuyama (1999), as we indicated above, $G$ is taken to be $\theta - 1 = 1.44$ while discussing plausibility. Our analysis thus reveals that the 2-cycles are a lot more "plausible".

and for any $G$-value in this interval, the two-cycle is stable. Since we do not know what the magnitude of the parameter $\sigma$ will be, let us take the range suggested in Matsuyama (1999), $[5, 22]$; we examine next, how the range of $G$-values which imply stability for the 2-cycle, $[\hat{G}, \theta - 1)$ behaves with a variation in the parameter $\sigma$.

**Claim 5.5.15** *Let $H$ be defined as the range of $G$-values which imply stability of the period 2-cycles, i.e., $H = \theta - 1 - \hat{G}$. Then $H$ is an increasing concave function of $\sigma$ in the range $[5, 22]$.*

The proof of this claim is contained in the Figure 5 below[48]

FIGURE 5 HERE

Thus so far as two-cycles are concerned, the range of $G$-values which ensure stability increase with the value of $\sigma$. Be that as it may, we shall show below, that we do not need to confine attention to 2-cycles alone to establish the fact that cycles are growth enhancing. In fact, the range of values for the parameter $G$ for which stable cycles are growth enhancing, as we indicate, cover a much wider range.

### 5.5.9 When 2-cycles fail to attract

When the 2-cycle is not attracting, we need to analyze further. In this section, we shall confine ourselves to the case when $1 < G < \theta - 1$. Given this restriction, we note that $\phi(G) < 1$. To proceed formally, note first of all,

[48]Given the nature of the functions involved, we have used Mathematica to generate the diagram which seems to be enough for the purpose at hand.

**Claim 5.5.16** $\phi : [0, G] \to [0, G]$.

Next, we note that[49]:

**Claim 5.5.17** $[\phi(G), G]$ *is an absorbing state (i.e.,* $\phi(G) \leq k \leq G \Rightarrow \phi(G) \leq \phi(k) \leq G$*).*

In addition,

**Claim 5.5.18** *For any initial point* $k^o \in [0, G]$*,* $k^o \neq 0, k^\star$*,* $\exists t$ *such that* $\phi^t(k^o) \in [\phi(G), G]$ *unless* $\phi^t(k^o) = 0$ *for some* $t$*.*

For, suppose to the contrary, that there is some $k^o \in [0, G]$ such that for no $t$ is $\phi^t(k^o) \in [\phi(G), G]$; then $\phi^t(k^o) \in (0, \phi(G)) \forall t$ i.e., $k_{t+1} = \phi^{t+1)}(k^o) = \phi(k_t) = G k_t^{1-1/\sigma} > k_t$. Thus $\{k_t\}$ is a monotonically increasing sequence bounded above and hence must converge to some $\overline{k}$, say, $\overline{k} \in [0, \phi(G)]$. Note that since $k_{t+1}$ and $k_t$ both converge to $\overline{k}$, $\phi(\overline{k}) = \overline{k} \Rightarrow \overline{k} = 0$ which is a contradiction, since 0 is an unstable equilibrium.

Thus the structure of an arbitrary trajectory is that apart from hitting the unstable equilibrium 0 accidentally, it will enter the interval $[\phi(G), G]$ in finite time and remain inside, thereafter. Limit points for the trajectory will thus exist and will be located in the interval $[\phi(G), G]$. Our next task is to locate these limit points, if possible. We note, however, that cyclical orbits, if these exist have a special structure:

**Claim 5.5.19** *Consider any cycle of period* $n$*,* $k_1, k_2, ..., k_n \in [0, G]$*, say, with* $k_1 = \min_{1 \leq j \leq n} k_j$ *must have* $k_1 < 1$ *and* $k_n > 1$*.*

[49] See, for instance, Matsuyama (1999), p. 342.

First of all note that $k_n > k^\star$ since otherwise $k_1 = \phi(k_n) \geq k_n$: a contradiction. Thus $1 < k^\star < k_n \leq G$. Hence, $\phi(1) = G > k^\star > k_1 > \phi(G)$. Suppose that $k_1 \geq 1$ that is, we have: $1 \leq k_1 < k^\star, k_n$. Then, $\phi(k_1) = k_2 > \phi(k^\star) = k^\star \Rightarrow k_3 = \phi(k_2) < k^\star$ but $k_3 > k_1$ since recall that $k_1$ was the minimum. Thus $k^\star > k_3 > k_1 \geq 1$ and $G > k_2 > k^\star$; proceeding in this manner, it may be concluded that $n$ must be even i.e., $n = 2s$ say and we must have

$$k_1 < k_3 < k_5 .... k^\star < k_{2s=n} < k_{2(s-1)} < ... k_2 \leq G$$

But now $k_1 \neq \phi(k_n)$: a contradiction. Hence $k_1 < 1$ as claimed.

Thus **any** $n$-cycle must spend at least one period in the Solow- Regime. This fact allows us to note the following property of growth rates along any $n$-period cycle. Consider any such cycle $k_1, k_2, ..., k_n$; we shall write $k_{n+1} = k_1$ and we note the following: If $k_j < 1$ then $k_{j+1} = G k_j^{1-1/\sigma}$; thus we have

$$\frac{k_{j+1}}{k_j} = G k_j^{-1/\sigma}$$

which means that

$$g_K = \frac{K_{j+1}}{K_j} = G k_j^{-1/\sigma} > G$$

since **no innovation occurs** and $N_{j+1} = N_j$. On the other hand, if $k_j > 1$, we have

$$\frac{k_{j+1}}{k_j} = \frac{G}{1 + \theta(k_j - 1)}$$

and further since **innovation takes place** , we have from equation (5.50)

$$g_N = \frac{N_{j+1}}{N_j} = 1 + \theta(k_j - 1)$$

consequently, we have:

$$\frac{K_{j+1}}{K_j} . \frac{N_j}{N_{j+1}} = \frac{g_K}{g_N} = \frac{G}{1 + \theta(k_j - 1)}$$

which means that $g_K = G$, given the expression for $g_N$ obtained above. With these preliminaries, let us return to the $n$-cycle $k_1, k_2, ..., k_n$ with $k_{n+1} = \phi(k_n) = k_1$ which means that

$$\frac{K_{n+1}}{N_{n+1}} = \frac{K_1}{N_1} \Rightarrow \frac{K_{n+1}}{K_1} = \frac{N_{n+1}}{N_1}$$

in other words, we have:

$$\frac{K_{n+1}}{K_n} ... \frac{K_2}{K_1} = \frac{N_{n+1}}{N_n} ... \frac{N_2}{N_1}$$

on the left hand side, each term is either $G$ if the corresponding $k_j \geq 1$ or is $G.k_j^{-1/\sigma}$ if the corresponding $k_j < 1$. Thus the product on the left hand side is greater than $G^n$, since we have shown that at least one of the $k_j$'s namely $k_1$ is less than 1. Consequently, the **average gross growth rate $g_K = g_Y = g_N$ along the cycle must be greater than $G$.** We note this in the form of the following claim:

**Claim 5.5.20** *Along any $n$-period cycle, the average gross growth rate $g_K = g_Y = g_N > G$.*

Let us return to the **Example 2** , where the 2-cycle had been shown to be unstable. It may be shown that in that context viz., $G = 1.07, \sigma = 5$: there is a stable 4-period cycle $k_1 < 1, k_2 > 1, k_3 < 1, k_4 > 1$, $\phi(k_i) = k_{i+1}, i = 1, 2, 3, 4$ with $k_5 = k_1$. $k_1 = 0.9888, k_2 = 1.060402, k_3 = 0.988814, k_4 = 1.0604139$; further it may be checked that $|\phi'(k_1)....\phi'(k_4)| = 0.000583969 < 1$ which signifies that the 4 period cycle is stable.

Figure 6 demonstrates the existence and convergence to such a 4-period cycle.

FIGURE 6 HERE

For the case $\sigma = 5$, it has not been possible to locate any more points of bifurcation; that is on the basis of simulation exercises, and the bifurcation diagram (Figure 7) it seems reasonable to conclude that there may be **at the most** a stable 4-period cycle.

FIGURE 7 HERE

The bifurcation diagram[50] plots for values of $G$ between $\theta - 1 = 1.4$ and 1, the last 100 points in an iteration of 1000 points from an arbitrary initial point. For $G$ between 1 and 1.4, beginning with the high $G$ values, we note the first approach to a two period cycle; for lower values of $G$, the two period cycle disappears and a stable four period cycle appears. This happens for values of $G$ close to 1 and it has not been possible to uncover other points of bifurcation[51]. Outside the range, we note convergence to a steady state, as the results indicate.

We consider next bifurcation diagrams for narrower ranges of the parameters but still for $\sigma = 5$: first consider the range of $G$ values between 1.4 and 1.48.

FIGURE 8 HERE

Notice the shift from convergence to the fixed point to convergence to the two-period cycle as the value of $G$ decreases. And consider next the range of $G$ values between 1 and 1.08, still for $\sigma = 5$.

[50] All diagrams were prepared on Mathematica Version 4.1 .

[51] The exercises we have conducted seem to indicate that bifurcations leading to cycles with period 8 and period 16 and more are non-existent, at least when $\sigma$ is small; see Matsuyama (1999), p.344. For such possibilities, $\sigma$ has to exceed 22. See below.

FIGURE 9 HERE

Notice here the shift from convergence to a two period cycle to convergence to a four period cycle as $G$ attains the value unity. Thus Figures 8 and 9 focus attention on the two extreme ranges of the parameter $G$ contained in Figure 7.

We present next, an example of topological chaos when $G$ is small and $\sigma$ is large. For this purpose it would be helpful to note the following:

**Claim 5.5.21** $\phi^2(1) = \phi(\phi(1)) < 1 \Leftrightarrow 1 < G < \theta - 1$.

Thus a necessary condition for topological chaos[52] is always satisfied, for every value of $G$ whenever, the steady state is unstable. But to clinch matters and to show the presence of complicated dynamics, we need to show that $\phi^3(1) < k^\star$ [53]

**Example 3** *Consider $\sigma = 22$ and $G = 1.001$; one may then check that $\phi^3(1) < k^\star$. We thus have topological chaos for this map. For values of $\sigma \geq 22$, low values of $G$ imply the existence of topological chaos*[54].

The bifurcation diagram for $\sigma = 22$ is provided below.

FIGURE 10 HERE

The above set of exercises allows us to draw the following conclusions:

[52] See, for instance, Mitra (2001), p. 140, (2.5).

[53] Mitra (2001), Proposition 2.3, p. 142.

[54] See Mitra (2001), p.142 where he looks at the case when $\sigma = 50, G = 1.01$ to apply his result.

- When the steady state is unstable, two-cycles exist; further, any $G$ value in the set $[\hat{G}, \theta - 1)$ implies that the two-cycle is stable. This range increases with $\sigma$ over the range $[5, 22]$ of plausible $\sigma$-values.

- When $\sigma$ is small (around 5), the steady state and the two period-cycle is unstable, there would be a stable cyclical orbit with period 4; cycles with higher periods do not appear possible.

- When $\sigma$ is large ($\geq 22$), possibilities for chaos exist particularly if the value of $G$ is small enough.

- Whenever the steady state is unstable and a stable cycle exists, the rate of growth along the cycle will be greater than that along the unstable steady state.

# References


Akerlof, G. A and Sitglitz, J. E., (1969), Capital, Wages and Structural Unemployment, **The Economic Journal,** 79, 269-281.

Anderson, C. M., Granat, S., Plott, C. R. and Shimomura, K-I., (2002), Global Instability in Experimental General Equilibrium: The Scarf Example, **Journal of Economic Theory,** (FORTHCOMING).

Andronov, A. A., Vitt, A.A., Khaikin, S.E.,(1966), **Theory of Oscillators,** (translated from Russian by F. Immirzi; translation edited and abridged by W. Fishwick), Pergamon Press: Oxford.

Arrow. K.J. and F.H. Hahn, (1971) **General Competitive Analysis,**Holden-Day: San Francisco.

Arrow, K.J., Hurwicz, L. and Uzawa, H., (1961), Constraint Qualifications in Maximization Problems, **Naval Research Logistics Quarterly,** VIII, 175-191.

Azariadis, C. (1993), **Intertemporal Macroeconomics,** Basil Blackwell, Oxford, U.K.

Bellman, R., (1960), **Introduction to Matrix Analysis,** McGraw-Hill: New York.

Bala, V., (1997), A Pitchfork Bifurcation in the Tatonnement Process, **Economic Theory,** 10, 521-530.

Bala, V., and Majumdar, M.(1992), Chaotic Tatonnement, **Economic Theory,** 2, 437-446.

Bala, V., Majumdar, M. and Mitra, T.(1998), A Note on Controlling a Chaotic Tatonnement, **Journal of Economic Behavior and Organization,**33, 411-420.

Bhaduri, A. and Harris, Donald J., (1987), The Complex Dynamics of the Ricardian System,

**Quarterly Journal of Economics,** 102, 893-902.

Boldrin, M. and Montrucchio, L., (1986), On the Indeterminacy of capital Accumulation Paths, **Journal of Economic Theory,** 40, 26-39.

Cass, D.,(1965), Optimum Growth in an Aggregative Model of Capital Accumulation, **Review of Economic Studies,** 32, 233-240.

Champsaur, P., J.H. Dreze and C. Henry, (1977), Stability Theorems with Economic Applications, **Econometrica,** 45, 273-294.

Day, R. H., (1983), The Emergence of Chaos from Classical Growth, **Quarterly Journal of Economics,**98, 201-213.

Day, R.H. and Pianigiani, G. (1991), Statistical Dynamics and Economics, **Journal of Economic Behavior and Organization,** 16, 37-83.

Debreu, G.,(1974), Excess Demand Functions, **Journal of Mathematical Economics,**1(1), 15-21.

Diamond, P., (1965), National Debt in a Neoclassical Growth Model, **American Economic Review,** 55, 1026-50.

Dreze, J.H.,(1975), Existence of an Equilibrium under Price-Rigidity and Quantity Rationing, **International Economic Review,**16, 301-320.

Flaschel, P., (1984) Some Stability Properties of Goodwin's Growth Cycle, A Critical Elaboration, **Zeitschrift für Nationalökonomie, Journal of Economics,** 44, 63-69.

Fujimoto, T. (1990), Stabilizing Scarf's Unstable Examples, **Okayama Economic Review, 22,** 67-71.

Fujimoto, T. and Ranade, R. R., (1988), The Weak Axiom of Revealed Preference for the Dominant Diagonal Case, **The Kagawa University Economic Review,** 16, No. 2, 75-83.

Gale, D., (1963), A Note on the Global Instability of Competitive Equilibrium, **Naval Research Logistics Quarterly,** 10, 81-87.

Gandolfo, G., (1997), **Economic Dynamics,** Study Edition, Springer Verlag: Berlin.

Gantmacher

Goodwin, R.M., (1967), A Growth Cycle, in **Socialism, Capitalism and Economic Growth** , Essays Presented to Maurice Dobb, C.H.Feinstein (ed.), Cambridge University Press: London.

Grandmont, J-M. (1988), Non-Linear Difference Equations, Bifurcations and Chaos: An Introduction, **CEPREMAP** Discussion Paper No. 8811.

Guckenheimer, J. and Holmes, P., (1983), **Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields** Springer-Verlag: Berlin.

Hahn, F. H., (1982), Stability, in **Handbook of Mathematical Economics,** Volume 2, Arrow, K.J. and Intrilligator, M. (eds.), North Holland: Amsterdam.

Hahn, F. H. and Negishi, T.,(1962), A Theorem on Non-Tatonnement Stability, **Econometrica,** 30.

Hicks, J.R., (1946), **Value and Capital (2nd Edition)** , Clarendon Press: Oxford.

Hildenbrand, W. and M. Jerison, (1989), The Demand Theory of the Weak Axiom of Revealed preference, **Economics Letters,**29, 209-213.

Hirota, M., (1981), On the Stability of Competitive Equilibrium and the Patterns of Initial Holdings: An Example, **International Economic Review,** 22, No. 2, 461-470.

Hirota, M., (1985), Global Stability in a Class of Markets with Three Commodities and Three Consumers, **Journal of Economic Theory,** 36, 186-192.

Hirsch, M.,(1982), Systems of Differential Equations which are Competitive or Cooperative, I: Limit Sets, **Siam Journal of Mathematical Analysis,**Volume 13, 167-179.

Hirsch, M.W. and Smale, S., (1974), **Differential Equations, Dynamical Systems and Linear Algebra,** Academic Press: New York.

Hurewicz, W. (1958), **Lectures on Ordinary Differential equations,** , MIT Press: Cambridge, Mass.

Ito, T. (1978), A Note on the Positivity Constraint in Olech's theorem, **Journal of Economic Theory,** 17, 312-318.

Jordan, J. (1983), Locally Stable Price Mechanisms, **Journal of Mathematical Economics,**11, 235-259.

Kamiya, K. (1990), A globally Stable Price Adjustment Process, **Econometrica,** 58, 1481-1486.

Kihlstrom, R., Mas-Colell, A. and Sonnenschein, H., (1976), The Demand Theory of the Weak Axiom of Revealed preference, **Econometrica,** , 44, 971-978.

Kind,C., (1999), Remarks on the Economic Interpretation of Hopf Bifurcations, **Economic Letters,**62, 147-154.

Koopmans, T. C., (1965), On the Concept of Optimal Economic Growth, Lauwerier, H.A., (1986), One dimensional Iterative Maps, in **Chaos,** ed. A. V. Holden, Manchester: Manchester University Press, 39-57.

Li, T., and Yorke, J.A. (1975), Period Three implies Chaos, **American Mathematical Monthly,** 82, 985-992.

Lorenz,H. W. (1993), **Nonlinear Dynamical Economics and Chaotic Motion,** , 2nd Edition, Springer Verlag: Berlin.

Majumdar, M., (1994) Chaotic Dynamical Systems: An Introduction, **Economic Theory,** 4, 641-648.

Majumdar, M. and Mitra, T. (1997), Complexities of Concrete Walrasian Systems, in **Issues in Economic Theory and Policy, Essays in Honor of Professor Tapas Majumdar,** eds., A. Bose et. al., Delhi: Oxford University Press, 41-63.

Majumdar, M. and Mitra, T. (1994), Periodic and Chaotic Programs of Optimal Intertemporal Allocation in an Aggregative Model with Wealth Effects,**Economic Theory,** 4, 649-676.

Malinvaud, E.,(1993), Existence proofs of General equilibrium, Looking Forty years Back, Anniversary lecture at CORE, June 3, 1993, commemorating the Arrow-Debreu-McKenzie contributions at the December 1952 Meetings of the Econometric Society.

Mas-Colell, A., (1985), **The Theory of General Economic Equilibrium, A Differentiable Approach,** Cambridge University Press, Cambridge.

May, R. L., (1976), Simple Mathematical Models with very Complicated Dynamics, **Nature,** 261, 459-467.

McKenzie, L. W., (1959), Matrices with Dominant Diagonals and Economic Theory, in **Mathematical Methods in Social Sciences 1959,**eds., Arrow, K.J., Karlin, S. and Suppes, P., Stanford University Press, Stanford, California, 47-62.

McKenzie, L. W., (1960), Stability of Equilibrium and the Value of Positive Excess Demand, **Econometrica,**28, 606-617.

McKenzie, L. W., (1994), Equilibrium, Trade and Capital Accumulation, Rochester Center for Economic Research, Working paper No. 395, November 1994, Reprinted in **Japanese Economic Review** , 50,4, December 1999, 371-397.

McKenzie, L.W., (2002), **Classical General Equilibrium Analysis** , Princeton University Press, New Jersey.

Misiurewicz, M., (1979), Horseshoes for mappings of the interval, **Bull. Acad. Polon. Sci. Ser. Sci. Math.** 27, 167-169.

Mitra, T., (2001), A Sufficient Condition for Topological Chaos with an Application to a Model of Endogenous Growth, **Journal of Economic Theory, 96** , 133-152.

Mitra and Nishimura eds. Special issue of JET

Montrucchio, L., (1992), Dynamical Systems that Solve Continuous-Time Concave Optimizing problems: Anything Goes, in **Cycles and Chaos in Equilibrium** ed., J. Benhabib, Princeton University Press, princeton, New Jersey, 277-288.

Mukherji, A., (1973), On the Sensitivity of Stability Results to the Choice of the Numeraire, **Review of Economic Studies,** , XL,427-433.

Mukherji, A., (1974), The Edgeworth-Uzawa Barter Stabilizes Prices, **International Eco-**

**nomic Review,** 15, 236-241.

Mukherji, A., (1974 a), Stability in an Economy with Production, in **Trade, Stability and Macroeconomics, Essays in honor of Lloyd Metzler** eds., G. Horwich and P. A. Samuelson, Academic Press, New York, 243-258.

Mukherji, A., (1975),The Hicksian Laws of Comparative Statics and the Correspondence Principle Keio Economic Studies, Vol. 12, No. 1, pp. 41-49.

Mukherji, A. (1989), On Tatonnement Processes, Working Paper, Centre for Economic Studies and Planning, Jawaharlal Nehru University, New Delhi; presented at the Far Eastern Meeting of the Econometric Society, Seoul, Korea, 1991.

Mukherji, A., (1990), **Walrasian and Non-Walrasian Equilibria: An Introduction to General Equilibrium Analysis**, Clarendon Press: Oxford.

Mukherji, A., (1990a), A Non-Taonnement Process, in **Economic Theory and Policy: Essays in honor of Professor D. Banerjee,**(eds.), Debraj Ray et. al., Oxford University press: Delhi.

Mukherji, A., (1999), A Simple Example of Complex Dynamics, **Economic Theory,** 14, No. 3, 741-749.

Mukherji, A., (1999a), The Robustness of Closed Orbits in a Class of Dynamic Economic Models, CESP-DSA Working Paper 05/2000, Presented to the World Congress of the Econometric Society, Seattle, August 2000.

Mukherji, A., (2000), Non-linear Dynamics with applications to economics: Stability of Competitive equilibrium reconsidered, **Journal of Quantitative Economics, 16** 93-144.

Mukherji, A., (2002), **An Introduction to General Equilibrium Analysis, Walrasian and Non-Walrasian Equilibria** , Oxford University Press, Delhi (an enlarged and revised edition of Mukherji (1990)).

Murata, Y., (1977), **Mathematics for Stability and Optimization of Economic Systems,** Academic Press: New York.

Negishi, T., (1962), Stability of a Competitive Economy, **Econometrica,** 30, 635-669.

Nishimura, K., (2001), Equilibrium Growth and Nonlinear Dynamics in Continuous-time Models, **Japanese Economic Review,** 52, 1-19.

Nishimura, K. and Sorger, G., (1996), Optimal Cycles and Chaos: A Survey, **Studies in Non-linear Dynamics and Econometrics,** , 1, 11-28.

Neuefeind, W., (1980), Notes on Existence of Equilibrium Proofs and Boundary Behavior of Supply, **Econometrica,** 48, 1831-1837.

Nishimura, K. and Yano, M. (1994), Optimal Chaos, Non-Linearity and Feasibility Conditions, **Economic Theory,** 4, 689-704.

Nishimura, K. and Sorger, G., (1996), Optimal Cycles and Chaos: A Survey, **Studies in Non-linear Dynamics and Econometrics,** 11-28.

Olech, C. (1963), On the Global Stability of an Autonomous System on the Plane, **Contributions to Differential Equations** Volume 1, 389-400.

Ortega, J.M. and Rheinboldt, W.C., (1970), **Iterative Solutions of Nonlinear Equations in Several Variables,** Academic press: New York.

Ramsey, F., (1928), A Mathematical Theory of Saving, **Economic Journal,** 38, 543-59.

Rose, H., (1967), On the Non-Linear Theory of the Employment Cycle, **Review of Economic Studies,** 34, 153-173.

Ruelle, D., (1979), Strange Attractors, **Mathematical Intelligencer** 2, 167-192.

Ruelle, D., (1988), Can Non-Linear Dynamics help Economists ? in **The Economy as and Evolving Complex System,** , Anderson, P., Arrow, K.J., and Pines, D. ( eds.), Santa-Fe Institute Studies in the Science of Complexity, Addison Wesley: Reading, Mass.

Samuelson, P. A., (1947), **Foundations of Economic Analysis,** Harvard University Press: Cambridge, Mass.

Samuelson, P. A., (1958), An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money, **Journal of Political Economy,** 66, 467-482.

Saari, D.G., (1991) Erratic Behavior in Economic Models, **Journal of Economic Behaviour and Organization,** 16, 3-36.

Saari, D. G. and Simon, C. T., (1978), Effective Price Mechanisms, **Econometrica** , 46, 1097-1125.

Scarf, H. (1960), Some Examples of Global Instability of the Competitive Equilibrium, **International Economic Review,** 1, 157-172.

Silvestre, J.,(1982), Fixprice Analysis in Exchange Economies, **Journal of Economic Theory,** 26, 28-58.

Smale, S., (1976), On the Differential Equations of Species in Competition, **Journal of Mathematical Biology,** 3, 5-7.

Smale, S., (1976 a), A Convergent Process of Price Adjustment and Global Newton Methods, **Journal of Mathematical Economics,** 3, 107-120.

Solow, R., (1956), A Contribution to the Theory of Economic Growth, **Quarterly Journal of Economics,** 70, 65-94.

Sonnenschein, H.,(1972), Market Excess Demand Functions, **Econometrica,**40, 549-563.

Swan, T., (1956), Economic Growth and capital Accumulation, **Economic Record,** 22, 334-361.

Uzawa, H.,(1958), Iterative Methods for Concave Programming, in **Studies in Linear and Non-linear Programming,**Arrow, K.J., Hurwicz, L. and Uzawa, H. (eds.), Stanford University Press: Stanford, California.

Uzawa, H., (1961), The Stability of Dynamic Processes, **Econometrica,** 29, 617-631.

Uzawa, H., (1962), On the Stability of Edgeworth's Barter Process, **International Economic Review,** 3, 218-232.

Younes, Y.,(1975), On the Role of Money in the Process of Exchange and the Existence of a Non-Walrasian equilibrium, **Review of Economic Studies,**42, 489-501.

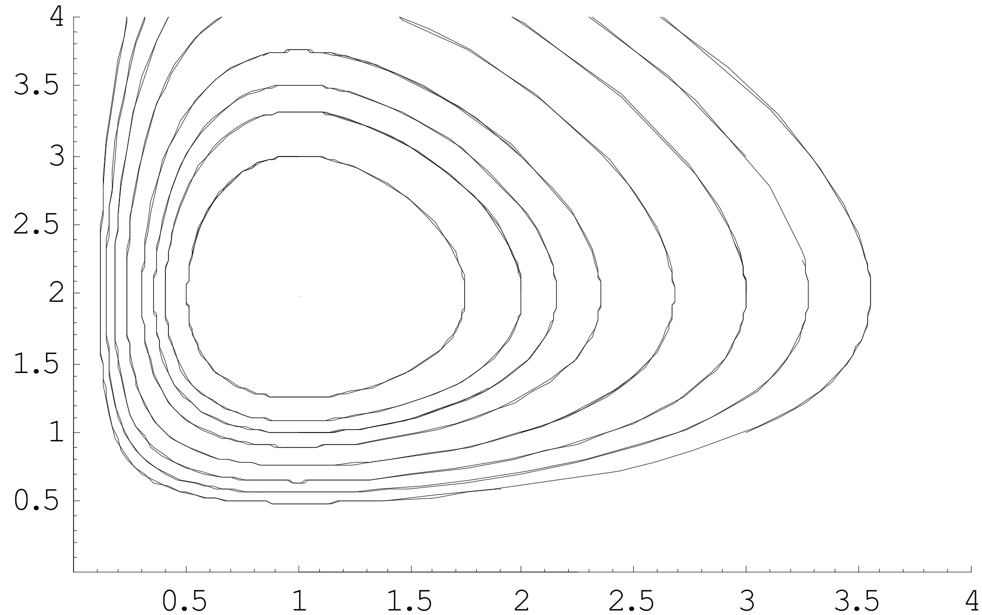


**Figure 1: The Predator-Prey Model (b, c, d = 1, a = 2)**

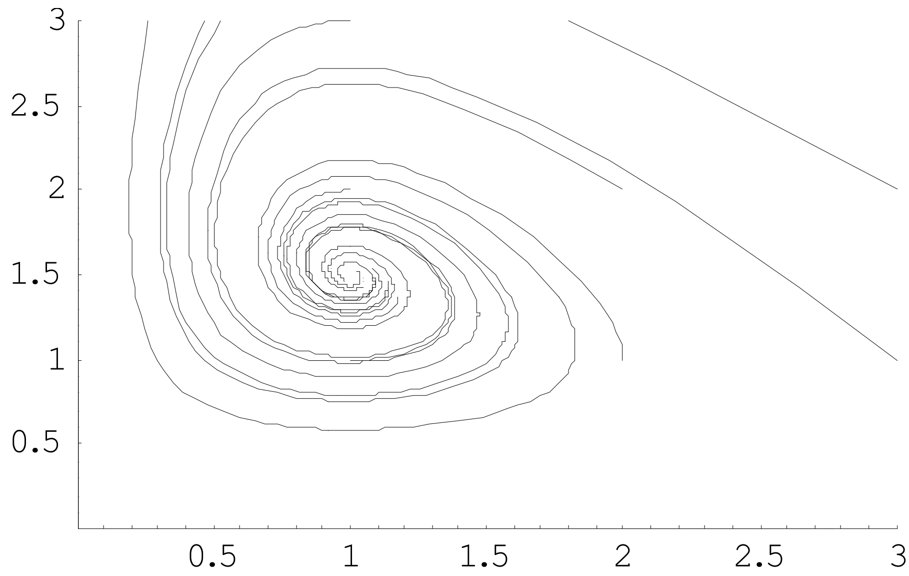


**Figure 2: The Predator-Prey Model (a, b, c, d as above; γ=-0.5)**

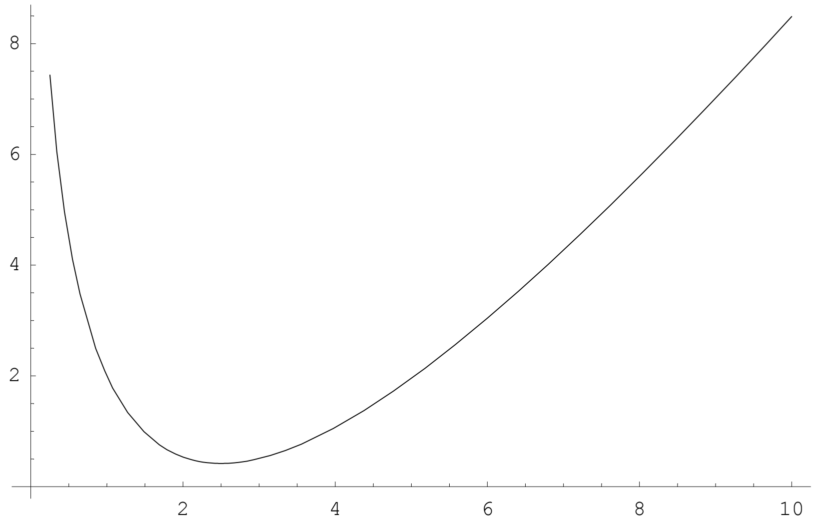


**Figure 3: The Function $\theta_1 x - \theta_2$ Log x, $\theta_i > 0$**

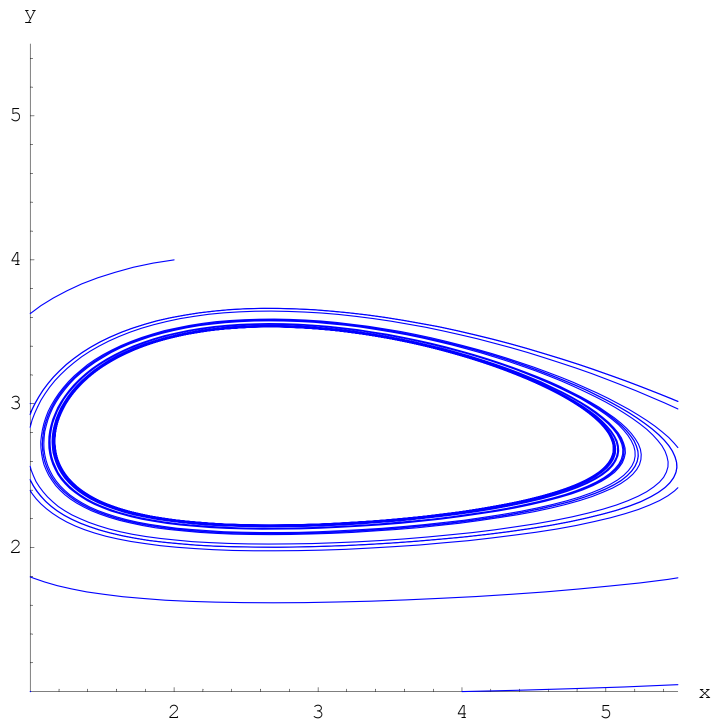


**Figure 4: Robust Periodic Behavior in a Lotka-Volterra Model**

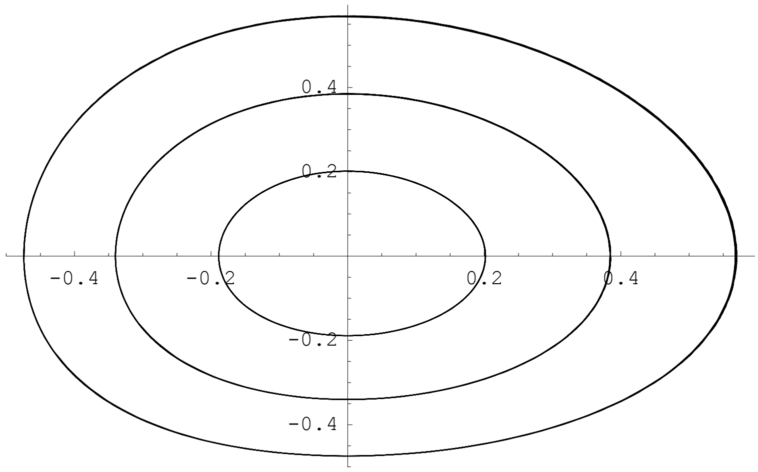


**Figure 5 : Closed Orbits of the Scarf Example**

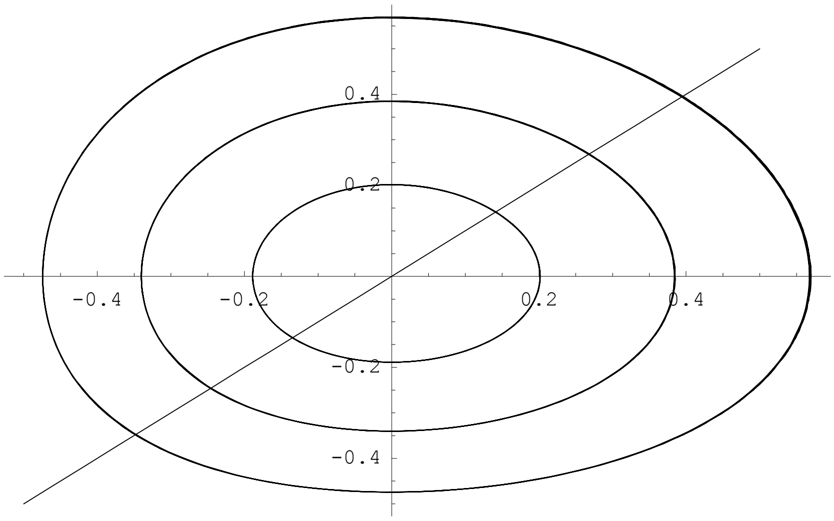


**Figure 6 : Scarf Example : The Set K**

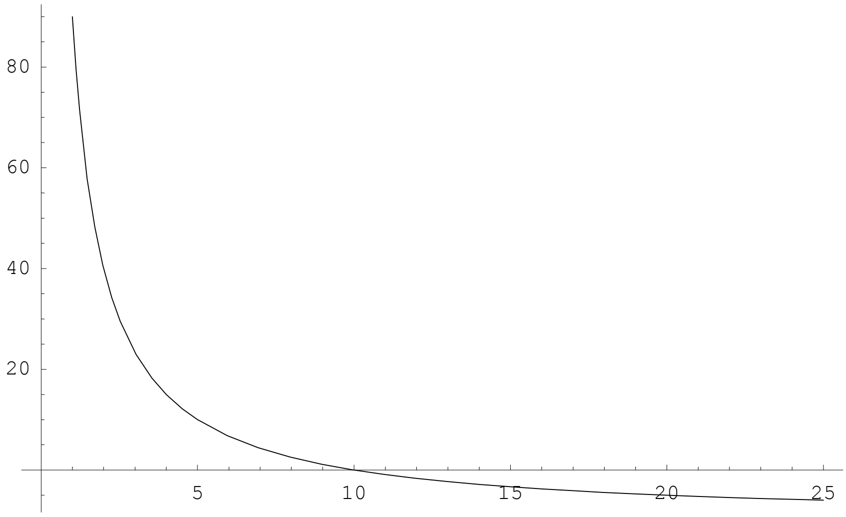


**Figure 7 : The Excess Demand Function**

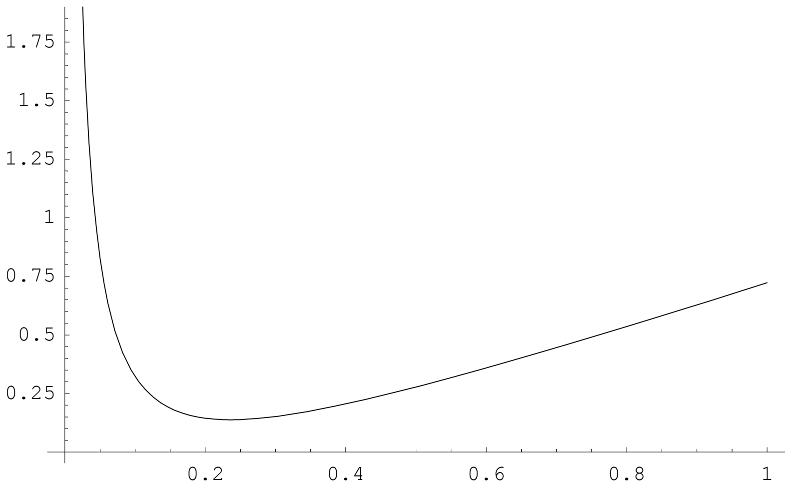


**Figure 8 : The f − map with K > 1**

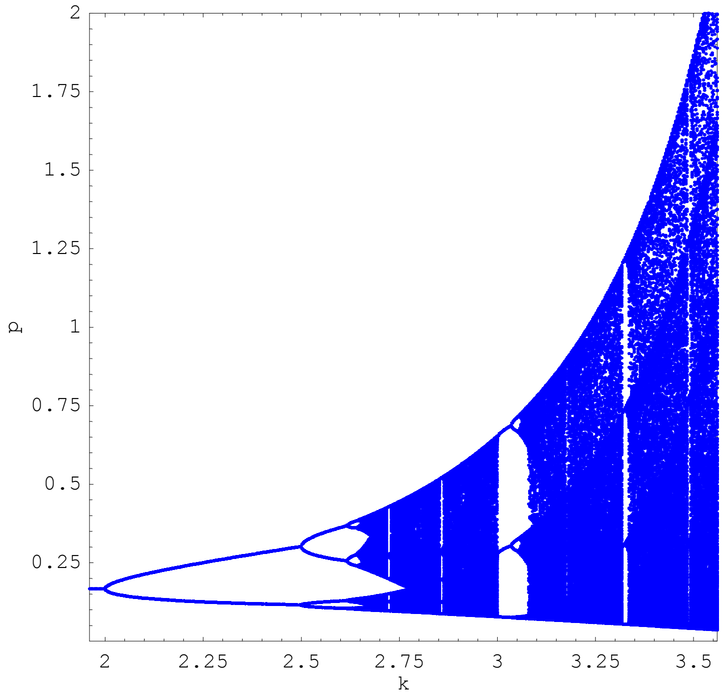


**Figure 9: The Bifurcation Map**

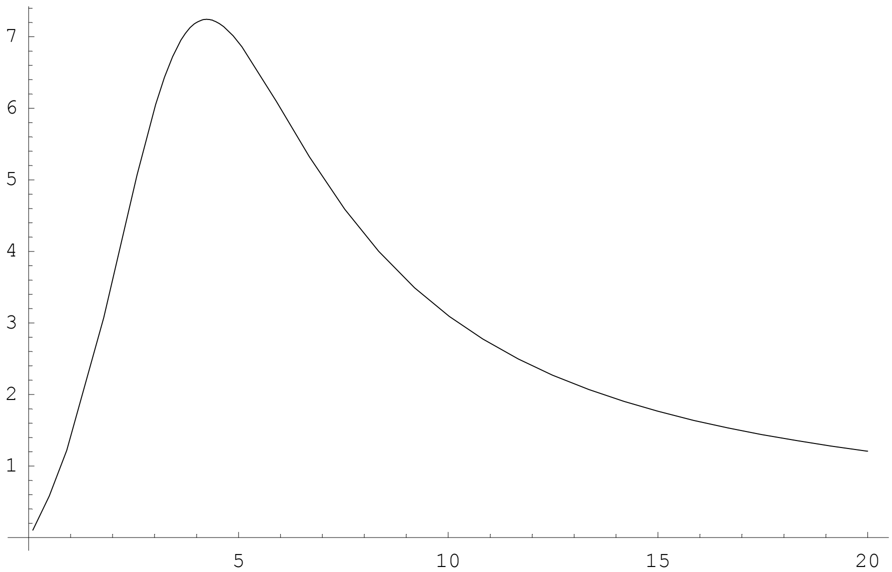


**Figure 10 : The g – Map**

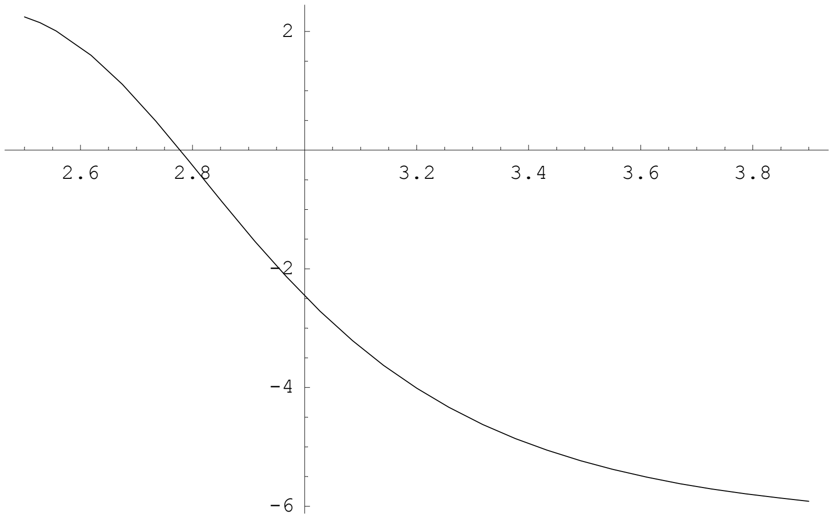


**Figure 11 : The Range for Topological Chaos**

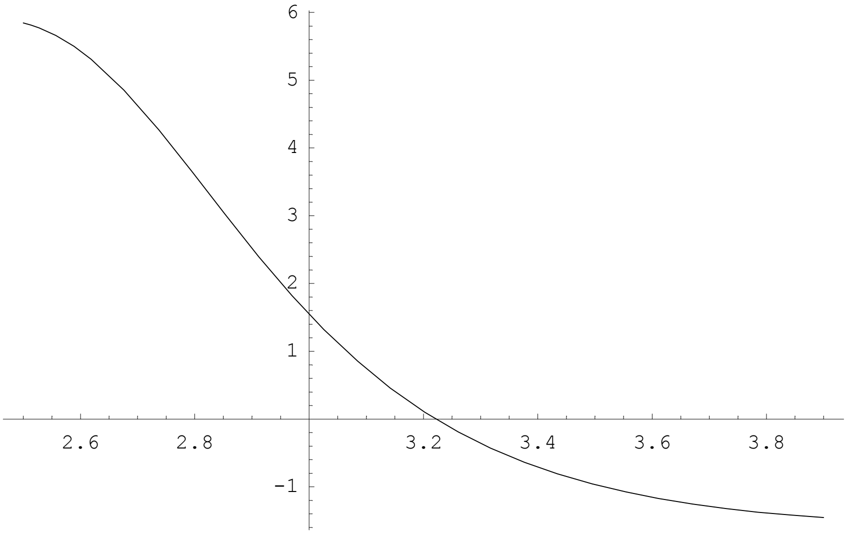


**Figure 12 : Ergodic Chaos**

# Diagrams for Chapter 5

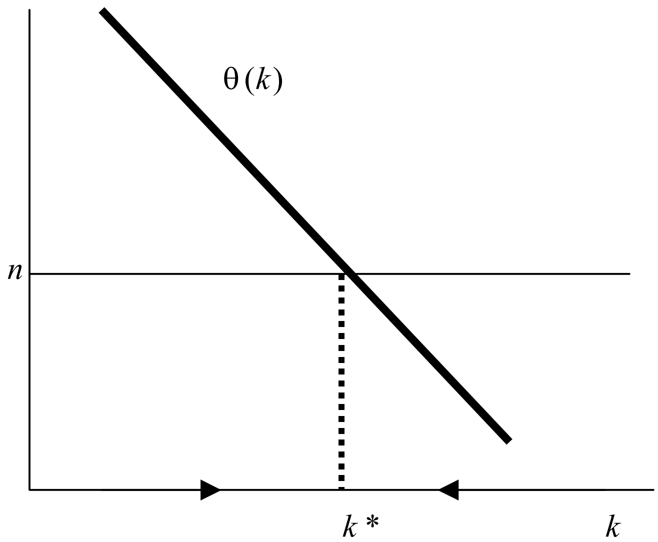


**Figure 1: Convergence to Steady State**

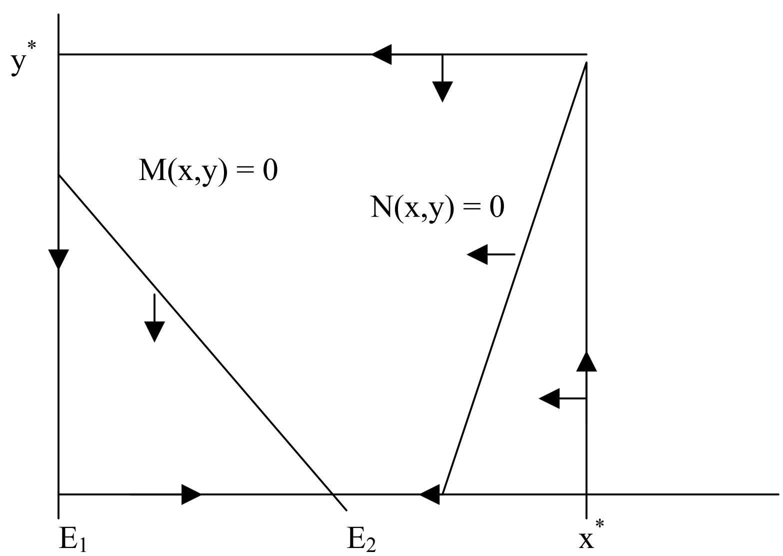


**Figure 2A: The Invariant Set When there is no $E_3$**

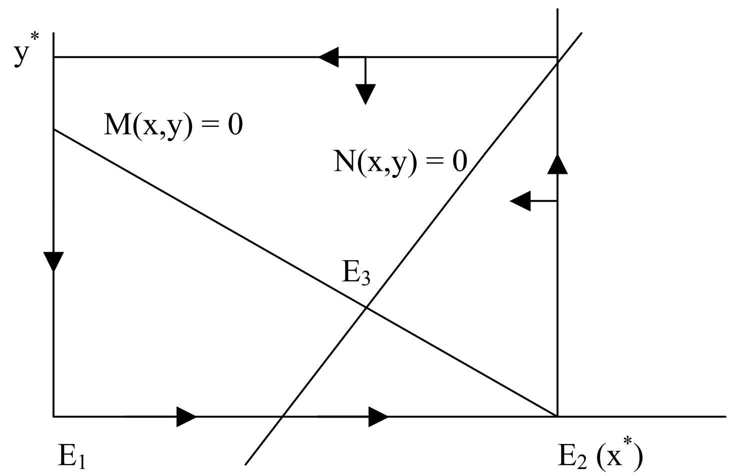


**Figure 2B: The Invariant Set when there is $E_3$**

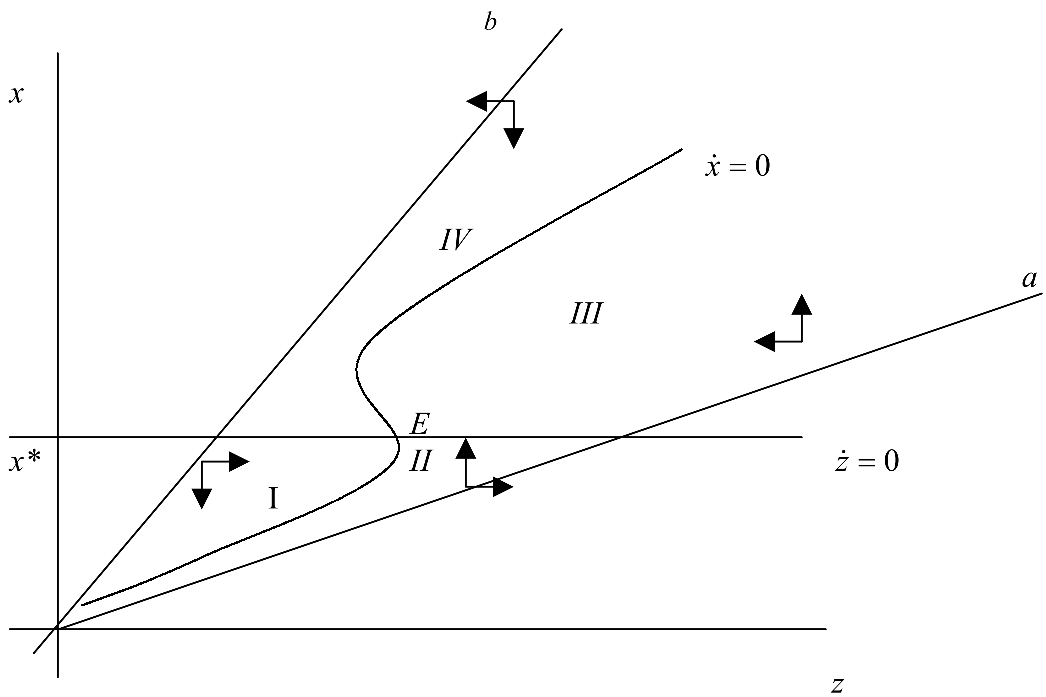


**Figure 2C: Phase Plane for the Rose Model**

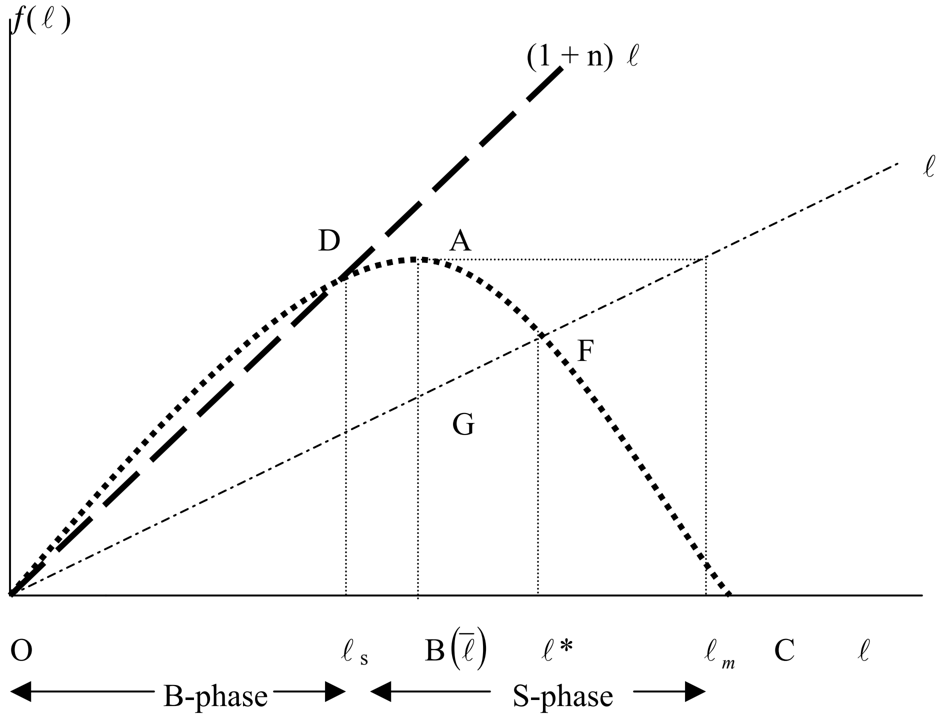


**Figure 3 Possibilities for Chaos**

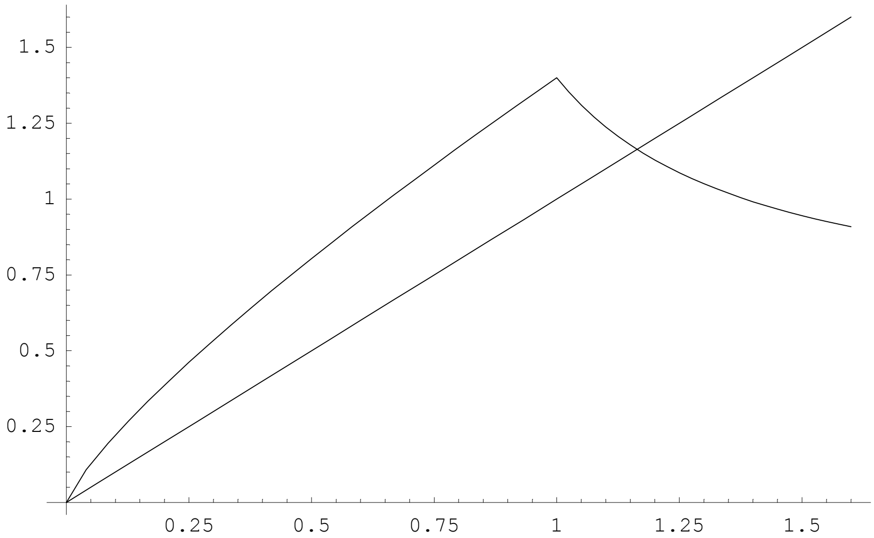


**Figure 4 A : The $\phi$ Map with K > 1**

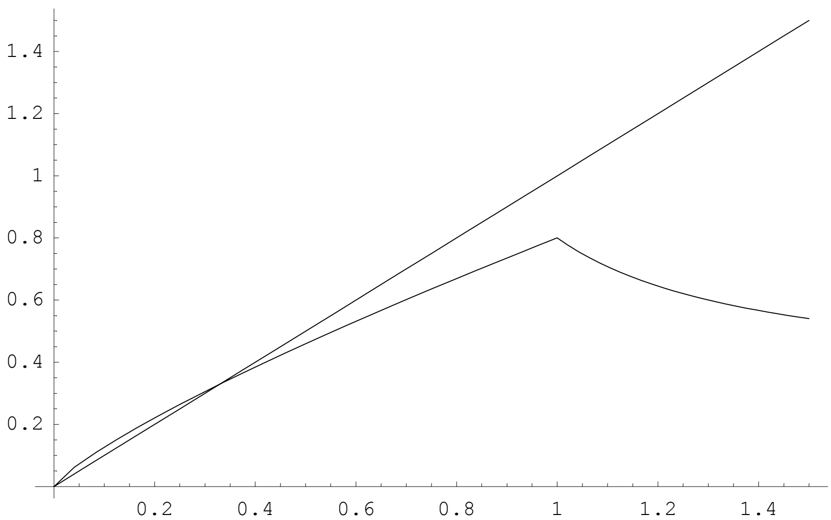


**Figure 4 B : The $\phi$ − Map with K < 1**

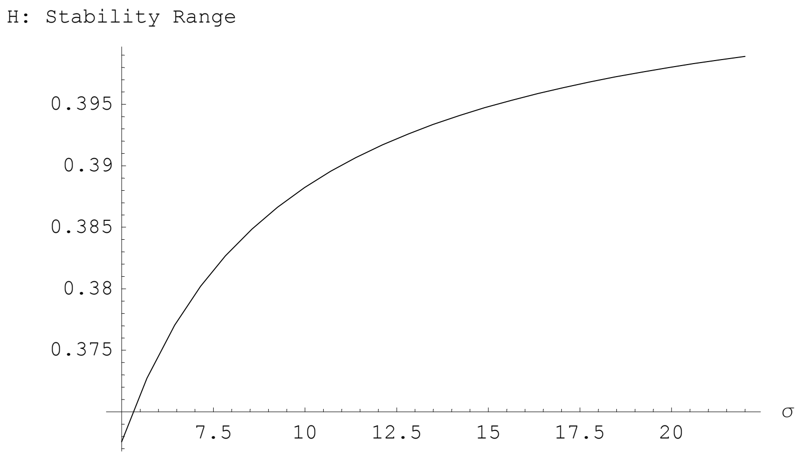


**Figure 5 : Range for Stable 2 – cycles**

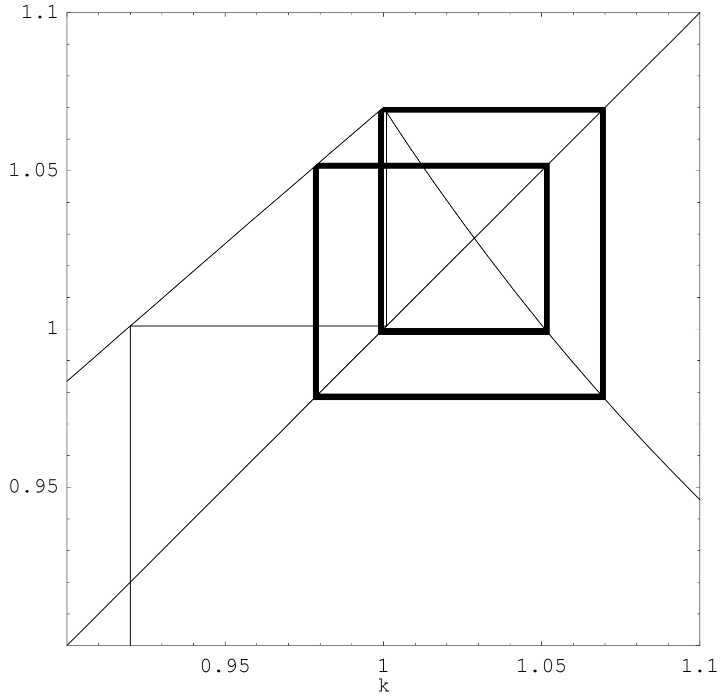


**Figure 6 : Existence and Convergence to a 4 – cycle**

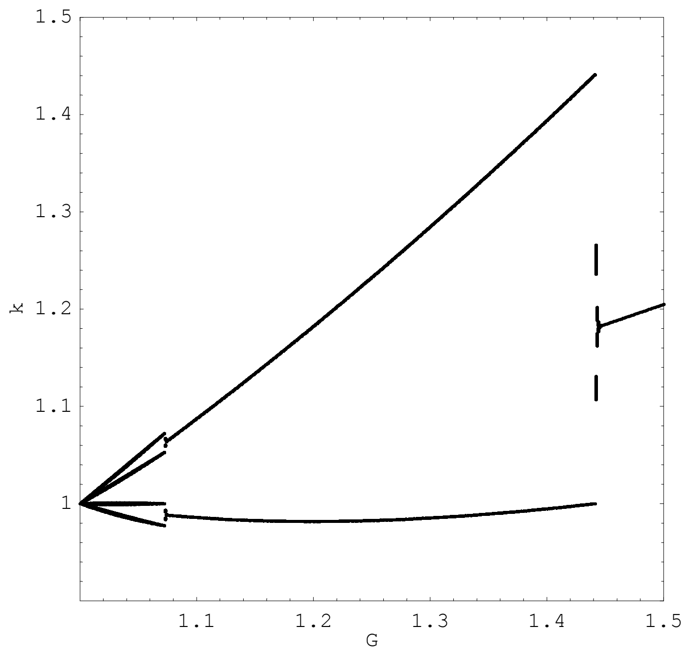


**Figure 7 : The Bifurcation Map $\sigma = 5$, $1.5 > G > 1$**

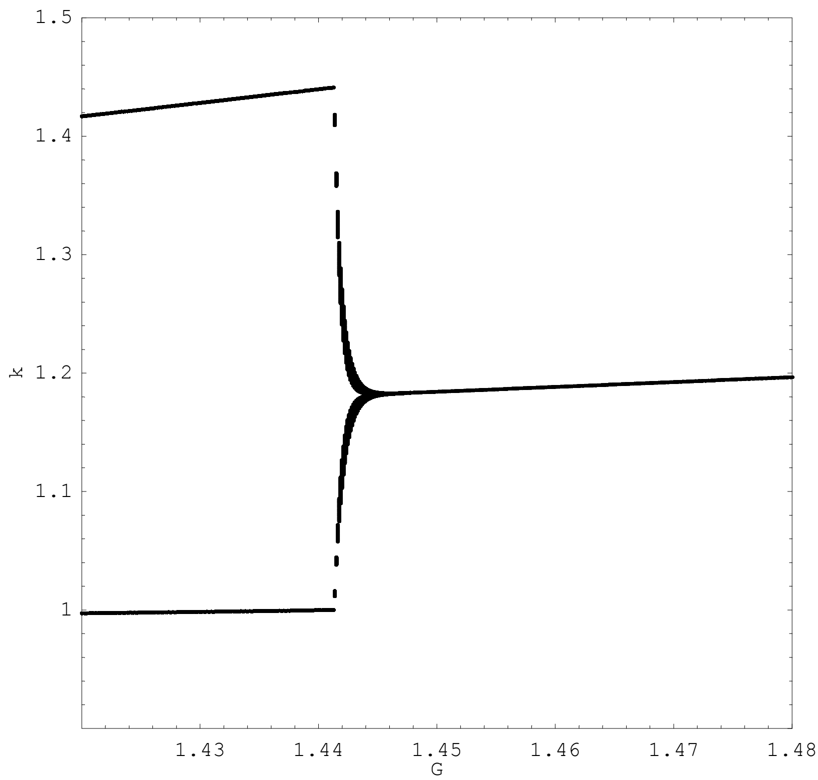


**Figure 8 : The Bifurcation Map $\sigma = 5$, $1.48 > G > 1.4$**

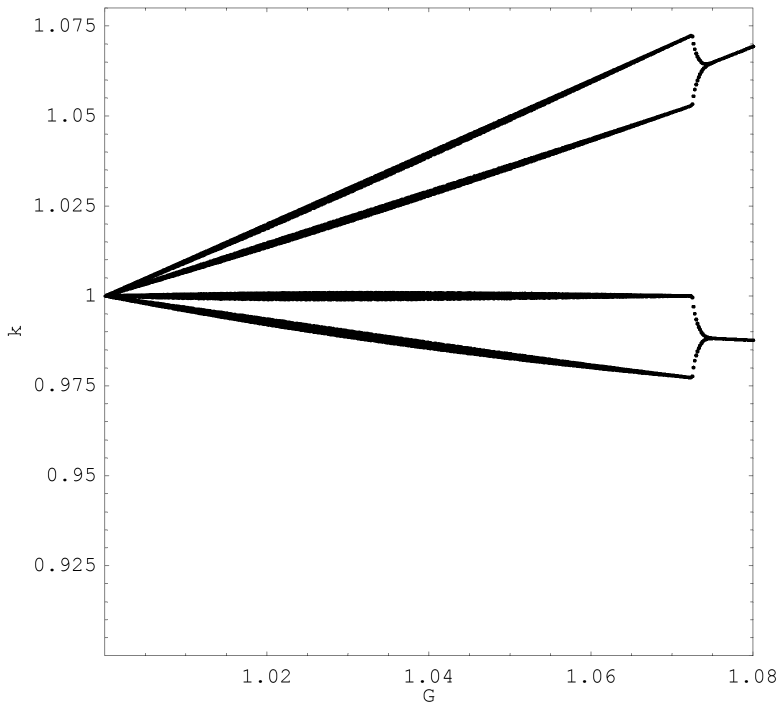


**Figure 9 : The Bifurcation Map $\sigma = 5,\ 1.08 > G > 1$**

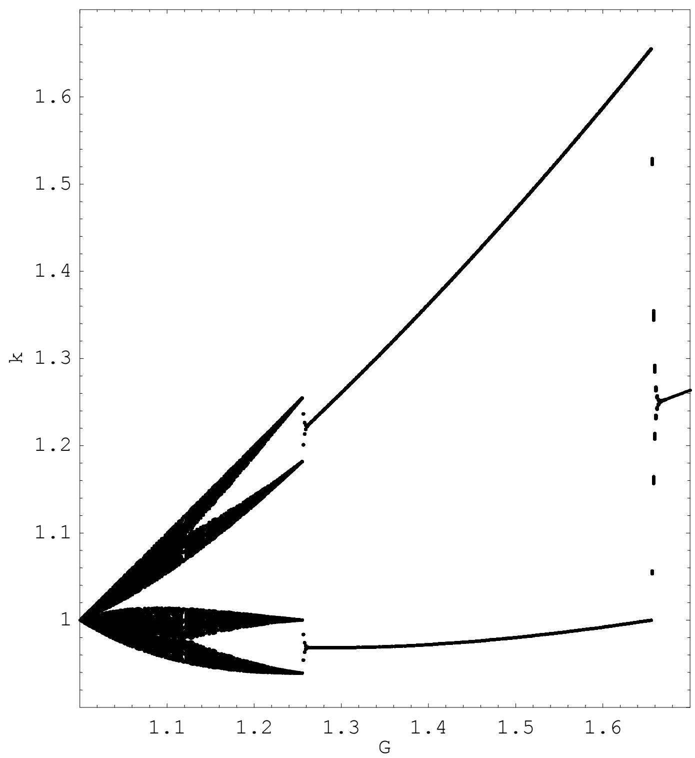


**Figure 10 : The Bifurcation Map $\sigma = 22$**