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**VICKEREY ALLOCATION RULE
WITH INCOME EFFECT**

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We consider situations where a society tries to efficiently allocate several homogeneous and indivisible goods among agents. Each agent receives at most one unit of the good. For example, suppose that a government wishes to allocate a fixed number of licenses to operate in its country to private companies with highest abilities to utilize the licenses. Usually companies with higher abilities can make more profits by licenses and are willing to pay higher prices for them. Thus, auction mechanisms are often employed to extract the information on companies' abilities and to allocate licenses efficiently. However, if prices are too high, they may damage companies' abilities to operate. Generally high prices may change the benefits agents obtain from the goods unless agents' preferences are quasi-linear, and we call it "income effect". In this paper, we establish that *on domains including nonquasi-linear preferences, that is, preferences exhibiting income effect, an allocation rule which satisfies Pareto-efficiency, strategy-proofness, individual rationality, and nonnegative payment uniquely exists and it is the Vickrey allocation rule.*

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1. INTRODUCTION

We consider situations where a society tries to efficiently allocate m homogeneous and indivisible goods among n agents. Each agent receives at most one unit of the good. For example, suppose that a government wishes to allocate a fixed number of licenses to operate in its country to private companies with highest abilities to utilize the licenses². Usually companies with higher abilities can make more profits by licenses and are willing to pay higher prices for them. Thus, auction mechanisms are often employed to extract the information on companies' abilities and to allocate licenses efficiently. However, if prices are too high, they may damage companies' abilities to operate. Generally high prices may change the benefits agents obtain from the goods unless agents' preferences are quasi-linear³, and we call it "income effect". The society must take this effect into account when it tries to efficiently allocate the goods. In this paper, we study rules allocating goods efficiently even if income effect exists.

An *allocation rule* is generally formulated as a function from the set of possible agents' preference profiles to the feasible set, and the set of possible agents' preference profiles is called "domain". Given an allocation rule, since agents' private preferences are not known to the others, there may be incentives for agents to misrepresent their preferences in order to manipulate the final outcomes to their favor. As a result, the actual outcomes may not constitute an efficient allocation relative to agents' true preferences. Thus allocation rules need to be immune to such strategic misrepresentation in order to surely attain an efficient allocation for agents' true preferences. If an allocation rule is immune to strategic behavior by any agent, that is, if it is a dominant strategy for each agent to announce his true preferences, then the allocation rule is said to be *strategy-proof*. To induce agents' voluntary participation, it is desirable for allocation rules to satisfy the property of *individual rationality*; it never assigns an allocation which makes some agent worse off than he would be if he receives no good and pays nothing. *Nonnegative payment* is also a natural requirement; it says that agents' payment should not be negative. Thus it is important to know what allocation rules satisfy Pareto-efficiency, strategy-proofness, individual rationality, and nonnegative payment.

The *Vickrey allocation rule* is the rule originally defined on the quasi-linear domain⁴ such that agents with m highest valuations⁵ of the goods receive the goods and pay the $(m + 1)$ -th valuation, and other agents pay nothing. The Vickrey allocation rule can be generalized to domains including nonquasi-linear preferences, that is, preferences exhibiting income effect. In this paper, we establish that *on these domains, an allocation rule which satisfies Pareto-efficiency, strategy-proofness, individual rationality, and nonnegative payment uniquely exists and it is the Vickrey allocation rule*.

Holmstrom (1979) studies public good models, and establish that an allocation rule is strategy-proof and provides the efficient level of the public good on the quasi-linear domain if and only if it is a Groves rule. The Vickrey allocation rule is the counterpart of the Groves rule in the auction model. Holmstrom's (1979) results can be also applied

²The frequency auctions recently implemented in many countries are such examples. See Klemperer (2001) for the details.

³A preference is *quasi-linear* if the benefits from consuming the good are independent of payments.

⁴The *quasi-linear domain* is the class of quasi-linear preference profiles

⁵If agents' preference are quasi-linear, since their benefits from consuming the good are independent of payments, their valuations of the goods are well-defined.

to auction models when agents have only quasi-linear preferences, and imply the same conclusion as ours.⁶

However, there are various cases in which preferences are not quasi-linear. In the case where agents are households, as the prices of goods are higher, they can afford to purchase less complements, and benefit less from the goods. Unless the prices of goods are small enough compared to their incomes, this effect will give rise to innegligible degree of nonquasi-linearity. The vehicle ownership license auction in Singapore is also one of such examples, where license prices are often as high as the annual income of average households. In the case where agents are private firms in auctions, high prices bring them difficulties in raising the funds to invest for the effective uses of the auctioned items. Che and Gale (1998) demonstrate that the imperfection of financial markets causes agents to have nonquasi-linear preferences by citing a variety of examples. Holmstrom's (1979) results do not cover these cases while our results do.

Whether an allocation rule satisfies certain requirements or not depends on its domain. A conclusion on one domain does not simply imply the same conclusion on other domains. Even though the same conclusion happens to hold on different domains, the same proof technique may not be applied to different domains. We emphasize that Holmstrom's (1979) proof technique does not work on domains including nonquasi-linear preferences.⁷ When preferences are quasi-linear, a Pareto-efficient allocation of goods is uniquely determined by net benefit maximization. Owing to this fact, Holmstrom (1979) actually characterizes payment rules satisfying his requirements. On the other hand, when preferences are not quasi-linear, whether an allocation of goods is Pareto-efficient or not depends on how much agents pay, and therefore, both of the allocation of goods and payments need to be analyzed simultaneously.

Demange, Gale and Sotomyer (1986) and Roth and Sotomyer (1990) also study the strategy-proofness of Vickrey auction with multiple units. Strengthening their results, Miyake (1998) shows that the Vickrey allocation rule is the unique allocation rule among "competitive allocation rules"⁸. Our characterization of the Vickrey allocation rule is more general in the sense that we show the uniques among the general class of allocation rules. Recently and independently Sakai (2005) establishes the same result as ours in the case in which there is one unit of the auctioned good.

Section 2 sets up the model, defines basic notions, and states main results. Section 3 provides the proofs of our main results.

2. THE MODEL AND MAIN RESULTS

We denote the set of agents by $N = \{1, 2, \dots, n\}$ ($n \geq 2$). There are $m < n$ indivisible and homogeneous goods to be allocated among agents. Each agent is admitted to receive

⁶Similar characterizations of Groves rules are previously established by Green and Laffont (1977), and Walker (1978). However, the characterizations of these two articles cannot be applied to auction models since they assume that the class of admissible preferences include preferences which are not admissible in auction models. Makowski and Ostroy (1987) apply Holmstrom's (1979) result to more general model, and obtain similar conclusion when preferences are quasi-linear. Chew and Serizawa (2004) also characterize the Vickrey allocation rule in the auction model with quasi-linear domains by induction logic.

⁷See Milgrom (2004) for the detailed explanation of Holmstrom's technique.

⁸An allocation rule is a *competitive allocation rule* if the rule assigns to each preference profile a competitive equilibrium allocation for the profile.

at most one unit of the good. A *good allocation* is a n -tuple $a = (a^1, \dots, a^n) \in \{0, 1\}^n$ such that $\sum a^i = m$. We denote the *set of good allocations* by A . We denote the money agent i pays by $t^i \in \mathbb{R}$. A *feasible allocation* is a pair $z = (a; t) \in A \times \mathbb{R}^n$, where $t = (t^1, \dots, t^n)$.⁹ We denote the set of feasible allocations by Z .

Agents have preferences on $\{0, 1\} \times \mathbb{R}$. We abuse language and identify a preference with its utility representation u^i .

DEFINITION: A preference u^i on $\{0, 1\} \times \mathbb{R}$ is *classical* if it satisfies the following properties:

(Continuity) For all $a^i \in \{0, 1\}$, $u^i(a^i, t^i)$ is continuous with respect to t^i .

(Strict Monotonicity) For all $a^i \in \{0, 1\}$, all $t^i \in \mathbb{R}$, and all $\hat{t}^i \in \mathbb{R}$,

$$u^i(1, t^i) > u^i(0, t^i) \quad \& \quad [\hat{t}^i > t^i \Rightarrow u^i(a^i, t^i) > u^i(a^i, \hat{t}^i)].$$

(Finiteness) For all $t^i \in \mathbb{R}$, there exist $d_1 \in \mathbb{R}_{++}$ and $d_2 \in \mathbb{R}_{++}$ such that $u^i(0, t^i) \geq u^i(1, t^i + d_1)$ and $u^i(0, t^i - d_2) \geq u^i(1, t^i)$.

We denote the class of agent i 's classical preferences by U_C^i , and $U_C = U_C^1 \times \dots \times U_C^n$.

DEFINITION: A classical preference u^i on $\{0, 1\} \times \mathbb{R}$ is *quasi-linear* if there is a *value* $v^i \in \mathbb{R}_{++}$ such that for all $a^i \in \{0, 1\}$ and all $t^i \in \mathbb{R}$, $u^i(a^i, t^i) = v^i \cdot a^i - t^i$.

We denote the class of agent i 's quasi-linear preferences by U_L^i , and $U_L = U_L^1 \times \dots \times U_L^n$.

Examples below illustrate the situations in which agents have nonquasi-linear preferences.

EXAMPLE 1: *Nonlinear cost of payments.* Suppose that agent i has budget I^i , and he has to incur interest cost if his payment exceeds his budget. Then, his cost $c^i(t^i)$ of payment t^i is t^i up to I^i and is $I^i + (1+r) \cdot (t^i - I^i)$ above I^i , where $r > 0$ is an interest rate. Given a value v^i and a budget I^i , agent i 's utility function $u^i(\cdot, \cdot; v^i, I^i)$ is the function on $\{0, 1\} \times \mathbb{R}$ such that for all $a^i \in \{0, 1\}$ and all $t^i \in \mathbb{R}$, $u^i(a^i, t^i; v^i, I^i, c^i) = v^i \cdot a^i - c^i(t^i)$. Unless $I^i = \infty$, $u^i(\cdot, \cdot; v^i, I^i)$ is not quasi-linear.

EXAMPLE 2: *Surplus depending values.* Suppose that agent i has budget I^i and his value of the good depends on the surplus $m^i = I^i - t^i$. For instance, imagine that the good is a license and the agent can use the license more effectively with more surplus, or that the agent benefits more from the good with more surplus since he can purchase more complements by the surplus. The agent has access to credit, and the surplus $m^i = I^i - t^i$ can be negative. Then, his value v^i of the good is a function from \mathbb{R} to \mathbb{R} . Assume that the value function $v^i : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. Given a value function v^i and a budget $I^i \in \mathbb{R}$, agent i 's utility function $u^i(\cdot, \cdot; v^i, I^i)$ is the function on $\{0, 1\} \times \mathbb{R}$ such that for all $a^i \in \{0, 1\}$ and all $t^i \in \mathbb{R}$, $u^i(a^i, t^i; v^i, I^i) = v^i(I^i - t^i) \cdot a^i + I^i - t^i$. Unless $v^i : \mathbb{R} \rightarrow \mathbb{R}$ is a constant function, $u^i(\cdot, \cdot; v^i, I^i)$ is not quasi-linear.

Given $u^i \in U_C^i$ and $t^i \in \mathbb{R}$, we define *willing to pay* $WP(u^i, t^i) \in \mathbb{R}$ by $u^i(1, t^i + WP(u^i, t^i)) = u^i(0, t^i)$, and *compensating transfer* $CT(u^i, t^i) \in \mathbb{R}$ by $u^i(0, t^i - CT(u^i, t^i)) = u^i(1, t^i)$. Note that $u^i \in U_C^i$ implies (i) the existences of $WP(u^i, t^i)$ and $CT(u^i, t^i)$ for all $t^i \in \mathbb{R}$, that continuity of u^i implies (ii) $WP(u^i, t^i)$ and $CT(u^i, t^i)$ are continuous with respect to t^i , and that strict monotonicity implies (iii) $WP(u^i, t^i) > 0$ and $CT(u^i, t^i) > 0$ for all $t^i \in \mathbb{R}$. Also note that $CT(u^i, t^i + WP(u^i, t^i)) = WP(u^i, t^i)$ and $WP(u^i, t^i - CT(u^i, t^i)) = CT(u^i, t^i)$ for all $t^i \in \mathbb{R}$. Thus, (iv) $WP(u^i, t^i)$ is increasing in t^i if and only if $CT(u^i, t^i)$ is increasing in t^i , (v) $WP(u^i, t^i)$ is decreasing in t^i if and only if

⁹Since $t \in \mathbb{R}^n$, negative payments are admissible for feasible allocations.

$CT(u^i, t^i)$ is decreasing in t^i , and (vi) $WP(u^i, t^i)$ is constant in t^i if and only if $CT(u^i, t^i)$ is constant in t^i .

The values of willing to pay are the maximum prices agents demand for the good. Income effect is the change of such values as the result of income change.

DEFINITION: A classical preference u^i on $\{0, 1\} \times \mathbb{R}$ exhibits *positive income effect* if for all $t^i \in \mathbb{R}$ and all $\hat{t}^i \in \mathbb{R}$, $t^i < \hat{t}^i$ implies $WP(u^i, t^i) > WP(u^i, \hat{t}^i)$.

DEFINITION: A classical preference u^i on $\{0, 1\} \times \mathbb{R}$ exhibits *nonnegative income effect* if for all $t^i \in \mathbb{R}$ and all $\hat{t}^i \in \mathbb{R}$, $t^i \leq \hat{t}^i$ implies $WP(u^i, t^i) \geq WP(u^i, \hat{t}^i)$.

DEFINITION: A classical preference u^i on $\{0, 1\} \times \mathbb{R}$ exhibits *negative income effect* if for all $t^i \in \mathbb{R}$ and all $\hat{t}^i \in \mathbb{R}$, $t^i < \hat{t}^i$ implies $WP(u^i, t^i) < WP(u^i, \hat{t}^i)$.

DEFINITION: A classical preference u^i on $\{0, 1\} \times \mathbb{R}$ exhibits *nonpositive income effect* if for all $t^i \in \mathbb{R}$ and all $\hat{t}^i \in \mathbb{R}$, $t^i \leq \hat{t}^i$ implies $WP(u^i, t^i) \leq WP(u^i, \hat{t}^i)$.

We denote the class of agent i 's preferences exhibiting nonnegative income effect by U_+^i , and $U_+ = U_+^1 \times \dots \times U_+^n$. We denote the class of agent i 's preferences exhibiting positive income effect by U_{++}^i , and $U_{++} = U_{++}^1 \times \dots \times U_{++}^n$. We denote the class of agent i 's preferences exhibiting nonpositive income effect by U_-^i , and $U_- = U_-^1 \times \dots \times U_-^n$. We denote the class of agent i 's preferences exhibiting negative income effect by U_{--}^i , and $U_{--} = U_{--}^1 \times \dots \times U_{--}^n$.

In Example 1 of nonlinear cost of payments, $U_L^i \subseteq \{u^i(\cdot, \cdot; v^i, I^i) : I^i \in \mathbb{R} \cup \{\infty\}, v^i \in \mathbb{R}_{++}\} \subseteq U_+^i$. In Examp1 2 of surplus depending values, given $v^i : \mathbb{R} \rightarrow \mathbb{R}$ and $I^i \in \mathbb{R}$, compensating transfer $CT^i(u^i, t^i)$ is the value such that $u^i(1, t^i; u^i, I^i) = v^i(I^i - t^i) + I^i - t^i = u^i(0, t^i - CT^i(u^i, t^i); v^i, I^i) = I^i - t^i + CT^i(u^i, t^i)$, i.e., compensating transfer $CT^i(u^i, t^i)$ coincides with the value $v^i(I^i - t^i)$. Denote the set of continuous, positive and increasing value functions by V_{++}^i , the set of continuous, positive and decreasing value functions by V_{--}^i , and the set of positive and constant value functions by V_0^i . Then, since $CT^i(u^i, t^i) = v^i(I^i - t^i)$ for all $t^i \in \mathbb{R}$, $\{u^i(\cdot, \cdot; v^i, I^i) : I^i \in \mathbb{R}, v^i \in V_{++}^i\} = U_{++}^i$, $\{u^i(\cdot, \cdot; v^i, I^i) : I^i \in \mathbb{R}, v^i \in V_{--}^i\} = U_{--}^i$, and $\{u^i(\cdot, \cdot; v^i, I^i) : I^i \in \mathbb{R}, v^i \in V_0^i\} = U_L^i$.

A *preference profile* is an n -tuple $u = (u^1, \dots, u^n)$ of agents' preferences. A class $U = U^1 \times \dots \times U^n$ of preference profiles is a set such that $U = U^1 \times \dots \times U^n \subseteq U_C$. The sets U_+ , U_{++} , U_- , U_{--} , and U_L are the examples of the class of preference profiles.

DEFINITION: Given a class U of preference profiles, an (*allocation*) *rule on U* is a function f from U to Z .

Given $N' \subseteq N$, let $U^{N'} = \prod_{j \in N'} U^j$. We denote generic elements of U , $U^{N'}$ and U^{-i} by u , $u^{N'}$ and u^{-i} respectively. If $u = (u^1, \dots, u^n) \in U$, $N' \subseteq N$, and $i \in N$ are given in advance, $u^{N'}$ denotes $(u^j)_{j \in N'}$ and u^{-i} denotes $(u^j)_{j \in N \setminus \{i\}}$.

Given a rule $f : U \rightarrow Z$ and $u \in U$, we denote agent i 's outcome consumption of the good by $f_a^i(u)$, and his outcome payment by $f_t^i(u)$, and we write:

$$f(u) = (f_a^1(u), \dots, f_a^n(u); f_t^1(u), \dots, f_t^n(u)), \quad f^i(u) = (f_a^i(u), f_t^i(u)), \quad f^{-i}(u) = f(u)_{j \neq i}.$$

Pareto-efficiency defined below takes the auctioneer's preference into account but assumes that he values the auctioned good at zero.

DEFINITION: An allocation $z \in Z$ is *Pareto-efficient for $u \in U$* if there is no allocation $\hat{z} \in Z$ such that $\sum_{i \in N} \hat{t}^i \geq \sum_{i \in N} t^i$, $u^i(\hat{z}^i) \geq u^i(z^i)$ for all $i \in N$, and $u^i(\hat{z}^i) > u^i(z^i)$ for some $i \in N$. An allocation rule f is *Pareto-efficient* if $f(u)$ is efficient for any $u \in U$.

REMARK: An allocation z is Pareto-efficient for $u \in U$ if and only if there is no allocation $\hat{z} \in Z$ such that $\sum_{i \in N} \hat{t}^i > \sum_{i \in N} t^i$, and $u^i(\hat{z}^i) \geq u^i(z^i)$ for all $i \in N$.

DEFINITION: A rule f is *strategy-proof* if for any $u \in U$, any $i \in N$, and any $\hat{u}^i \in U^i$,

$$u^i(f^i(u)) \geq u^i(f^i(\hat{u}^i, u^{-i})).$$

DEFINITION: A rule f is *individually rational* if for any $u \in U$ and any $i \in N$,

$$u^i(f^i(u)) \geq u^i(0, 0).$$

DEFINITION: A rule f satisfies *nonnegative payment* if for any $u \in U$ and any $i \in N$, $f_t^i(u) \geq 0$.

Given $u \in U$, we rank agents' values, $WP(u^i, 0)$, $i \in N$, of willing to pay, and denote the agent with 1st highest willing to pay by $i(u, 1)$, the agent with 2nd highest willing to pay by $i(u, 2)$, and so on. Ties are broken arbitrarily. Under the Vickrey allocation rule defined below, agents with m highest values of willing to pay receive the goods and pay the $(m + 1)$ -th highest value, $WP(u^{i(u, m+1)}, 0)$, and other agents pay nothing.

DEFINITION: The *Vickrey allocation rule* is a rule g such that for any $u \in U$, $g_a(u) \in \arg \max \{ \sum_{i \in N} a^i \cdot WP(u^i, 0) : a = (a^i)_{i \in N} \in A \}$, and

$$g_t^i(u) = \begin{cases} WP(u^{i(u, m+1)}, 0) & \text{if } g_a^i(u) = 1, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

To be precise, the Vickrey allocation rule is not unique, since the way to break tie is not unique. In other words, there are as many Vickrey allocation rules as the ways to break ties. However, in view of Remark below, we treat a Vickrey allocation rule as if it is unique.

REMARK: Let g and \hat{g} be Vickrey rules. Then

- (i) $\sum_{i \in N} g_t^i(u) = \sum_{i \in N} \hat{g}_t^i(u)$ for any $u \in U$.
- (ii) $u^i(g^i(u)) = u^i(\hat{g}^i(u))$ for any $u \in U$ and any $i \in N$.

The following theorems characterize the Vickrey allocation rule as the unique allocation rule satisfying Pareto-efficiency, strategy-proofness, nonnegative payment, and individual rationality on various domains.

THEOREM 1: *Let U be a superset of the class U_L of quasi-linear preferences and be a subset of the class U_C of classical preferences, that is, $U_L \subseteq U \subseteq U_C$. The Vickrey allocation rule is the unique allocation rule that satisfies Pareto-efficiency, strategy-proofness, nonnegative payment, and individual rationality on U .*

Theorem 1 implies the characterizations of the Vickrey allocation rule on the class U_L of quasi-linear preferences, the class U_+ of preferences exhibiting non-negative income effect, the class U_- of preferences exhibiting non-positive income effect, etc. Theorem 1 can be also applied to the situation described in Example 1. However, Theorem 1 cannot be applied to the class U_{++} of preferences exhibiting positive income effect or the class U_{--} of preferences exhibiting negative income effect since these domains do not include U_L . Theorems 2 and 3 characterize the Vickrey allocation rules on the two domains.

THEOREM 2: *The Vickrey allocation rule is the unique allocation rule that satisfies Pareto-efficiency, strategy-proofness, nonnegative payment, and individual rationality on the class U_{++} of preferences exhibiting positive income effect.*

THEOREM 3: *The Vickrey allocation rule is the unique allocation rule that satisfies Pareto-efficiency, strategy-proofness, nonnegative payment, and individual rationality on the class U_{--} of preferences exhibiting negative income effect.*

A *mechanism* is a pair (S, h) of an action space S and an outcome function h such that $S = S^1 \times \dots \times S^n$, S^i is agent i 's action space, and h is a function from S to Z . A mechanism (S, h) *implements* an allocation rule $f : U \rightarrow Z$ in dominant strategies if there is a dominant strategy $s^i : U^i \rightarrow S^i$ for each agent $i \in N$ and $h(s(u)) = f(u)$ for all $u \in U$, where $s(u) = (s^1(u^1), \dots, s^n(u^n))$. A *(closed) auction mechanism* is a mechanism (S, h) such that $S^i = \mathbb{R}_+$ for all $i \in N$, and agents who choose m highest values of s^i receive the goods. In auction mechanisms, an action $a^i \in S^i = \mathbb{R}_+$ is called a ‘‘bid’’.

The *Vickrey auction* is the auction mechanism such that agents receiving the goods pay the $(m + 1)$ -th highest bid, and others pay nothing. In the Vickrey auction, it is a dominant strategy for each agent to bid his willing to pay $WP(u^i, 0)$. Thus, the Vickrey auction implements the Vickrey allocation rule in dominant strategies.

3. PROOF OF THEOREMS

This section is devoted to the proofs of Theorems 1, 2 and 3. First we prove that the Vickrey rule g satisfies efficiency, strategy-proofness, nonnegative payment and individual rationality on the class U_C of classical preferences (Proposition 1). This result implies that the Vickrey rule also satisfies the four properties on any subclass U of U_C . The proofs of our three theorems have the same structure. Thus, we prove only Theorem 1 in detail, and explain how Theorem 1's proof needs to be modified for Theorems 2 and 3.

PROPOSITION 1: *The Vickrey rule g satisfies efficiency, strategy-proofness, nonnegative payment and individual rationality on the class U_C of classical preferences.*

PROOF OF PROPOSITION 1: Since individual rationality and nonnegative payment are trivial, we show efficiency and strategy-proofness only.

Pareto-Efficiency. Let $u \in U_C$. By contradiction, suppose that there is an allocation z that Pareto-dominates $g(u)$. Without loss of generality, assume that $WP(u^1, 0) \geq \dots \geq WP(u^n, 0)$. Let $N_0 = \{i \in N : g_a^i(u) = z_a^i = 0\}$, $N_1 = \{i \in N : g_a^i(u) = 1 \& z_a^i = 0\}$, $N_2 = \{i \in N : g_a^i(u) = 0 \& z_a^i = 1\}$, and $N_3 = \{i \in N : g_a^i(u) = z_a^i = 1\}$. Note that $\sum_{i \in N} g_t^i(u) = m \cdot WP(u^{m+1}, 0)$. Also note that for all $i \in N$, if $u^i(z^i) \geq u^i(g^i(u))$,

$$\begin{aligned} i \in N_0 &\Rightarrow z_t^i \leq 0 \\ i \in N_1 &\Rightarrow z_t^i \leq 0 \\ i \in N_2 &\Rightarrow z_t^i \leq WP(u^i, 0) \\ i \in N_3 &\Rightarrow z_t^i \leq g_t^i(u) = WP(u^{m+1}, 0) \end{aligned}$$

and if $u^i(z^i) > u^i(g^i(u))$, the inequalities above are strict. Since z Pareto-dominates $g(u)$, $\sum_{i \in N} g_t^i(u) \leq \sum_{i \in N} z_t^i$, $u^i(z^i) \geq u^i(g^i(u))$ for all $i \in N$, and $u^i(z^i) > u^i(g^i(u))$ for some

$i \in N$. Thus, we have

$$\begin{aligned} \sum_{i \in N} z_t^i &< \sum_{i \in N_2} WP(u^i, 0) + (\#N_3) \cdot WP(u^{m+1}, 0) \\ &\leq (\#N_2 + \#N_3) \cdot WP(u^{m+1}, 0) \\ &\leq m \cdot WP(u^{m+1}, 0) = \sum_{i \in N} g_t^i(u). \end{aligned}$$

This contradicts to $\sum_{i \in N} g_t^i(u) \leq \sum_{i \in N} z_t^i$.

Strategy-Proofness. Let $u \in U_C$, $i \in N$, and $\hat{u}^i \in U_C^i$. We show $u^i(g^i(u)) \geq u^i(g^i(\hat{u}^i, u^{-i}))$. Let $N_1 = \{j \in N : g_a^j(u) = 1\}$ and $N_0 = \{j \in N : g_a^j(u) = 0\}$.

Case 1: $i \in N_1$. Note that $g_t^i(u) = \max\{WP(u^j, 0) : j \in N_0\}$. By individual rationality, $u^i(g^i(u)) \geq u^i(0, 0)$. If $WP(\hat{u}^i, 0) > g_t^i(u)$, then $g^i(\hat{u}^i, u^{-i}) = g^i(u)$, and so $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(g^i(u))$. If $WP(\hat{u}^i, 0) < g_t^i(u)$, then $g^i(\hat{u}^i, u^{-i}) = 0$, and so $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(0, 0) \leq u^i(g^i(u))$. If $WP(\hat{u}^i, 0) = g_t^i(u)$ and $g_a^i(\hat{u}^i, u^{-i}) = 1$, then $g^i(\hat{u}^i, u^{-i}) = g^i(u)$, and so $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(g^i(u))$. If $WP(\hat{u}^i, 0) = g_t^i(u)$, and $g_a^i(\hat{u}^i, u^{-i}) = 0$, then $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(0, 0) \leq u^i(g^i(u))$.

Case 2: $i \in N_0$. Note that $g^i(u) = (0, 0)$, and $WP(u^i, 0) \leq \min\{WP(u^j, 0) : j \in N_1\}$. If $WP(\hat{u}^i, 0) < \min\{WP(u^j, 0) : j \in N_1\}$, then $g^i(\hat{u}^i, u^{-i}) = (0, 0)$, and so $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(g^i(u))$. If $WP(\hat{u}^i, 0) > \min\{WP(u^j, 0) : j \in N_1\}$, then $g^i(\hat{u}^i, u^{-i}) = (1, \min\{WP(u^j, 0) : j \in N_1\})$, and so by $WP(u^i, 0) \leq \min\{WP(u^j, 0) : j \in N_1\}$, $u^i(g^i(\hat{u}^i, u^{-i})) \leq u^i(g^i(u))$. If $WP(\hat{u}^i, 0) = \min\{WP(u^j, 0) : j \in N_1\}$, then $u^i(g^i(\hat{u}^i, u^{-i})) = u^i(g^i(u))$ whether $g_a^i(\hat{u}^i, u^{-i}) = 0$ or $g_a^i(\hat{u}^i, u^{-i}) = 1$.

We have established Proposition 1.

PROOF OF THEOREM 1:

Owing to Proposition 1, we only need to show uniqueness. Let $U_L \subseteq U = U^1 \times \dots \times U^n \subseteq U_C$. Let f be a rule on U that satisfies Pareto-efficiency, strategy-proofness, individual rationality and nonnegative payment. We prove $f = g$. Fact 1.1 below directly follows from individual rationality and nonnegative payment.

FACT 1.1: For all $i \in N$ and all $u \in U$, if $f_a^i(u) = 0$, $f_t^i(u) = 0$, and if $f_a^i(u) = 1$, $f_t^i(u) \leq WP(u^i, 0)$.

FACT 1.2: For all $i, j \in N$ and all $u \in U$,

$$[WP(u^i, 0) < WP(u^j, 0), u^i \in U_L, \text{ and } f_a^i(u) = 1] \Rightarrow f_a^j(u) = 1.$$

PROOF OF FACT 1.2: By contradiction, suppose $f_a^j(u) = 0$. By Fact 1.1, $f_t^j(u) = 0$. Fact 1.1 also implies $f_t^i(u) \leq WP(u^i, 0)$. Then the allocation \hat{z} such that $\hat{z}^i = (0, f_t^i(u) - WP(u^i, 0))$, $\hat{z}^j = (1, WP(u^j, 0))$, and $\hat{z}^k = f^k(u)$ for all $k \neq i, j$ Pareto-dominates $f(u)$ since $\sum_{k \in N} \hat{z}^k = \sum_{k \in N} f^k(u) + WP(u^j, 0) - WP(u^i, 0)$.¹⁰ This is a contradiction to Pareto-efficiency. Q.E.D.

FACT 1.3: For all $i \in N$ and all $u \in U$, if $f_a^i(u) = 1$, $f_t^i(u) \geq WP(u^{i(u, m+1)}, 0)$.

PROOF OF FACT 1.3: Suppose the contrary so that $f_a^i(u) = 1$ and $f_t^i(u) < WP(u^{i(u, m+1)}, 0)$ for some $i \in N$ and $u \in U$. Let $\hat{u}^i \in U_L$ be such that $f_t^i(u) < WP(\hat{u}^i, 0) < WP(u^{i(u, m+1)}, 0)$.

¹⁰It may be the case that agent i 's payment $f_t^i(u) - WP(u^i, 0)$ of the allocation \hat{z} is negative. Such an allocation is admissible since we only assume the nonnegative payment of the allocation rule f .

If $f_a^i(\hat{u}^i, u^{-i}) = 1$, then by Fact 1.2, for all $j \in N$ with $WP(u^j, 0) \geq WP(u^{i(u, m+1)}, 0)$, $f_a^j(\hat{u}^i, u^{-i}) = 1$. This implies $\sum_{j \in N} f_a^j(\hat{u}^i, u^{-i}) > m$, contradicting feasibility. Thus, we have $f_a^i(\hat{u}^i, u^{-i}) = 0$, and so by Fact 1.1, $f_t^i(\hat{u}^i, u^{-i}) = 0$. Strategy-proofness then implies $0 = \hat{u}^i(f^i(\hat{u}^i, u^{-i})) \geq \hat{u}^i(f^i(u)) = WP(\hat{u}^i, 0) - f_t^i(u)$, i.e., $f_t^i(u) \geq WP(\hat{u}^i, 0)$. This is a contradiction to $f_t^i(u) < WP(\hat{u}^i, 0)$.

Q.E.D.

FACT 1.4: For all $i \in N$ and all $u \in U$,

$$WP(u^i, 0) = WP(u^{i(u, m+1)}, 0) \Rightarrow u^i(f^i(u)) = u^i(0, 0).$$

PROOF OF FACT 1.4: If $f_a^i(u) = 0$, then the result follows from Fact 1.1. Let $f_a^i(u) = 1$. Since $WP(u^i, 0) = WP(u^{i(u, m+1)}, 0)$, Fact 1.1 and Fact 1.3 imply $WP(u^{i(u, m+1)}, 0) \geq f_t^i(u) \geq WP(u^{i(u, m+1)}, 0)$. Thus we have $f_t^i(u) = WP(u^{i(u, m+1)}, 0)$. Therefore, $u^i(f^i(u)) = u^i(0, 0)$.

Q.E.D.

Let $u \in U$ be given. Without loss of generality, we assume that $WP(u^1, 0) \geq \dots \geq WP(u^n, 0)$, and we define the sets N_+ , N_0 , and N_- by

$$\begin{aligned} N_+ &= \{i \in N : WP(u^i, 0) > WP(u^{m+1}, 0)\}, \\ N_0 &= \{i \in N : WP(u^i, 0) = WP(u^{m+1}, 0)\}, \text{ and} \\ N_- &= \{i \in N : WP(u^i, 0) < WP(u^{m+1}, 0)\}. \end{aligned}$$

In order to show that f is a Vickrey rule, we have only to show that

$$\begin{aligned} f^i(u) &= \begin{cases} (1, WP(u^{m+1}, 0)) & \text{if } i \in N_+, \\ (0, 0) & \text{if } i \in N_-, \text{ and} \end{cases} \\ u^i(f^i(u)) &= u^i(0, 0) \quad \text{if } i \in N_0. \end{aligned}$$

FACT 1.5: For all $i \in N_-$, $f^i(u) = (0, 0)$.

PROOF OF FACT 1.5: Let $i \in N_-$. Suppose on the contrary $f_a^i(u) = 1$. By Facts 1.1 and 1.3, $WP(u^{m+1}, 0) \leq f_t^i(u) \leq WP(u^i, 0)$. This is a contradiction to $i \in N_-$. Thus $f_a^i(u) = 0$. Then, Fact 1.1 implies $f^i(u) = (0, 0)$.

Q.E.D.

FACT 1.6: For all $i \in N_+$, $f^i(u) = (1, WP(u^{m+1}, 0))$.

PROOF OF FACT 1.6: We proceed this proof by two steps.

Step 1: $f_a^i(u) = 1$ for all $i \in N_+$. Suppose on the contrary that there is $i \in N_+$ such that $f_a^i(u) = 0$. By Fact 1.1, $f_t^i(u) = 0$. Note that there is $j \in N_0 \cup N_-$ such that $f_a^j(u) = 1$. Fact 1.5 implies $j \in N_0$. Moreover, Fact 1.4 implies $f_t^j(u) = WP(u^{m+1}, 0)$. Then the allocation \hat{z} such that $\hat{z}^i = (1, WP(u^i, 0))$, $\hat{z}^j = (0, 0)$, and $\hat{z}^k = f^k(u)$ for all $k \neq i, j$ Pareto-dominates $f(u)$ since $\sum_{k \in N} \hat{z}^k = \sum_{k \in N} f^k(u) + WP(u^i, 0) - WP(u^{m+1}, 0) > \sum_{k \in N} f^k(u)$. This is a contradiction.

Step 2: $f_t^i(u) = WP(u^{m+1}, 0)$ for all $i \in N_+$. Let $i \in N_+$. By Step 1, $f_a^i(u) = 1$. Fact 1.3 implies $f_t^i(u) \geq WP(u^{m+1}, 0)$. By contradiction, suppose that $f_t^i(u) > WP(u^{m+1}, 0)$. Let $\hat{u}^i \in U_L$ be such that $f_t^i(u) > WP(\hat{u}^i, 0) > WP(u^{m+1}, 0)$. Then, Step 1 implies $f_a^i(\hat{u}^i, u^{-i}) = 1$. Thus by Fact 1.1, $f_t^i(\hat{u}^i, u^{-i}) \leq WP(\hat{u}^i, 0) < f_t^i(u)$. This is a contradiction to strategy-proofness.

Q.E.D.

Fact 1.4, Fact 1.5, and Fact 1.6 imply that f is a Vickrey rule.
We have established Theorem 1.

PROOFS OF THEOREMS 2 AND 3:

Let f be a rule on U_{++} that satisfies Pareto-efficiency, strategy-proofness, individual rationality and nonnegative payment to prove Theorem 2. (Let f be a rule on U_{--} that satisfies Pareto-efficiency, strategy-proofness, individual rationality and nonnegative payment to prove Theorem 3.) We prove $f = g$. Facts 2.1 and 3.1 below are the counterparts of Fact 1 on U_{++} and U_{--} respectively. They directly follow from individual rationality and nonnegative payment.

FACT 2.1: *For all $i \in N$ and all $u \in U_{++}$, if $f_a^i(u) = 0$, $f_t^i(u) = 0$, and if $f_a^i(u) = 1$, $f_t^i(u) \leq WP(u^i, 0)$.*

FACT 3.1: *For all $i \in N$ and all $u \in U_{--}$, if $f_a^i(u) = 0$, $f_t^i(u) = 0$, and if $f_a^i(u) = 1$, $f_t^i(u) \leq WP(u^i, 0)$.*

Facts 2.2 and 3.2 below are the counterparts of Fact 1.2 on U_{++} and U_{--} respectively.

FACT 2.2: *For all $i, j \in N$ and all $u \in U_{++}$,*

$$[f_a^i(u) = 1, \text{ and } \forall t^i \in [0, WP(u^i, 0)], CT(u^i, t^i) < WP(u^j, 0)] \Rightarrow f_a^j(u) = 1.$$

FACT 3.2: *For all $i, j \in N$ and all $u \in U_{--}$,*

$$[f_a^i(u) = 1, \text{ and } \forall t^i \in [0, WP(u^i, 0)], CT(u^i, t^i) < WP(u^j, 0)] \Rightarrow f_a^j(u) = 1.$$

PROOF OF FACT 2.2: By contradiction, suppose that $f_a^j(u) = 0$. By Fact 2.1, $f_t^j(u) = 0$. Moreover, Fact 2.1 together with nonnegative payment implies $0 \leq f_t^i(u) \leq WP(u^i, 0)$, so that $CT(u^i, f_t^i(u)) < WP(u^j, 0)$. Then the allocation \hat{z} such that $\hat{z}^i = (0, f_t^i(u) - CT(u^i, f_t^i(u)))$, $\hat{z}^j = (1, WP(u^j, 0))$, and $\hat{z}^k = f^k(u)$ for all $k \neq i, j$ Pareto-dominates $f(u)$ since $\sum_{k \in N} \hat{z}^k = \sum_{k \in N} f^k(u) + WP(u^j, 0) - CT(u^i, f_t^i(u)) > \sum_{k \in N} f^k(u)$. This is a contradiction to Pareto-efficiency. Q.E.D.

The proof of Fact 3.2 is the same as Fact 2.2 except that Fact 2.1 is replaced by Fact 3.1.

Facts 2.3 and 3.3 below are the counterparts of Fact 1.3 on U_{++} and U_{--} respectively.

FACT 2.3: *For all $i \in N$ and all $u \in U_{++}$, if $f_a^i(u) = 1$, $f_t^i(u) \geq WP(u^{i(u, m+1)}, 0)$.*

FACT 3.3: *For all $i \in N$ and all $u \in U_{--}$, if $f_a^i(u) = 1$, $f_t^i(u) \geq WP(u^{i(u, m+1)}, 0)$.*

PROOF OF FACT 2.3: Suppose the contrary so that $f_a^i(u) = 1$ and $f_t^i(u) < WP(u^{i(u, m+1)}, 0)$ for some $i \in N$ and $u \in U_{++}$. Let $\hat{u}^i \in U_{++}^i$ be such that $f_t^i(u) < WP(\hat{u}^i, 0) < WP(u^{i(u, m+1)}, 0)$, and for all $t^i \in [0, WP(u^i, 0)]$ and for all $j \neq i$ with $WP(u^j, 0) \geq WP(u^{i(u, m+1)}, 0)$, $CT(\hat{u}^i, t^i) < WP(u^j, 0)$. If $f_a^i(\hat{u}^i, u^{-i}) = 1$, then by Fact 2.2, for all $j \in N$ with $WP(u^j, 0) \geq WP(u^{i(u, m+1)}, 0)$, $f_a^j(\hat{u}^i, u^{-i}) = 1$. This implies $\sum_{j \in N} f_a^j(\hat{u}^i, u^{-i}) > m$, contradicting feasibility. Thus, we have $f_a^i(\hat{u}^i, u^{-i}) = 0$, and so by Fact 2.1, $f_t^i(\hat{u}^i, u^{-i}) = 0$. The definition of WP and strategy-proofness then imply $\hat{u}^i(1, WP(\hat{u}^i, 0)) = \hat{u}^i(0, 0) = \hat{u}^i(f^i(\hat{u}^i, u^{-i})) \geq \hat{u}^i(f^i(u)) = \hat{u}^i(1, f_t^i(u))$. Thus, we have $WP(\hat{u}^i, 0) \leq f_t^i(u)$. This is a contradiction to $f_t^i(u) < WP(\hat{u}^i, 0)$. Q.E.D.

The proof of Fact 3.3 is the same as Fact 2.3 except that Fact 2.1, Fact 2.2, U_{++} and U_{++}^i are replaced by Fact 3.1, Fact 3.2, U_{--} and U_{--}^i respectively.

Facts 2.4 and 3.4 below are the counterparts of Fact 1.4 on U_{++} and U_{--} respectively. Similarly to Fact 1.4, Fact 2.4 follows from Facts 2.1 and 2.3, and Fact 3.4 follows from Facts 3.1 and 3.3.

FACT 2.4: For all $i \in N$ and all $u \in U_{++}$, if $WP(u^i, 0) = WP(u^{i(u,m+1)}, 0)$, $u^i(f^i(u)) = u^i(0, 0)$.

FACT 3.4: For all $i \in N$ and all $u \in U_{--}$, if $WP(u^i, 0) = WP(u^{i(u,m+1)}, 0)$, $u^i(f^i(u)) = u^i(0, 0)$.

Let $u \in U_{++}$ be given to prove Theorem 2. (Let $u \in U_{--}$ be given to prove Theorem 3.) Without loss of generality, we assume that $WP(u^1, 0) \geq \dots \geq WP(u^n, 0)$, and we define the sets N_+ , N_0 , and N_- as in the proof of Theorem 1.

Similarly to Fact 1.5, Fact 5 below directly follows from Facts 2.1 and 2.3 in the case of $u \in U_{++}$. (Fact 5 follows from Facts 3.1 and 3.3 in the case of $u \in U_{--}$.)

FACT 5: For all $i \in N_-$, $f^i(u) = (0, 0)$.

Similarly to Fact 1.6, Fact 6 below follows from Facts 2.1, 2.3, 2.4 and 2.6 in the case of $u \in U_{++}$. (Fact 6 follows from Facts 2.1, 2.3, 2.4 and 2.6 in the case of $u \in U_{--}$.)

FACT 6: For all $i \in N_+$, $f^i(u) = (1, WP(u^{m+1}, 0))$.

Facts 2.4, 5 and 6 imply $f = g$ on U_{++} . Facts 3.4, 5 and 6 imply $f = g$ on U_{--} .

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