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COALITIONALLY STRATEGY-PROOF RULES IN ALLOTMENT ECONOMIES OF HOMOGENEOUS INDIVISIBLE OBJECTS

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Coalitionally Strategy-Proof Rules in Allotment Economies of Homogeneous Indivisible Objects*

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Abstract

We consider the allotment problems of homogeneous indivisible objects among agents with single-peaked and risk-averse von Neumann-Morgenstern expected utility functions. We establish that the rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule. By constructing an example, we show that if same-sideness is replaced by respect for unanimity, the statement does not hold even with additional requirements of peaks-onlyness and continuity.

Keywords: coalitional strategy-proofness, homogeneous indivisible objects, single-peakedness, risk-averseness, uniform probabilistic rule

JEL Classification Numbers: C72, D71, D81

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1 Introduction

This article analyzes axiomatically the probabilistic allotment problems of homogeneous indivisible objects among agents with single-peaked and risk-averse von Neumann-Morgenstern expected utility functions, especially by the axiom of coalitional strategy-proofness.

Sprumont (1991) initiates an axiomatic analysis of the allotment problems. He analyzes the deterministic model with a perfectly divisible object, that is, the model in which there is a perfectly divisible object and allocation rules are deterministic. In this model, he assumes that agents have "single-peaked" preferences over their consumption levels, and characterizes "the uniform rule"[1]. A preference is *single-peaked* if there is some point called a "peak", and consumption levels closer to the peak are preferred. *The uniform rule* is the rule such that agents are allowed to choose their preferred consumption subject to a common upper or lower bound, which is chosen to obtain feasibility. Sprumont (1991) shows that the uniform rule is a unique allocation rule satisfying strategy-proofness[2], Pareto-efficiency, and anonymity[3]. Many authors follow Sprumont (1991) in analyzing the uniform rule from different perspectives by employing various axioms.[4]

In the real world, the objects to be allocated are often not perfectly divisible. For example, consider the situation where a professor assigns her graduate students several units of teaching-assistant work. Although time is perfectly divisible in principle, teaching assistants' working time is often institutionally restricted by units of one hour or so on. Sasaki (1997) investigates the problem of allocating finite units of indivisible objects probabilistically. He assumes that agents have single-peaked and "risk-averse" utility functions satisfying von Neumann-Morgenstern expected utility, and establishes a counterpart of Sprumont (1991); "the uniform probabilistic rule"[5], a probabilistic variant of the uniform rule, is a unique rule satisfying strategy-proofness, Pareto-efficiency, and anonymity. Several authors also follow Sasaki (1997).[6]

In his recent work, Serizawa (2006) shows that in the deterministic model of a perfectly divisible object, the uniform rule is a unique rule satisfying "effectively pairwise strategy-proofness", "respect for unanimity" and sym-

[1]The *uniform rule* is first considered by Benassy (1982) for the analysis of a fixed price economy.

[2]*Strategy-proofness* is a frequently employed incentive compatibility property. It requires that it is a weakly dominant strategy for each agent to represent her true preference.

[3]*Anonymity* requires that the name of each agent does not matter for the outcome allocation.

[4]For example, Ching (1994), Thomson (1994a, 1994b, 1995), Barberà, Jackson and Neme (1997), Chun (2006), and Klaus (2006).

[5]Sasaki (1997) himself calls this rule "randomized uniform allocation mechanisms".

[6]For example, Kureishi (2000), Ehlers and Klaus (2003), and Kureishi and Mizukami (2007).

metry[7]. *Effective pairwise strategy-proofness* requires that rules are strategy-proof and that no pair of agents has an incentive to manipulate the rule in such a way that no agent of the pair has the incentive to betray her partner. *Respect for unanimity* requires that if the sum of agents' peaks equals the endowment, all agents receive their peak consumptions. This property is much weaker than Pareto-efficiency. Because, as shown in Sasaki (1997), the counterpart of Sprumont (1991) holds in the probabilistic model, it is natural to conjecture that the counterpart of Serizawa (2006) also holds in the probabilistic model.

However, in this article, we find that the counterpart of Serizawa (2006) does not hold in the probabilistic model. That is, although the uniform probabilistic rule satisfies effectively pairwise strategy-proofness, respect for unanimity, and symmetry, it is not a unique rule satisfying the three properties. This is true even though symmetry is strengthened to "strong symmetry"[8], and effective pairwise strategy-proofness is strengthened to a much stronger concept of "coalitional strategy-proofness". *Coalitional strategy-proofness* requires that by coalitional manipulation, no coalition can increase the utility of any member in the coalition without decreasing the utility of some other member in it. In situations where planners cannot observe agents' preferences and agents can cooperate in manipulation, the property of coalitional strategy-proofness is beneficial for making agents reveal their true preferences certainly.

Furthermore, we find that even with additional requirements of "peaks-onlyness" [9] and "continuity"[10], the uniqueness of the uniform probabilistic rule does not hold, that is, the uniform probabilistic rule is not a unique allocation rule satisfying peaks-onlyness and continuity in addition to coalitional strategy-proofness, strong symmetry, and respect for unanimity. Theses results demonstrate the differences between the deterministic model with a perfectly divisible object and the probabilistic model with indivisible objects.

Owing to these differences, we impose "same-sideness"[11], together with coalitional strategy-proofness, on allocation rules, which is an efficiency

[7] *Symmetry* requires that whenever two agents have the same preferences, they receive the indifferent consumptions.

[8] *Strong symmetry* requires that whenever two agents have the same preferences, the objects are distributed to them by the same probability distribution.

[9] *Peaks-onlyness* requires that the outcome allocation depends only on the peak profile.

[10] *Continuity* requires that small changes in the utility profile cause only small changes in the outcome allocation.

[11] *Same-sideness* requires that if the sum of the amount of the peak profile is in excess demand, any agent receives an amount less than or equal to her peak. Similarly if there is in excess supply, any agent receives an amount greater than or equal to her peak. Same-sideness is a weaker property than Pareto-efficiency in the probabilistic model of homogeneous indivisible objects, even though it is equivalent to Pareto-efficiency in the model of a perfectly divisible object.

property stronger than respect for unanimity, but still weaker than Pareto-efficiency. In this article, we show that the uniform probabilistic rule is a unique rule satisfying coalitional strategy-proofness, same-sideness and strong symmetry.

This paper organized as follows: Section 2 describes the model and the results, Section 3 presents the proofs, and Section 4 concludes the paper.

2 The model and the results

There are $k \in \mathbb{Z}_{++}$[12] units of homogeneous indivisible objects. We consider the problem of alloting k units of the objects to a set of agents $N = \{1, \cdots, n\}$. Let $K = \{0, 1, \cdots, k\}$. We call $a = (x_1, \cdots, x_n) \in K^n$ a *feasible allocation* if $\sum_{i \in N} x_i = k$. Let A denote the set of all feasible allocations.

A (probability) distribution over A is interpreted as a lottery on A. For $A = \{a^1, \cdots, a^{|A|}\}$[13], we denote such a distribution over A by $[\tilde{p}^1 \circ a^1, \cdots, \tilde{p}^{|A|} \circ a^{|A|}]$ where for all $l \in \{1, \cdots, |A|\}$, $\tilde{p}^l \in [0, 1]$ is the probability of a^l, and $\sum_{l=1}^{|A|} \tilde{p}^l = 1$. For convenience, to express a distribution, we write only feasible allocations a^l that occur with a strictly positive probability $\tilde{p}^l > 0$. For example, instead of $[\frac{1}{2} \circ a^1, \frac{1}{2} \circ a^2, 0 \circ a^3, \cdots, 0 \circ a^{|A|}]$, we write $[\frac{1}{2} \circ a^1, \frac{1}{2} \circ a^2]$. Let $\tilde{P}$ denote the set of all distributions over A.

Let P_i denote the set of all marginal (probability) distributions for $i \in N$ over her allotments in K, induced by all $\tilde{p} \in \tilde{P}$. Each agent $i \in N$ only cares for her marginal distribution $p_i \in P_i$ on K. Given $p_i \in P_i$ and $K' \subseteq K$, $p_i(K')$ denotes the probability that the marginal distribution p_i places over K'. If $K' = \{x\}$, we write simply $p_i(x)$ instead of $p_i(K')$ to refer to the probability that agent i receives x units through the marginal distribution p_i.

Each agent $i \in N$ has a utility function $u_i : K \to \mathbb{R}$, which satisfies the von Neumann-Morgenstern expected utility property. Given a marginal distribution $p_i \in P_i$, we denote the expected utility by

$$E(p_i; u_i) = \sum_{x \in K} p_i(x) \cdot u_i(x).$$

We introduce two properties of the utility function.

Definition. A utility function u_i is *single-peaked* if there exists a unique peak $b(u_i) \in K$ such that for all $x, y \in K$ with $x > y \geq b(u_i)$ or $b(u_i) \geq y > x$, $u(y) > u(x)$.

Definition. A utility function u_i is *risk-averse* if for all $x \in K \backslash \{0, k\}$, $u_i(x) - u_i(x-1) > u_i(x+1) - u_i(x)$.

[12] $\mathbb{Z}_{++}$ is the set of positive integers and $\mathbb{Z}_{+}$ is the set of nonnegative integers.
[13] $|A|$ is the number of feasible allocations.

Let U denote the class of all single-peaked and risk-averse von Neumann-Morgenstern utility functions.[14] Let U^n denote the set of all von Neumann-Morgenstern utility profiles $u = (u_i)_{i \in N}$ such that for all $i \in N$, $u_i \in U$.

Note that two distributions need not be equal even though their marginal distributions are all the same, as illustrated by Example 1 below.

Example 1 (Ehlers and Klaus, 2003)**.** Let $N = \{1, 2, 3\}$, $k = 9$, $\tilde{p} = [\frac{1}{3} \circ (3, 6, 0), \frac{1}{3} \circ (0, 3, 6), \frac{1}{3} \circ (6, 0, 3)]$, and $\tilde{p}' = [\frac{1}{3} \circ (3, 0, 6), \frac{1}{3} \circ (6, 3, 0), \frac{1}{3} \circ (0, 6, 3)]$. Let p_i and p'_i be the marginal distributions for $i \in N$ induced by $\tilde{p}$ and $\tilde{p}'$. Then, for all $i \in N$, $p_i = p'_i$, but $\tilde{p} \neq \tilde{p}'$.

If two distributions $\tilde{p}, \tilde{p}' \in \tilde{P}$ have the same marginal distributions, *i.e.*, $p_i = p'_i$ for all $i \in N$, then $\tilde{p}$ and $\tilde{p}'$ are equivalent from the viewpoint of agents. Thus, we focus on marginal distribution profiles instead of distributions on A. A marginal distribution profile $p = (p_1, \cdots, p_n) \in \prod_{i \in N} P_i$ is *feasible* if there is a probability distribution $\tilde{p} \in \tilde{P}$ such that for all $i \in N$, p_i is induced by $\tilde{p}$. We denote by P the set of all feasible marginal distribution profiles.

We define two properties of marginal distribution profiles related to efficiency.

Definition. A marginal distribution profile $p \in P$ satisfies *Pareto-efficiency* with respect to $u \in U^n$ if there is no $p' \in P$ such that for all $i \in N$, $E(p'_i; u_i) \geq E(p_i; u_i)$ and for some $j \in N$, $E(p'_j; u_j) > E(p_j; u_j)$.

Definition. A marginal distribution profile $p \in P$ satisfies *same-sideness* with respect to $u \in U^n$ if $\sum_{i \in N} b(u_i) \geq k$ implies that for all $i \in N$, $p_i([0, b(u_i)]) = 1$, and $\sum_{i \in N} b(u_i) \leq k$ implies that for all $i \in N$, $p_i([b(u_i), k]) = 1$.

Pareto-efficiency implies same-sideness, but Example 2 illustrates that the inverse implication is not true.

Example 2. Let $N = \{1, 2\}$ and $k = 2$. Let $u \in U^2$ be such that $u_1 = u_2$, $u_1(0) = 0$, $u_1(1) = 2$ and $u_1(2) = 3$. Let $p \in P$ be such that $p_i(0) = p_i(2) = \frac{1}{2}$ for $i = 1, 2$ and $p' \in P$ be such that $p_i(1) = \frac{1}{2}$ for $i = 1, 2$.

Then, $E(p_i; u_i) = 1.5$ and $E(p'_i; u_i) = 2$ for $i = 1, 2$. p is same-sided with respect to u but is not Pareto-efficient with respect to u.

We introduce a property called "at most binary". It says that each agent has strictly positive probabilities over at most two adjacent elements of K. Fact 1 below implies that this property has an important role in this model.

[14] If a utility function exhibits risk-aversion, then it is *weak single-peaked*; i.e., there exist at most two adjacent peaks $b(u_i), b(u_i)+1 \in K$, and for all $x, y \in K$, if $x > y \geq b(u_i)+1$ or $b(u_i) \geq y > x$, then $u(y) > u(x)$. However, (strict) single-peakedness and risk-averseness are independent.

Definition. A marginal distribution profile $p \in P$ satisfies *at most binary* if for all $i \in N$, there exists $x \in K \backslash \{k\}$ such that $p_i(x) + p_i(x+1) = 1$.

Fact 1 (Sasaki, 1997)**.** A marginal distribution profile $p \in P$ satisfies Pareto-efficiency with respect to u if and only if it satisfies same-sideness with respect to u and at most binary.

A *probabilistic (allocation) rule* is a function $f : U^n \to P$. Given a probabilistic rule f, $u \in U^n$ and $i \in N$, $f_i(u)$ denotes the marginal distribution of agent i when the utility profile is u under the rule f. Given $K' \subseteq K$, $f_i(u)(K')$ denotes the probability that $f_i(u)$ places over K', and if $K' = \{x\}$, let $f_i(u)(x)$ denote $f_i(u)(K')$.

We introduce several properties of f. The first four properties are related to the efficiency of f.

Definition. A probabilistic rule f satisfies *Pareto-efficiency* if for all $u \in U^n$, $f(u)$ is Pareto-efficient with respect to u.

Definition. A probabilistic rule f satisfies *same-sideness* if for all $u \in U^n$, $f(u)$ is same-sided with respect to u.

Definition. A probabilistic rule f satisfies *at most binary* if for all $u \in U^n$, $f(u)$ satisfies at most binary..

Definition. A probabilistic rule f satisfies *respect for unanimity* if for all $u \in U^n$ such that $\sum_{i \in N} b(u_i) = k$, and all $i \in N$, $f_i(u)(b(u_i)) = 1$.

Note that a probabilistic rule f satisfies Pareto-efficiency if and only if it satisfies same-sideness and at most binary, due to Fact 1. In addition, note that same-sideness implies respect for unanimity.

The next two properties are related to the incentive compatibility for agents to reveal their true utility functions. "Strategy-proofness" requires that no agent can increase her utility by manipulating her revealed utility. "Coalitional strategy-proofness" is a stronger condition; it requires that no coalition can increase the utility of any member in the coalition via coalitional manipulation without decreasing the utility of some other member in the coalition.

Definition. A probabilistic rule f satisfies *strategy-proofness* if for all $u \in U^n$, for all $i \in N$, and all $\hat{u}_i \in U$, $E(f_i(u); u_i) \geq E(f_i(\hat{u}_i, u_{-i}); u_i)$.

Definition. A probabilistic rule f satisfies *coalitional strategy-proofness* if for all $u \in U^n$, all $N' \subseteq N$, and all $\hat{u}_{N'} \in U^{N'}$, whenever there is $i \in N'$ such that $E(f_i(\hat{u}_{N'}, u_{-N'}); u_i) > E(f_i(u); u_i)$, there exists $j \in N'$ such that $E(f_j(u); u_j) > E(f_j(\hat{u}_{N'}, u_{-N'}); u_j)$.

In addition, we introduce properties related to fairness. "Anonymity" requires that the name of each agent does not matter. "Strong symmetry" requires that agents with the same utility functions have the same marginal distributions. "Symmetry" requires that agents with the same utility functions obtain the same expected utilities. Note that anonymity implies strong symmetry, and strong symmetry implies symmetry.

Definition. Let Π^n be the class of all permutations on N. For all $u \in U^n$ and all $\pi \in \Pi^n$, let $u^\pi = (u_{\pi(i)})_{i \in N}$. A probabilistic rule f satisfies *anonymity* if for all $u \in U^n$, all $\pi \in \Pi^n$, and all $i \in N$, $f_{\pi(i)}(u) = f_i(u^\pi)$.

Definition. A probabilistic rule f satisfies *strong symmetry* if for all $u \in U^n$ and all $i, j \in N$ such that $u_i = u_j$, $f_i(u) = f_j(u)$.

Definition. A probabilistic rule f satisfies *symmetry* if for all $u \in U^n$ and all $i, j \in N$ such that $u_i = u_j$, $E(f_i(u); u_i) = E(f_j(u); u_j)$.

We define the uniform probabilistic rule. The uniform probabilistic rule assigns each utility profile the marginal distribution profile that depends on the common bound. Let $\lambda : U^n \to \mathbb{R}_+$ be the function such that if $\sum_{i \in N} b(u_i) \geq k$, $\sum_{i \in N} \min\{b(u_i), \lambda(u)\} = k$, and if $\sum_{i \in N} b(u_i) < k$, $\sum_{i \in N} \max\{b(u_i), \lambda(u)\} = k$. Let $x_\lambda : U^n \to K$ be the function such that if $\sum_{i \in N} b(u_i) \geq k$, $\lambda(u) \in [x_\lambda(u), x_\lambda(u) + 1)$ and if $\sum_{i \in N} b(u_i) < k$, $\lambda(u) \in (x_\lambda(u), x_\lambda(u) + 1]$.

Definition (Sasaki, 1997)**.** The *uniform probabilistic rule* is the probabilistic rule f such that for all $u \in U^n$, the following holds:
(i) If $\sum_{i \in N} b(u_i) > k$ (*excess demand*) , then for all $i \in N$,

$$b(u_i) \leq x_\lambda(u) \Longrightarrow f_i(u)(b(u_i)) = 1, \qquad \text{and}$$

$$b(u_i) \geq x_\lambda(u) + 1 \Longrightarrow \begin{cases} f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u) \\ f_i(u)(x_\lambda(u)) = 1 - (\lambda(u) - x_\lambda(u)). \end{cases}$$

(ii) If $\sum_{i \in N} b(u_i) = k$ (*balanced demand*), then for all $i \in N$, $f_i(u)(b(u_i)) = 1$.
(iii) If $\sum_{i \in N} b(u_i) < k$ (*excess supply*), then for all $i \in N$,

$$b(u_i) \geq x_\lambda(u) + 1 \Longrightarrow f_i(u)(b(u_i)) = 1, \qquad \text{and}$$

$$b(u_i) \leq x_\lambda(u) \Longrightarrow \begin{cases} f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u) \\ f_i(u)(x_\lambda(u)) = 1 - (\lambda(u) - x_\lambda(u)). \end{cases}$$

Example 3 below provides illustrations of the uniform probabilistic rule.

Example 3 (Sasaki, 1997)**.** (i) Let $N = \{1, 2, 3, 4\}$ and $k = 15$. Assume $b(u_1) = 1$, $b(u_2) = b(u_3) = 2$, and $b(u_4) = 5$. In this case, $\sum_{i \in N} b(u_i) < k$. Calculate $\lambda = \frac{10}{3}$. The uniform probabilistic rule f induces for all $i \in$

$\{2,3,4\}$, $f_i(u)(3) = \frac{2}{3}$, $f_i(u)(4) = \frac{1}{3}$, and $f_4(u)(5) = 1$.
(ii) Let $N = \{1,2,3,4\}$ and $k = 12$. Assume $b(u_1) = 4$, $b(u_2) = 2$, $b(u_3) = 10$, and $b(u_4) = 3$. In this case, $\sum_{i \in N} b(u_i) > k$. Calculate $\lambda = \frac{7}{2}$. The uniform probabilistic rule f induces for all $i \in \{1,3\}$, $f_i(u)(3) = \frac{1}{2}$, $f_i(u)(4) = \frac{1}{2}$, $f_2(u)(2) = 1$, and $f_4(u)(3) = 1$.

In the previous investigation of the probabilistic model, Sasaki (1997) shows that the uniform probabilistic rule is the only rule satisfying strategy-proofness, Pareto-efficiency, and anonymity. Kureishi (2000) weakens anonymity to symmetry and shows the uniqueness of the uniform probabilistic rule satisfying the properties. These results for the probabilistic model are parallel to those of Sprumont (1991) and Ching (1994), respectively, who originally studied a deterministic model in which the objects are perfectly divisible.

In the deterministic model with a perfectly divisible object, Serizawa (2006) recently showed that the uniform rule is the only rule satisfying effectively pairwise strategy-proofness, respect for unanimity, and symmetry. Thus, it is an interesting question whether a result parallel to Serizawa (2006) also holds in the probabilistic model. However, Example 4 below illustrates that Serizawa's uniqueness result does not hold in the probabilistic model even though effectively pairwise strategy-proofness and symmetry are respectively strengthened to coalitional strategy-proofness and strong symmetry.

Example 4. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as below:
If $u \in U^3$ is such that for one agent, say i, $b(u_i) = 1$ and for any other agent $j \in N \backslash \{i\}$, $b(u_j) = 0$, then, (i) in the case of $u_i(1) - u_i(0) \geq u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(1) = \frac{18}{20}, f_i(u)(2) = \frac{2}{20} \\ f_j(u)(0) = \frac{11}{20}, f_j(u)(1) = \frac{9}{20} \end{cases}$$

and (ii) in the case of $u_i(1) - u_i(0) < u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(0) = \frac{2}{20}, f_i(u)(1) = \frac{18}{20} \\ f_j(u)(0) = \frac{9}{20}, f_j(u)(1) = \frac{11}{20}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, although the probabilistic rule f satisfy coalitional strategy-proofness, respect for unanimity, and strong symmetry, it is not the uniform probabilistic rule.

"Peaks-onlyness" requires that the outcome marginal distribution profile depends only on the peak profile. If a rule satisfies peaks-onlyness, we can reduce the necessary information for a planner to the peak profile. "Continuity" requires that small changes in the utility profile cause only small changes in the outcome allocation.

Definition. A probabilistic rule f satisfies *peaks-onlyness* if for all $u, u' \in U^n$ such that for all $i \in N$, $b(u_i) = b(u'_i)$, $f(u) = f(u')$.

Definition. A probabilistic rule f satisfies *continuity* if for all $u \in U^n$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u' \in U^n$,

$$\begin{aligned} &[\forall i \in N, \forall x \in K, \| u_i(x) - u'_i(x) \| < \delta] \\ \Longrightarrow &[\forall i \in N, \forall x \in K, \| f_i(u)(x) - f_i(u')(x) \| < \epsilon]. \end{aligned}$$

In the deterministic model with a perfectly divisible object, these two properties are standard and are often obtained from strategy-proofness with auxiliary properties. However, note that the rule in Example 4 does not satisfy peaks-onlyness or continuity, even though it satisfies coalitional strategy-proofness, respect for unanimity, and strong symmetry. Thus, in the probabilistic model, these three properties do not imply peak-onlyness or continuity.

In this model, peaks-onlyness implies continuity.

Fact 2. If a probabilistic rule f satisfies peaks-onlyness, then it satisfies continuity.

The proof of Fact 2 is in the Appendix.

Example 5 below illustrates that even though we impose peaks-onlyness as well as the previous three properties, we cannot characterize the uniform probabilistic rule as a unique rule satisfying such properties. In addition, owing to Fact 2, adding continuity with these properties has no effect.

Example 5. Let $n = 4$ and $k = 2$. We define a probabilistic rule f as below:
If $u \in U^4$ is such that for one agent, say i, $b(u_i) = 0$, and for any other agent $j \in N \backslash \{i\}$, $b(u_j) \geq 1$, then

$$\begin{cases} f_i(u)(0) = \frac{27}{30},\ f_i(u)(1) = \frac{3}{30} \\ f_j(u)(0) = \frac{11}{30},\ f_j(u)(1) = \frac{19}{30}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, althoug the rule f satisfies the four properties; coalitional strategy-proofness, respect for unanimity, strong symmetry, and peaks-onlyness, it is not the uniform probabilistic rule.

In these probabilistic allotment economies, to characterize the uniform probabilistic rule, we need a stronger efficiency property than respect for unanimity. Our main characterization employs same-sideness instead of respect for unanimity.

Theorem. *A probabilistic rule f satisfies coalitional strategy-proofness, same-sideness and strong symmetry if and only if it is the uniform probabilistic rule.*

Since same-sideness is weaker than Pareto-efficiency in the probabilistic model, this characterization is independent from Sasaki (1997), Kureishi (2000), and Ehlers and Klaus (2003).

Although coalitional strategy-proofness is stronger than strategy-proofness, we emphasize that coalitional strategy-proofness and same-sideness do not imply at most binary. This fact is illustrated by Example 6 below.

Example 6. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as below:
For all $u \in U^3$, if $b(u_1) = 2$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(2) = \frac{14}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{14}{15}, f_2(u)(1) = f_3(u)(1) = \frac{1}{15}, \end{cases}$$

and if $b(u_1) = 1$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(1) = \frac{14}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{7}{15}, f_2(u)(1) = f_3(u)(1) = \frac{8}{15}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, the rule f satisfies coalitional strategy-proofness and same-sideness, even though it violates at most binary.

3 Proof of the Theorem

This section is devoted to the proof of the theorem in Section 2. It is easy to check the *if* part of the theorem. Here, we show the *only if* part. First we introduce three lemmas.

Lemma 1. For all $u \in U^n$, if $p, p' \in P$ are both Pareto-efficient with respect to u, and for all $i \in N$, $E(p_i; u_i) = E(p'_i; u_i)$, then $p = p'$.

Proof of Lemma 1. Let $u \in U^n$, let $p, p' \in P$ be Pareto-efficient with respect to u, and let $E(p_i; u_i) = E(p'_i; u_i)$ for all $i \in N$. We show $p = p'$.

Suppose, on the contrary, that there exists $i \in N$ such that $p_i \neq p'_i$, and we derive a contradiction.

Since both p and p' are Pareto-efficient with respect to u, Fact 1 implies that p and p' satisfy same-sideness with respect to u and at most binary.

From at most binary, there exist $x \in K$ such that $p_i(x) > 0$ and $p_i(x) + p_i(x+1) = 1$, and $y \in K$ such that $p'_i(y) > 0$ and $p'_i(y) + p'_i(y+1) = 1$.[15]

[15] In the case of $x = k$, $p_i(x) = 1$. Similarly, in the case of $y = k$, $p_i(y) = 1$.

CASE 1: $x \neq y$.

Without loss of generality, assume $x > y$. If $\sum_{i \in N} b(u_i) \geq k$, by same-sideness, $y < y+1 \leq x < x+1 \leq b(u_i)$.[16] Then by single-peakedness and $p'_i(y) > 0$, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) > p'_i(y) \cdot u(y) + p'_i(y+1) \cdot u(y+1) = E(p_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

If $\sum_{i \in N} b(u_i) < k$, by same-sideness, $b(u_i) \leq y < y+1 \leq x$. Then by single-peakedness and $p'_i(y) > 0$, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) < p'_i(y) \cdot u(y) + p'_i(y+1) \cdot u(y+1) = E(p'_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

CASE 2: $x = y$.

Without loss of generality, assume $p_i(x) > p'_i(x)$. If $\sum_{i \in N} b(u_i) \geq k$, then by same-sideness and single-peakedness, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) < p'_i(x) \cdot u(x) + p'_i(x+1) \cdot u(x+1) = E(p'_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

If $\sum_{i \in N} b(u_i) < k$, then by same-sideness and single-peakedness, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) > p'_i(y) \cdot u(y) + p'_i(y+1) \cdot u(y+1) = E(p'_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

From Cases 1 and 2, we have $p = p'$. □

Lemma 2. Let f be a rule satisfying coalitional strategy-proofness and symmetry. For all $u \in U^n$ such that $u_1 = \cdots = u_n$ and all $u' \in N$ such that for all $i \in N$, $b(u'_i) = b(u_i)$, if $f(u)$ is Pareto-efficient with respect to u, then $f(u) = f(u')$.

Proof of Lemma 2. Let $u, u' \in U^n$ be such that $u_1 = \cdots = u_n$, for all $i \in N$, $b(u'_i) = b(u_i)$, and $f(u)$ is Pareto-efficient with respect to u. We show $f(u) = f(u')$ by mathematical induction.

STEP A: If $u'_1 = \cdots = u'_n$, then $f(u) = f(u')$.

By symmetry, $E(f_1(u); u_1) = \cdots = E(f_n(u); u_n)$ and $E(f_1(u'); u'_1) = \cdots = E(f_n(u'); u'_n)$. Since $f(u)$ is Pareto-efficient with respect to u, Fact 1 implies that $f(u)$ satisfies same-sideness with respect to u and at most binary. Since $b(u_i) = b(u'_i)$ for all $i \in N$, $f(u)$ also satisfies same-sideness with respect to u'. Thus, $f(u)$ is Pareto-efficient with respect to u'.

If for some $j \in N$, $E(f_j(u); u'_j) < E(f_j(u'); u'_j)$, then by symmetry, for all $i \in N$, $E(f_i(u); u'_i) < E(f_i(u'); u'_i)$. It contradicts Pareto-efficiency of $f(u)$ with respect to u'. Thus, for all $i \in N$, $E(f_i(u); u'_i) \geq E(f_i(u'); u'_i)$.

If for some $j \in N$, $E(f_j(u); u'_j) > E(f_j(u'); u'_j)$, then by symmetry, for all $i \in N$, $E(f_i(u); u'_i) > E(f_i(u'); u'_i)$. Then, the coalition of all agents N

[16] If $x = b(u_i)$, then same-sideness implies $p_i(x) = 1$ and $p_i(x+1) = 0$ even though $b(u_i) < x+1$. Thus, the proof still works.

with profile u' manipulates the rule via u and increases the utilities of all members. It is a contradiction to coalitional strategy-proofness.

Therefore, for all $i \in N$, $E(f_i(u); u'_i) = E(f_i(u'); u'_i)$. By Lemma 1, $f(u) = f(u')$.

STEP B: Let $h \in N$. Assume that if $u'_1 = \cdots = u'_h$, $f(u) = f(u')$. Then, if $u'_1 = \cdots = u'_{h-1}$, $f(u) = f(u')$.

Let $u' \in U^n$ be such that $u'_1 = \cdots = u'_{h-1}$. Then by symmetry, $E(f_1(u'); u'_1) = \cdots = E(f_{h-1}(u'); u'_{h-1})$. Thus, if for some $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u'_i) > E(f_i(u'); u'_i)$, then for all $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u'_i) > E(f_i(u'); u'_i)$. Then, the coalition $\{1, \cdots, h-1\}$ with $u'_{\{1,\cdots,h-1\}}$ manipulates the rule via $\hat{u}_{\{1,\cdots,h-1\}}$ such that for all $i \in \{1, \cdots, h-1\}$, $\hat{u}_i = u'_h$. Then, any $i \in \{1, \cdots, h-1\}$ obtains $f_i(u)$ and increases her utility by the induction hypothesis. It is a contradiction to coalitional strategy-proofness. Therefore, for all $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u'_i) \leq E(f_i(u'); u'_i)$.

If, for some $j \in \{h, \cdots, n\}$, $E(f_j(u); u'_j) > E(f_j(u'); u'_j)$, then j with u'_j manipulates the rule via $\hat{u}_j = u'_1$ and obtains $f_j(u)$ by the induction hypothesis. It is a contradiction to strategy-proofness. Thus, for all $j \in \{h, \cdots, n\}$, $E(f_j(u); u'_j) \leq E(f_j(u'); u'_j)$.

Therefore, for all $i \in N$, $E(f_i(u); u'_i) \leq E(f_i(u'); u'_i)$. Similarly to STEP A, We can show that $f(u)$ is Pareto-efficient with respect to u'. Thus, for all $i \in N$, $E(f_i(u); u'_i) = E(f_i(u'); u'_i)$. Therefore, by Lemma 1, $f(u) = f(u')$.

From STEP A and B, we have the statement of the lemma. □

Lemma 3. If f satisfies same-sideness, then it respects unanimity.

Proof of Lemma 3. By same-sideness, $\sum_{i \in N} b(u_i) = k$ implies that for all $i \in N$, $f_i(u)([0, b(u_i)]) = 1$ and $f_i(u)([b(u_i), k]) = 1$. Thus for all $i \in N$, $f_i(u)(b(u_i)) = 1$. □

We prove the theorem by five steps. Hereafter, let f be a rule satisfying coalitional strategy-proofness, same-sideness, and strong symmetry.

Step 1. For all $u \in U^n$ such that $\sum_{i \in N} b(u_i) = k$ and all $i \in N$, $f_i(u)(b(u_i)) = 1$.

Proof of Step 1. By Lemma 3, the statement is directly implied. □

Step 2. Let $x \in K$ be such that $\frac{k}{n} \in [x, x+1)$. Let $u \in U^n$ be such that for all $i \in N$, $b(u_i) = x$. Then for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x+1) = \frac{k}{n} - x$.

Proof of Step 2. For all $z \in K$ such that $x + 2 \leq z \leq k$, let $r_z(u_i) \in \mathbb{R}$ be such that $r_z(u_i) \cdot [u_i(x) - u_i(x+1)] = u_i(x+1) - u_i(z)$. Note that by single-peakedness of u_i with $b(u_i) = x$, for all $z \in K$ such that $x + 2 \leq z \leq k - 1$,

$r_z(u_i) < r_{z+1}(u_i)$. By single-peakedness and risk-averseness, we also have that for all $z \in K$ such that $x+2 \leq z \leq k-1$,

$$0 < r_z(u_i) - [z-(x+1)] < r_{z+1}(u_i) - [(z+1)-(x+1)]. \tag{1}$$

[Figure 1 enters here.]

Let $p \in P$ be such that for all $i \in N$, $p_i(x) = x+1-\frac{k}{n}$ and $p_i(x+1) = \frac{k}{n} - x$. We show $f(u) = p$.

Case A: $n \cdot x = k$.

By Step 1, for all $i \in N$, $f_i(u)(x) = 1$ and $f_i(u)(x+1) = 0$. Thus the statement holds.

Case B: $n \cdot x < k$

Step B-1. First, we consider the case where $u_1 = \cdots = u_n$. By same-sideness, for all $i \in N$, $f_i(u)([x,k]) = 1$. Let $u' \in U^n$ be such that $b(u'_1) = \cdots = b(u'_{k-nx}) = x+1$ and $b(u'_{k-nx+1}) = \cdots = b(u'_n) = x$. Then, we have $\sum_{i\in N} b(u'_i) = k$. Thus, by Step 1, for all $i \in \{1, \cdots, k-nx\}$, $f_i(u')(x+1) = 1$ and for all for all $i \in \{k-nx+1, \cdots, k\}$, $f_i(u')(x) = 1$.

Thus, coalitional strategy-proofness and symmetry imply that for all $i \in \{1,\cdots,k-nx\}$, $E(f_i(u);u_i) \geq E(f_i(u');u_i) = u_i(x+1)$. By symmetry, for all $i \in N$, $E(f_i(u);u_i) \geq u_i(x+1)$. Note that

$$\begin{aligned}
& E(f_i(u);u_i) \geq u_i(x+1) \\
\iff & \sum_{z\in[x,k]} f_i(u)(z)\cdot u_i(z) \geq u_i(x+1) \qquad \text{(by same-sideness)} \\
\iff & \sum_{z\in[x,k]} f_i(u)(z)\cdot [u_i(z) - u_i(x+1)] \geq 0 \\
\iff & f_i(u)(x)\cdot[u_i(x) - u_i(x+1)] \\
& \quad - \sum_{z\in[x+2,k]} f_i(u)(z)\cdot[u_i(x+1) - u_i(z)] \geq 0.
\end{aligned} \tag{2}$$

By using the notation r, we rewrite (2) as: for all $i \in N$,

$$f_i(u)(x) - \sum_{z\in[x+2,k]} f_i(u)(z)\cdot r_z(u_i) \geq 0. \tag{3}$$

We show that for all $i \in N$, $f_i(u)([x+2,k]) = 0$ by mathematical induction.

Step B-1-1: For all $i \in N$, $f_i(u)(k) = 0$.

Suppose, on the contrary, for some $j \in N$, $f_i(u)(k) > 0$. Then, by strong symmetry and $u_1 = \cdots u_n$, for all $i \in N$, $f_i(u)(k) > 0$. We derive a contradiction.

Let $\hat{u} \in U^n$ be such that $\hat{u}_1 = \cdots = \hat{u}_n$, for all $i \in N$, $b(\hat{u}_i) = x$, and $r_{x+2}(\hat{u}_i) > \frac{1}{f_i(u)(k)}$. By strong symmetry, $f_1(\hat{u}) = \cdots = f_n(\hat{u})$.

Suppose for some $j \in N$, $f_j(\hat{u})([x+2,k]) \geq f_j(u)(k)$. Then,

$$\begin{aligned}
& f_j(\hat{u})(x) - \sum_{z \in [x+2,k]} f_j(\hat{u})(z) \cdot r_z(\hat{u}_j) \\
\leq\ & f_j(\hat{u})(x) - \sum_{z \in [x+2,k]} f_j(\hat{u})(z) \cdot r_{x+2}(\hat{u}_j) \\
& \text{(by } r_{x+2}(\hat{u}_j) \leq r_z(\hat{u}_j) \text{ for all } z \in [x+2,k]) \\
=\ & f_j(\hat{u})(x) - f_j(\hat{u})([x+2,k]) \cdot r_{x+2}(\hat{u}_j) \\
\leq\ & f_j(\hat{u})(x) - f_j(u)(k) \cdot r_{x+2}(\hat{u}_j) \\
<\ & f_j(\hat{u})(x) - 1 \qquad \text{(by } r_{x+2}(\hat{u}_j) > \frac{1}{f_j(u)(k)}) \\
\leq\ & 0.
\end{aligned}$$

It is a contradiction since $\hat{u} \in U^n$ also has to satisfy (3). Thus, for all $i \in N$,

$$\sum_{z \in [x+2,k]} f_i(\hat{u})(z) < f_i(u)(k). \tag{4}$$

By feasibility, $\sum_{i \in N} \sum_{z \in K} f_i(\hat{u})(z) \cdot z = \sum_{i \in N} \sum_{z \in K} f_i(u)(z) \cdot z = k$. Thus, by strong symmetry, for all $i \in N$,

$$\sum_{z \in K} f_i(\hat{u})(z) \cdot z = \sum_{z \in K} f_i(u)(z) \cdot z = \frac{k}{n}.$$

Then, by same-sideness, $\sum_{z \in [x,k]} f_i(\hat{u})(z) \cdot z = \sum_{z \in [x,k]} f_i(u)(z) \cdot z$. Note that

$$\begin{aligned}
& \sum_{z \in [x,k]} f_i(\hat{u})(z) \cdot z = \sum_{z \in [x,k]} f_i(u)(z) \cdot z \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z = -[f_i(\hat{u})(x+1) - f_i(u)(x+1)] \cdot (x+1) \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z \\
& = -\{[1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(\hat{u})(z)] - [1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(u)(z)]\} \cdot (x+1) \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z
\end{aligned}$$

$$= \{ \sum_{z \in [x,k] \setminus \{x+1\}} f_i(\hat{u})(z) - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(u)(z)\} \cdot (x+1)$$

$$\iff \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)] = 0$$

$$\iff \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)] = f_i(\hat{u})(x) - f_i(u)(x). \quad (5)$$

Thus,

$$\begin{aligned}
& E(f_i(\hat{u}); u_i) - E(f_i(u); u_i) \\
= & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot u_i(z) \\
& + \{[1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(\hat{u})(z)] - [(1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(u)(z)]\} \cdot u_i(x+1) \\
= & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [u_i(z) - u_i(x+1)] \\
= & \{ \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)]\} \cdot [u_i(x) - u_i(x+1)] \\
& + \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [u_i(z) - u_i(x+1)] \qquad \text{(by (5))} \\
= & \{ \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)]\} \cdot [u_i(x) - u_i(x+1)] \\
& + \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot \{-r_z(u_i) \cdot [u_i(x) - u_i(x+1)]\} \\
& \qquad \text{(by the definition of } r\text{)} \\
= & [u_i(x) - u_i(x+1)] \cdot \{ \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1) - r_z(u_i)]\} \\
= & [u_i(x) - u_i(x+1)] \\
& \cdot \{ \sum_{z \in [x+2,k]} [f_i(u)(z) - f_i(\hat{u})(z)] \cdot [r_z(u_i) - \{z - (x+1)\}]\} \qquad (6)
\end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{z \in [x+2,k]} [f_i(u)(z) - f_i(\hat{u})(z)] \cdot [r_z(u_i) - \{z - (x+1)\}] \\
= & [f_i(u)(k) - f_i(\hat{u})(k)] \cdot [r_k(u_i) - \{k - (x+1)\}] \\
& + \sum_{z \in [x+2,k-1]} [f_i(u)(z) - f_i(\hat{u})(z)] \cdot [r_z(u_i) - \{z - (x+1)\}] \\
\geq & [f_i(u)(k) - f_i(\hat{u})(k)] \cdot [r_k(u_i) - \{k - (x+1)\}]
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{z\in[x+2,k-1]} f_i(\hat{u})(z)\cdot[r_z(u_i)-\{z-(x+1)\}] \\
&\quad \text{(by (1), for all } z\in[x+2,k-1], r_z(u_i)-\{z-(x+1)\}>0) \\
&\geq [f_i(u)(k)-f_i(\hat{u})(k)]\cdot[r_k(u_i)-\{k-(x+1)\}] \\
&\quad - \sum_{z\in[x+2,k-1]} f_i(\hat{u})(z)\cdot[r_k(u_i)-\{k-(x+1)\}] \qquad \text{(by (1))} \\
&= [f_i(u)(k) - \sum_{z\in[x+2,k]} f_i(\hat{u})(z)]\cdot[r_k(u_i)-\{k-(x+1)\}] \\
&> 0 \qquad \text{(by (4) and (1).)} \qquad (7)
\end{aligned}
$$

Then (6) and (7) together imply that for all $i\in N$, $E(f_i(\hat{u});u_i)-E(f_i(u);u_i)>0$. It is a contradiction to coalitional strategy-proofness. Thus, for all $i\in N$, $f_i(u)(k)=0$.

Step B-1-2. Let $y\in K$ be such that $y\geq x+2$. Assume that for all $i\in N$, $f_i(u)([y+1,k])=0$. Then, for all $i\in N$, $f_i(u)([y,k])=0$.

By same-sideness and the induction hypothesis, for all $i\in N$, $f_i([x,y])=1$. Then we apply a similar argument to Step B-1-1 by replacing k with y, and we have that for all $i\in N$, $f_i(u)([y,k])=0$.

Now, we have for all $u\in U^n$ such that $u_1=\cdots=u_n$ and $b(u_i)=x$, $f_i(u)([x,x+1])=1$. Symmetry and feasibility imply that for all $i\in N$, $f_i(u)(x)=x+1-\frac{k}{n}$ and $f_i(u)(x+1)=\frac{k}{n}-x$, *i.e.*, $f(u)=p$.

Step B-2. Note that for all $u\in U^n$ such that for all $i\in N$, $b(u_i)=x$, p is Pareto-efficient with respect to u. Thus by Lemma 2 and Step B-1, for all $u\in U^n$ such that for all $i\in N$, $b(u_i)=x$, $f(u)=p$. We finish Case B.

From Cases A and B, the statement is established. □

Step 3. Let $x\in K$ be such that $\frac{k}{n}\in[x,x+1)$. Let $u\in U^n$ be such that $b(u_1)=\cdots=b(u_n)$. Then for all $i\in N$, $f_i(u)(x)=x+1-\frac{k}{n}$ and $f_i(u)(x+1)=\frac{k}{n}-x$.

Proof of Step 3. Step A. First, we consider the case where $u_1=\cdots=u_n$.

Let $p\in P$ be such that for all $i\in N$, $p_i(x)=x+1-\frac{k}{n}$ and $p_i(x+1)=\frac{k}{n}-x$. Then, p satisfies same-sideness with respect to u and at most binary. Thus, it is Pareto-efficient with respect to u.

Let $\hat{u}\in U^n$ be such that for all $i\in N$, $b(\hat{u}_i)=x$. Then, by Step 2, $f(\hat{u})=p$. Since p is Pareto-efficient with respect to u, and symmetry implies $E(f_1(u);u_1)=\cdots=E(f_n(u);u_n)$, it follows that for all $i\in N$, $E(f_i(\hat{u});u_i)=E(p_i;u_i)\geq E(f_i(u);u_i)$. If $E(f_i(\hat{u});u_i)>E(f_i(u);u_i)$, it is a contradiction to coalitional strategy-proofness. Thus, for all $i\in N$, $E(f_i(u);u_i)=E(f_i(\hat{u});u_i)=E(p_i,u_i)$.

Therefore, by Lemma 1, $f(u) = p$.

STEP B. From Lemma 2 and Step A, for all $u \in U^n$ such that $b(u_1) = \cdots = b(u_n)$, we have $f(u) = p$. $\square$

Step 4. Let $x \in K$ be such that $\frac{k}{n} \in [x, x+1)$. Let $u \in U^n$ be such that for all $i \in N$, $b(u_i) \leq x$, or for all $i \in N$, $b(u_i) \geq x+1$. Then for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x+1) = \frac{k}{n} - x$.

Proof of Step 4. Assume that for all $i \in N$, $b(u_i) \leq x$, since the other case can be treated symmetrically. Let $p \in P$ be such that for all $i \in N$, $p_i(x) = x+1-\frac{k}{n}$ and $p_i(x+1) = \frac{k}{n} - x$. We prove $f(u) = p$ by mathematical induction.

STEP A: If $u_1 = \cdots = u_n$, then $f(u) = p$.

The statement is from Step 3.

STEP B: Let $h \in N$. Assume that for all $u' \in U^n$ such that for all $u'_1 = \cdots = u'_h$, we have $f(u) = p$. Then, if u is such that $u_1 = \cdots = u_{h-1}$, we have $f(u) = p$.

Let $u_1 = \cdots u_{h-1}$. By symmetry, $E(f_1(u); u_1) = \cdots = E(f_{h-1}(u); u_{h-1})$. Thus, if, for some $j \in \{1, \cdots, h-1\}$, $E(p_j(u); u_j) > E(f_j(u); u_j)$, then for all $i \in \{1, \cdots, h-1\}$, $E(p_i(u); u_i) > E(f_i(u); u_i)$. Then, coalition $\{1, \cdots, h-1\}$ with $u_{\{1,\cdots,h-1\}}$ manipulates the rule via $\hat{u}_{\{1,\cdots,h-1\}}$ such that for all $i \in \{1, \cdots, h-1\}$, $\hat{u}_i = u_h$, and any $i \in \{1, \cdots, h-1\}$ obtains p_i by the induction hypothesis and increases her utility. It is a contradiction to coalitional strategy-proofness. Therefore, for all $i \in \{1, \cdots, h-1\}$, $E(p_i; u_i) \leq E(f_i(u); u_i)$.

On the other hand, if, for some $j \in \{h, \cdots, n\}$, $E(p_j(u); u_j) > E(f_j(u'); u_j)$, then j with u_j manipulates the rule via $\hat{u}_j = u_1$. Then, j obtains p_j by induction hypothesis and increases her utility. It is a contradiction to strategy-proofness. Therefore, for all $j \in \{h, \cdots, n\}$, $E(p_j; u_j) \leq E(f_j(u); u_j)$.

Thus, for all $i \in N$, $E(p_i; u_i) \leq E(f_i(u); u_i)$. Since p satisfies same-sideness with respect to u and at most binary, Fact 1 implies that p is Pareto-efficient with respect to u. Therefore, for all $i \in N$, $E(f_i(u); u'_i) = E(p_i; u_i)$. By Lemma 1, $f(u) = p$.

By STEP A and STEP B, we have the statement of this step. $\square$

Step 5. (i) For all $u \in U^n$, if $\sum_{i \in N} b(u_i) < k$, then for all $i \in N$ such that $b(u_i) \geq x_\lambda(u)+1$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u)+1) - \lambda(u)$.
(ii) For all $u \in U^n$, if $\sum_{i \in N} b(u_i) > k$, then for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \geq x_\lambda(u)+1$, $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u)+1) - \lambda(u)$.

Proof of Step 5. Given $u \in U^n$, let $\overline{N}(u) = \{i \in N : b(u_i) \geq x_\lambda(u) + 1\}$ and $\underline{N}(u) = \{i \in N : b(u_i) \leq x_\lambda(u)\}$. In addition, let $\overline{n}(u)$ be the number of agents in $\overline{N}(u)$, and $\underline{n}(u)$ be the number of agents in $\underline{N}(u)$. Note that $\overline{N}(u) \cup \underline{N}(u) = N$.

Without loss of generality, assume $u \in U^n$ is such that $\sum_{i \in N} b(u_i) < k$, since the other case is symmetrically proved. We prove this step by mathematical induction on $\overline{n}(u)$.

Step A: For all $u \in U^n$, if $\overline{n}(u) = 0$, then for all $i \in \underline{N}(u) = N$, $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$.

In this case, $\lambda(u) = \frac{k}{n}$. Thus by Step 4, the statement is directly implied.

Step B: Let $l \in N \setminus \{0, n\}$.[17] Assume that for all $u \in U^n$, if $\overline{n}(u) \leq l - 1$, then for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u) + 1 - \lambda(u)$ and $f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u)$. Then for all $u \in U^n$, if $\overline{n}(u) = l$, for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u) + 1 - \lambda(u)$ and $f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u)$.

First, we show that for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$. Here, we start a new mathematical induction within Step B. Note that by same-sideness, we have that for all $i \in \overline{N}(u)$, $f_i(u)([b(u_i), k]) = 1$.

Step B-i: For all $i \in \overline{N}(u)$, $f_i(u)(k) = 0$.

Suppose, on the contrary, that for some $i \in \overline{N}(u)$, $f_i(u)(k) > 0$. We derive a contradiction.

Let $u'_i \in U$ be such that $b(u'_i) = 0$. Then, $f_i(u'_i, u_{-i})(x_\lambda(u'_i, u_{-i})) = x_\lambda(u'_i, u_{-i}) + 1 - \lambda(u'_i, u_{-i})$ and $f_i(u'_i, u_{-i})(x_\lambda(u'_i, u_{-i}) + 1) = \lambda(u'_i, u_{-i}) - x_\lambda(u'_i, u_{-i})$ by the induction hypothesis of Step B. Note that by the definition of x_λ, $b(u_i) \geq x_\lambda(u'_i, u_{-i}) + 1$.[18]

[Figure 2 enters here.]

Since $f_i(u)(k) > 0$, there is $\hat{u}_i \in U$ such that $b(\hat{u}_i) = b(u_i)$ and $\hat{u}_i(0) > \{1 - f_i(u)(k)\} \cdot \hat{u}_i(b(u_i)) + f_i(u)(k) \cdot \hat{u}_i(b(u_i)+1)$. Suppose $f_i(\hat{u}_i, u_{-i})([b(u_i)+1, k]) \geq f_i(u)(k)$. Then, by same-sideness, we have that

$$f_i(\hat{u}_i, u_{-i})(b(u_i)) \leq 1 - f_i(u)(k). \tag{8}$$

[17] We need not consider the case of $l = n$ since if $\sum_{i \in N} b(u_i) < k$, $\overline{n}(u)$ cannot be equal to n.

[18] Suppose, on the contrary, that $b(u_i) \leq x_\lambda(u'_i, u_{-i})$. Then, $x_\lambda(u'_i, u_{-i}) = x_\lambda(u)$ by the definition of x_λ in the case $\sum_{i \in N} b(u_i) < k$. This contradicts the assumption that $b(u_i) \geq x_\lambda(u) + 1$. Thus, $b(u_i) \geq x_\lambda(u'_i, u_{-i}) + 1$.

Then,

$$
\begin{aligned}
& E(f_i(u_i', u_{-i}); \hat{u}_i) \\
&= f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i})) \cdot \hat{u}_i(x_\lambda(u_i', u_{-i})) \\
&\quad + f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i}) + 1) \cdot \hat{u}_i(x_\lambda(u_i', u_{-i}) + 1) \\
&> \hat{u}_i(0) \qquad \text{(by } b(u_i) \geq x_\lambda(u_i', u_{-i}) + 1 \text{ and single-peakedness)} \\
&> \{1 - f_i(u)(k)\} \cdot \hat{u}_i(b(u_i)) + f_i(u)(k) \cdot \hat{u}_i(b(u_i) + 1) \\
&\quad \text{(by the definition of } \hat{u}_i) \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot \hat{u}_i(b(u_i)) + f_i(\hat{u}_i, u_{-i})([b(u_i) + 1, k]) \cdot \hat{u}_i(b(u_i) + 1) \\
&\quad \text{(by (8) and } \hat{u}_i(b(u_i)) > \hat{u}_i(b(u_i) + 1)) \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot \hat{u}_i(b(u_i)) + \sum_{z \in [b(u_i)+1, k]} f_i(\hat{u}_i, u_{-i})(z) \cdot \hat{u}_i(z) \\
&\quad \text{(by single-peakedness)} \\
&= E(f_i(\hat{u}_i, u_{-i}); \hat{u}_i)
\end{aligned}
$$

It is a contradiction to strategy-proofness. Thus

$$f_i(\hat{u}_i, u_{-i})([b(u_i) + 1, k]) < f_i(u)(k). \tag{9}$$

Then,

$$
\begin{aligned}
& E(f_i(\hat{u}_i, u_{-i}); u_i) \\
&= f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot u_i(b(u_i)) + \sum_{z \in [b(u_i)+1, k]} f_i(\hat{u}_i, u_{-i})(z) \cdot u_i(z) \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot u_i(b(u_i)) + f_i(\hat{u}_i, u_{-i})([b(u_i) + 1, k]) \cdot u_i(k) \\
&\quad \text{(by single-peakedness)} \\
&> \{1 - f_i(u)(k)\} \cdot u_i(b(u_i)) + f_i(u)(k) \cdot u_i(k) \qquad \text{(by (9))} \\
&\geq \sum_{z \in [b(u_i), k-1]} f_i(u)(z) \cdot u_i(z) + f_i(u)(k) \cdot u_i(k) \qquad \text{(by single-peakedness)} \\
&= E(f_i(u); u_i).
\end{aligned}
$$

It is a contradiction to strategy-proofness. Thus, we have $f_i(u)(k) = 0$ for all $i \in \overline{N}(u)$.

Step B-ii: Let $x \in K$ be such that $b(u_i) + 1 \leq x \leq k - 1$. Assume that for all $i \in \overline{N}(u)$, $f_i(u)([x + 1, k]) = 0$. Then for all $i \in \overline{N}(u)$, $f_i(u)([x, k]) = 0$.

Suppose, on the contrary, that for some $i \in \overline{N}(u)$, $f_i(u)([x, k]) > 0$, and we derive a contradiction. By $f_i(u)([x + 1, k]) = 0$ (induction hypothesis), $f_i(u)([b(u_i), x]) > 0$. Then we apply a similar argument Step B-ii by replacing k with x, and we have that for all $i \in N$, $f_i(u)([x, k]) = 0$.

Now, we have that for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$. Next, we show that for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u)+1-\lambda(u)$ and $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$.

Let $k' = k - \sum_{i\in\overline{N}(u)} b(u_i)$. Since for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$, $\sum_{i\in\underline{N}(u)} \sum_{x_i\in K} f_i(u)(x_i)x_i = k'$. Note that $\lambda(u) = \frac{k'}{\underline{n}(u)}$ and for all $i \in \underline{N}(u)$, $b(u_i) < \lambda(u)$.

Then we can use a similar argument to Step 4 by replacing k with k' and $\frac{k}{n}$ by $\frac{k'}{\overline{n}(u)}$. We omit the detailed proof.

By Steps A and B, we have for all $i \in N$ such that $b(u_i) \geq x_\lambda(u) + 1$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$. □

Finally, by Step 1 and Step 5, a probabilistic rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule.

4 Concluding Remarks

We have established that a rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule. This result implies that the uniform probabilistic rule retains a very important role in the probabilistic model of homogeneous indivisible objects when a planner wishes to coalitionally strategy-proof property, similarly to the consequence in the deterministic model. We also show, by constructing examples, that if same-sideness is replaced by respect for unanimity, the statement does not hold even with additional requirements of peaks-onlyness and continuity. This fact emphasizes the difference between the probabilistic model and the deterministic model. We hope that this paper will encourage further studies on the similarities and differences of these two models.

Appendix

Proof of Fact 2. We introduce a lemma at first, and then prove the fact.

Lemma 4. For all $u, u' \in U^n$ and all $i \in N$, if $b(u_i) \neq b(u'_i)$, then there exists $x \in K$ such that $\| u_i(x) - u'_i(x) \| > [u_i(b(u_i)) - u_i(b(u'_i))]/2$.

Proof of Lemma 4. Let $u, u' \in U^n$, $i \in N$ and $b(u_i) \neq b(u'_i)$. First, we show that

$$\begin{aligned} &\| u_i(b(u_i)) - u'_i(b(u_i)) \| + \| u_i(b(u'_i)) - u'_i(b(u'_i)) \| \\ &> u_i(b(u_i)) - u_i(b(u'_i)). \end{aligned} \tag{10}$$

Case 1. $u_i(b(u'_i)) \geq u'_i(b(u'_i))$

Note that $u_i(b(u_i)) - u'_i(b(u_i)) > u_i(b(u_i)) - u'_i(b(u'_i)) \geq u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (10) holds.

Case 2. $u_i(b(u_i)) \leq u'_i(b(u_i))$

Note that $u'_i(b(u'_i)) - u_i(b(u'_i)) > u'_i(b(u_i)) - u_i(b(u'_i)) \geq u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (10) holds.

Case 3. $u_i(b(u'_i)) < u'_i(b(u'_i))$ and $u_i(b(u_i)) > u'_i(b(u_i))$

In this case, $\| u_i(b(u_i)) - u'_i(b(u_i)) \| + \| u_i(b(u'_i)) - u'_i(b(u'_i)) \| = u_i(b(u_i)) - u'_i(b(u_i)) + u'_i(b(u'_i)) - u_i(b(u'_i)) > u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (10) holds.

By the above three cases, we have (10). Thus $\max\{\| u_i(b(u_i)) - u'_i(b(u_i)) \|, \| u_i(b(u'_i)) - u'_i(b(u'_i)) \|\} > [u_i(b(u_i)) - u_i(b(u'_i))]/2$ and we have the statement. □

Let a probabilistic rule f satisfy peaks-onlyness. Let $u \in U^n$ and $\epsilon > 0$. Given $i \in N$, let $y_i = \arg\max_{y \in K \setminus b(u_i)} u_i(y)$. Let $\delta > 0$ be such that for all $i \in N$, $\delta \leq [u_i(b(u_i)) - u_i(y_i)]/2$, and let $u' \in U^n$ be such that for all $i \in N$ and all $x \in K$, $\| u_i(x) - u'_i(x) \| < \delta$. We show that for all $i \in N$ and all $x \in K$, $\| f_i(u)(x) - f_i(u')(x) \| < \epsilon$.

First, we show that for all $i \in N$, $b(u_i) = b(u'_i)$. Suppose there exists $i \in N$ such that $b(u_i) \neq b(u'_i)$, and we derive a contradiction. By the assumption, for all $x \in K$, $\| u_i(x) - u'_i(x) \| < \delta \leq [u_i(b(u_i)) - u_i(y_i)]/2$. By the definition of y_i, $[u_i(b(u_i)) - u_i(y)]/2 \leq [u_i(b(u_i)) - u_i(b(u'_i))]/2$. Thus we have that for all $x \in K$, $\| u_i(x) - u'_i(x) \| < [u_i(b(u_i)) - u_i(b(u'_i))]/2$. It is a contradiction to Lemma 4. Thus, we have that for all $i \in N$, $b(u_i) = b(u'_i)$.

Therefore, peaks-onlyness implies $f(u) = f(u')$, and we have the statement of the fact. □

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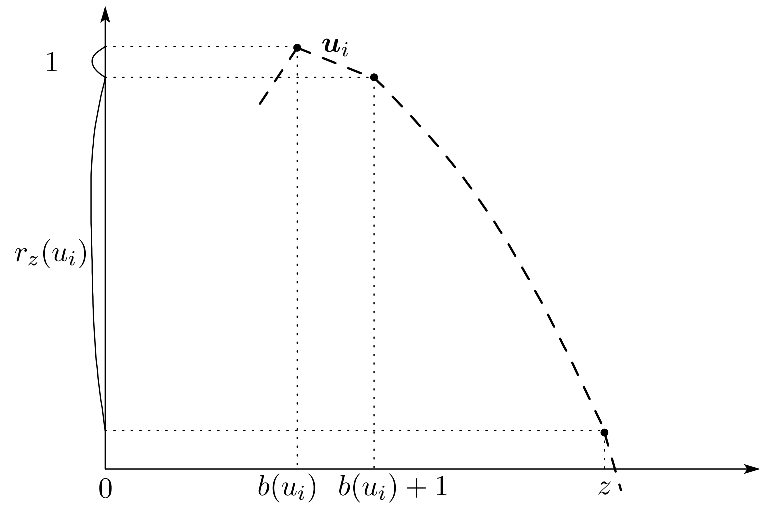

Figure 1. Illustration of $r_z(u_i)$ in the proof of Step 2.
In this figure, $u_i(b(u_i)) - u_i(b(u_i)+1)$ is normalized to be one.

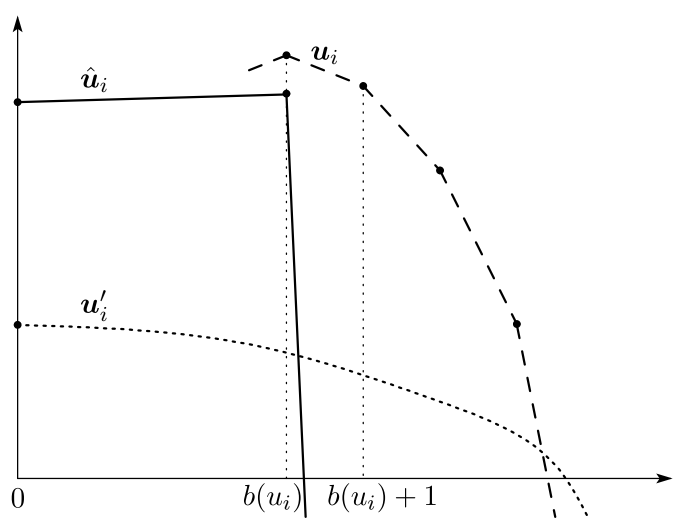

Figure 2. Illustration of u_i, u_i' and $\hat{u}_i$ in the proof of Step B-i of Step 5