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# MONOPOLY SALE OF A NETWORK GOOD

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**Abstract**

This paper studies the problem of a monopolist who sells a network good through a price posting scheme. The scheme posts a price of every possible allocation for each buyer, who are then asked to report their private information to the seller. The seller then implements the allocation based on the reports. The social choice functions that are ex post implementable through such a sales scheme are characterized, and the conditions are identified under which the revenue maximizing scheme has the property that the price of a larger network is more affordable than that of a smaller network.

Key words: network externalities, ex post equilibrium, revenue maximization.
Journal of Economic Literature Classification Numbers: C72, D82.

## 1 Introduction

Goods have network externalities when their value to any consumer depends on the consumption decision of other consumers. A classical example of a good with network externalities, or more simply a network good, is a telecommunication device whose value depends directly on the number of other people using the device. Other leading examples of network goods include the operating system (OS) of PC's, fuel-cell vehicles, social networking services, industrial parks, and so on. The nature of network externalities may be purely physical as in the case of the telecommunication device, but may also be market-based or psychological. Market-based externalities arise when more users of a good induces the market to provide complementary goods

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that enhance the value of the good. More users of a fuel-cell vehicle, for example, encourages entry into the market of charge stations, which leads to the increased value of such vehicles. On the other hand, much of bandwagon consumption in the fashion, toy and electronic industries can be explained through psychological externalities where consumers' tastes for a particular good are directly influenced by the size of its consumption. When all types of externalities are accounted for, it would be no exaggeration to say that a substantial fraction of consumption goods have network properties.

Despite their importance, network goods have received relatively little attention in economic theory.[1] Analysis of network goods in the literature has mostly been focused on the resolution of the coordination problem arising from the multiplicity of equilibria. When every consumer expects others to adopt the good, its expected value is high enough to render adoption a rational decision (at least for some price). On the other hand, when every consumer expects no other consumers to adopt, then its low expected value makes no adoption rational. Expectation is self-fulfilling in both cases, leading to multiple, Pareto-ranked equilibria. A subsidy scheme as proposed by Dybvig and Spatt (1983) is one way to eliminate the problem by promising to compensate the adopters when the number of adoptions is below some threshold. The existence of Pareto-ranked equilibria is also the main focus of the analysis of intertemporal patterns of adoption of a network good.[2] In contrast, the problem of revenue maximization by a monopolist has been analyzed only partially either through the analysis of subsidy schemes under the implicit assumption that higher participation implies higher revenue, or through the analysis of introductory prices, a common practice of setting a low price for early adopters and a higher, regular price for others.[3] In contrast, our objective in this paper is to directly explore the revenue maximization problem in the incomplete information environment.

In the present context, an allocation is the list of all buyers' adoption/non-adoption decisions. Each buyer $i$'s valuation function $v_i$ depends on an allocation, and also is an increasing function of his private signal distributed over the unit interval. A price-posting scheme is described as follows: The seller first posts a price of every possible allocation for each buyer. The buyers then report their private signals to the seller. An allocation is determined by the reports through

[1] Rofles (1974) is the first to give a theoretical analysis of network goods.

[2] See Gale (1995, 2001), Ochs and Park (2009) and Shichijo and Nakayama (2009).

[3] See Cabral *et al.* (1999). Sekiguchi (2009) examines the monopolist's revenue in the dynamic setup as in Gale (1995) when the price is held constant over time and across consumers.

an allocation rule, and offered to the buyers at the originally posted price. Finally, each buyer who is supposed to adopt in the proposed allocation chooses whether to accept the offer or not.[4]

We analyze a revenue maximizing price posting scheme that is strategy-proof and ex post individually rational. Our analysis focuses on the "regularity" property defined as follows: We say that for buyer $i$, price $p$ of an allocation $a$ is *more affordable* than price $p'$ of allocation $a'$ if for some signal $s_i$, $i$'s valuation of $a$ is above $p$ but his valuation of $a'$ is below $p'$. In other words, buyer $i$ is willing to accept $a$ at $p$ whenever he is willing to accept $a'$ at $p'$. A price-posting scheme is *regular* if (1) whenever allocation $a$ is larger than allocation $a'$ (*i.e.,* $a$ has more adopters than $a'$), the price of $a$ is more affordable than the price of $a'$ for every buyer, and (2) the allocation rule chooses the largest allocation as permitted by individual rationality. When the buyers' private signals are independently distributed, we find that the optimal scheme is regular when there are only two buyers. For a general number of ex ante symmetric buyers, we demonstrate the optimality of a regular scheme among the class of symmetric schemes when the externalities are sufficiently strong. We also show that a regular scheme is coalitionally strategy-proof in the sense that no group deviations are profitable, and that there exists a regular scheme that is optimal among the class of symmetric coalitionally strategy-proof schemes. The latter findings indicate the robustness of the optimality of a regular price-posting scheme against buyer collusion.[5]

The idea of price-posting schemes is most closely related to the concept of an inducement scheme proposed by Park (2004). An inducement scheme, which is itself a generalization of the subsidy schemes discussed above to the incomplete information environment, is a sales mechanism in which the transfer between the seller and buyers depends on the final allocation. We may think of an inducement scheme as first posting a price of each allocation, and then letting the buyers simultaneously decide whether to adopt or not. Hence, an inducement scheme is a subclass of price-

[4] In the sense that the adopters may be a subset of agents, the network good problem is related to the problem of excludable public goods, where agents can be excluded from the use of the good. However, the value of the public good depends on the amount of contributions from the agents rather than their adoption decisions, and the focus of analysis is on the efficient cost sharing rather than revenue maximization. See, for example, Moulin (1994), Deb and Razzolini (1999a, b), and Bag and Winter (1999).

[5] For example, potential buyers of an industrial park may be from the same industry and know each other well.

posting schemes in which the buyers' adoption decisions are made independently of one another. In contrast, a price-posting scheme coordinates their decisions through the reported signal profile.

The paper is organized as follows: The next section introduces a price posting scheme. Ex post implementable schemes are characterized in Section 3. We study the problem with two buyers in Section 4, and optimal symmetric schemes with a general number of ex ante symmetric buyers in Sections 5 and 6. Section 5 analyzes the case of strong externalities, and Section 6 analyzes coalitionally strategy-proof schemes. We conclude in Section 7. All the proofs are collected in the Appendix.

# 2 Model

There are $I$ potential buyers of a network good indexed by $i \in I = \{1, \ldots, I\}$. Buyer $i$'s decision is either to buy the good ($a_i = 1$), or not ($a_i = 0$). An *allocation* (or a *network*) is a profile of adoption decisions $a = (a_i)_{i \in I}$, and an element of the set $A = \{0,1\}^I$. Let $A_i$ be the set of allocations in which buyer $i$ buys the good: $A_i = \{a \in A : a_i = 1\}$. The value of the good to buyer $i$, denoted $v_i(a, s_i)$, depends on the allocation $a$ as well as his own private signal $s_i$. The signal profile $s = (s_i)_{i \in I}$ has a strictly positive joint density $g$ over $S = \prod_{i \in I} S_i$, where $S_i$ is the unit interval $[0,1] \subset \boldsymbol{R}_+$.

A *social choice function* determines the allocation of the good and monetary transfer from each buyer as a function of the private signal profile. Formally, a social choice function is a pair $(f, \tau)$ of an allocation rule $f : S \to A$ and a transfer rule $\tau = (\tau_1, \ldots, \tau_I) : S \to \boldsymbol{R}^I$: $f(s) \in A$ is the allocation under the signal profile $s \in S$, and $\tau_i(s) \in \boldsymbol{R}$ is the monetary transfer from buyer $i$ under $s$. A social choice function $(f, \tau)$ is *strategy-proof* if

$$v_i(f(s_i, s_{-i}), s_i) - \tau_i(s_i, s_{-i}) \geq v_i(f(s'_i, s_{-i}), s_i) - \tau_i(s'_i, s_{-i}) \qquad \text{for every } i,\ s_i,\ s'_i \text{ and } s_{-i},$$

and *ex post individually rational* if

$$v_i(f(s_i, s_{-i}), s_i) - \tau_i(s_i, s_{-i}) \geq 0 \quad \text{for any } i,\ s_i, \text{ and } s_{-i}.$$

A social choice function $(f, \tau)$ is *ex post implementable* if it is both strategy-proof and ex post individually rational.

Given the concern for the multiplicity of equilibria in the network good problems, strategy-proofness is a particularly suitable requirement compared with Bayesian incentive compatibility, which does not address the multiplicity issue.[6] Ex post individual rationality also adequately handles the possibility of withdrawal by a buyer after they update the value of the good upon learning the seller's recommendation.[7]

The social choice function $(f, \tau)$ is *simple* if for any $s$, $s' \in S$, $f(s) = f(s')$ implies $\tau(s) = \tau(s')$. Under a simple social choice function, hence, the transfer depends on the signal profile only through the allocation. When $(f, \tau)$ is simple, we express it as $(f, t)$, where $t : A \to \boldsymbol{R}^I$ is a function of the allocation $a$. A *price posting scheme* is a mechanism described as follows: For every buyer $i$ and every allocation $a \in A$, the seller first posts price $t_i(a)$ charged to buyer $i$ when allocation $a$ is realized. Facing the price schedule, the buyers report their private signals to the seller. The seller then determines and announces the allocation $f(s) \in A$ as a function of the report profile $s$. Finally, every buyer $i$ responds by either accepting or rejecting the proposed allocation. When any buyer rejects, no transaction takes place and every buyer receives the reservation utility of zero. Formally, a price posting scheme can be described by $(S, \{0,1\}^I, f, t)$, where $S$ is the set of message profiles, $\{0,1\}$ is the set of each buyer's decisions to accept ($d_i = 1$) or reject ($d_i = 0$) the proposed allocation, $t = (t_i(a))_{a \in A,\, i \in I}$ is the posted price schedule, and $f = (f_i)_{i \in I} : S \to A$ is the allocation rule.

A strategy for buyer $i$ in the price-posting mechanism is a pair $(\rho_i, \sigma_i)$, where $\rho_i : S_i \to S_i$ determines the reported signal given his true signal $s_i$, and $\sigma_i : S_i \times A \to \{0,1\}$ determines whether to accept or reject the proposed allocation as a function of the proposal $a$ and the own signal $s_i$. The participation strategy $(\rho_i^*, \sigma_i^*)$ of buyer $i$ is honest and obedient if $\rho_i^*(s_i) = s_i$ and $\sigma_i^*(s_i, a) = a_i$ for every $s_i \in S_i$ and $a \in A$. Note in particular that buyer $i$ cannot adopt if the proposed allocation $f(s) \notin A_i$. The profile of the honest and obedient strategies $(\rho^*, \sigma^*)$ is an *ex post equilibrium* if

$$v_i(f(s_i, s_{-i}), s_i) - t_i(f(s_i, s_{-i})) \geq \max \left\{ \max_{s_i' \in S_i} v_i(f(s_i', s_{-i}), s_i) - t_i(f(s_i', s_{-i})),\ 0 \right\}$$

for every $i$, $s_i$, and $s_{-i}$.

[6] Park (2004) presents an analysis of Bayesian implementable sales mechanisms for a network good.

[7] That is, when buyer $i$ with signal $s_i$ has reported $s_i$ and is recommended to adopt for the payment of $x_i$, his updated utility equals

$$E_{s_{-i}}[v_i(f(s_i, s_{-i}), s_i) - x_i \mid s_i, f_i(s_i, s_{-i}) = 1, \tau_i(s_i, s_{-i}) = x_i].$$

Ex post IR guarantees that the above is non-negative whereas interim IR condition $E_{s_{-i}}[v_i(f(s_i, s_{-i}), s_i) - \tau_i(s_i, s_{-i}) \mid s_i] \geq 0$ does not.

The first term in the parentheses on the right-hand side corresponds to $i$'s payoff when he makes a false report and then accepts the proposed allocation, and the second term to his payoff when he rejects the proposal. The following result is immediate.

**Proposition 1** *Let $(f, t)$ be a simple social choice function. Then $(f, t)$ is ex post implementable if and only if honesty and obedience is an ex post equilibrium of the price posting scheme $(S, \{0, 1\}^I, f, t)$.*

This proposition allows us to identify a simple ex post implementable social choice function with a price posting scheme that has honesty and obedience as an ex post equilibrium. In what follows, hence, we call such a social choice function $(f, t)$ an ex post implementable price posting scheme.

As seen, a price-posting scheme determines the price of the good only as a function of the publicly observable final allocation. As such, it leaves little room for the seller to deviate from his announced mechanism compared with more general mechanisms in which the transfer may vary with the reports even when the allocation is the same.[8] Let the seller's expected revenue per buyer under a price posting scheme $(f, t)$ be defined by

$$R(f, t) = \frac{1}{I} \sum_{i \in I} E_s[t_i(f(s))].$$

An ex post implementable price posting scheme $(f, t)$ is *optimal* if it maximizes the seller's expected revenue:

$$R(f, t) = \max \{R(f', t') : (f', t') \text{ is simple and ex post implementable}\}.$$

## 3 Characterization of Ex Post Implementability

In this section, we present a basic characterization of ex post implementability that will later be used in the analysis of optimal schemes. We make the following assumptions on the valuation function $v_i : A \times S_i \to \boldsymbol{R}_+$:

**Assumption 1** *For any $i \in I$ and $a \in A$,*

[8] Suppose that buyer $i$'s payment is higher when some other buyer $j$ reports $s_j$ than when he reports $s'_j$. Since $j$'s report is privately solicited by the seller and unknown to $i$, the seller may pretend to $i$ that $j$ has reported $s_j$ when in fact he reported $s'_j$ in order to demand the higher payment. A similar problem arises in a sealed-bid second-price auction.

1. $v_i(a,0) = 0$,

2. $a \notin A_i \Rightarrow v_i(a,\cdot) \equiv 0$,

3. $a \in A_i \Rightarrow \frac{\partial v_i}{\partial s_i}(a,\cdot) > 0$,

4. $v_i(a,\cdot) \neq v_i(b,\cdot) \Rightarrow \frac{\partial v_i}{\partial s_i}(a,\cdot) > \frac{\partial v_i}{\partial s_i}(b,\cdot)$ *or* $\frac{\partial v_i}{\partial s_i}(b,\cdot) > \frac{\partial v_i}{\partial s_i}(a,\cdot)$.

That is, the value of the good equals zero (1) to a buyer at the lowest margin, and (2) to a non-adopter. Moreover, (3) the value is strictly increasing with the private signal, and (4) when the two allocations are not equivalent to any buyer, the rate of increase in his value is strictly higher for one of them. We introduce some notation as follows. First, let

$$C_i(a) = \{a' \in A : v_i(a',\cdot) = v_i(a,\cdot)\}$$

be the set of allocations among which buyer $i$ is indifferent. For example, when the level of externalities depends only on the size of an allocation defined by $|a| = \sum_{i \in I} a_i$, then $C_i(a) = \{a' \in A_i : |a'| = |a|\}$ for $a \in A_i$. Next, fix any $s_{-i} \in S_{-i}$ and let

$$B_i(s_{-i}) = \{f(s_i, s_{-i}) : s_i \in S_i\}$$

be the set of possible allocations that buyer $i$ can achieve by changing his report when the signal profile of other buyers is fixed at $s_{-i}$. Further, for any allocation $a \in A$ and profile $s_{-i} \in S_{-i}$, let

$$L_i(a, s_{-i}) = \mathrm{cl}\,\{s_i \in S_i : f(s_i, s_{-i}) = a\}$$

be the (closure of the) set of $i$'s signals that would lead to allocation $a$ when others' signal profile is fixed at $s_{-i}$, and for any allocation $a \in A$,

$$L_a = \mathrm{cl}\,\{s \in S : f(s) = a\}$$

be the (closure of the) set of signal profiles that induce allocation $a$.

Now suppose that $(f,t)$ is a price-posting scheme. Given any allocation $a \in A_i$, define $y_i^a \in [0,1]$ to be the marginal signal at which buyer $i$ is indifferent between accepting allocation $a$ at price $t_i(a)$, and not accepting:

$$v_i(a, y_i^a) - t_i(a) = 0.$$

Such a signal $y_i^a$ is unique by Assumption 1 if it exists. If $v_i(a,0) - t_i(a) > 0$, then let $y_i^a = 0$ and if $v_i(a,1) - t_i(a) < 0$, then let $y_i^a = 1$. Likewise, given any pair of allocations $a$, $b \in A_i$ such that $v_i(a,\cdot) > v_i(b,\cdot)$, define $y_i^{ab} = y_i^{ba} \in [0,1]$ to be the marginal signal at which buyer $i$ is indifferent between allocation $a$ at price $t_i(a)$ and allocation $b$ at price $t_i(b)$:

$$v_i(a, y_i^{ab}) - t_i(a) = v_i(b, y_i^{ab}) - t_i(b). \tag{1}$$

Again, such a signal $y_i^{ab}$ is unique if it exists. If $v_i(a,0) - t_i(a) > v_i(b,0) - t_i(b)$, set $y_i^{ab} = 0$ and if $v_i(a,1) - t_i(a) < v_i(b,1) - t_i(b)$, set $y_i^{ab} = 1$.

For each $i \in I$ and $a \in A_i$, we may restrict attention to the price $t_i(a)$ such that $0 \leq t_i(a) \leq v_i(a,1)$. Since there is a one-to-one correspondence between $t_i(a)$ and $y_i^a$ for any such $t_i(a)$, we will interchangeably use the profile $y = (y_i^a)_{i \in I, a \in A_i}$ and the transfer rule $t$ in what follows.

**Proposition 2** *A price posting scheme $(f,t)$ is ex post implementable if and only if the following holds. For any $i$ and $s_{-i}$, if $a^1, \ldots, a^n \in A$ are all distinct allocations such that*

1. $\frac{\partial v_i}{\partial s_i}(a^1,\cdot) < \cdots < \frac{\partial v_i}{\partial s_i}(a^n,\cdot)$, *and*
2. $\{a^1, \ldots, a^n\} \subset B_i(s_{-i}) \subset \bigcup_{k=1}^{n} C_i(a^k)$,

*then for $k = 1, \ldots, n$,*

1. $t_i(a) = t_i(a^k)$ *if* $a \in C_i(a^k) \cap B_i(s_{-i})$,
2. $t_i(a^1) \leq 0$,
3. $t_i(a^1) \leq \cdots \leq t_i(a^n)$.
4. $\displaystyle\bigcup_{a \in C_i(a^k) \cap B_i(s_{-i})} L_i(a, s_{-i}) = \left[y_i^{a^{k-1}a^k},\, y_i^{a^k a^{k+1}}\right]$, *where* $y_i^{a^0 a^1} = 0$.

The above proposition can be illustrated as follows: Fix the signal profile of buyers other than $i$. The allocations that may be chosen for different reports of $i$'s signal should be lined up in the order of the marginal values $\frac{\partial v_i}{\partial s_i}(a, s_i)$: The allocation $a$ with the $k$th smallest $\frac{\partial v_i}{\partial s_i}(a, s_i)$ is chosen for the $k$th partition interval of reports. (Figure 1). Any two allocations with the same $\frac{\partial v_i}{\partial s_i}$ are equivalent to $i$,

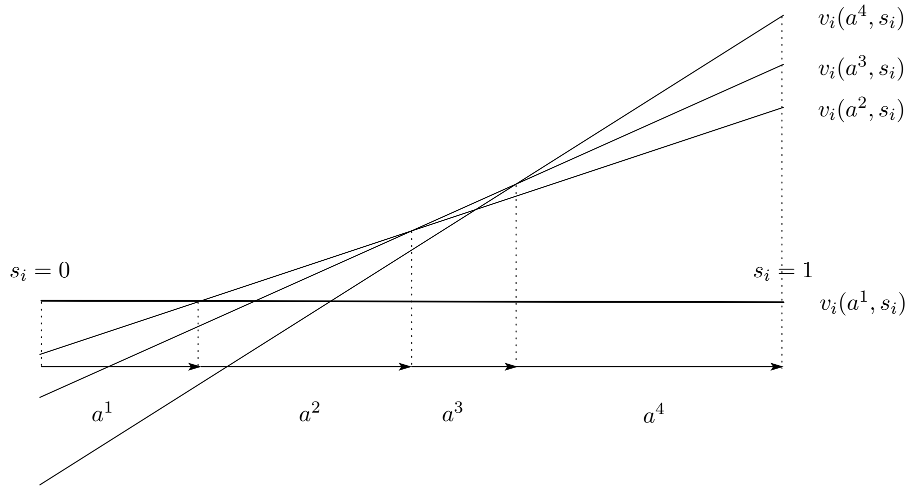


Figure 1: Ex post implementable allocation rule

and the relative ordering between them is indeterminate. It is also clear from the figure that

$$\left[y_i^{a^{k-1}a^k},\, y_i^{a^k a^{k+1}}\right] = \left\{ s_i : v_i(a^k, s_i) - t_i(a^k) = \max_{a \in B_i(s_{-i})} v_i(a, s_i) - t_i(a) \right\}. \qquad (2)$$

# 4 Optimal Schemes against Two Buyers

Suppose now that there are only two buyers $I = \{1, 2\}$. The set of possible allocations in this case is given by

$$A = \{11, 10, 01, 00\},$$

where

$$11 = (1,1),\ 10 = (1,0),\ 01 = (0,1) \text{ and } 00 = (0,0).$$

We assume positive network externalities as follows.

**Assumption 2** *For each $a \in A_i$, $v_i(a, 0) = 0$. Furthermore,*

$$\frac{\partial v_1}{\partial s_1}(11, \cdot) > \frac{\partial v_1}{\partial s_1}(10, \cdot), \quad \frac{\partial v_2}{\partial s_2}(11, \cdot) > \frac{\partial v_2}{\partial s_2}(01, \cdot).$$

The following theorem characterizes the optimal schemes in a general environment with two buyers.

**Theorem 1** *If $(f, t)$ is an optimal ex post implementable price posting scheme against two buyers under Assumption 2, then it takes one of the following forms.*

$$\textit{(A) } y_1^{11},\ y_2^{11} < 1,\ y_1^{10},\ y_2^{01} \in (0,1),\ \begin{cases} \textit{(A0) } y_1^{11} \le y_1^{10},\ y_2^{11} \le y_2^{01} \\ \textit{(A1) } y_1^{11} \le y_1^{10},\ y_2^{11} > y_2^{01}, \\ \textit{(A2) } y_1^{11} > y_1^{10},\ y_2^{11} \le y_2^{01} \end{cases}$$

$$\begin{cases} L_{11} = [y_1^{11},\ 1] \times [y_2^{11},\ 1] \\ L_{10} = [y_1^{10},\ 1] \times [0,\ y_2^{11}] \\ L_{01} = [0, y_1^{11}] \times [y_2^{01}, 1]. \end{cases}$$

$$\textit{(B1) } 0 < y_1^{10} < y_1^{11} < y_1^{11,10} < 1,\ y_2^{11} < 1,\ y_2^{01} \in (0,1),$$

$$\begin{cases} L_{11} = [y_1^{11,10},\ 1] \times [y_2^{11},\ 1] \\ L_{10} = [y_1^{10},\ 1] \times [0, 1] \setminus \operatorname{int} L_{11} \\ L_{01} = [0, y_1^{10}] \times [y_2^{01}, 1]. \end{cases}$$

$$\textit{(B2) } 0 < y_2^{01} < y_2^{11} < y_2^{11,01} < 1,\ y_1^{10} \in (0,1),\ y_1^{11} < 1,$$

$$\begin{cases} L_{11} = [y_1^{11},\ 1] \times [y_2^{11,01},\ 1] \\ L_{10} = [y_1^{10},\ 1] \times [0,\ y_2^{01}] \\ L_{01} = [0, 1] \times [y_2^{01}, 1] \setminus \operatorname{int} L_{11}. \end{cases}$$

$$\textit{(C1) } y_1^{10} < y_1^{11} < 1,\ 0 < y_2^{01} < y_2^{11} < 1,$$

$$\begin{cases} L_{11} = [y_1^{11},\ 1] \times [y_2^{11},\ 1] \\ L_{10} = [y_1^{10},\ 1] \times [0,\ y_2^{11}] \\ L_{01} = [0, y_1^{10}] \times [y_2^{01}, 1]. \end{cases}$$

$$\textit{(C2) } 0 < y_1^{10} < y_1^{11} < 1,\ y_2^{01} < y_2^{11} < 1,$$

$$\begin{cases} L_{11} = [y_1^{11},\ 1] \times [y_2^{11},\ 1] \\ L_{10} = [y_1^{10},\ 1] \times [0,\ y_2^{01}] \\ L_{01} = [0, y_1^{11}] \times [y_2^{01}, 1]. \end{cases}$$

These configurations are depicted in Figures 2, 3 and 4. As seen, an optimal scheme permits various allocation rules. Which one of these is optimal depends on the specific distribution of signal profiles.

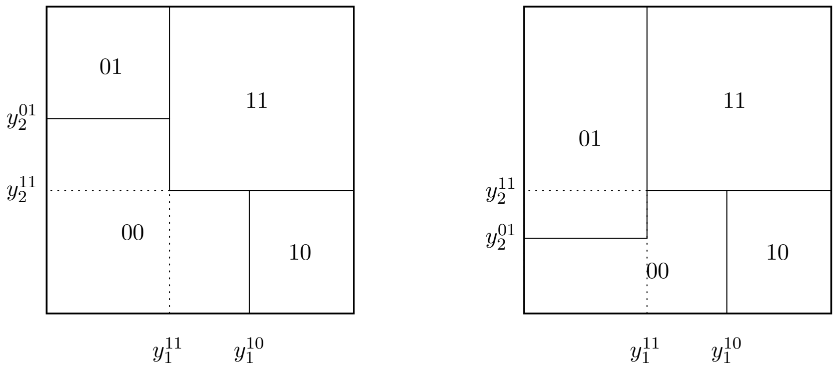


Figure 2: Configurations (A0) (left) and (A1) (right)

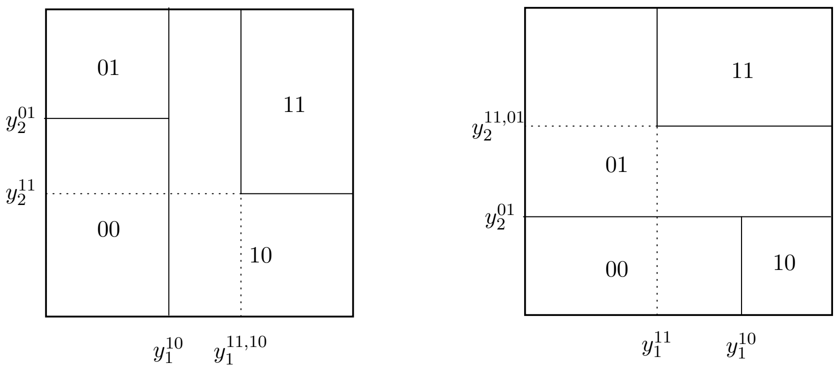


Figure 3: Configurations (B1) (left) and (B2) (right)

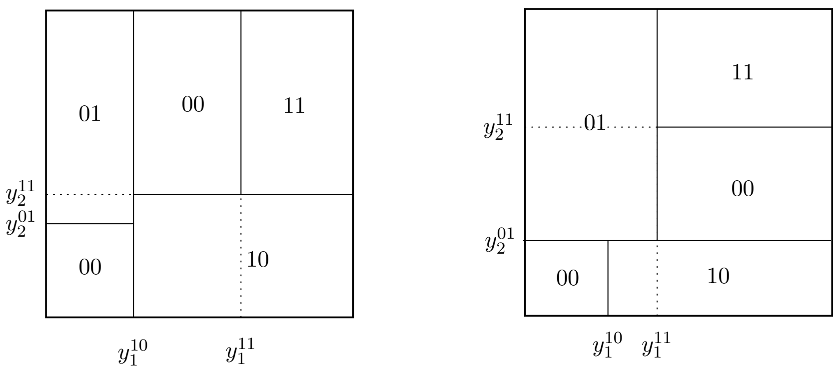


Figure 4: Configurations (C1) (left) and (C2) (right)

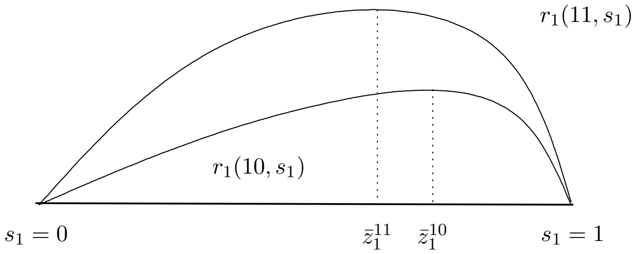


Figure 5: Functions $r_1(10, \cdot)$ and $r_1(11, \cdot)$.

## 4.1 Independent Signals

A more precise characterization of an optimal scheme becomes possible when we make some additional assumptions on the valuation functions and the signal distribution. Assume specifically that the signals $s_1$ and $s_2$ are independent. Let $G_i$ be the cumulative distribution function of $s_i$, and for $i \in I$, $a \in A$ and $s_i \in S_i$, define

$$r_i(a, s_i) = \{1 - G_i(s_i)\}\, v_i(a, s_i).$$

to be the seller's expected revenue from buyer $i$ when he offers allocation $a$ for price $v_i(a, s_i)$. We make the following assumptions.

**Assumption 3** *1. $v_i(a, \cdot)$ is strictly log-concave for each $a \in A_i$.*[9]

*2. $\frac{v_1(11,\cdot)}{v_1(10,\cdot)}$ and $\frac{v_2(11,\cdot)}{v_2(01,\cdot)}$ are weakly decreasing.*

*3. $\frac{g_i(\cdot)}{1-G_i(\cdot)}$ is strictly increasing.*

The first two conditions hold, for example, when $v_i(a, s_i) = \gamma(a)\, h(s_i)$ for some functions $\gamma : A \to \boldsymbol{R}_+$ and $h : [0, 1] \to \boldsymbol{R}_+$ such that $h$ is strictly log-concave. The increasing hazard rate condition in the third line is known to hold for most distributions. As summarized by the following lemma and depicted in Figure 5, the above assumption implies that the graph of $r_1(a, \cdot)$ has a single peak when $a = 11$ or 10 and that the peak of $r_1(11, \cdot)$ is located to the left of that of $r_1(10, \cdot)$.

**Lemma 1** *Suppose that Assumptions 2 and 3 hold. Then*

1. *For each $a \in A_i$, $r_i(a, \cdot)$ is strictly log-concave with the (unique) maximizer $\bar{z}_i^a$ which satisfies $\bar{z}_1^{11} \leq \bar{z}_1^{10}$ and $\bar{z}_2^{11} \leq \bar{z}_2^{01}$.*

[9] That is, log $v_i(a, \cdot)$ is strictly concave for each $a$.

*2.*

$$\frac{\partial r_1}{\partial s_1}(11, s_1) < \frac{\partial r_1}{\partial s_1}(10, s_1) \quad \textit{for } s_1 > \bar{z}_1^{10},$$
$$\frac{\partial r_2}{\partial s_2}(11, s_2) < \frac{\partial r_2}{\partial s_2}(01, s_2) \quad \textit{for } s_2 > \bar{z}_2^{01}.$$

We say that a price posting scheme $(f, t)$ against two buyers is *regular* if

1. $y_1^{11} \leq y_1^{10}$, $y_2^{11} \leq y_2^{01}$, and
2. $f_1(s) = \begin{cases} 1 & \text{if } s_1 \geq y_1^{10}, \text{ or } s \geq (y_1^{11}, y_2^{11}), \\ 0 & \text{otherwise,} \end{cases}$ and

   $f_2(s) = \begin{cases} 1 & \text{if } s_2 \geq y_2^{01}, \text{ or } s \geq (y_1^{11}, y_2^{11}), \\ 0 & \text{otherwise.} \end{cases}$

Under a regular scheme, hence, the price of a larger network $a = 11$ is more affordable than that of a smaller network $a = 10$ or $01$, and the network size is maximized subject to the individual rationality constraints. The second property can also be interpreted as saying that the good is allocated to a single buyer only when the other buyer's signal is too low for joint adoption. Configuration (A0) in Figure 1 corresponds to a regular scheme. If a regular scheme is optimal, then $(y_1^{10}, y_2^{01}) = (\bar{z}_1^{10}, \bar{z}_2^{01})$ and

$$\begin{aligned}(y_1^{11}, y_2^{11}) \in \operatorname{argmax} \{1 - G_2(y_2^{11})\}\, r_1(11, y_1^{11}) + \{1 - G_1(y_1^{11})\}\, r_2(11, y_2^{11}) \\ + G_2(y_2^{11})\, r_1(10, \bar{z}_1^{10}) + G_1(y_1^{11})\, r_2(01, \bar{z}_2^{01}).\end{aligned} \tag{3}$$

**Theorem 2** *Suppose that $(s_1, s_2)$ is independent. If $(f, t)$ is an optimal ex post implementable price posting scheme against two buyers under Assumptions 2 and 3, then it is regular.*

**Proof.** See the Appendix. ■

When $y_1^{11} < y_1^{10}$ and $y_2^{11} < y_2^{01}$, it is impossible to replicate the allocation rule $f$ of a regular scheme by any scheme in which the buyers' decisions are based only on their own signals or on the decisions of other buyers: In any such scheme, at least one buyer's decision (*e.g,* the first-mover's decision) must be independent of other buyers' signals.

**Example:** Suppose that $s_i$ has the uniform distribution $G_i(s_i) = s_i$, and that the buyers' valuation functions are given by

$$\begin{array}{ll} v_1(10, s_1) = \gamma s_1 & v_2(01, s_2) = \gamma s_2 \\ v_1(11, s_1) = \delta s_1, & v_2(11, s_2) = \delta s_2, \end{array}$$

where $0 < \gamma < \delta$. Given that the optimal scheme is regular, the marginal value for the single adoption 10 or 01 equals $y_1^{10} = y_2^{01} = \frac{1}{2}$. By (3) and symmetry, the marginal value $y_1^{11} = y_2^{11}$ for the joint adoption 11 solves

$$y_1^{11} = y_2^{11} \in \underset{x}{\operatorname{argmax}} \; \delta x(1-x)^2 + \frac{\gamma}{4} x.$$

Solving this, we get[10]

$$y_1^{11} = y_2^{11} = \frac{1}{3\delta} \left\{ 2\delta - \sqrt{\delta^2 - \frac{3}{4}\gamma\delta} \right\}.$$

We can confirm that $y_1^{11} = y_2^{11} < \frac{1}{2} = y_1^{10} = y_2^{01}$ if and only if $\gamma < \delta$. Consider now the price of each allocation associated with these marginal values. They are given by

$$t_1(10) = t_2(01) = \frac{\gamma}{2}, \quad \text{and} \quad t_1(11) = t_2(11) = \frac{1}{3} \left\{ 2\delta - \sqrt{\delta^2 - \frac{3}{4}\gamma\delta} \right\}.$$

From these, we can check that the price of the size 2 network 11 is higher than that of the size 1 network if and only if

$$\frac{\delta}{\gamma} > \frac{3}{4}.$$

In other words, when the network externalities are strong, the actual price of the larger network is higher than that of the smaller network, and vice versa.

# 5 Optimal Symmetric Schemes

With more than two buyers, the problem of identifying all the ex post implementable schemes becomes intractable. In this section, we focus on an optimal symmetric scheme when the buyers are ex ante symmetric. We show that the optimal scheme

[10] As seen, the explicit derivation of the marginal values is possible only under very limited specifications of the distribution and values.

is regular when the network externalities are strong, or when a stronger notion of incentive compatibility is imposed.

Suppose that the signals $s_1, \ldots, s_I$ are independent and identically distributed, and denote by $g$ the density of $s_i$ and by $G$ the corresponding cumulative distribution. The valuation functions are symmetric in the sense that

$$v_i(a, s) = v_j(a, s')$$

for any $a \in A$, $i \neq j$, and $s$, $s' \in S$ such that $(s_i, s_j, s_{-i-j}) = (s'_j, s'_i, s_{-i-j})$. The symmetry condition implies that the network externalities depend only on the *size* of the allocation $a \in A$, defined by $|a| = \sum_{j \in I} a_j$. For this reason, we denote by $v^n : [0,1] \to \boldsymbol{R}_+$ the valuation function of any single buyer when he adopts an allocation of size $n \in N \equiv \{1, \ldots, I\}$.[11]

We say that a price posting scheme $(f, t)$ is *symmetric* if for any $i \neq j$,

$$(f_i(s), f_j(s), f_{-i-j}(s)) = (f_j(s'), f_i(s'), f_{-i-j}(s'))$$

for any $s$, $s' \in S$ such that $(s_i, s_j, s_{-i-j}) = (s'_j, s'_i, s_{-i-j})$, and

$$(t_i(a), t_j(a), t_{-i-j}(a)) = (t_j(a'), t_i(a'), t_{-i-j}(a'))$$

for any $a$, $a' \in A$ such that $(a_i, a_j, a_{-i-j}) = (a'_j, a'_i, a'_{-i-j})$. That is, when $(f, t)$ is symmetric, swapping the private signals of any pair of buyers results in the swapping of their allocations but does not affect those of any other buyers.[12] When the scheme $(f, t)$ is symmetric, the transfer depends on the allocation only through its size. That is, $t_i(a) = t_j(a')$ for any $i$, $j \in I$ and any $a \in A_i$, $a' \in A_j$ such that $|a| = |a'|$. Hence, we let $t^n$ denote the transfer required of any single buyer when he is one of $n$ adopters of the good. The following assumption is a symmetric generalization of that in the previous section.

**Assumption 4** *1.* $v^1(0) = \cdots = v^I(0) = 0$ *and* $(v^1)'(\cdot) < \cdots < (v^I)'(\cdot)$.

*2.* $\frac{g(\cdot)}{1-G(\cdot)}$ *is strictly increasing.*

*3.* $v^1, \ldots, v^I$ *are strictly log-concave.*

*4.* $\frac{v^n(\cdot)}{v^m(\cdot)}$ *is weakly decreasing if* $m < n$.

[11] Although the set of sizes of positive networks equals the set $I = \{1, \ldots, I\}$ of buyers, we use different notation $N$ to avoid confusion.

[12] In the social choice literature, this property is often called anonymity.

Recall from (1) that for any $a$, $a' \in A_i$, $y_i^a$ denotes the signal at which buyer $i$ is indifferent between allocation $a$ priced at $t_i(a)$ and no-adoption, and $y_i^{a,a'}$ denotes the signal at which he is indifferent between $a$ and $a'$. For any $m$, $n \in I$, $m \neq n$, and $a$, $a' \in A_i$ such that $|a| = m$ and $|a'| = n$, we let $y^m = y_i^a$, and $y^{mn} = y_i^{a,a'}$. In the present context, $y^n$ is defined by

$$v^n(y^n) = t^n \text{ if } t^n \in [0, v^n(1)],$$

and $y^n = 0$ if $t^n < 0$, and $y^n = 1$ if $t^n > v^n(1)$. Likewise, for $m < n$, $y^{mn} = y^{nm}$ is defined by

$$v^n(y^{mn}) - t^n = v^m(y^{mn}) - t^m \text{ if } t^n - t^m \in [0, v^n(1) - v^m(1)],$$

and $y^{mn} = 0$ if $t^n - t^m < 0$, and $y^{mn} = 1$ if $t^n - t^m > v^n(1) - v^m(1)$. Just as in the general formulation of Section 3, restricting the range of the transfer rule $t^n$ to $[0, v^n(1)]$ for each $n \in N$ entails no loss of generality as far as the expected revenue is concerned. Given the one-to-one correspondence between such a transfer rule $t = (t^1, \ldots, t^n)$ and the profile of marginal signals $y = (y^1, \ldots, y^I)$, we again use $t$ and $y$ interchangeably when describing a price-posting scheme.

Let $\lambda^0 = \lambda^0_{I-1} = 1$, and for each $k = 1, \ldots, I-1$, let $\lambda^k = \lambda^k_{I-1}$ be the $k$th highest value among $I-1$ signals $s_{-i} = (s_j)_{j \neq i}$. A symmetric price-posting scheme $(f, t)$ is *regular* if

1. $y^I \leq \cdots \leq y^1$, and
2. $f_i(s) = \begin{cases} 1 & \text{if } s_i \geq y^n \text{ and } \lambda^{n-1} \geq y^n \text{ for some } n \in N, \\ 0 & \text{otherwise.} \end{cases}$

Again, a regular scheme (1) sets a more affordable price for a larger network, and (2) maximizes the network size subject to individual rationality. The second property is implied if for any $k \in N$, $|f(s)| = k$ if and only if $|\{i \in I : s_i \geq y^k\}| = k$: To see the "only if" part, suppose that $|f(s)| = k$. Then IR implies that $|\{i \in I : s_i \geq y^k\}| \geq k$. If the inequality is strict, then take any $i$ such that $s_i \geq y^k$. For this $i$, $\lambda^k \geq y^k \geq y^{k+1}$ so that $|f(s)| \geq k+1$ must hold by definition, a contradiction. It is also not difficult to see from Proposition 2 that a regular scheme is strategy-proof.[13]

As in the case with two buyers, we consider the seller's expected revenue from a single buyer $i$. Specifically, take any set $K \subset N$ and write $K = \{k_1, \ldots, k_m\}$

[13] Proposition 4 below proves that it satisfies a stronger condition of coalitional strategy-proofness.

for $k_1 < \cdots < k_m$. Let also the marginal signals $y = (y^1, \ldots, y^I) \in [0,1]^I$ be given. Suppose now that the seller simultaneously offers buyer $i$ an allocation of size $k_1$ for price $v^{k_1}(y^{k_1})$, an allocation of size $k_2$ for price $v^{k_2}(y^{k_2})$, and so on. Letting $y^K = (y^k)_{k \in K}$, we will denote by $r^K(y^K)$ the seller's expected revenue from these offers to buyer $i$.[14] When $K = \{k\}$, we denote $r^K(y^K) = r^k(y^k)$, and when $K = \{k, \ell\}$, we denote $r^K(y^K) = r^{k\ell}(y^k, y^\ell)$. Under Assumption 4, we have:

**Lemma 2** *Suppose that Assumption 4 holds. Then the following hold.*

1. *For $m < n$, $\frac{(v^n)'(\cdot)}{v^n(\cdot)} \leq \frac{(v^m)'(\cdot)}{v^m(\cdot)}$.*
2. *For each $n \in I$, $r^n$ is strictly log-concave with the (unique) maximizer $\bar{z}^n$ which satisfies $1 > \bar{z}^1 \geq \cdots \geq \bar{z}^I > 0$.*
3. *$\frac{(r^m)'(s)}{(v^m)'(s)} > \frac{(r^n)'(s')}{(v^n)'(s')}$ if $m < n$ and $s < s'$.*
4. *If $m < n$, $y^m < y^n$, and $(r^n)'(y^{mn}) \geq (r^m)'(y^{mn})$, then $r^n(y^n) > r^{mn}(y^m, y^n)$.*

The last observation above says that for the seller, offering two allocations is dominated by offering just one of them when $y^{mn}$ is not so large.[15] Let the marginal signals $y = (y^1, \ldots, y^I)$ and set $K \subset N$ be given. Define $L^K(y)$ by

$$L^K(y) = \{s_{-i} \in S_{-i} : \min_{k \in K} (\lambda^{k-1} - y^k) \geq 0, \max_{k \notin K} (\lambda^{k-1} - y^k) < 0\}.$$

The interpretation of $L^K(y)$ is as follows. $L^K(y)$ is the set of signal profiles of buyers other than $i$ such that when $s_{-i} \in L^K(y)$, an ex post IR price-posting scheme $(f, t)$ may assign an allocation of size $k \in K$ to $(s_i, s_{-i})$ if $s_i \geq y^k$, but not an allocation of size $k \notin K$ for any $s_i$. Let

$$Q^K(y) = P(s_{-i} \in L^K(y)).$$

## 5.1 Optimal Symmetric Scheme under Strong Externalities

We assume that the network externalities are sufficiently large in the sense specified below.

[14] An explicit formula for $r^K(y^K)$ is presented in the Appendix.

[15] Note that $(r^n)'(y^{mn}) \geq (r^m)'(y^{mn})$ holds only if $y^{mn} < \bar{z}^n$.

**Assumption 5** *1. For any $m < n$ and $y^m$, $y^n \in (0,1)$, if $r^n(y^n) \leq r^m(y^m)$ and $y^n < y^m$, then*

$$\frac{(r^n)'(y^n)}{r^n(y^n)}\{G(y^m) - G(y^n)\} > g(y^n).$$

*2. Let $\bar{\mu}^n = \frac{v^{n-1}(1)}{v^n(1)}$ and $\underline{\mu}^n = \lim_{s\to 0} \frac{v^{n-1}(s)}{v^n(s)}$ for $n = 2, \ldots, I$. If*

$$\bar{\mu}^n < \frac{1 + (n-2)\prod_{k=2}^{n-1} \underline{\mu}^k}{n-1},$$

*for $n = 2, \ldots, I-1$,*

The following proposition verifies that Assumption 5 is a requirement on the degree of network externalities for the linear valuation functions.[16]

**Proposition 3** *Suppose that $v^n(s_i) = \rho^n s_i$ for every $n = 1, \ldots, I$, where $0 < \rho^1 < \cdots < \rho^I$ are constants. Suppose also that the density $g$ is continuous and strictly positive. Then there exists $\varepsilon > 0$ such that Assumption 5 holds when $\max_{2\leq k\leq I} \frac{\rho^{k-1}}{\rho^k} < \varepsilon$.*

Now define $w : [0,1]^I \to \boldsymbol{R}_+$ by

$$w(y) = \sum_{1\in K\subset N} Q^K(y) \max_{\emptyset\neq L\subset K} r^L(y^L).$$

$w(y)$ is the maximal revenue that the seller can raise from any single buyer $i$ when he only takes in account (i) IC of buyer $i$, and (ii) ex post IR of all buyers. It hence gives an upper bound on the expected revenue under a symmetric, ex post implementable price-posting scheme $(f, y)$. We can also make the following observation. Suppose that $(f, y)$ is a regular scheme. Suppose further that $s_{-i} \in L^K(y)$ for some $i$ and $K \subset N$ with $1 \in K$. Since $y^{\max K} \leq y^k$ for any $k \in K$, if $s_i < y^{\max K}$, then $s_i < y^k$ for any $k \in K$ so that $f_i(s_i, s_{-i}) = 0$. On the other hand, since the network size is maximized subject to IR, if $s_i \geq y^{\max K}$, then $|f(s_i, s_{-i})| = \max K$. Therefore, the expected revenue from buyer $i$ conditional on $s_{-i} \in L^K(y)$ equals $r^{\max K}(y^{\max K})$, and the unconditional expected revenue from buyer $i$ equals

$$R(f, y) = \sum_{1\in K\subset N} Q^K(y)\, r^{\max K}(y^{\max K}). \tag{4}$$

[16] For a more general class of valuation functions, the first condition in Assumption 5 also requires some condition on their derivative.

If $r^{\max K}(y^{\max K}) = \max_{\emptyset \neq L \subset K} r^L(y^L)$ for any such $K$, hence, we have $w(y) = R(f, y)$ by the definition of $w(y)$ and (4). The following lemma summarizes this observation.

**Lemma 3** *Let $y \in [0,1]^I$ be such that $y^I \leq \cdots \leq y^1$, and*

$$r^{\max K}(y^{\max K}) = \max_{\emptyset \neq L \subset K} r^L(y^L) \textit{ for any } 1 \in K \subset N.$$

*If $(f, y)$ is a regular scheme, then $R(f, y) = w(y)$.*

The following theorem proves the optimality of a regular scheme by showing that any maximizer $y$ of $w$ satisfies the conditions of Lemma 3.

**Theorem 3** *Suppose that Assumptions 4 and 5 hold. Then the optimal symmetric price-posting scheme is regular.*

When the externalities are positive but weak, preliminary analysis indicates that an optimal ex post implementable symmetric scheme is not regular. Full characterization of an optimal scheme in such an environment appears extremely difficult as it entails a very complex allocation rule. As seen in the next section, however, requiring a stronger version of incentive compatibility recovers the regularity of an optimal scheme for any positive degree of externalities.

## 5.2 Optimal Symmetric Scheme under Coalitional Implementability

Given a price-posting scheme $(f, t)$, a subset $J \subset I$ of buyers, and signal profiles $s = (s_J, s_{-J})$ and $\hat{s}_J$, $\hat{s}_J$ is a *profitable deviation* for the coalition $J$ at $s$ if

$$v_i(f(\hat{s}_J, s_{-J}), s_i) - t_i(f(\hat{s}_J, s_{-J})) \geq v_i(f(s), s_i) - t_i(f(s)) \text{ for every } i \in J, \text{ and}$$
$$v_i(f(\hat{s}_J, s_{-J}), s_i) - t_i(f(\hat{s}_J, s_{-J})) > v_i(f(s), s_i) - t_i(f(s)) \text{ for some } i \in J.$$

$(f, t)$ is *coalitionally strategy-proof* if no coalition of buyers has a profitable deviation at any signal profile. Coalitional strategy-proofness is hence a strong robustness requirement since even if there exists a group of buyers who share the information about their private signals and jointly misreport them, the deviation is not profitable. $(f, t)$ is *coalitionally ex post implementable* if it is coalitionally strategy-proof and ex post individually rational.

**Proposition 4** *A regular scheme $(f, t)$ is coalitionally ex post implementable.*

Given the marginal signals $y = (y^1, \dots, y^I)$, define

$$M(y) = \{m : m = 1, \dots, I-1, \, y^m < \max_{\ell > m} y^\ell\}.$$

$M(y)$ is the set of sizes of a network whose marginal value is smaller than the marginal value for some larger network. Also, let

$$K(f) = \{n \in N : |f(s)| = n \text{ for some } s \in S\}$$

be the set of network sizes that may be achieved under an allocation rule $f$. If $(f, y)$ is a regular scheme, then $K(f) = N$, and also $M(y) = \emptyset$ since a larger network is always more affordable ($y^I \leq \dots \leq y^1$). Hence, $M(y) \cap K(f) = \emptyset$ for a regular scheme. For $K \subset N$ and $y \in [0,1]^I$, let $w(K, y)$ be defined by

$$w(K, y) = \begin{cases} \sum_{k \in K} P\left(\lambda^{k-1} \geq y^k, \, \max_{\substack{\ell > k \\ \ell \in K}} (\lambda^{\ell-1} - y^\ell) < 0\right) r^k(y^k) & \text{if } K \cap M(y) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

The following proposition shows that for any coalitionally ex post implementable scheme, (1) $y^m \leq y^k$ holds whenever $k$, $m \in K(f)$ and $k < m$, (2) its expected revenue is bounded above by $w(K(f), y)$, and (3) when it is regular, the expected revenue $R(f, y)$ equals $w(K(f), y) = w(N, y)$.

**Lemma 4** *Let $(f, y)$ be a symmetric, coalitionally ex post implementable price-posting scheme. Then*

1. *$M(y) \cap K(f) = \emptyset$.*
2. *$R(f, y) \leq w(K(f), y)$.*
3. *If $(f, y)$ is regular, then $R(f, y) = w(N, y)$.*

Given that the expected revenue from a regular scheme $(f, y)$ equals an upper bound $w(N, y)$, it is optimal if the function $w(K, y)$ is itself maximized at some $(N, y)$. The following theorem shows that this in fact holds.

**Theorem 4** *Suppose that Assumption 4 holds. Then there exists a regular price-posting scheme that is optimal in the class of symmetric, coalitionally ex post implementable price-posting schemes.*

## 6 Conclusion

Most of the sales schemes for network goods proposed in the literature specify a fixed price or transfer for each allocation but do not coordinate the buyers' adoption decisions. A price-posting scheme maintains a one-to-one correspondence between the price and allocation and allows the seller to coordinate the buyers' adoption decisions through the reported signals. As such, hence, it presents a reasonable generalization of many sales schemes studied in the literature. The ex post implementability eliminates the multiplicity of equilibria, a central issue in the network adoption problems. We identify the conditions under which the optimal scheme is regular. In a regular scheme, a more affordable price is set for a larger network, and given those prices, the network size is maximized as allowed by individual rationality. Given that regularity is defined in terms of the private signals, it has no direct implication on the actual price levels for different allocations. As observed in the example in Section 4.1, it is consistent with a lower price for a smaller network when the network externalities are strong, and a lower price for a larger network when the externalities are not so strong. The observation in the first case corresponds to a refund from the seller to the adopters when the number of adoptions is below some threshold.

In this paper, we have only looked at positive network externalities. It would be interesting to study optimal sales schemes under negative externalities as seen in the case of snob consumption, or more complex forms of externalities based on graph structure.[17] Network goods are often supplied competitively as in the case of cellular phones or PC operating systems. While some aspects of such competition have been analyzed by Katz and Shapiro (1985, 1986), much remains to be understood.

## Appendix

**Proof of Proposition 2** (Necessity) 1. We first show that if $f(s_i, s_{-i}) = a^k$ and $f(s_i', s_{-i}) = a^m$ for $k < m$, then $s_i < s_i'$. Since $(f, t)$ is strategy-proof,

$$\begin{aligned}
&v_i(a^m, s_i') - t_i(a^m) \\
&= v_i(f(s_i', s_{-i}), s_i') - t_i(f(s_i', s_{-i})) \\
&\geq v_i(f(s_i, s_{-i}), s_i') - t_i(f(s_i, s_{-i})) \\
&= v_i(a^k, s_i') - t_i(a^k),
\end{aligned}$$

[17] See Sundararajan (2007) for one such formulation.

and

$$\begin{aligned}
&v_i(a^k, s_i) - t_i(a^k) \\
&= v_i(f(s_i, s_{-i}), s_i) - t_i(f(s_i, s_{-i})) \\
&\geq v_i(f(s_i', s_{-i}), s_i) - t_i(f(s_i', s_{-i})) \\
&= v_i(a^m, s_i) - t_i(a^m).
\end{aligned}$$

It hence follows that

$$v_i(a^m, s_i') - v_i(a^k, s_i') \geq t_i(a^m) - t_i(a^k) \geq v_i(a^m, s_i) - v_i(a^k, s_i).$$

This further implies that

$$\begin{aligned}
\int_{s_i}^{s_i'} \frac{\partial v_i}{\partial s_i}(a^m, s_i)\, ds_i &= v_i(a^m, s_i') - v_i(a^m, s_i) \\
&\geq v_i(a^k, s_i') - v_i(a^k, s_i) = \int_{s_i}^{s_i'} \frac{\partial v_i}{\partial s_i}(a^k, s_i)\, ds_i.
\end{aligned}$$

Since $\frac{\partial v_i}{\partial s_i}(a^m, \cdot) > \frac{\partial v_i}{\partial s_i}(a^k, \cdot)$ by assumption, this implies that $s_i < s_i'$.

2. If $t_i(a) > t_i(a^k)$ for $a \in C_i(a^k) \cap B_i(s_{-i})$, then $v_i(f(s_i, s_{-i}), s_i) - t_i(f(s_i, s_{-i})) < v_i(f(s_i', s_{-i}), s_i) - t_i(f(s_i', s_{-i}))$ for $s_i$ and $s_i'$ such that $f(s_i, s_{-i}) = a$ and $f(s_i', s_{-i}) = a^k$, contradicting the strategy-proofness of $(f, t)$.

3. Ex post IR requires that $v_i(a^1, 0) - t_i(a^1) = -t_i(a^1) \geq 0$.

4. For $s_i$ and $s_i'$ such that $f(s_i, s_{-i}) = a^k$ and $f(s_i', s_{-i}) = a^{k+1}$, we have

$$\begin{aligned}
v_i(a^k, s_i) - t_i(a^k) &= v_i(f(s_i, s_{-i}), s_i) - t_i(f(s_i, s_{-i})) \\
&\geq v_i(f(s_i', s_{-i}), s_i) - t_i(f(s_i', s_{-i})) \\
&= v_i(a^{k+1}, s_i) - t_i(a^{k+1}).
\end{aligned}$$

Hence,

$$t_i(a^{k+1}) - t_i(a^k) \geq v_i(a_{k+1}, s_i) - v_i(a^k, s_i) \geq 0.$$

(Sufficiency) Fix $i \in I$ and $s_{-i} \in S_{-i}$.
Strategy-proofness:

Suppose that $s_i \in [y_i^{a^{k-1}a^k}, y_i^{a^k a^{k+1}}]$ and that $s_i' \in [y_i^{a^{\ell-1}a^\ell}, y_i^{a^\ell a^{\ell+1}}]$ for some $k \neq \ell$. Then

$$\begin{aligned}
v_i(f(s_i, s_{-i}), s_i) - t_i(f(s_i, s_{-i})) &= v_i(a^k, s_i) - t_i(a^k) \\
&\geq v_i(a^\ell, s_i) - t_i(a^\ell) \\
&= v_i(f(s_i', s_{-i}), s_i) - t_i(f(s_i', s_{-i})),
\end{aligned}$$

where the inequality follows from (2).

Ex post IR:

Since for $s_i \in [y_i^{a^k a^{k-1}}, y_i^{a^{k+1} a^k}]$, we have

$$
\begin{aligned}
& v_i(a^k, s_i) - t_i(a^k) \\
& \geq v_i(a^k, y_i^{a^k a^{k-1}}) - t_i(a^k) = v_i(a^{k-1}, y_i^{a^k a^{k-1}}) - t_i(a^{k-1}) \\
& \geq v_i(a^{k-1}, y_i^{a^{k-1} a^{k-2}}) - t_i(a^{k-1}) = v_i(a^{k-2}, y_i^{a^{k-1} a^{k-2}}) - t_i(a^{k-2}) \\
& \geq \cdots \\
& \geq -t_i(a^1) \geq 0.
\end{aligned}
$$

**Proof of Theorem 1** We begin with the following lemma.

**Lemma 5** *Suppose that $(f, t)$ is an optimal ex post implementable price posting scheme against two buyers under Assumption 2. Then*

1. *There exist no $0 \leq \alpha_1 < \beta_1 \leq 1$ such that $f(s) = 0$ for every $s \in (\alpha_1, \beta_1) \times (y_2^{01}, 1]$.*
2. *There exist no $0 \leq \alpha_2 < \beta_2 \leq 1$ such that $f(s) = 0$ for every $s \in (y_1^{10}, 1] \times (\alpha_2, \beta_2)$.*
3. *$L_{11}$ is a rectangle with a non-empty interior such that $(1, 1) \in L_{11}$ and $(0, 0) \notin L_{11}$.*

**Proof.**

1. Suppose that there exist such $\alpha_1$ and $\beta_1$ and denote $D = (\alpha_1, \beta_1) \times (y_2^{01}, 1]$. We will show that $(f, t)$ is suboptimal. If $y_2^{01} = 0$ or 1, let $(\hat{f}, \hat{t})$ be such that $\hat{y}_1 = y_1$, $(\hat{y}_2^{01}, \hat{y}_2^{11}) = (\frac{1}{2}, y_2^{11})$, and

$$
\hat{f}(s) = \begin{cases} 01 & \text{if } s \in (\alpha_1, \beta_1) \times (\frac{1}{2}, 1], \\ f(s) & \text{otherwise.} \end{cases}
$$

Then $(\hat{f}, \hat{t})$ is ex post implementable and raises a strictly positive expected revenue $P(s \in (\alpha_1, \beta_1) \times (\frac{1}{2}, 1])\, v_2(01, \frac{1}{2})$ from $D$. When $y_2^{01} \in (0, 1)$, let $(\hat{f}, \hat{t})$ be such that $\hat{y} = y$ and

$$
\hat{f}(s) = \begin{cases} 01 & \text{if } s \in D, \\ f(s) & \text{otherwise.} \end{cases}
$$

Again, $(\hat{f}, \hat{t})$ is ex post implementable and raises a strictly positive expected revenue $P(s \in D)\, v_2(01, y_2^{01})$ from $D$. In both cases, $R(\hat{f}, \hat{t}) > R(f, t)$.

3. If $L_{11} \neq \emptyset$, then it contains $(1,1)$ by Assumption 2 and Proposition 2. Suppose that int $L_{11} = \emptyset$. The optimality of $(f, t)$ would then imply that $(1,1) \in L_{10} \cup L_{01}$. Assume without loss of generality that $(1,1) \in L_{10}$. We will show that $(f, t)$ is dominated by an alternative scheme $(\hat{f}, \hat{t})$ defined as follows:

$$\hat{f}(s) = \begin{cases} 11 & \text{if } s \in [y_1^{10}, 1] \times [0,1], \\ f(s) & \text{otherwise.} \end{cases} \qquad (\hat{y}_1^{10}, \hat{y}_1^{11}) = (y_1^{10}, y_1^{10}), \quad (\hat{y}_2^{01}, \hat{y}_2^{11}) = (y_2^{01}, 0)$$

Then $(\hat{f}, \hat{t})$ is ex post implementable. Furthermore, the expected revenue under $(\hat{f}, \hat{t})$ from $[y_1^{10}, 1] \times [0,1]$ equals

$$P(L_{11})\, v_1(11, y_1^{10}).$$

This is strictly greater than the expected revenue under $(f, t)$ from the same set since the latter is bounded above by

$$P(L_{11})\, v_1(10, y_1^{10}),$$

and $v_1(11, y_1^{10}) > v_1(10, y_1^{10})$ by Assumption 2. The expected revenue under $(\hat{f}, \hat{t})$ and that under $(f, t)$ are the same elsewhere. We hence conclude that $R(f, t) < R(\hat{f}, \hat{t})$.

Next, we show that $L_{11}$ is a rectangle. If $y_1^{10} \geq y_1^{11}$, then $L_{11} = [y_1^{11}, 1] \times [y_2^{11}, 1]$ or $L_{11} = [y_1^{11}, 1] \times [y_2^{11,01}, 1]$. It is also a rectangle if $y_2^{01} \geq y_2^{11}$. Suppose then that $y_1^{11,10} > y_1^{11}$ and $y_2^{11,01} > y_2^{11}$. $L_{11}$ may fail to be a rectangle only if $L_{11} = [y_1^{11}, 1] \times [y_2^{11}, 1] \setminus [y_1^{11}, y_1^{11,10}] \times [y_2^{11}, y_2^{11,01}]$. However, if $f(s) = 10$ for $s \in [y_1^{11}, y_1^{11,10}] \times [y_2^{11}, y_2^{11,01}]$, $f$ is not ex post IC since for $s_1 \in (y_1^{11}, y_1^{11,10})$, $L_2(10, s_1) = [y_2^{11}, y_2^{11,01}]$ and $L_2(11, s_1) = [y_2^{11,01}, 1]$ and violates Proposition 2. Likewise, $f(s) \neq 01, 00$ for $s \in [y_1^{11}, y_1^{11,10}] \times [y_2^{11}, y_2^{11,01}]$. Therefore, $L_{11}$ is a rectangle in all cases. Finally, $(0,0) \notin L_{11}$ since otherwise, $L_{11} = [0,1]^2$ and the expected revenue under $(f, t)$ would equal zero. ■

We now return to the proof of the theorem.

Case 1) $y_1^{11} \leq y_1^{10}$ and $y_2^{11} \leq y_2^{01}$. For $s \ll (y_1^{11}, y_2^{11})$, $f(s) = 00$ by ex post IR. For $s \in [0, y_1^{10}) \times [0, y_2^{11})$, $f(s) = 00$ by ex post IR. It then follows from Lemma 5.1 that $y_1^{10} < 1$ and that $f(s) = 10$ for $s \in (y_1^{10}, 1] \times [0, y_2^{11})$. The symmetric argument shows that $y_1^{01} < 1$, $f(s) = 00$ for $s \in [0, y_1^{11}) \times (y_2^{11}, y_2^{01})$, and $f(s) = 01$

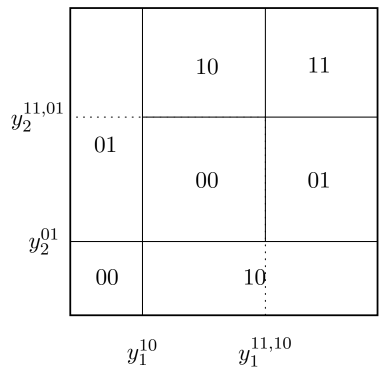


Figure 6: Configuration (D)

for $s \in [0, y_1^{11}) \times (y_2^{01}, 1]$. By Lemma 5.3, it must be the case that $f(s) = 11$ for $s \in (y_1^{10}, 1] \times (y_2^{01}, 1]$. By ex post IC, we must then have $f(s) = 11$ for $s \in (y_1^{11}, y_1^{10}] \times (y_2^{01}, 1]$ and $s \in (y_1^{10}, 1] \times (y_2^{11}, y_2^{01})$. Since $L_{11}$ is a rectangle, we can then conclude that $L_{11} = [y_1^{11}, 1] \times [y_2^{11}, 1]$.

Case 2) $y_1^{10} < y_1^{11} < y_1^{11,10}$ and $y_2^{01} < y_2^{11} < y_2^{11,01}$. For $s \ll (y_1^{10}, y_2^{01})$, $f(s) = 00$ by ex post IR. For $s \in (y_1^{10}, 1] \times (0, y_2^{01})$, $f(s) \in \{00, 10\}$ by ex post IR, and hence $f(s) = 10$ by Proposition 2 and Lemma 5.1. The symmetric argument shows that $f(s) = 01$ for $s \in [0, y_1^{10}) \times (y_2^{01}, 1]$. We now proceed by separately considering possible configurations of $L_{11}$. Since $L_{11}$ is a rectangle containing $(1, 1)$ by Lemma 5.3, there are four possible cases as follows:

1. $L_{11} = [y_1^{11,10}, 1] \times [y_2^{11,01}, 1]$. Proposition 2 shows that $f(s) = 10$ for $s \in (y_1^{10}, y_1^{11,10}) \times (y_2^{11,01}, 1]$ and that $f(s) = 01$ for $s \in (y_1^{11,10}, 1] \times (y_2^{01}, y_2^{11,01})$. Proposition 2 further implies that $f(s) = 00$ for $s \in (y_1^{10}, y_1^{11,10}) \times (y_2^{01}, y_2^{11,01})$. This configuration, called (D), is depicted in Figure 6.

   Now consider configurations (B1) and (B2) which have the same $t$ as (D) above. We show that (D) is dominated by (B1) if $t_1(10) \geq t_2(01)$ and dominated by (B2) if $t_1(10) \leq t_2(01)$. To see this, note that the expected revenue under (D) minus that under (B1) is written as

$$\begin{aligned} R^F - R^B = {} & P([y_1^{10}, y_1^{11,10}] \times [y_2^{01}, y_2^{11,01}]) \{-t_1(10)\} \\ & + P([y_1^{11,10}, 1] \times [y_2^{01}, y_2^{11}]) \{t_2(01) - t_1(10)\} \\ & + P([y_1^{11,10}, 1] \times [y_2^{11}, y_2^{11,01}]) \{t_2(01) - t_1(11) - t_2(11)\}. \end{aligned}$$

   Since $y_1^{11} > y_1^{10} \geq 0$ implies $t_1(11) = v_1(11, y_1^{11}) > v_1(10, y_1^{10}) = t_1(10) \geq 0$, this difference is strictly negative if $t_1(10) \geq t_2(01)$. Likewise, the expected revenue under (D) minus that under (B2) is written as

$$\begin{aligned}R^F - R^C &= P([y_1^{10}, y_1^{11,10}] \times [y_2^{01}, y_2^{11,01}]) \{-t_2(01)\} \\ &+ P([y_1^{10}, y_1^{11}] \times [y_2^{11,01}, 1]) \{t_1(10) - t_2(01)\} \\ &+ P([y_1^{11}, y_1^{11,10}] \times [y_2^{11,01}, 1]) \{t_1(10) - t_1(11) - t_2(11)\}.\end{aligned}$$

Since $y_2^{11} > y_2^{01} \geq 0$ implies $t_2(11) = v_2(11, y_2^{11}) > v_2(01, y_2^{01}) = t_2(01) \geq 0$, the difference is strictly negative if $t_2(01) \leq t_1(10)$. Hence, (D) is never optimal.

2. $L_{11} = [y_1^{11,10}, 1] \times [y_2^{11}, 1]$. By Proposition 2, $f(s) = 10$ for $s \in (y_1^{10}, y_1^{11,10}) \times (y_2^{11}, 1]$. Furthermore, Lemma 5.1 shows that $f(s) = 10$ for $s \in (y_1^{10}, 1] \times (y_2^{01}, y_2^{11})$. This yields (B1).

3. $L_{11} = [y_1^{11}, 1] \times [y_2^{11,01}, 1]$. A similar reasoning as above shows that $f(s) = 01$ for $(y_1^{10}, 1] \times (y_2^{01}, 1] \setminus L_{11}$. This yields (B2).

4. $L_{11} = [y_1^{11}, 1] \times [y_2^{11}, 1]$. In this case, we have two possibilities:

   (a) $f(s) = 10$ for $s \in (y_1^{10}, 1] \times (y_2^{01}, y_2^{11})$ and $f(s) = 00$ for $s \in (y_1^{10}, y_1^{11}) \times (y_2^{11}, 1]$. This yields (C1).

   (b) $f(s) = 01$ for $s \in (y_1^{10}, 1] \times (y_2^{01}, y_2^{11})$ and $f(s) = 00$ for $s \in (y_1^{10}, y_1^{11}) \times (y_2^{11}, 1]$. This yields (C2).

Case 3) $y_1^{10} < y_1^{11} < y_1^{11,10}$ and $y_2^{11} < y_2^{01}$.

By ex post IR and Lemma 5.1, $f(s) = 00$ for $s \in [0, y_1^{10}) \times [0, y_2^{01})$, $f(s) = 01$ for $s \in [0, y_1^{10}) \times (y_2^{01}, 1]$, and $f(s) = 10$ for $s \in [y_1^{10}, 1] \times [0, y_2^{11})$. By Lemma 5.3, $L_{11}$ can be either (i) $[y_1^{11,10}, 1] \times [y_2^{11}, 1]$ or (ii) $s \in [y_1^{11}, 1] \times [y_2^{11}, 1]$. In case (i), it must be the case that $f(s) = 10$ for $s \in (y_1^{10}, y_1^{11,10}) \times (y_2^{11}, 1]$. Hence we obtain configuration (B1). In case (ii), $f(s) = 00$ for $s \in (y_1^{10}, y_1^{11}) \times (y_2^{11}, y_2^{01})$ by Proposition 2. Proposition 2 also implies that that $f(s) \in \{10, 00\}$ for $s \in (y_1^{10}, y_1^{11}) \times (y_2^{01}, 1]$. However, Lemma 5.1 implies that $f(s) = 10$. This yields (A).

Case 4) $y_1^{11} < y_1^{10}$ and $y_2^{01} < y_2^{11} < y_2^{11,01}$.

The reasoning similar to that of Case 3 above yields (A) and (B2).

**Proof of Theorem 2** We first examine the optimality of configuration (B1), which requires $y_1^{10} < y_1^{11} < y_1^{11,10} < 1$. Since $y_1^{11,10}$ is uniquely determined as a function of $y_1 = (y_1^{10}, y_1^{11})$ in this case, we can use the pair of variables $(y_1^{10}, y_1^{10,11})$ instead of $y_1$ to express the seller's expected revenue.

$$
\begin{aligned}
& R^B(y_1^{11,10}, y_1^{10}, y_2^{11}, y_2^{01}) \\
& = \{1 - G_2(y_2^{11})\}\{1 - G_1(y_1^{11,10})\} \\
& \times \left\{v_1(11, y_1^{11,10}) - v_1(10, y_1^{11,10}) + v_1(10, y_1^{10}) + v_2(11, y_2^{11})\right\} \\
& + \left[1 - G_1(y_1^{10}) - \{1 - G_2(y_2^{11})\}\{1 - G_1(y_1^{11,10})\}\right] v_1(10, y_1^{10}) \\
& + G_1(y_1^{10})\{1 - G_2(y_2^{01})\} v_2(01, y_2^{01}) \\
& = \{1 - G_2(y_2^{11})\}\left\{r_1(11, y_1^{11,10}) - r_1(10, y_1^{11,10}) + \{1 - G_1(y_1^{11,10})\} v_2(11, y_2^{11})\right\} \\
& + r_1(10, y_1^{10}) + G_1(y_1^{10}) r_2(01, y_2^{01}).
\end{aligned}
$$

Differentiation of $R^B$ with respect to $y_1^{10}$ yields:

$$
\frac{\partial R^B}{\partial y_1^{11,10}}(y_1^{11,10}, y_1^{10}, y_2^{11}, y_2^{01}) = \frac{\partial r_1}{\partial s_1}(10, y_1^{10}) + g_1(y_1^{10}) r_2(01, y_2^{01}).
$$

If $y_1^{10} < \bar{z}_1^{10}$, then $\frac{\partial r_1}{\partial s_1}(10, y_1^{10}) > 0$ by Assumption 3 and hence the above partial derivative is strictly positive. It follows that the optimal $y_1^{10}$ must satisfy $y_1^{10} \geq \bar{z}_1^{10}$. Next, differentiation of $R^B$ with respect to $y_1^{11,10}$ yields:

$$
\begin{aligned}
& \frac{\partial R^B}{\partial y_1^{11,10}}(y_1^{11,10}, y_1^{10}, y_2^{11}, y_2^{01}) \\
& = \{1 - G_2(y_2^{11})\}\left\{\frac{\partial r_1}{\partial s_1}(11, y_1^{11,10}) - \frac{\partial r_1}{\partial s_1}(10, y_1^{11,10}) - g_1(y_1^{11,10}) v_2(11, y_2^{11})\right\}.
\end{aligned}
$$

Since $y_1^{11,10} > y_1^{10} \geq \bar{z}_1^{10}$, $\frac{\partial r_1}{\partial s_1}(11, y_1^{11,10}) < \frac{\partial r_1}{\partial s_1}(10, y_1^{11,10})$ by Assumption 3. It follows that

$$
\frac{\partial R^B}{\partial y_1^{11,10}}(y_1^{11,10}, y_1^{10}, y_2^{11}, y_2^{01}) < 0 \quad \text{for } y_1^{11,10} > y_1^{10},
$$

suggesting that (B1) cannot be optimal. The symmetric discussion shows that (B2) is also suboptimal. Consider next configuration (C1) which requires $y_1^{10} < y_1^{11} < 1$.

The expected revenue can be written as:

$$
\begin{aligned}
&R^D(y_1^{11}, y_1^{10}, y_2^{11}, y_2^{01}) \\
&= \{1 - G_2(y_2^{11})\}\{1 - G_1(y_1^{11})\}\Big\{v_1(11, y_1^{11}) + v_2(11, y_2^{11})\Big\} \\
&+ \{1 - G_1(y_1^{10})\}\, G_2(y_2^{11})\, v_1(10, y_1^{10}) \\
&+ G_1(y_1^{10})\{1 - G_2(y_2^{01})\}\, v_2(01, y_2^{01}) \\
&= \{1 - G_2(y_2^{11})\}\, r_1(11, y_1^{11}) + \{1 - G_1(y_1^{11})\}\, r_2(11, y_2^{11}) \\
&+ G_2(y_2^{11})\, r_1(10, y_1^{10}) + G_1(y_1^{10})\, r_2(01, y_2^{01}).
\end{aligned}
\tag{5}
$$

Differentiation of $R^D$ with respect to $y_1^{10}$ yields

$$\frac{\partial R^D}{\partial y_1^{10}}(y_1^{11}, y_1^{10}, y_2^{11}, y_2^{01}) = G_2(y_2^{11})\, \frac{\partial r_1}{\partial s_1}(10, y_1^{10}).$$

Hence, the optimal $y_1^{10}$ should equal $\bar{z}_1^{10}$. Differentiation of $R^D$ with respect to $y_1^{11}$ on the other hand yields

$$\frac{\partial R^D}{\partial y_1^{11}}(y_1^{11}, y_1^{10}, y_2^{11}, y_2^{01}) = \{1 - G_2(y_2^{11})\}\, \frac{\partial r_1}{\partial s_1}(11, y_1^{11}) - g_1(y_1^{11})\, r_2(11, y_2^{11}).$$

Since $y_1^{11} > y_1^{10} = \bar{z}_1^{10}$, $\frac{\partial r_1}{\partial s_1}(11, y_1^{11}) < \frac{\partial r_1}{\partial s_1}(10, y_1^{11}) < 0$ by Assumption 3. Therefore, the derivative is strictly negative and (C1) cannot be optimal. That (C2) cannot be optimal is shown by a symmetric argument. We are then left with configurations in (A), which require either $y_1^{11} \leq y_b$ or $y_2^{11} \leq y_2^{01}$. The expected revenue under each one of (A) has the same expression as that under (B2) in (5). It then follows from the discussion there that the optimal values satisfy $y_1^{10} = \bar{z}_1^{10}$, $y_2^{01} = \bar{z}_2^{01}$, $y_1^{11} \leq \bar{z}_1^{10}$ and $y_2^{11} \leq \bar{z}_2^{10}$. The optimal scheme is hence (A0), which is regular.

**Formula for $r^K(y^K)$:** We can verify that the seller's expected revenue from these price offers equals

$$
\begin{aligned}
&r^K(y^K) \\
&= \max\Big\{0, G(\min\{y^{k_1 k_2}, \ldots, y^{k_1 k_m}\}) - G(y^{k_1})\Big\}\, v^{k_1}(y^{k_1}) \\
&+ \sum_{n=2}^{m-1} \max\Big\{0, G(\min\{y^{k_n k_{n+1}}, \ldots, y^{k_n k_m}\}) \\
&- G(\max\{y^{k_n}, y^{k_1 k_n}, \ldots, y^{k_{n-1} k_n}\})\Big\}\, v^{k_n}(y^{k_n}) \\
&+ \max\Big\{0, 1 - G(\max\{y^{k_m}, y^{k_1 k_m}, \ldots, y^{k_{m-1} k_m}\})\Big\}\, v^{k_m}(y^{k_m}).
\end{aligned}
$$

When $y^{k_\ell k_n} < y^{k_m k_n}$ for every $\ell < m$ and $n$, we can express $r^K(y^K)$ as

$$r^K(y^K) = \sum_{n=2}^{m} \{r^{k_n}(y^{k_{n-1}k_n}) - r^{k_{n-1}}(y^{k_{n-1}k_n})\} + r^{k_1}(y^{k_1}). \tag{6}$$

**Proof of Lemma 4** (i) For $m < n$, $\frac{(v^n)'(\cdot)}{v^n(\cdot)} \leq \frac{(v^m)'(\cdot)}{v^m(\cdot)}$.

This readily follows from Assumption 4(iv), which implies that $\left(\frac{v^n(s)}{v^m(s)}\right)' \leq 0$ when $m < n$.

(ii) For each $n \in I$, $r^n$ is strictly log-concave with the (unique) maximizer $\bar{z}^n$ which satisfies $1 > \bar{z}^1 \geq \cdots \geq \bar{z}^I > 0$.

Note that

$$\begin{aligned}
(r^n)'(s) &= -g(s)\, v^n(s) + \{1 - G(s)\}\, (v^n)'(s) \\
&= \{1 - G(s)\}\, v^n(s) \left\{ -\frac{g(s)}{1 - G(s)} + \frac{(v^n)'(s)}{v^n(s)} \right\} \\
&= r^n(s) \left\{ -\frac{g(s)}{1 - G(s)} + \frac{(v^n)'(s)}{v^n(s)} \right\}.
\end{aligned}$$

Since $\frac{g(s)}{1-G(s)}$ is strictly increasing and $\frac{(v^n)'(s)}{v^n(s)}$ is strictly decreasing, $\frac{(r^n)'(\cdot)}{r^n(\cdot)}$ is strictly decreasing, implying that $r^n$ is strictly log-concave. Hence, the maximizer $\bar{z}^n$ of $r^n$ is unique and satisfies $\bar{z}^n \in (0,1)$ as $r^n(0) = r^n(1) = 0$. For $m < n$, $\bar{z}^m$ and $\bar{z}^n$ satisfy

$$\frac{(v^m)'(\bar{z}^m)}{v^m(\bar{z}^m)} = \frac{g(\bar{z}^m)}{1 - G(\bar{z}^m)} \quad \text{and} \quad \frac{(v^n)'(\bar{z}^n)}{v^n(\bar{z}^n)} = \frac{g(\bar{z}^n)}{1 - G(\bar{z}^n)}.$$

If $\bar{z}^m < \bar{z}^n$, then

$$\frac{g(\bar{z}^m)}{1 - G(\bar{z}^m)} < \frac{g(\bar{z}^n)}{1 - G(\bar{z}^n)},$$

and hence

$$\frac{(v^m)'(\bar{z}^m)}{v^m(\bar{z}^m)} < \frac{(v^n)'(\bar{z}^n)}{v^n(\bar{z}^n)},$$

which contradicts (i).

(iii) $\frac{(r^m)'(s)}{(v^m)'(s)} > \frac{(r^n)'(s')}{(v^n)'(s')}$ if $m < n$ and $s < s'$.

The inequality is equivalent to

$$\{1 - G(s')\} \left[1 - \frac{g(s')}{1 - G(s')} \frac{v^n(s')}{(v^n)'(s')}\right] < \{1 - G(s)\} \left[1 - \frac{g(s)}{1 - G(s)} \frac{v^m(s)}{(v^m)'(s)}\right].$$

Since $s < s'$, this holds if $\frac{g(\cdot)}{1-G(\cdot)}$ is (strictly) increasing, and $\frac{(v^n)'(s')}{v^n(s')} \leq \frac{(v^m)'(s)}{v^m(s)}$. By the log-concavity of $v^m$, the latter inequality holds if $\frac{(v^n)'(s')}{v^n(s')} \leq \frac{(v^m)'(s')}{v^m(s')}$, which is true by (i).

(iv) If $m < n$, $s < s'$, $s'' = \varphi^{mn}(s, s')$, and $(r^n)'(s'') \geq (r^m)'(s'')$, then $r^n(s') > r^{mn}(s, s')$.

We first show that $(r^n)'(s) \geq (r^m)'(s)$ implies that $(r^n)'(s)$, $(r^m)'(s) \geq 0$. Note that $(r^n)'(s) \geq (r^m)'(s)$ is equivalent to

$$\frac{(v^n)'(s) - (v^m)'(s)}{v^n(s) - v^m(s)} \geq \frac{g(s)}{1 - G(s)}. \tag{7}$$

and that $(r^n)'(s) \geq 0$ is equivalent to

$$\frac{(v^n)'(s)}{v^n(s)} \geq \frac{g(s)}{1 - G(s)}. \tag{8}$$

Furthermore, since $\frac{(v^n)'(\cdot)}{v^n(\cdot)} \leq \frac{(v^m)'(\cdot)}{v^m(\cdot)}$ by (i),

$$\frac{(v^n)'(s)}{v^n(s)} \geq \frac{(v^n)'(s) - (v^m)'(s)}{v^n(s) - v^m(s)}. \tag{9}$$

(8) then follows from (9) and (7). That $(r^m)'(s) > 0$ can be obtained in a similar manner.

Now, since $(r^n)'(s'')$, $(r^m)'(s'') \geq 0$, we have $(r^m)'(s) > 0$ and $(r^n)'(s') > 0$ for any $s$, $s' < s''$ by the strict log-concavity of $r^m$ and $r^n$. It hence follows from (iii) that for any such $s$ and $s'$,

$$\frac{(v^m)'(s)}{(v^n)'(s')} < \frac{(r^m)'(s)}{(r^n)'(s')}. \tag{10}$$

Now fix $s''$ such that $r^n(s'') > r^m(s'')$, and consider the following functions of $s \in [0, s'']$:

$$s' = (v^n)^{-1}\left(v^m(s) + v^n(s'') - v^m(s'')\right),$$

and

$$s' = (r^n)^{-1}\left(r^m(s) + r^n(s'') - r^m(s'')\right).$$

Both functions are differentiable over the domain, and the graph of the former lies above that of the latter since both of them go through $(s'', s'')$ and have a single crossing point because of (10), which shows that the latter has a steeper slope than the former at any point of intersection between the two. Hence, for any $s < s''$, we have

$$(v^n)^{-1}\left(v^m(s) + v^n(s'') - v^m(s'')\right) > (r^n)^{-1}\left(r^m(s) + r^n(s'') - r^m(s'')\right).$$

In other words, whenever $v^n(s') = v^m(s) + v^n(s'') - v^m(s'')$, $r^n(s') > r^m(s) + r^n(s'') - r^m(s'')$. Equivalently, we have $r^n(s') > r^m(s) + r^n(s'') - r^m(s'')$ when $s'' = \varphi^{mn}(s, s')$, and $s < s'$. The desired conclusion then follows since by (6),

$$r^{mn}(s, s') = r^n(s'') - r^m(s'') + r^m(s).$$

**Proof of Proposition 3** Since the density $g$ is continuous and strictly positive over $[0, 1]$, we have

$$\frac{G(y^m) - G(y^n)}{y^m - y^n} \geq \beta\, g(y^n),$$

where $\beta = \frac{\min g(s_i)}{\max g(s_i)} > 0$. Hence, Assumption 5.1 holds if

$$\beta(y^m - y^n)\frac{(r^n)'(y^n)}{r^n(y^n)} > 1.$$

Since $r^m(y^m) \geq r^n(y^n)$ and $y^m > y^n$ imply $v^m(y^m) > v^n(y^n)$, using the linearity of the value functions, we see that the above inequality holds if

$$\beta\left(\frac{\rho^n}{\rho^m} - 1\right) y^n\, \frac{(r^n)'(y^n)}{r^n(y^n)} > 1.$$

Substituting

$$\begin{aligned}\frac{(r^n)'(y^n)}{r^n(y^n)} &= \frac{(v^n)'(y^n)}{v^n(y^n)} - \frac{g(y^n)}{1 - G(y^n)}\\ &= \frac{1}{y^n} - \frac{g(y^n)}{1 - G(y^n)}\end{aligned}$$

into the above, we obtain

$$\beta\left(\frac{\rho^n}{\rho^m} - 1\right)\left(1 - y^n\, \frac{g(y^n)}{1 - G(y^n)}\right) > 1.$$

Since $y^n < \frac{\rho^m}{\rho^n} < \varepsilon$, the increasing hazard rate condition then implies that the above holds if

$$\beta\left(\frac{1}{\varepsilon} - 1\right)\left(1 - \varepsilon\, \frac{g(\varepsilon)}{1 - G(\varepsilon)}\right) > 1,$$

which is true for a small $\varepsilon > 0$.

For Assumption 5.2, the assumed linearity of $v^n$ implies that $\bar{\mu}^n = \underline{\mu}^n = \frac{\rho^{n-1}}{\rho^n} < \varepsilon$. The inequality then holds if

$$\varepsilon < \frac{1 + (n-2)\varepsilon^{n-2}}{n - 1},$$

which is true for a small $\varepsilon > 0$.

**Proof of Theorem 3** The proof of the theorem begins with Lemmas 6, 7 and 8.

**Lemma 6** *If* $z \in \operatorname{argmax}_{y\in[0,1]^I} w(y)$, *then there exists a permutation* $\pi_1, \ldots, \pi_I$ *of* $1, \ldots, I$ *such that for each* $n = 1, \ldots, I$,

$$r^{\pi_n}(z^{\pi_n}) = \max_{\emptyset \neq L \subset \Pi_n} r^L(z^L),$$

*where* $\Pi_I = N$ *and* $\Pi_n = N \setminus \{\pi_{n+1}, \ldots, \pi_I\}$. *In particular,* $r^{\pi_I}(z^{\pi_I}) \geq \cdots \geq r^{\pi_1}(z^{\pi_1})$.

**Proof.**
Step 1. $r^{\pi_I}(z^{\pi_I}) = \max_{\emptyset \neq J \subset I} r^J(z^J)$ for some $\pi_I \in I$.

Suppose to the contrary that there exists $K$ such that $K \subset I$, $|K| \geq 2$, and

$$r^K(z^K) = \max_{\emptyset \neq J \subset I} r^J(z^J). \tag{11}$$

If there exists more than one such set that satisfies (11), choose any one with the smallest cardinality $|K|$. Write $K = \{\kappa_1, \ldots, \kappa_n\}$ for some $2 \leq n \leq I$ and $\kappa_1 < \cdots < \kappa_n$. Now consider $y$ such that

$$0 < y^{\kappa_1} < \cdots < y^{\kappa_n} < 1, \text{ and } y^{\kappa_{n-1}} < y^{\kappa_n} < y^{\kappa_{n-1}\kappa_n} < 1. \tag{12}$$

Given that $K$ has the smallest cardinality, $z$ must satisfy (12): Otherwise, there is redundancy in $K$ and we can find a strictly smaller set $\hat{K} \subset K$ such that $r^{\hat{K}}(z^{\hat{K}}) = r^K(z^K)$. Write $y^{\kappa_1\kappa_2}, \ldots, y^{\kappa_{n-1}\kappa_n}$ as functions of $y^{\kappa_1}, \ldots, y^{\kappa_n}$ as follows:

$$y^{\kappa_1\kappa_2} = y^{\kappa_1\kappa_2}(y^{\kappa_1}, y^{\kappa_2}), \ldots, y^{\kappa_{n-1}\kappa_n} = y^{\kappa_{n-1}\kappa_n}(y^{\kappa_{n-1}}, y^{\kappa_n}).$$

Let

$$z^{\kappa_1\kappa_2} = y^{\kappa_1\kappa_2}(z^{\kappa_1}, z^{\kappa_2}), \ldots, z^{\kappa_{n-1}\kappa_n} = y^{\kappa_{n-1}\kappa_n}(z^{\kappa_{n-1}}, z^{\kappa_n}).$$

$$z^1 = \varphi^{\kappa_1\kappa_2}(z^{\kappa_1}, z^{\kappa_2}), \ldots, \zeta^{n-1} = \varphi^{\kappa_{n-1}\kappa_n}(z^{\kappa_{n-1}}, z^{\kappa_n}).$$

By our choice of $K$, we must have $0 < \zeta^1 < \zeta^2 < \cdots < \zeta^{n-1} < 1$, $z^{\kappa_1} < z^{\kappa_2} < \zeta^1$, $z^{\kappa_2} < z^{\kappa_3} < \zeta^2$,…,$z^{\kappa_{n-1}} < z^{\kappa_n} < \zeta^{n-1}$.

Now define $\hat{w}$ by

$$\hat{w}(y) = \sum_{\substack{1 \in J \subset N \\ \kappa_n \notin J}} Q^J(y) \max_{\emptyset \neq L \subset J} r^L(y^L) + r^K(y^K) \sum_{\substack{1 \in J \subset N \\ K \subset J}} Q^J(y).$$

Note that $L^J(z) = \emptyset$ for any $J \subset N$ such that $K \not\subset J$ and $\kappa_n \in J$: Suppose to the contrary that $s_{-i} \in L^J(z)$ for such a $J$. Since $K \not\subset J$, there exists $k \in K$ such that $\lambda^{k-1} < z^k$. However, since $\lambda^{\kappa_n - 1} \leq \lambda^{k-1} < z^k \leq z^{\kappa_n}$ for any such $s_{-i}$, we must have $\lambda^{\kappa_n-1} < z^{\kappa_n}$, contradicting the assumption that $\kappa_n \in J$. Hence, $\sum_{\substack{1 \in J \subset N \\ \kappa_n \in J, K \not\subset J}} Q^J(z) = 0$ so that

$$\hat{w}(z) = \sum_{\substack{1 \in J \subset N \\ K \not\subset J}} Q^J(z) \max_{\emptyset \neq L \subset J} r^L(z^L) + r^K(z^K) \sum_{\substack{1 \in J \subset N \\ K \subset J}} Q^J(z).$$

This suggests that $\hat{w}(y) \leq w(y)$ for any $y$, and $\hat{w}(z) = w(z)$ by our hypothesis. From the definition of $\hat{w}$, we have

$$\begin{aligned} \frac{\partial \hat{w}}{\partial y^{\kappa_n}}(z) = &\sum_{\substack{1 \in J \subset N \\ \kappa_n \notin J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z) \max_{\emptyset \neq L \subset J} r^L(z^L) \\ &+ r^K(z^K) \sum_{\substack{1 \in J \subset N \\ K \subset J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z) + \frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) \sum_{\substack{1 \in J \subset N \\ K \subset J}} Q^J(z). \end{aligned}$$

Using

$$\sum_{\substack{1 \in J \subset N \\ \kappa_n \notin J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z) = - \sum_{\substack{1 \in J \subset N \\ K \subset J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z),$$

we observe that the FOC $\frac{\partial \hat{w}}{\partial y^{\kappa_n}}(z) = 0$ is given by

$$\sum_{\substack{1 \in J \subset N \\ \kappa_n \notin J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z^J) \left\{ \max_{\emptyset \neq L \subset J} r^L(z^L) - r^K(z^K) \right\} + \frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) \sum_{\substack{1 \in J \subset N \\ K \subset J}} Q^J(z) = 0.$$

Note that the bracketed term is negative and that $\sum_{\substack{1 \in J \subset N \\ K \subset J}} Q^J(z) > 0$ since $z^{\kappa_1} < \cdots < z^{\kappa_n} < 1$. It follows that this equation holds only if

$$\frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) \geq 0.$$

Recall from (6) that

$$r^K(z^K) = r^{\kappa_1}(z^{\kappa_1}) + \sum_{\ell=2}^{n} \left\{ r^{\kappa_\ell}(z^{\kappa_{\ell-1}\kappa_\ell}) - r^{\kappa_{\ell-1}}(z^{\kappa_{\ell-1}\kappa_\ell}) \right\}.$$

The derivative of $r^K$ is hence given by

$$\frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) = \left\{ (r^{\kappa_n})'(z^{\kappa_{n-1}\kappa_n}) - (r^{\kappa_{n-1}})'(z^{\kappa_{n-1}\kappa_n}) \right\} \frac{\partial y^{\kappa_{n-1}\kappa_n}}{\partial y^{\kappa_n}}(z^{\kappa_{n-1}}, z^{\kappa_n}).$$

Since $\frac{\partial y^{\kappa_{n-1}\kappa_n}}{\partial y^{\kappa_n}}(z^{\kappa_{n-1}}, z^{\kappa_n}) > 0$, $(r^{\kappa_n})'(z^{\kappa_{n-1}\kappa_n}) - (r^{\kappa_{n-1}})'(z^{\kappa_{n-1}\kappa_n}) \geq 0$. Since $z^{\kappa_\ell - 1} < z^{\kappa_\ell} < z^{\kappa_{\ell-1}\kappa_\ell} \leq z^{\kappa_{n-1}\kappa_n}$ for each $\ell \leq n$, we have by Lemma 4

$$r^{\kappa_\ell}(z^{\kappa_\ell}) > r^{\kappa_{\ell-1}\kappa_\ell}(z^{\kappa_{\ell-1}}, z^{\kappa_\ell}) = r^{\kappa_{\ell-1}}(z^{\kappa_{\ell-1}}) + r^{\kappa_\ell}(z^{\kappa_{\ell-1}\kappa_\ell}) - r^{\kappa_{\ell-1}}(z^{\kappa_{\ell-1}\kappa_\ell}).$$

Substituting this into (6), we obtain

$$\begin{aligned}
&r^K(z^K)\\
&= r^{\kappa_1}(z^{\kappa_1}) + \sum_{\ell=2}^{n} \Big\{ r^{\kappa_\ell}(z^{\kappa_{\ell-1}\kappa_\ell}) - r^{\kappa_{\ell-1}}(z^{\kappa_{\ell-1}\kappa_\ell}) \Big\}\\
&< r^{\kappa_1}(z^{\kappa_1}) + \sum_{\ell=2}^{n} \Big\{ r^{\kappa_\ell}(z^{\kappa_\ell}) - r^{\kappa_{\ell-1}}(z^{\kappa_{\ell-1}}) \Big\}\\
&= r^{\kappa_n}(z^{\kappa_n}).
\end{aligned}$$

This however contradicts our original supposition.

Step 2.

As an induction hypothesis, suppose that for $m = \mu + 1, \ldots, I$, there exists $\pi_m \in \Pi_m$ such that

$$r^{\pi_m}(z^{\pi_m}) = \max_{\emptyset \neq L \subset \Pi_m} r^L(z^L).$$

We will show that

$$r^K(z^K) < \max_{\emptyset \neq J \subset \Pi_\mu} r^J(z^J)$$

for any $K$ such that $K \subset \Pi_\mu$ and $|K| \geq 2$. Suppose to the contrary that $r^K(z^K) = \max_{\emptyset \neq J \subset \Pi_\mu} r^J(z^J)$ for some $K = \{\kappa_1, \ldots, \kappa_n\}$ such that $K \subset \Pi_\mu$ and $n \geq 2$. Define

$$\begin{aligned}
\hat{w}(y) = &\sum_{\substack{1 \in J \subset \Pi_\mu \\ \kappa_n \notin J}} Q^J(y) \max_{\emptyset \neq L \subset J} r^L(y^L) + r^K(y^K) \sum_{\substack{1 \in J \subset \Pi_\mu \\ K \subset J}} Q^J(y)\\
&+ \sum_{m=\mu+1}^{I} r^{\pi_m}(y^{\pi_m}) \sum_{\substack{1 \in J \subset \Pi_m \\ \pi_m \in J}} Q^J(y).
\end{aligned}$$

As in Step 1, we observe that $L^J(z) = \emptyset$ for any $J$ such that $\kappa_n \in J$ and $K \not\subset J$.[18] Then $\hat{w}(y) \leq w(y)$ for any $y$ and $\hat{w}(z) = w(z)$ by the induction hypothesis. Since

$$\sum_{\substack{1 \in J \subset \Pi_m \\ \pi_m \in J}} Q^J(y) = P\Big(\lambda^{\pi_m - 1} \geq y^{\pi_m}, \max_{\ell > m} (\lambda^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big),$$

[18]The reasoning is the same as that following the definition of $\hat{w}$ in Step 1.

the third term in the definition of $\hat{w}$ is independent of $y^{\Pi_\mu}$. It follows that

$$\frac{\partial \hat{w}}{\partial y^{\kappa_n}}(z) = \sum_{\substack{1\in J\subset\Pi_\mu \\ \kappa_n\notin J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z) \max_{\emptyset\neq L\subset J} r^L(z^L) + r^K(z^K) \sum_{\substack{1\in J\subset\Pi_\mu \\ K\subset J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z)$$
$$+ \frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) \sum_{\substack{1\in J\subset\Pi_\mu \\ K\subset J}} Q^J(z).$$

Noting that $\sum_{\substack{1\in J\subset\Pi_\mu \\ \kappa_n\notin J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z) = -\sum_{\substack{1\in J\subset\Pi_\mu \\ K\subset J}} \frac{\partial Q^J}{\partial y^{\kappa_n}}(z)$, we conclude as before that $\frac{\partial r^K}{\partial y^{\kappa_n}}(z^K) \geq 0$. Using the same logic as in Step 1, we can then derive the contradiction that $r^K(z^K) < r^{\kappa_n}(z^{\kappa_n})$. This advances the induction step and completes the proof. ■

**Lemma 7** *If $z \in \operatorname{argmax}_{y\in[0,1]^I} w(y)$, then $z \in (0,1)^I$.*

**Proof.** Suppose not and take the largest $n$ for which $z^{\pi_n} = 0$ or 1, where $\pi_1, \ldots, \pi_I$ are as defined in Lemma 6. It would then follow that $r^{\pi_n}(z^{\pi_n}) = 0$ and hence that $r^{\pi_\ell}(z^{\pi_\ell}) = 0$ for every $\ell < n$ as well. Define $\hat{z}$ to be such that

$$\hat{z}^{\pi_\mu} = \begin{cases} z^{\pi_\mu} & \text{if } \mu \neq n, \\ \frac{1}{2} & \text{if } \mu = n. \end{cases}$$

We then have

$$\begin{aligned} w(\hat{z}) &\geq \sum_{\mu=1}^{I} r^{\pi_\mu}(\hat{z}^{\pi_\mu}) \sum_{\substack{1\in J\subset\Pi_\mu \\ \pi_\mu\in J}} Q^J(\hat{z}) \\ &= r^{\pi_n}(\hat{z}^{\pi_n})\, P\Big(\lambda^{\pi_n - 1} \geq \hat{z}^{\pi_n},\, \max_{\ell>n}\,(\lambda^{\pi_\ell - 1} - z^{\pi_\ell}) < 0\Big) \\ &+ \sum_{\mu=n+1}^{I} r^{\pi_\mu}(z^{\pi_\mu})\, P\Big(\lambda^{\pi_\mu - 1} \geq z^{\pi_\mu},\, \max_{\ell>\mu}\,(\lambda^{\pi_\ell - 1} - z^{\pi_\ell}) < 0\Big) \\ &> \sum_{\mu=n+1}^{I} r^{\pi_\mu}(z^{\pi_\mu})\, P\Big(\lambda^{\pi_\mu - 1} \geq z^{\pi_\mu},\, \max_{\ell>\mu}\,(\lambda^{\pi_\ell - 1} - z^{\pi_\ell}) < 0\Big) \\ &= w(z), \end{aligned}$$

where the inequality holds since $r^{\pi_n}(\frac{1}{2}) > 0$ and $z^{\pi_\ell} > 0$ for $\ell > n$. This is a contradiction. ■

**Lemma 8** *Suppose that Assumptions 4 and 5 hold. If* $z \in \operatorname{argmax}_{y\in[0,1]^I} w(y)$*, then* $\pi_\mu = \mu$ *for* $\mu = 1, \ldots, I$*, or equivalently,*

$$r^1(z^1) \le \cdots \le r^I(z^I).$$

**Proof.** Note first that $\operatorname{argmax}_y w(y) \neq \emptyset$ since $w$ is a continuous function over the compact domain $[0,1]^I$. Let $z = (z^1, \ldots, z^I) \in \operatorname{argmax}_y w(y)$ be any maximizer. We prove the claim by induction over $\mu = 1, \ldots, I$.

Step 1. $r^I(z^I) = \max_{\emptyset\neq J\subset I} r^J(z^J)$.

Given the conclusion of Lemma 6, the claim is equivalent to $\pi_I = I$, where $\pi_1, \ldots . \pi_I$ are as defined there. Suppose to the contrary that $\pi_I < I$, and take $n < I$ such that $\pi_n = I$. If we define

$$\begin{aligned}\hat{w}(y) &= \sum_{\mu=1}^{I} r^{\pi_\mu}(y^{\pi_\mu}) \sum_{\substack{1\in J\subset\Pi_\mu \\ \pi_\mu\in J}} Q^J(y) \\ &= \sum_{\mu=1}^{I} r^{\pi_\mu}(y^{\pi_\mu})\, P\Big(\lambda^{\pi_\mu - 1} \ge y^{\pi_\mu},\, \max_{\ell>\mu} (\lambda^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big),\end{aligned}$$

then $\hat{w}(y) \le w(y)$ for any $y$ and $\hat{w}(z) = w(z)$. Differentiating $\hat{w}$ with respect to $y^{\pi_n} = y^I$, we obtain

$$\begin{aligned}\frac{\partial \hat{w}}{\partial y^{\pi_n}}(y) = &\sum_{\mu=1}^{n-1} r^{\pi_\mu}(y^{\pi_\mu}) \sum_{\substack{1\in J\subset\Pi_\mu \\ \pi_\mu\in J}} \frac{\partial Q^J}{\partial y^{\pi_n}}(y) \\ &+ (r^{\pi_n})'(y^{\pi_n}) \sum_{\substack{1\in J\subset\Pi_n \\ \pi_n\in J}} Q^J(y) \\ &+ r^{\pi_n}(y^{\pi_n}) \sum_{\substack{1\in J\subset\Pi_n \\ \pi_n\in J}} \frac{\partial Q^J}{\partial y^{\pi_n}}(y).\end{aligned} \tag{13}$$

Since $z \in (0,1)^I$ by Lemma 7, the FOC $\frac{\partial \hat{w}}{\partial y^{\pi_n}}(z) = 0$ holds at $y = z$. Furthermore, since $r^{\pi_\mu}(z^{\pi_\mu}) \le r^{\pi_n}(z^{\pi_n})$ for every $\mu < n$, and

$$\sum_{\substack{1\in J\subset\Pi_\mu \\ \pi_\mu\in J}} \frac{\partial Q^J}{\partial y^{\pi_n}}(z) = - \sum_{\substack{1\in J\subset\Pi_n \\ \pi_n\in J}} \frac{\partial Q^J}{\partial y^{\pi_n}}(z) > 0, \tag{14}$$

the sum of the first and third terms on the right-hand side of (13) evaluated at $z$ is $\leq 0$, implying that $(r^{\pi_n})'(z^{\pi_n}) \geq 0$. This and $r^{\pi_\ell}(z^{\pi_\ell}) \geq r^{\pi_n}(z^{\pi_n})$ for $\ell > n$ together imply that $z^{\pi_\ell} > z^{\pi_n}$ for $\ell > n$. Let $z^{\pi_m}$ be the smallest among $z^{\pi_{n+1}}, \ldots, z^{\pi_I}$.

For any $y$ such that $y^{\pi_n} < y^{\pi_m} < \min_{\ell > n, \ell \neq m} y^{\pi_\ell}$, we now evaluate the probability $\sum_{\substack{1 \in J \subset \Pi_n \\ \pi_n \in J}} Q^J(y)$ appearing in (13) in two ways. First, assigning just one of $I-1$ signals to the interval $[y^{\pi_n}, y^{\pi_m}]$, we see that

$$\begin{aligned}\sum_{\substack{1 \in J \subset \Pi_n \\ \pi_n \in J}} Q^J(y) &= P\Big(\lambda^{\pi_n - 1} \geq y^{\pi_n},\ \max_{\ell > n} (\lambda^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big) \\ &= \binom{I-1}{1} \{G(y^{\pi_m}) - G(y^{\pi_n})\} \\ &\quad \times P\Big(\lambda_{I-2}^{\pi_n - 2} \geq y^{\pi_n},\ \max_{\ell > n} (\lambda_{I-2}^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big),\end{aligned} \tag{15}$$

where $\lambda_{I-2}^k$ is the $k$th largest value among $I-2$ signals, and $\lambda_k^\ell = 0$ whenever $k < \ell$. Second, suppose we assign $p$ of $I-1$ signals to $[y^{\pi_n}, y^{\pi_m}]$ and the remaining $q = I-1-p$ signals to $(y^{\pi_m}, 1]$. In this case, $q < \pi_m - 1$ must hold since $\lambda^{\pi_m - 1} < y^{\pi_m}$. Hence,

$$\begin{aligned}\sum_{\substack{1 \in J \subset \Pi_n \\ \pi_n \in J}} Q^J(y) &= P\Big(\lambda^{\pi_n - 1} \geq y^{\pi_n}, \max_{\ell > n} (\lambda^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big) \\ &= \sum_{\substack{p+q = I-1 \\ q < \pi_m - 1}} \binom{I-1}{p} \{G(y^{\pi_m}) - G(y^{\pi_n})\}^p \\ &\quad \times P\Big(\max_{\ell > n, \ell \neq m} (\lambda_q^{\pi_\ell - 1} - y^{\pi_\ell}) < 0,\ \lambda_q^q \geq y^{\pi_m}\Big).\end{aligned} \tag{16}$$

Differentiating (16) with respect to $y^{\pi_n} = y^I$ and rearranging, we obtain

$$\begin{aligned}&\sum_{\substack{1 \in J \subset \Pi_n \\ \pi_n \in J}} \frac{\partial Q^J}{\partial y^{\pi_n}}(y) \\ &= -(I-1)\, g(y^{\pi_n})\, P\Big(\lambda_{I-2}^{\pi_n - 2} \geq y^{\pi_n},\ \max_{\ell > n} (\lambda_{I-2}^{\pi_\ell - 1} - y^{\pi_\ell}) < 0\Big).\end{aligned} \tag{17}$$

Now substitute (15) and (17) into (13) and set $y = z$ to get

$$\begin{aligned}&\frac{\partial \hat{w}}{\partial y^{\pi_n}}(z) \\ &\geq P\Big(\lambda_{I-2}^{\pi_n - 2} \geq z^{\pi_n},\ \max_{\ell > n} (\lambda_{I-2}^{\pi_\ell - 1} - z^{\pi_\ell}) < 0\Big) \\ &\times (I-1) \Big[-g(z^{\pi_n})\, r^{\pi_n}(z^{\pi_n}) + (r^{\pi_n})'(z^{\pi_n}) \{G(z^{\pi_m}) - G(z^{\pi_n})\}\Big] \\ &> 0.\end{aligned}$$

where the first inequality follows from the fact that the first term on the right-hand side of (13) is positive (*i.e.,* (14)), and the second from Assumption 1 along with the fact that $z^{\pi_n} < z^{\pi_m}$ and $r^{\pi_m}(z^{\pi_m}) \geq r^{\pi_n}(z^{\pi_n})$. We have hence derived a contradiction to the fact that $z$ is an interior maximizer.

Step 2. For $n = 1, \dots, I-1$, $r^n(z^n) = \max_{\ell \leq n} r^\ell(z^\ell)$.

As an induction hypothesis, suppose that the statement holds for $n+1, \dots, I$. Define

$$\begin{aligned}
\hat{w}_n(y) &= \sum_{k=1}^{n} r^{\pi_k}(y^{\pi_k}) \sum_{\substack{1 \in J \subset \Pi_k \\ \pi_k \in J}} Q^J(y) + \sum_{k=n+1}^{I} r^k(y^k) \sum_{\substack{1 \in J \subset N \\ \max J = k}} Q^J(y) \\
&= \sum_{k=1}^{n} r^{\pi_k}(y^{\pi_k}) \sum_{\substack{1 \in J \subset \Pi_k \\ \pi_k \in J}} Q^J(y) \\
&+ \sum_{k=n+1}^{I} P\Big(\lambda^{k-1} \geq y^k, \max_{\ell > k}(\lambda^{\ell-1} - y^\ell) < 0\Big) r^k(y^k).
\end{aligned}$$

We then have $\hat{w}_n(y) \leq w(y)$ for any $y$, and by the induction hypothesis, $\hat{w}_n(z) = w(z)$. Hence, since $z$ is a maximizer of $w$, it is a maximizer of $\hat{w}_n$ as well. Note that the second term on the right-hand side above is independent of $(y^1, \dots, y^n)$, and the first term has the same form as $\hat{w}$ in Step 1 with the only exception that $n$ replacing $I$. This implies that the same reasoning as that in step 1 proves

$$r^n(z^n) = \max_{\ell \leq n} r^\ell(z^\ell).$$

■

We now return to the proof of the theorem. We will show that any maximizer $z$ of $w : \boldsymbol{R}^I \to \boldsymbol{R}$ satisfies $z^I \leq \cdots \leq z^1$ under Assumption 5.2. For $n = 1, \dots, I$ and $y \in [0,1]^I$, define

$$R^n(y) = \sum_{\substack{1 \in J \subset I_n \\ n \in J}} Q^J(y) = P(\lambda^{n-1} \geq y^n, \max_{\ell > n}(\lambda^{\ell-1} - y^\ell) < 0).$$

For any $y$ such that $y^I > y^{I-1}$, we have

$$\frac{\partial R^1}{\partial y^I}(\cdot) = \cdots = \frac{\partial R^{I-2}}{\partial y^I}(\cdot) = 0.$$

Furthermore, since

$$R^I(y) = \{1 - G(y^I)\}^{I-1},$$

and

$$
\begin{aligned}
&R^{I-1}(y)\\
&= P(\lambda^{I-2} \geq y^{I-1},\, \lambda^{I-1} < y^{I})\\
&= (I-1)\{1-G(y^{I-1})\}^{I-2} G(y^{I-1})\\
&+ \{1-G(y^{I-1})\}^{I-1} - \{1-G(y^{I})\}^{I-1},
\end{aligned}
$$

we have

$$
\frac{\partial R^{I}}{\partial y^{I}}(y) = -(I-1)\{1-G(y^{I})\}^{I-2} g(y^{I}),
$$

and

$$
\frac{\partial R^{I-1}}{\partial y^{I-1}}(y) = -(I-1)(I-2)\{1-G(y^{I-1})\}^{I-3} G(y^{I-1}) g(y^{I-1}).
$$

Suppose now that there exists a maximizer $z$ of $w$ such that $z^I > z^{I-1}$. Since $z \in (0,1)^I$ by Lemma 7, $z$ satisfies the FOC's:

$$
\begin{aligned}
\frac{\partial w}{\partial y^{I}}(z) = {} & \frac{\partial R^{I-1}}{\partial y^{I}}(z)\, r^{I-1}(z^{I-1})\\
& + \frac{\partial R^{I}}{\partial y^{I}}(z)\, r^{I}(z^{I}) + R^{I}(z)\,(r^{I})'(z^{I}) = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial w}{\partial y^{I-1}}(z) = {} & \sum_{n=1}^{I-1} \frac{\partial R^{n}}{\partial y^{I-1}}(z)\, r^{n}(z^{n})\\
& + R^{I-1}(z)\,(r^{I-1})'(z^{I-1}) = 0.
\end{aligned}
$$

Noting that $\frac{\partial R^{I-1}}{\partial y^{I}}(z) = -\frac{\partial R^{I-1}}{\partial y^{I}}(z) > 0$, and $\sum_{n=1}^{I-2} \frac{\partial R^{n}}{\partial y^{I-1}}(z) = -\frac{\partial R^{I-1}}{\partial y^{I-1}}(z) > 0$, we obtain

$$
\begin{aligned}
\frac{(r^{I})'(z^{I})}{r^{I}(z^{I})} &= -\frac{1}{R^{I}(z)}\,\frac{\partial R^{I}}{\partial y^{I}}(z)\left\{1 - \frac{r^{I-1}(z^{I-1})}{r^{I}(z^{I})}\right\},\\
&= (I-1)\,\frac{g(z^{I})}{1-G(z^{I})}\left\{1 - \frac{r^{I-1}(z^{I-1})}{r^{I}(z^{I})}\right\},
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
\frac{(r^{I-1})'(z^{I-1})}{r^{I-1}(z^{I-1})} &= \frac{1}{R^{I-1}(z)}\left\{-\frac{\partial R^{I-1}}{\partial y^{I-1}}(z) - \sum_{n=1}^{I-2} \frac{\partial R^{n}}{\partial y^{I-1}}(z)\,\frac{r^{n}(z^{n})}{r^{I-1}(z^{I-1})}\right\}\\
&\leq -\frac{1}{R^{I-1}(z)}\,\frac{\partial R^{I-1}}{\partial y^{I-1}}(z)\left\{1 - \frac{r^{1}(z^{1})}{r^{I-1}(z^{I-1})}\right\}.
\end{aligned}
\tag{19}
$$

Let $\hat{R}^{I-1}(y) = R^{I-1}(\hat{y})$, where $\hat{y}^k = y^k$ for $k \neq I$ and $\hat{y}^I = y^{I-1}$. Then $\hat{R}^{I-1}(y) \leq R^{I-1}(y)$, and

$$-\frac{1}{\hat{R}^{I-1}(z)} \frac{\partial R^{I-1}}{\partial y^{I-1}}(z) = (I-2)\, \frac{g(z^{I-1})}{1-G(z^{I-1})}.$$

Since $z^{I-1} < z^1 = \bar{z}^1$ and $(r^1)'(y^1) > 0$ for $y^1 < \bar{z}^1$, we have

$$\frac{r^1(z^1)}{r^{I-1}(z^{I-1})} > \frac{r^1(z^{I-1})}{r^{I-1}(z^{I-1})} = \frac{v^1(z^{I-1})}{v^{I-1}(z^{I-1})} \geq \prod_{k=2}^{I-1} \underline{\mu}^k.$$

Hence, (19) implies

$$\frac{(r^{I-1})'(z^{I-1})}{r^{I-1}(z^{I-1})} \leq (I-2)\, \frac{g(z^{I-1})}{1-G(z^{I-1})} \left\{ 1 - \prod_{k=2}^{I-1} \underline{\mu}^k \right\}. \tag{20}$$

Since $(r^I)'(z^I) > 0$, $z^{I-1} < z^I$ and $(r^{I-1})'(y) > 0$ whenever $(r^I)'(y) > 0$, we also have

$$\frac{r^{I-1}(z^{I-1})}{r^I(z^I)} < \frac{r^{I-1}(z^I)}{r^I(z^I)} = \frac{v^{I-1}(z^I)}{v^I(z^I)} \leq \bar{\mu}^I.$$

substituting this into (18), we get

$$\frac{(r^I)'(z^I)}{r^I(z^I)} > (I-1)\, \frac{g(z^I)}{1-G(z^I)}\, (1-\bar{\mu}^I). \tag{21}$$

Combining (20) and (21) together, we see that

$$\begin{aligned}
&(I-1)\, \frac{g(z^I)}{1-G(z^I)}\, (1-\bar{\mu}^I) \\
&< \frac{(r^I)'(z^I)}{r^I(z^I)} \leq \frac{(r^I)'(z^{I-1})}{r^I(z^{I-1})} \leq \frac{(r^{I-1})'(z^{I-1})}{r^{I-1}(z^{I-1})} \\
&\leq (I-2)\, \frac{g(z^{I-1})}{1-G(z^{I-1})} \left\{ 1 - \prod_{k=2}^{I-1} \underline{\mu}^k \right\},
\end{aligned}$$

where the inequalities in the second line hold because $\frac{(r^I)'}{r^I}$ is decreasing, $z^I > z^{I-1}$. and $\frac{(r^I)'(s_i)}{r^I(s_i)} \leq \frac{(r^{I-1})'(s_i)}{r^{I-1}(s_i)}$ for any $s_i$. Furthermore, given the increasing hazard rate, we must have

$$(I-1)\,(1-\bar{\mu}^I) < (I-2) \left\{ 1 - \prod_{k=2}^{I-1} \underline{\mu}^k \right\}.$$

This, however, contradicts Assumption 5.2.

As an induction hypothesis, suppose that $z^I \leq \cdots \leq z^n$. Suppose that $z^n > z^{n-1}$. For any $y$ such that $y^I \leq \cdots \leq y^n$ and $y^n > y^{n-1}$, we have $\frac{\partial R^1}{\partial y^n}(y) = \cdots = \frac{\partial R^{n-1}}{\partial y^n}(y) = 0$, and

$$\begin{aligned}
&R^n(y)\\
&= P(\lambda^{n-1} \geq y^n, \lambda^n < y^{n+1}, \ldots, \lambda^{I-1} < y^I)\\
&= \binom{I-1}{n-1} \{1 - G(y^n)\}^{n-1} P(\lambda^1_{I-n} < y^{n+1}, \ldots, \lambda^{I-n}_{I-n} < y^I),
\end{aligned}$$

where $\lambda^k_{I-n}$ is the $k$th largest value among $I - n$ signals. Hence,

$$\begin{aligned}
&\frac{\partial R^n}{\partial y^n}(y)\\
&= -g(y^n)(n-1)\{1 - G(y^n)\}^{n-2} \binom{I-1}{n-1} P(\lambda^1_{I-n} < y^{n+1}, \ldots, \lambda^{I-n}_{I-n} < y^I),
\end{aligned}$$

and

$$\frac{1}{Q^n(y)} \frac{\partial Q^n}{\partial y^n}(y) = -(n-1) \frac{g(y^n)}{1 - G(y^n)}.$$

The first-order condition $\frac{\partial w}{\partial y^n}(z) = 0$ then yields

$$\begin{aligned}
\frac{(r^n)'(z^n)}{r^n(z^n)} &= -\frac{1}{Q^n} \frac{\partial Q^n}{\partial y^n} \left\{1 - \frac{r^{n-1}(z^{n-1})}{r^n(z^n)}\right\}\\
&= (n-1) \frac{g(z^n)}{1 - G(z^n)} \left\{1 - \frac{r^{n-1}(z^{n-1})}{r^n(z^n)}\right\}.
\end{aligned} \tag{22}$$

On the other hand, suppose $y^{n+1} \leq y^{n-1} < y^n$. Other cases can be treated in a similar manner.

$$\begin{aligned}
&R^{n-1}(y)\\
&= P(\lambda^{n-2} \geq y^{n-1}, \lambda^{n-1} < y^n, \lambda^n < y^{n+1}, \ldots, \lambda^{I-1} < y^I)\\
&= \binom{I-1}{n-2} \{1 - G(y^{n-1})\}^{n-2}\\
&\times \Big[(I-n+1)\{G(y^{n-1}) - G(y^{n+1})\} P(\lambda^1_{I-n} < y^{n+1}, \ldots, \lambda^{I-n}_{I-n} < y^I)\\
&+ P(\lambda^1_{I-n+1} < y^{n+1}, \ldots, \lambda^{I-n+1}_{I-n+1} < y^I)\Big]\\
&+ \binom{I-1}{n-1} \Big[\{1 - G(y^{n-1})\}^{n-1} - \{1 - G(y^n)\}^{n-1}\Big] P(\lambda^1_{I-n} < y^{n+1}, \ldots, \lambda^{I-n}_{I-n} < y^I).
\end{aligned}$$

Differentiating $R^{n-1}$ with respect to $y^{n-1}$, we obtain

$$\begin{aligned}
&\frac{\partial R^{n-1}}{\partial y^{n-1}}(y) \\
&= -g(y^{n-1}) \binom{I-1}{n-2} (n-2)\{1 - G(y^{n-1})\}^{n-3} \\
&\times \Big[(I-n+1)\{G(y^{n-1}) - G(y^{n+1})\} P(\lambda^1_{I-n} < y^{n+1}, \ldots, \lambda^{I-n}_{I-n} < y^I) \\
&+ P(\lambda^1_{I-n+1} < y^{n+1}, \ldots, \lambda^{I-n+1}_{I-n+1} < y^I)\Big].
\end{aligned}$$

Let $\hat{R}^{n-1}(y) = R^{n-1}(\hat{y})$, where $\hat{y}^k = y^k$ for $k \neq n$ and $\hat{y}^n = y^{n-1}$. We have $\hat{R}^{n-1}(y) \leq R^{n-1}(y)$ and can also verify that

$$\frac{1}{\hat{R}^{n-1}(y)} \frac{\partial R^{n-1}}{\partial y^{n-1}}(y) = -(n-2) \frac{g(y^{n-1})}{1 - G(y^{n-1})}.$$

The first-order condition $\frac{\partial w}{\partial y^{n-1}}(z) = 0$ then yields

$$\begin{aligned}
&\frac{(r^{n-1})'(z^{n-1})}{r^{n-1}(z^{n-1})} \\
&= -\frac{1}{R^{n-1}(z)} \left[\sum_{k=1}^{n-1} \frac{\partial R^k}{\partial y^{n-1}}(z) \frac{r^k(z^k)}{r^{n-1}(z^{n-1})}\right] \\
&\leq (n-2) \frac{g(y^n)}{1 - G(y^n)} \left(1 - \frac{r^1(z^1)}{r^{n-1}(z^{n-1})}\right) \qquad (23) \\
&< (n-2) \frac{g(y^n)}{1 - G(y^n)} \left(1 - \frac{r^1(z^{n-1})}{r^{n-1}(z^{n-1})}\right) \\
&\leq (n-2) \frac{g(y^n)}{1 - G(y^n)} \left(1 - \prod_{k=2}^{n-1} \underline{\mu}^k\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{(r^n)'(z^n)}{r^n(z^n)} &= (n-1) \frac{g(z^n)}{1 - G(z^n)} \left\{1 - \frac{r^{n-1}(z^{n-1})}{r^n(z^n)}\right\} \\
&> (n-1) \frac{g(z^n)}{1 - G(z^n)} \left\{1 - \frac{r^{n-1}(z^n)}{r^n(z^n)}\right\} \qquad (24) \\
&> (n-1) \frac{g(z^n)}{1 - G(z^n)} \left\{1 - \bar{\mu}^n\right\}.
\end{aligned}$$

Just as in Step 1, we can combine (23) and (24) to yield a contradiction to Assumption 5.2, which is equivalent to

$$(n-1) \left\{1 - \bar{\mu}^n\right\} \geq (n-2) \left(1 - \prod_{k=2}^{n-1} \underline{\mu}^k\right).$$

This advances the induction step and completes the proof.

**Proof of Lemma 1** Take any $m \in M(y)$. Let $n$ be such that $n > m$ and $y^m < y^n$. Take $s$ such that $y^m < s_1 = \cdots = s_n < y^n$ and $s_{n+1} = \cdots = s_I = 0$. Symmetry and ex post IR then imply that $f(s)$ equals either $(\underbrace{0,\ldots,0}_{n},\underbrace{1,\ldots,1}_{I-n})$, $(0,\ldots,0)$, or $(1,\ldots,1)$. When $f(s) = (0,\ldots,0,1,\ldots,1)$, $t^{I-n} = 0$ should hold by ex post IR, and when $f(s) = (1,\ldots,1)$, then $t^I = 0$ should hold by ex post IR. Suppose now that $m \in K(f)$, or that there exists $\hat{s}$ such that $|f(\hat{s})| = m$. By symmetry, we can take $\hat{s}$ such that $f(\hat{s}) = (\underbrace{1,\ldots,1}_{m},0,\ldots,0)$.

If $f(s) = (0,\ldots,0)$. then $\hat{s}$ is a profitable deviation for the coalition $J = I$ at $s$: For $i = 1,\ldots,m$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = v^m(s_i) - t^m > 0 = v_i(f(s), s_i) - t_i(s),$$

where the inequality follows from the fact that $s_i > y^m$. Furthermore, for $i = m+1,\ldots,I$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = 0 = v_i(f(s), s_i) - t_i(s).$$

If $f(s) = (\underbrace{0,\ldots,0}_{n},1,\ldots,1)$, then $t^{I-n} = 0$ as noted above and $\hat{s}$ is a profitable deviation for the coalition $J = I$ at $s$: For $i = 1,\ldots,m$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = v^m(s_i) - t^m > 0 = v_i(f(s), s_i) - t_i(f(s)),$$

for $i = m+1,\ldots,n$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = 0 = v_i(f(s), s_i) - t_i(f(s)),$$

and for $i = n+1,\ldots,I$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = 0 = v^{I-n}(0) - t^{I-n} = v_i(f(s), s_i) - t_i(f(s)).$$

If $f(s) = (1,\ldots,1)$, then $y^I = 0$ as noted above and $s$ is a profitable deviation for the coalition $J = I$ at $\hat{s}$: For $i = 1,\ldots,m$,

$$v_i(f(s), \hat{s}_i) - t_i(f(s)) = v^I(\hat{s}_i) - t^I > v^m(\hat{s}_i) - t^m = v_i(f(\hat{s}), \hat{s}_i) - t_i(f(\hat{s})),$$

and for $i = m+1,\ldots,I$,

$$v_i(f(s), \hat{s}_i) - t_i(f(s)) = v^I(\hat{s}_i) - t^I > 0 = v_i(f(\hat{s}), \hat{s}_i) - t_i(f(\hat{s})).$$

Therefore, $(f,t)$ is not coalitionally strategy-proof.

**Proof of Lemma 2** For $K$ and $y$ such that $K \cap M(y) = \emptyset$, $w$ can alternatively be written as

$$w(K, y) = \sum_{k \in K} P(\lambda^{k-1} \geq y^k, \max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0)\, r^k(y^k).$$

Let $(f, y)$ be such that $K(f) = K$. Fix $k \in K$ and $s_{-i}$ such that $\lambda^{k-1} \geq y^k$ and $\max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0$. By our choice of $s_{-i}$, $|f(s_i, s_{-i})| \leq k$ for any $s_i$ by ex post IR. Moreover, if $s_i < y^k$, then $s_i < y^m$ for any $m < k$ so that $|f_i(s_i, s_{-i})| = 0$. In what follows, we show that $|f(s_i, s_{-i})| = k$ whenever $s_i > y^k$. If this holds, then

$$\begin{aligned}
E_{s_i}[t_i(f(s_i, s_{-i})) \mid s_{-i}] &= P(s_i < y^k)\, E_{s_i}[t_i(f(s_i, s_{-i})) \mid s_i < y^k, s_{-i}] \\
&\quad + P(s_i > y^k)\, E_{s_i}[t_i(f(s_i, s_{-i})) \mid s_i > y^k, s_{-i}] \\
&= P(s_i > y^k)\, t^k \\
&= r^k(y^k).
\end{aligned}$$

This in turn implies that

$$\begin{aligned}
R(f, t) &= E_s[t_i(f(s))] \\
&= E_{s_{-i}}\left[E_{s_i}[t_i(f(s_i, s_{-i})) \mid s_{-i}]\right] \\
&= \sum_{k \in K} r^k(y^k)\, P(\lambda^{k-1} \geq y^k,\ \max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0) \\
&= w(K, y).
\end{aligned}$$

Suppose that $s_i > y^k$ and denote $s = (s_i, s_{-i})$. We will derive a contradiction when $m = |f(s)| < k$. Let $J \subset I$ be such that $i \in J$, $|J| = k$, and

$$\{j : f_j(s) = 1\} \subset J \subset \{j : s_j \geq y^k\}.$$

Such a set $J$ exists since $f_j(s) = 1$ implies that $s_j \geq y^m \geq y^k$ by ex post IR. Since $k \in K = K(f)$, take $\hat{s}$ such that $|f(\hat{s})| = k$ and $\{j : f_j(\hat{s}) = 1\} = J$. Such a signal profile $\hat{s}$ exists by symmetry. When $s_j \geq y^k$, note that

$$v^k(s_j) - v^m(s_j) \geq v^k(y^k) - v^m(y^k) \geq v^k(y^k) - v^m(y^m),$$

where the first inequality follows from $(v^k)' > (v^m)'$, and the second from $y^k \leq y^m$. We will show that $\hat{s}$ is a profitable deviation for $I$ at $s$:
For $j \in J \cap \{j : f_j(s) = 1\}$,

$$\begin{aligned}
v_j(f(\hat{s}), s_j) - t_j(f(\hat{s})) &= v^k(s_j) - v^k(y^k) \\
&\geq v^m(s_j) - v^m(y^m) \\
&= v_j(f(s), s_j) - t_j(f(s)).
\end{aligned} \tag{25}$$

For $j \in J \cap \{j : f_j(s) = 0\}$,

$$v_j(f(\hat{s}), s_j) - t_j(f(\hat{s})) = v^k(s_j) - t^k \geq 0 = v_j(f(s), s_j) - t_j(f(s)). \tag{26}$$

For $j \in I \setminus J$,

$$v_j(f(\hat{s}), s_j) - t_j(f(\hat{s})) = 0 = v_j(f(s), s_j) - t_j(f(s)).$$

Since $s_i > y^k$, if $f_i(s) = 1$, then (25) holds with strict inequality for $j = i$, and if $f_i(s) = 0$, then (26) holds with strict inequality for $j = i$. Hence, $(f, t)$ is not strategy-proof.

**Proof of Lemma 4** Take any $J \subset I$, $s = (s_J, s_{-J})$ and $\hat{s}_J$. Denote $\hat{s} = (\hat{s}_J, s_{-J})$ and $k = |f(s)|$. If $|f(\hat{s})| = m > k$, then $|\{i : s_i \geq z^k\}| = k < m$ and $|\{i : \hat{s}_i \geq z^m\}| = m$ by the definition of a regular scheme. Hence, there exists at least one buyer $i \in J$ for whom $s_i < z^m$, $\hat{s}_i \geq z^m$ and $f_i(\hat{s}) = 1$. It follows that for this $i$,

$$v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) = v^m(s_i) - t^m < 0 \leq v_i(f(s), s_i) - t_i(f(s)),$$

suggesting that $\hat{s}_J$ is not a profitable deviation of $J$ at $s$. If $|f(\hat{s})| = m < k$, take any $i \in J$ for whom $f_i(\hat{s}) = 1$. If there exists no such $j \in J$, then $\hat{s}$ is not a profitable deviation for $J$. Since $z^m \geq z^k$, we have $v^k(z^m) - v^m(z^m) \geq v^k(z^k) - v^m(z^m)$. Furthermore, since $(v^k)' \geq (v^m)'$, $v^k(s_i) - v^m(s_i) \geq v^k(z^m) - v^m(z^m)$ for any $s_i \geq z^m$. It follows that $v^k(s_i) - v^m(s_i) \geq v^k(z^k) - v^m(z^m)$ for any $s_i \geq z^m$. In other words, if $f_i(\hat{s}) = 1$ and $s_i \geq z^m$, then

$$\begin{aligned} v_i(f(\hat{s}), s_i) - t_i(f(\hat{s})) &= v^m(s_i) - t^m \\ &= v^m(s_i) - v^m(z^m) \\ &\leq v^k(s_i) - v^k(z^k) \\ &= v_i(f(s), s_i) - t_i(f(s)). \end{aligned}$$

This implies that $\hat{s}$ is not a profitable deviation for $J$ at $s$.

**Proof of Theorem 4** We will show that when $K \neq N$ and $y \in [0,1]^I$, $w(K, y) \leq w(I, \hat{y})$ for some $\hat{y}$ such that $\hat{y}^I \leq \cdots \leq \hat{y}^1$. Since $w(N, \cdot)$ is continuous over the compact set $\{y : y^I \leq \cdots \leq y^1\}$, it achieves a maximum at some $z$ in this set. Hence, $w(N, z) = \max_{K \subset N, y \in [0,1]^I} w(K, y)$, and by Proposition 4, there exists a regular scheme $(f, t)$ such that $R(f, t) = w(N, z)$.

When $y^k = 1$ for some $k \in K$, then $w(K, y) = w(K \setminus \{k\}, y)$ so that we may restrict attention to the case where $\max_{k \in K} y^k < 1$. Suppose that $K \neq N$ and fix any $y$ such that $K \cap M(y) = \emptyset$. We have

$$w(K, y) = \sum_{\ell \in K} P(\lambda^{\ell-1} \geq y^\ell,\ \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0)\, r^\ell(y^\ell).$$

Let $n = \min N \setminus K$ and $\hat{K} = K \cup \{n\}$. If $n = 1$, then let $\hat{y}$ be such that $\hat{y}^1 = \max_{k \in K} y^k$ and $\hat{y}^k = y^k$ for $k > 1$. Then $\hat{K} \cap M(\hat{y}) = \emptyset$ and $w(\hat{K}, \hat{y})$ is given by

$$\begin{aligned}
w(\hat{K}, \hat{y}) &= \sum_{k \in \hat{K}} P(\lambda^{k-1} \geq y^k,\ \max_{\substack{\ell \in \hat{K} \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0)\, r^k(y^k) \\
&= P(\max_{\ell \in K} (\lambda^{\ell-1} - y^\ell) < 0)\, r^1(y^1) \\
&+ \sum_{k \in K} P(\lambda^{k-1} \geq y^k,\ \max_{\substack{\ell \in K \\ \ell > k}} (\lambda^{\ell-1} - y^\ell) < 0)\, r^k(y^k) \\
&\geq w(K, y).
\end{aligned}$$

If $n > 1$, then $n - 1 \in K$ and let $\hat{y}$ be such that

$$\hat{y}^k = \begin{cases} y^k & \text{if } k \neq n, \\ y^{n-1} & \text{if } k = n. \end{cases}$$

Since $\hat{K} \cap M(\hat{y}) = \emptyset$, $w$ is given by

$$\begin{aligned}
w(\hat{K}, \hat{y}) &= \sum_{\ell \in \hat{K}} P\Big(\lambda^{\ell-1} \geq \hat{y}^\ell,\ \max_{\substack{m \in \hat{K} \\ m > \ell}} (\lambda^{m-1} - \hat{y}^m) < 0\Big)\, r^\ell(\hat{y}^\ell) \\
&= \sum_{\substack{\ell \in K \\ \ell < n}} P\Big(\lambda^{\ell-1} \geq y^\ell,\ \lambda^{n-1} < y^{n-1},\ \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\Big)\, r^\ell(y^\ell) \\
&+ P\Big(\lambda^{n-1} \geq y^{n-1},\ \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\Big)\, r^n(y^{n-1}) \\
&+ \sum_{\substack{\ell \in K \\ \ell > n}} P\Big(\lambda^{\ell-1} \geq y^\ell,\ \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\Big)\, r^\ell(y^\ell).
\end{aligned}$$

Noting that $\lambda^{n-2} < y^{n-1}$ implies $\lambda^{n-1} < y^{n-1}$, we can reduce the above to

$$
\begin{aligned}
w(\hat{K}, \hat{y}) = & \sum_{\substack{\ell \in K \\ \ell < n-1}} P\Big(\lambda^{\ell-1} \geq y^{\ell}, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\Big) r^{\ell}(y^{\ell}) \\
& + P\Big(\lambda^{n-2} \geq y^{n-1}, \lambda^{n-1} < y^{n-1}, \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\Big) r^{n-1}(y^{n-1}) \\
& + P\Big(\lambda^{n-1} \geq y^{n-1}, \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\Big) r^{n}(y^{n-1}) \\
& + \sum_{\substack{\ell \in K \\ \ell > n}} P\Big(\lambda^{\ell-1} \geq y^{\ell}, \max_{\substack{m \in K \\ m > \ell}} (\lambda^{m-1} - y^m) < 0\Big) r^{\ell}(y^{\ell}).
\end{aligned}
$$

Using $r^n(y^{n-1}) \geq r^{n-1}(y^{n-1})$, and

$$
\begin{aligned}
& P\Big(\lambda^{n-2} \geq y^{n-1}, \lambda^{n-1} < y^{n-1}, \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\Big) \\
& + P\Big(\lambda^{n-1} \geq y^{n-1}, \max_{\substack{m \in K \\ m > n}} (\lambda^{m-1} - y^m) < 0\Big) \\
& = P\Big(\lambda^{n-2} \geq y^{n-1}, \max_{\substack{m \in K \\ m > n-1}} (\lambda^{m-1} - y^m) < 0\Big),
\end{aligned}
$$

we obtain $w(\hat{K}, \hat{y}) \geq w(K, y)$. Iteration of this process shows that $w(I, \hat{y}) \geq w(K, y)$ for some $\hat{y}$.

# References


[1] Parimal Kanti Bag, and Eyal Winter (1999), "Simple Subscription Mechanisms for Excludable Public Goods," Journal of Economic Theory 87, 72-94.

[2] Luis M. B. Cabral, David J. Salant, and Glenn A. Woroch (1999), "Monopoly pricing with network externalities," International Journal of Industrial Organization, 17, 199-214.

[3] Rajat Deb, and Laura Razzolini (1999a), "Voluntary cost sharing for an excludable public project," Mathematical Social Science, 37, 123-138.

[4] Rajat Deb, and Laura Razzolini (1999b), "Auction-like mechanisms for pricing excludable public goods," Journal of Economic Theory, 88, 340-368.

[5] Philip H. Dybvig, and Chester S. Spatt (1983), " Adoption externalities as public goods, " Journal of Public Economics, 20, 231-247.

[6] Douglas Gale (1995),"Dynamic coordination games, "Economic Theory, 5, 1-18.

[7] Douglas Gale (2001), "Monotone games with positive spillovers," Games and Economic Behavior, 37, 295-320.

[8] Michael L. Katz, and Carl Shapiro (1985), "Network externalities, competition, and compatibility," American Economic Review, 75, 424-440.

[9] Michael L. Katz, and Carl Shapiro (1986), "Technology adoption in the presence of network externalities," Journal of Political Economy, 94, 822-841.

[10] Herve Moulin (1994), "Serial cost sharing of excludable public goods," Review of Economic Studies, 61, 305-325.

[11] Jack Ochs, and In-Uck Park (2009), "Overcoming the coordination problem: dynamic formation of networks," working paper.

[12] In-Uck Park (2004), "A simple inducement scheme to overcome adoption externalities," Contributions to Theoretical Economics, 4(1), Article 3.

[13] Jeffrey H. Rohlfs (1974), "A theory of interdependent demand for a communications service," Bell Journal of Economics 5 (1), 16-37.

[14] Tadashi Sekiguchi (2009), "Pricing of durable network goods under dynamic coordination failure," working paper.

[15] Tatsuhiro Shichijo, and Yuji Nakayama (2009) "A two-step subsidy scheme to overcome network externalities in a dynamic game," The B.E. Journal of Theoretical Economics, 9(1) (Topics), Article 4.

[16] Arun Sundararajan (2007) " Local network effects and complex network structure, "The B.E. Journal of Theoretical Economics, 7(1) (Contributions), Article 46.