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AUCTIONS VERSUS NEGOTIATIONS: THE ROLE OF PRICE DISCRIMINATION

Chia-Hui Chen
Junichiro Ishida

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Auctions Versus Negotiations: The Role of Price Discrimination

Chia-Hui Chen* and Junichiro Ishida†

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Abstract

Auctions are a popular and prevalent form of trading mechanism, despite the restriction that the seller cannot price-discriminate among potential buyers. To understand why this is the case, we consider an auction-like environment in which a seller with an indivisible object negotiates with two asymmetric buyers to determine who obtains the object and at what price. The trading process resembles the Dutch auction, except that the seller is allowed to offer different prices to different buyers. We show that when the seller can commit to a price path in advance, the optimal outcome can generally be implemented. When the seller lacks such commitment power, however, there instead exists an equilibrium in which the seller's expected payoff is driven down to the second-price auction level. Our analysis suggests that having the discretion to price discriminate is not necessarily beneficial for the seller, and even harmful under plausible conditions, which could explain the pervasive use of auctions in practice.

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Keywords: Dutch auction, second-price auction, negotiation, commitment, price discrimination, asymmetric buyers.

*Institute of Economics, Academia Sinica. Email: chchen@econ.sinica.edu.tw
†Institute of Social and Economic Research, Osaka University. Email: jishida@iser.osaka-u.ac.jp

1 Introduction

Auctions, in various formats, are a very popular and prevalent form of trading mechanism when a seller has an indivisible object up for sale. At a glance, however, the reason for their popularity is not so obvious because it is well known that standard auctions fail to realize the optimal (revenue-maximizing) outcome in many instances. The case in point is when buyers are *ex ante* asymmetric with respect to their valuations for the object: in a case like this, the seller must bias against "strong buyers" and level the playing field to achieve the optimal outcome (Myerson, 1981).[1] Since buyers in most trading environments are in fact heterogenous in terms of readily observable characteristics, such as age, gender, occupation and so on, it remains to be seen why sellers usually prefer auctions to more flexible forms of negotiation where they can apparently retain a greater degree of discretion in price setting, including the ability to price discriminate among potential buyers.

To address this issue, this paper considers a less structured negotiation process in which a seller with an indivisible object negotiates with two buyers to determine who obtains the object and at what price. The two buyers are *ex ante* asymmetric in that their private valuations for the object are drawn independently from different distributions. The seller offers a pair of prices at the beginning and gradually lowers them over time until one of the buyers accepts or the seller terminates the negotiation without selling the object. The trading environment resembles the Dutch auction (and in fact encompasses it as a special case), except that the seller is allowed to offer different prices to different buyers at any point in time. Within this environment, we evaluate the value of price discrimination, i.e., the extent to which the seller can benefit from having the discretion to offer different prices to different buyers.

We largely obtain two results in this setup. First, we show that the seller can implement any individually rational and incentive compatible mechanism when she can commit to a pair of price sequences, or simply a "price path," in advance, even though she is restricted to offer weakly descending price sequences (Theorem 1). This result immediately implies that the seller can implement Myerson's optimal outcome with full commitment power, suggesting that the value of price discrimination cannot be negative in general and is strictly positive in

[1] Between two buyers, a buyer is weaker if his value distribution is first-order stochastically dominated by the other buyer's.

the face of asymmetric buyers. The optimal outcome necessarily involves price discrimination between the two buyers, and is not efficient in that the seller might not sell the object to the buyer with the highest valuation. By carefully tailoring a price path, the seller can improve upon the Dutch auction in which she is required to call out a single price for both of the buyers.

In a typical negotiation environment, however, it is usually prohibitively costly, or simply infeasible, for the seller to fully commit to a particular price path in advance. We thus shift our attention to the case where the seller lacks such commitment power, and all the price offers must be sequentially rational. We show that, in the case without commitment, there instead exists an equilibrium whose allocation coincides with that of the second-price auction (Theorem 2). Since the seller's payoff in the second-price auction is lower than in the Dutch auction under many plausible circumstances (Vickrey (1961) and Maskin and Riley (2000)), this result implies that the value of price discrimination can even be negative in the absence of commitment power. Moreover, we also show that with some reasonable restrictions on the strategy space and the buyers' type distributions, this outcome is the unique equilibrium in this environment (Theorem 3). In light of these results, we argue that the seller would lose very little, and even gain under plausible conditions, by giving up the discretion to price discriminate, which could explain the pervasive use of auctions in practice.

To see why the lack of commitment may drive the equilibrium payoff down to the second-price auction level, recall that when there are two asymmetric buyers, Myerson's optimal mechanism must be designed in a way that even when the weak buyer's realized valuation is slightly lower than the strong buyer's, the object is still allocated to the weak buyer. The optimal mechanism is, however, vulnerable to the commitment problem because the seller inevitably gains additional information about the buyers' valuations from their rejections to which she cannot resist reacting. The key here is that after each rejection, the buyers become more and more "symmetric" from the seller's point of view, which diminishes the seller's incentive to favor the weak buyer and forces the seller to deviate from the initially intended optimal path. Knowing this, the buyers no longer bid as aggressively as they would under the optimal mechanism. We show that in the limit case where the seller can incorporate new information and revise the price offers continuously, the equilibrium eventually converges to the second-price auction outcome in which the buyer with the higher valuation always obtains the object.

Related Literature: The paper is in spirit most closely related to the literature which compares the performances of auctions and other trading mechanisms. Wang (1993) compares standard auctions with posted-price selling in an environment where buyers arrive stochastically over time and shows that auctions are generally superior absent any auctioning costs. Manelli and Vincent (1995) consider a sequential offer process in which the order of buyers with whom the seller negotiates and the prices offered are determined in advance, and the seller receives, at most, one chance to negotiate with each buyer.[2] They then find that the negotiation of this form outperforms the second-price auction from the seller's perspective under certain conditions.[3] Bulow and Klemperer (1996) compare an English auction with no reserve price and an optimally-structured negotiation with one less bidder and show that the auction is always preferable under plausible assumptions. Bulow and Klemperer (2009) consider a sequential negotiation process in which potential buyers in turn decide whether to enter the bidding and compare this with a standard English auction. They find that although the sequential negotiation is always more efficient, the auction usually generates higher revenue because it is more conducive to entry. The current paper also addresses a similar question, comparing an (asymmetric) auction with a particular form of negotiation, but approaches from a different perspective with emphasis on the role of price discrimination.

Since our model describes a dynamic trading process without commitment, it also has an inherent connection with the literature on durable goods monopoly. There is a critical difference, however, between the durable-goods problem and the current setup. In the prototype durable-goods problem, the monopolist has unlimited supply of the good, making it impossible to stop at the right moment. In contrast, an important feature of our model is that the seller only has limited supply (one unit) compared to potential demand (two buyers), which works as a strong commitment device just like in auction settings. Due to this feature, the seller can have the buyers compete against one another and extract rents from them even when they are infinitely patient as we assume here.

In the sense that the seller has only limited supply, the paper is more closely related to the

[2]To avoid confusion, we consistently apply the terms as specified in our model by referring to the informed party as the buyers and the uninformed party as the seller.

[3]In this paper, we show that with independent private value, if the number of chances to negotiate is unlimited and the price path is determined in advance, Myerson's optimal outcome can always be implemented, meaning that the negotiation always outperforms the second-price auction. This result thus indicates that the key driving force behind their work is the limited number of chances to negotiate in the independent private-value setting.

so-called revenue management problem which examines the optimal pricing strategy when the seller has finitely many goods to sell before a deadline. While much of the literature assumes perfect commitment power, several recent works (Horner and Samuelson, 2011; Chen, 2012; Dilme and Li, 2012) analyze this problem when the seller lacks the ability to commit to any price path in advance. Aside from the fact that our model has no exogenously imposed deadline, the critical departure from this strand of literature is that potential buyers are *ex ante* symmetric in those previous works, so that there is no inherent need to price discriminate among the buyers.[4] In contrast, the two buyers in our setup are *ex ante* asymmetric, where their valuations (or their "types") are possibly drawn from different distributions, so that the ability to price discriminate is supposed to be highly valuable.

On the more technical side, to formalize the negotiation environment of our interest, we employ a continuous-time model in which the seller continuously adjusts the prices offered to the two buyers. The technique adopted to analyze the model is related to the theory of differential games originated by Isaacs (1954). Most applications of this technique involve complete-information games, on issues such as oligopoly games with dynamic prices, R&D competition, and capital accumulation (e.g., Dockner et al., 2000). Recently, the technique has also been applied to settings under incomplete information as in ours (see, e.g., Bolton and Harris, 1999; Bergemann and Valimaki, 1997, 2000, 2002; Decamps and Mariotti, 2004).

The remainder of the paper is organized as follows. Section 2 describes the model. Section 3 characterizes the optimal outcome that the seller can achieve when commitment is possible. Section 4 characterizes the equilibrium without commitment and proves that this equilibrium is unique under some mild conditions. Section 5 concludes the paper. All the proofs are relegated to Appendix A.

2 The Model

Environment: Time is continuous and extends from zero to infinity. We consider an open-ended process in which a seller with an indivisible object negotiates with two asymmetric buyers, denoted by $i = 1, 2$, to determine who obtains the object and at what price. All parties are risk neutral and there is no time discounting. Each buyer has a private valuation X_i for the object, which is drawn from some distribution F_i over the interval $W_i \equiv [\underline{w}_i, \overline{w}_i]$, $\underline{w}_i \geq 0$

[4] Dilme and Li (2012) consider a model with two types, high and low, where high-type buyers flow into the market at a constant rate.

with its corresponding density f_i. We make the following assumption on the distribution functions.

Assumption 1 *For* $i = 1, 2$, $\psi_i(x) \equiv x - \frac{1-F_i(x)}{f_i(x)}$*, i.e., the virtual valuation, is strictly increasing in* x *and* $f_i(x) > 0$ *for all* $x \in W_i$*.*

Note that the buyers are possibly asymmetric in that their valuations might be drawn from different distributions. Letting x_i denote the realized valuation and m_i the payment to the seller, buyer i's payoff is given by $z_i x_i - m_i$ where $z_i = 1$ if buyer i obtains the object and $z_i = 0$ otherwise. The object yields no value to the seller who simply maximizes the expected payment she collects from the buyers.

Negotiation process: At each instance t, the seller offers a pair of (possibly asymmetric) prices $p(t) \equiv (p_1(t), p_2(t))$, where $p_i(t)$ denotes the price offered to buyer i: strictly for expositional purposes, we refer to each $p_i(t)$ as a *price sequence* and to $p(t)$ as a *price path.* Offers are made in public, so that each buyer knows not only the price offered to him but also the price offered to the other buyer. The negotiation process ends either when one of the buyers accepts or when the seller decides to terminate the process without selling the object. If both of the buyers accept at the same time, each of them obtains the object with probability one half. We assume that the seller adjusts the price offers continuously over time, so that each price sequence $p_i(t)$ cannot make discrete jumps.

Assumption 2 *The price sequence for each buyer,* $p_i(t)$*, is continuous and weakly decreasing in* t*.*

The negotiation process described above includes the Dutch auction as a special case, in which case $p_1(t) = p_2(t)$ is required for all t.[5] Within this environment, we ask whether and to what extent the seller can benefit from this larger degree of discretion in price setting, i.e., the discretion to price discriminate, both with and without commitment. We say that the value of price discrimination is positive (negative) if the seller's expected payoff when she is allowed to price discriminate increases (decreases) from the payoff level which she can attain by running the Dutch auction.

[5] As long as $p_1(t) = p_2(t)$ is satisfied for all t, any pair of weakly descending price sequences would replicate the Dutch-auction outcome.

3 Optimal Outcome with Commitment

In order to investigate the role of commitment power in this negotiation environment, we first consider the case, as a benchmark, where the seller can fully commit to any price path before the negotiation begins. The allocation rule of this trading environment specifies who obtains the object for each pair of realized valuations (x_1, x_2). In particular, the allocation rule of any incentive compatible and individually rational mechanism can be written as $\hat{x}_2(x_1)$, where $\hat{x}_2(x_1)$ indicates the cutoff type of buyer 2 as a function of buyer 1's valuation, i.e., given x_1, buyer 2 obtains the object if his valuation exceeds $\hat{x}_2(x_1)$. Given this formulation, we set up the the seller's problem as the one to find the optimal pair of (cutoff) buyer type sequences rather than the one to find the pair of price sequences.[6] Since the optimal mechanism may assign zero probability to types in the lower end, we let the domain of $\hat{x}_2(x_1)$ be $[\underline{x}_1, \overline{w}_1]$, $\underline{x}_1 \geq \underline{w}_1$. Note that in the absence of time discounting, the timing of transaction is payoff-irrelevant as in most standard auction environments and is hence not included in the allocation rule. To be more precise, a price path $\{\tilde{p}_1(\alpha t), \tilde{p}_2(\alpha t)\}$, with or without commitment, results in the same allocation for any $\alpha > 0$. Throughout the analysis, therefore, we abstract away from the time dimension.

To derive the optimal path, we first establish in Theorem 1 that, with commitment, the seller can implement any incentive compatible and individually rational mechanism by appropriately tailoring a pair of weakly descending price sequences. The fact that the seller can implement Myerson's optimal outcome with full commitment power then directly follows from this result.

Theorem 1 *With commitment, the seller can implement any incentive compatible and individually rational mechanism.*

The optimal allocation rule characterized by Myerson (1981) is to allocate the object to the buyer whose virtual valuation is the greatest and also positive. A direct corollary from Theorem 1 is that Myerson's optimal outcome can be implemented with commitment. Since the optimal outcome cannot be implemented by standard auctions in the presence of asymmetric buyers, the result implies that the value of price discrimination is strictly positive when the buyers are *ex ante* asymmetric.

[6]Given the allocation function which pairs the two buyer types, it is relatively straightforward to derive the associated pair of price sequences which implements this allocation. See the proof of Theorem 1 for detail.

Corollary 1 *By committing to a specifically designed price path, the seller can implement Myerson's optimal outcome. With full commitment, the value of price discrimination cannot be negative in general and is strictly positive when the buyers are ex ante asymmetric.*

It is fairly straightforward to construct the optimal price path from the allocation function $\hat{x}_2(x_1)$. The seller's task here is simply to design a price path along which the buyer with the higher virtual valuation always accepts earlier. Without loss of generality, assume that $\psi_1(\underline{w}_1) \geq \psi_2(\underline{w}_2)$. Following the notations defined above, let $\underline{x}_1 = \psi_1^{-1}(0)$ if $\psi_1(\underline{w}_1) < 0$, and $\underline{x}_1 = \underline{w}_1$ otherwise, so that the virtual valuation of buyer 1 is positive if his valuation is greater than $\underline{x}_1$. Finally, let the function characterizing the allocation rule be $\hat{x}_2(x_1) \equiv \psi_2^{-1}(\psi_1(x_1))$. Then, buyer 1 with valuation x_1 has the same virtual valuation as buyer 2 with valuation $\hat{x}_2(x_1)$, i.e., $\psi_1(x_1) = \psi_2(\hat{x}_2(x_1))$, and given $\hat{x}_2(x_1)$, by committing to the price path constructed in the proof of Theorem 1, the seller can implement Myerson's optimal outcome.

Theorem 1 suggests that, with full commitment, the seller can accomplish various goals, including revenue maximization, by carefully designing a pair of weakly descending price sequences. However, as a growing body of literature on mechanism design emphasizes (see, e.g., McAfee and Vincent, 1997; McAdams and Schwarz, 2007; Skreta, 2006, 2011; Horner and Samuelson, 2011; Chen, 2012; Vartiainen, 2013), it is often prohibitively costly for the seller to make full commitment in advance. It is hence crucial to see how much the seller can gain from her discretion to offer asymmetric prices when she lacks such commitment power at her disposal. We dedicate the next section to investigate this issue.

4 Equilibrium without Commitment

4.1 Preliminaries

We now examine the case in which the seller cannot commit to any price path, or any "mechanism," in advance. The problem is now substantially more complicated because every price offer must be sequentially rational at any continuation game. In particular, to implement the optimal outcome, the seller must have an incentive not to deviate from the initially intended optimal path *after each rejection*.

Before we formally define the equilibrium concept for the non-commitment case, it is helpful to establish some equilibrium properties which allow us to narrow down the class of strategies we need to consider. To be more precise, we show that any equilibrium of this game has the following properties:

- Each buyer's purchasing decision is characterized by a cutoff strategy; [7]
- The seller does not terminate the negotiation until the object is sold.

In what follows, we prove these properties in turn.

We first show that each buyer's strategy attains some sort of monotonicity. The first property directly follows from the next lemma and corollary.

Lemma 1 *Buyer i's equilibrium strategy has the property that, given any history, belief, and current price offer, if buyer i with valuation x_i accepts, then buyer i with valuation higher than x_i also accepts.*

The lemma implies that given any history, a player's belief about buyer i's valuation can be characterized by a cutoff representing the supremum of buyer i's valuation. Throughout the analysis, therefore, we generically denote a player's belief at any continuation game by $\omega \equiv (w_1, w_2) \in W \equiv W_1 \times W_2$ where w_i is the supremum of buyer i's valuation. This property in particular implies that the equilibrium allocation of this non-commitment case can also be characterized by an allocation function $\hat{x}_2(x_1)$ which pairs the two buyer types. We will hence attempt to characterize the seller's strategy in terms of buyer types rather than of prices, as we did for the commitment case. Given the set of equilibrium strategies, it is relatively straightforward to construct the associated price path which implements the equilibrium allocation.

We next show that although we allow the seller to terminate the negotiation at any point, this option is never exercised in equilibrium when the seller lacks commitment power.

Lemma 2 *In equilibrium, the seller never terminates the negotiation process when $w_1 > \underline{w}_1$ and $w_2 > \underline{w}_2$. Therefore, the object is sold with probability one.*

[7] In a dynamic bargaining game where a seller makes offers to a buyer who discounts the future and has private information about his value, the buyer also adopts a cutoff strategy in equilibrium. In that setting, time functions as a screening device since different types of buyer value time differently. In our model, instead of time, buyers are screened by the fear that they will lose the trade to a competitor, who might accept the current price.

The second property suggests that the seller cannot help lowering at least one of the prices p_i to the lower bound of buyer i's valuation $\underline{w}_i$, so that the object is sold with probability one. This draws a clear contrast to the full-commitment case where the seller can set a reserve price at which she terminates the negotiation without selling the object. More precisely, this property alone suggests that Myerson's optimal outcome cannot be implemented without commitment if $\underline{w}_1 - \frac{1-F_1(\underline{w}_1)}{f_1(\underline{w}_1)} < 0$ and $\underline{w}_2 - \frac{1-F_2(\underline{w}_2)}{f_2(\underline{w}_2)} < 0$, in which case the object should be left unsold with some positive probability in the optimal mechanism.

The equilibrium concept we adopt is the Markov perfect equilibrium, with the posterior belief ω as the state variable. The two properties mentioned above, along with the notion of the Markov perfect equilibrium, allow us to characterize the equilibrium in a clearer manner. The strategies of the game can be defined as follows.

Buyer: Each buyer's strategy is characterized by a set of functions $\{P_{i,\omega}(x_i)\}_{\omega \in W}$, where $P_{i,\omega}(x_i)$ indicates the maximum price that buyer i with valuation x_i is willing to accept, given the current belief ω. Define buyer i's *marginal strategy* by $P_i(w_i, w_j) \equiv P_{i,\omega}(w_i)$, i.e., the maximum price that the cutoff type is willing to accept. Although $P_i(w_i, w_j)$ does not fully describe buyer i's strategy, it provides enough information to characterize the equilibrium of this game, due to our focus on continuous strategies in which the seller can adjust the prices only continuously.[8] In what follows, therefore, we simply refer to $P_i(x_i, x_j)$ as buyer i's strategy wherever it is not confusing.

Seller: The seller's strategy is represented by a set of functions $\{x_{2,\omega}(x_1)\}_{\omega \in W}$, where $x_{2,\omega}(x_1)$ indicates the pair of buyer types that are induced to accept at the same time, given the current belief ω.[9] Let $x_{1,\omega}(x_2) \equiv x_{2,\omega}^{-1}(x_2) = \inf\{x_1 \in [\underline{w}_1, w_1] \mid x_{2,\omega}(x_1) \geq x_2\}$.

4.2 The Buyer's Problem

We begin with the buyer's problem. Taking the seller's strategy $\{x_{2,\omega}(x_1)\}_{\omega \in W}$ as given, to satisfy buyer 1's incentive compatibility constraints for all types, the following equation must

[8] Note that along the seller's equilibrium strategy, buyer i's strategy, i.e., $P_{i,\omega}(x)$ for $x \in W_i$, can be fully derived. We will make this point when we solve the buyer's problem in Section 4.2.

[9] It is certainly possible to represent the seller's strategy by a pair of prices to be offered. An advantage of our current approach is that the domain of the optimization problem is clearly defined. On the other hand, if we set up the seller's problem as the one to choose a price path, the relevant domain is not clearly defined because there exists a path along which no buyer type would accept.

hold:

$$F_2(w_2)[w_1 - P_1(w_1, w_2)] = \int_{\underline{w}_1}^{w_1} F_2(x_{2,\omega}(x))\,dx + F_2(x_{2,\omega}(\underline{w}_1))[\underline{w}_1 - P_1(\underline{w}_1, x_{2,\omega}(\underline{w}_1))]. \quad (1)$$

Equation (1) comes from the revenue equivalence principle.[10] The left-hand side is the expected payoff of buyer 1 with valuation w_1. On the right-hand side,

$$F_2(x_{2,\omega}(\underline{w}_1))(\underline{w}_1 - P_1(\underline{w}_1, x_{2,\omega}(\underline{w}_1)))$$

indicates the expected payoff of buyer 1 with the lowest valuation, and $\int_{\underline{w}_1}^{w_1} F_2(x_{2,\omega}(x))\,dx$ is the additional information rent received by buyer 1 with valuation w_1. If the expected payoff of buyer 1 with the lowest valuation $\underline{w}_1$ is zero, (1) is reduced to

$$P_1(w_1, w_2) = w_1 - \frac{\int_{\underline{w}_1}^{w_1} F_2(x_{2,\omega}(x))\,dx}{F_2(w_2)}.$$

Similarly, $P_2(w_2, w_1)$ must satisfy

$$F_1(w_1)[w_2 - P_2(w_2, w_1)] = \int_{\underline{w}_2}^{w_2} F_1(x_{1,\omega}(x))\,dx + F_1(x_{1,\omega}(\underline{w}_2))[\underline{w}_2 - P_2(\underline{w}_2, x_{1,\omega}(\underline{w}_2))]. \quad (2)$$

We assume that both P_1 and P_2 are continuous and weakly increasing in their respective arguments, so that the seller can implement $\{x_{2,\omega}(x_1)\}_{\omega\in W}$ by offering a pair of descending price sequences.

While we only use the marginal strategies to characterize the equilibrium, it is conceptually straightforward to construct each buyer's fully specified strategy, i.e., $P_{i,\omega}(x_i)$ for all x_i. Given the current belief ω, and $P_1(x_1, x_{2,\omega}(x_1))$ and $P_2(x_{2,\omega}(x_1), x_1)$ satisfying equations (1) and (2) for all $x_1 \in [\underline{w}_1, w_1]$, we can prove that the following strategy is optimal for buyer i: buyer i with valuation $x_i \geq w_i$ accepts the current price $P_i(w_i, w_j)$, and buyer i with value $x_i < w_i$ waits and accepts at price $P_i(x_i, x_{j,\omega}(x_i))$.[11] This implies that given the current belief ω, the maximum price that buyer i with valuation $x_i \geq w_i$ is willing to accept is $P_i(w_i, w_j)$, and the maximum price that buyer i with valuation $x_i < w_i$ is willing to accept

[10] In our model, given an incentive compatible mechanism, the revenue equivalence principle implies that

$$q_i(x_i)\,x_i - m_i(x_i) = q_i(\underline{w}_i)\,\underline{w}_i - m_i(\underline{w}_i) + \int_{\underline{w}_i}^{x_i} q_i(t)\,dt,$$

where $m_i(x_i)$ is the expected payment of buyer i with valuation x_i, $q_i(x_i)$ is the probability that the buyer obtains the object, and therefore, $q_i(x_i)\,x_i - m_i(x_i)$ is the expected payoff for the buyer.

[11] To prove this, we can apply the same argument as in the proof of Theorem 1.

is $x_i - \frac{F_j(x_{j,\omega}(x_i))}{F_j(w_j)}\left[x_i - P_i\left(x_i, x_{j,\omega}\left(x_i\right)\right)\right]$.[12]

Finally, we can show that the buyers' strategies, derived from (1) and (2), are robust to the seller's deviation from the equilibrium strategy. To see this, suppose that the seller deviates and unexpectedly reaches some belief $\omega' = (w_1', w_2')$. Even in this contingency, since the deviation is made by a player who does not have any private information, the belief about the buyers' valuations cannot be arbitrary and must be consistent with the buyers' continuation strategies which maximize their payoffs given their expectations about the seller's continuation strategy, i.e., $x_{2,\omega'}(x)$. Therefore, $P_1(x_1, x_{2,\omega'}(x_1))$ and $P_2(x_{2,\omega'}(x_1), x_1)$ derived from (1) and (2) for all $x_1 \in [\underline{w}_1, w_1']$ continue to be optimal and characterize the two buyers' strategies for a continuation game starting with any given ω'.

4.3 The Seller's Problem

We now turn to the seller's problem, which is far more complicated in the absence of commitment power because every price offer must now be sequentially rational given the current belief. The seller's problem is defined as choosing a function $x_2(x_1)$ which maximizes her expected payoff given the current belief ω and the buyers' strategies $P_i(x_i, x_j)$, $i = 1, 2$, under the restriction that the function be weakly increasing.[13] Fixing the buyers' strategies, the principle of optimality then ensures that if some function $x_{2,\omega}(x_1)$ is the optimal path for a game starting with ω, then for any continuation game starting with $(x, x_{2,\omega}(x))$ where $x \in (\underline{w}_1, w_1)$, $x_{2,\omega}(x_1)$ for $x_1 \le x$ is also the optimal path, so that it gives a sequentially rational solution.

More precisely, given the current belief ω and each buyer's strategy $P_i(x_i, x_j)$, the seller's optimal strategy $x_{2,\omega}(x_1)$ is obtained as the solution to the following problem:

$$\begin{aligned} x_{2,\omega}(x_1) = \ & \arg \max_{x_2(x_1):[\underline{w}_1, w_1]\to[\underline{w}_2, w_2]} \int_{\underline{w}_1}^{w_1} P_1(x_1, x_2(x_1))\, F_2(x_2(x_1))\, f_1(x_1)\, dx_1 \\ & + \int_{\underline{w}_2}^{w_2} P_2(x_2, x_1(x_2))\, F_1(x_1(x_2))\, f_2(x_2)\, dx_2 \qquad (P1) \\ & s.t. \quad \frac{dx_2}{dx_1} \ge 0. \end{aligned}$$

[12]Given the price $p = x_i - \frac{F_j(x_{j,\omega}(x_i))}{F_j(w_j)}\left[x_i - P_i\left(x_i, x_{j,\omega}(x_i)\right)\right]$, buyer i with value x_i feels indifferent between accepting now and accepting later at price $P_i(x_i, x_{j,\omega}(x_i))$, i.e., $F_j(w_j)(x_i - p) = F_j(x_{j,\omega}(x_i))\left[x_i - P_i(x_i, x_{j,\omega}(x_i))\right]$.

[13]The restriction that the seller's strategy must be weakly increasing comes from the fact that the buyer with the higher valuation accepts earlier.

The first and the second integrals are the seller's expected payoffs received from buyer 1 and buyer 2, respectively. The first integrand represents the seller's payoff increment from buyer 1 when he decreases x_1 by dx_1: the object is sold to buyer 1 at price $P_1(x_1, x_2(x_1))$ if buyer 2's value is below $x_2(x_1)$ and buyer 1's value is x_1, which occurs with probability $F_2(x_2(x_1)) f_1(x_1) dx_1$. Similar reasoning applies to the second integrand. We derive the seller's optimal strategy $x_{2,\omega}(x_1)$ from (P1) for all beliefs $\omega \in W$, including those off the equilibrium path. Given the solution to this problem, we can also straightforwardly construct the actual price path submitted by the seller (see Appendix B for more detail).

4.4 A Markov perfect equilibrium

We are now ready to obtain an equilibrium of this game. The set of strategies ($P_1(x_1, x_2)$, $P_2(x_2, x_1)$, $\{x_{2,\omega}(x_1)\}_{\omega \in W}$) constitutes a Markov perfect equilibrium if:

- Taking the seller's strategy $\{x_{2,\omega}(x_1)\}_{\omega\in W}$ as given, $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$ solve (1) and (2), respectively, for all $(x_1, x_2) \in W$.
- Taking the buyers' strategies as given, $x_{2,\omega}(x_1)$ solves (P1) for all $\omega \in W$;

We now establish the next result which constitutes the main contention of the paper.

Theorem 2 *When the seller cannot commit to a price path in advance, there always exists a Markov perfect equilibrium whose allocation is the same as that of the second-price auction.*

The theorem states that there always exists an equilibrium in which the seller's expected profit is driven down to the second-price auction level. The equilibrium is efficient, but is more likely to favor the buyers rather than the seller.[14] It is now well known that, with asymmetric buyers, the seller's payoff in the Dutch auction (or the first-price auction) is greater than that in the second-price auction in a range of circumstances, including the case where the buyers' valuations are distributed uniformly (Vickrey, 1961; Maskin and Riley, 2000) . A striking corollary of this result, combined with this conclusion in the literature, is

[14] Vartiainen (2013) considers an auction setting in which both the seller and the buyers lack commitment power and shows that the only implementable mechanism is the English auction because it reveals just the right amount of information to the seller. Both Vartiainen's and our results seem to suggest that mechanisms resulting in efficient allocations are generally robust against commitment problems. Although it is only a speculation and a lot remains to be seen at this stage, it is of some interest to explore the link between efficiency and commitment in other contexts as well.

that the seller could end up with a lower payoff than in the Dutch auction in the absence of commitment power even though her strategy space is strictly larger.

Corollary 2 *In the absence of commitment power, the value of price discrimination can be negative.*

To understand the intuition behind this result, consider a simple setting in which the valuations of the two buyers are uniformly distributed on $[0, \overline{w}_1]$ and $[0, \overline{w}_2]$, respectively. Assume $\overline{w}_2 > \overline{w}_1$, so that buyer 1 (buyer 2) is the weak (strong) buyer. To implement the optimal mechanism in this specification, the seller must submit prices in a way to pair buyer 1 with valuation w_1 with buyer 2 with valuation $\frac{w_1+w_2}{2}$. Suppose that the seller starts the negotiation with the prices which pair buyer 1 with valuation $\overline{w}_1$ with buyer 2 with valuation $\overline{w}_2' \equiv \frac{\overline{w}_1+\overline{w}_2}{2}$. If buyer 2 rejects this offer, the seller's belief about buyer 2 (buyer 2's highest possible value) is then updated to $\overline{w}_2'$, while she gains no relevant information about buyer 1 from the rejection, with her belief remaining at $\overline{w}_1$. At the next instance, therefore, the seller faces a different problem in which buyer 2's valuation is distributed on $[0, \overline{w}_2']$, rather than on $[0, \overline{w}_2]$, in which he needs to pair buyer 1 with valuation $\overline{w}_1$ with buyer 2 with valuation $\frac{\overline{w}_1+\overline{w}_2'}{2}$. By repeating this process over and over, the situation eventually converges to the symmetric case where the valuations of both players are distributed on $[0, \overline{w}_1]$. Knowing this, the strong buyer would not bid as aggressively as under the optimal mechanism because there is virtually no risk of losing the trade to the weak buyer. Theorem 2 shows that this intuition generally holds in the current trading environment.

4.5 Uniqueness of the equilibrium

In the previous subsection, we established that there always exists an equilibrium whose allocation coincides with the second-price auction outcome when the seller cannot commit to a price path in advance. This result does not necessarily imply, though, that the seller cannot do any better than in the second-price auction because we do not rule out other equilibria as possible outcomes. What is especially critical from the seller's viewpoint is whether there exists any other Markov perfect equilibrium in which the seller can benefit from the discretion to price discriminate.

One way to address this question is to find a set of conditions under which the equilibrium identified in Theorem 2 is the unique equilibrium outcome of this negotiation process. Although establishing the uniqueness of equilibrium is in general a far more challenging problem to deal with, and especially so in our inherently dynamic setup with incomplete information, we can nonetheless establish the uniqueness by imposing two mild (and purely technical) restrictions, one for the seller and the other for the buyers, on the feasible sets of strategies, as summarized below.

Assumption 3 *(i) Given any belief ω, the seller's strategy $x_{2,\omega}(x_1)$ is continuous, piecewise differentiable on $[\underline{w}_1, w_1]$, and strictly increasing on the interval $\left(\underline{x}_{1,\omega}, \overline{x}_{1,\omega}\right)$ where*

$$\overline{x}_{1,\omega} \equiv \sup\left\{x_1 \in [\underline{w}_1, w_1] \mid x_{2,\omega}(x_1) < w_2\right\} \text{ and } \underline{x}_{1,\omega} \equiv \inf\left\{x_1 \in [\underline{w}_1, w_1] \mid x_{2,\omega}(x_1) > \underline{w}_2\right\};$$

(ii) The two functions characterizing the buyers' strategies, $C_1(w_1, w_2) \equiv \int_{\underline{w}_1}^{w_1} F_2(x_{2,\omega}(x))\,dx$ and $C_2(w_1, w_2) \equiv \int_{\underline{w}_2}^{w_2} F_1(x_{1,\omega}(x))\,dx$, are continuous, and the first derivatives of $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$ with respect to x_1 and x_2 exist.

These restrictions are not at all stringent. The first restriction implies that the probability that a buyer gets the object increases continuously with his valuation. The restriction does not entail much loss of generality, because this is an equilibrium property presenting itself in many mechanisms, including the first-price and second-price auctions. The second restriction is also a natural one to impose, as it simply states that the function $x_{2,\omega}(x)$ does not change much when the belief ω is slightly modified, so $\int_{\underline{w}_1}^{w_1} F_2(x_{2,\omega}(x))\,dx$ is continuous in w_1 and w_2. Figure 1 shows several examples of this class of paths.

[Figure 1 about here]

Without loss of generality, we focus on the equilibria in which buyer 1 with valuation $\underline{w}_1$ and buyer 2 with valuation $\underline{w}_2$ get no information rent. We can then obtain the following result which establishes the uniqueness of the equilibrium.

Theorem 3 *Suppose that the two buyers' valuations are uniformly distributed on $[\underline{w}_1, \overline{w}_1]$ and $[\underline{w}_2, \overline{w}_2]$ respectively. Then, under Assumption 3, the efficient equilibrium is the unique equilibrium.*

As is well known, with asymmetric buyers, the seller's expected payoff in the second-price auction is lower than in the Dutch auction when the buyers' valuations are uniformly distributed. The theorem thus implies that without commitment, the seller is *necessarily* made worse off in this negotiation environment when she is endowed with the discretion to price discriminate at will. Conversely speaking, this is where auctions can be especially valuable, as they instantaneously provide credible commitment not to price discriminate, which renders the buyers bid more aggressively and consequently raises the expected profit.

Corollary 3 *Under Assumption 3, the value of price discrimination is strictly negative if the two buyers' valuations are uniformly distributed.*

We use a simple example to illustrate what the equilibrium price path looks like and how different types of buyer accept along the path. Suppose that the two buyers' valuations are uniformly distributed on $[0, \overline{w}_1]$ and $[0, \overline{w}_2]$, respectively. In this case, under Assumption 3, it is straightforward to show that the following set of strategies constitutes the unique equilibrium:

$$x_{2,\omega}(x_1) = \begin{cases} x_1, \text{ for } 0 \leq x_1 \leq \min\{w_1, w_2\} \\ w_{2,} \text{ for } \min\{w_1, w_2\} < x_1 \leq w_1 \end{cases} \quad \text{and } P_i(w_i, w_j) = \begin{cases} w_i - \frac{w_i^2}{2w_j}, \text{ if } w_j \geq w_i \\ \frac{w_j}{2}, \text{ if } w_j < w_i \end{cases}.$$

Figure 2 shows the equilibrium price path (p_1, p_2) submitted by the seller and the corresponding cutoff path (x_1, x_2) when $\overline{w}_2 > \overline{w}_1$. Note that $x_2(x_1) = x_1$ for $x_i \in [0, \overline{w}_1]$, so that the equilibrium allocation is efficient.

[Figure 2 about here]

4.6 Why do sellers prefer auctions?

We would like to conclude the analysis by revisiting our motivating question concerning the pervasive use of auctions, i.e., why sellers prefer auctions to more flexible forms of negotiation. The most distinctive feature of standard auctions arguably lies in their simplicity, with the trading process characterized by a minimal set of rules which offers both costs and benefits. An apparent benefit of auctions is that they are easy to implement, and therefore entail low implementation costs. There is also a drawback, however, because the simplicity of auctions necessarily restricts the seller's freedom to pursue strategies to maximize profit. To justify the use of auctions, therefore, the "benefit of simplicity" must be traded off against the potential loss of profit which stems from those restrictions.

The current paper provides a framework to evaluate the potential cost of one such restriction that the seller must offer one price for all buyers and therefore cannot price discriminate among them.[15] As we have seen, this restriction can be quite costly if the seller is endowed with the ability to fully commit to a price path in advance. In a more realistic environment where the seller cannot make any credible commitment, however, the restriction is not costly after all and can even be beneficial under many plausible circumstances. In particular, under the conditions obtained in section 4.5, the value of price discrimination must be negative, meaning that there is no loss on the seller's part to give up the discretion to price discriminate even when the benefit of simplicity is negligibly small. We argue that these findings could explain the deep-rooted popularity of auctions even in environments where potential buyers are apparently heterogeneous.

5 Conclusion

This paper studies the environment in which a seller with an indivisible object negotiates with two asymmetric buyers to determine who obtains the object and at what price. The seller repeatedly submits price offers to the two buyers until one of them accepts. Unlike the Dutch auction, the two prices offered to the two buyers can be different. We show that if the seller can commit to a price path in advance, the payoff realized in Myerson's optimal mechanism is achievable. However, if commitment is not possible, there instead exists an equilibrium in which the seller's expected profit is driven down to the second-price auction level. The result suggests that having the discretion to price discriminate is not beneficial after all, and even harmful in many cases, which could explain the pervasive use of auctions in practice.

As a final note, one important extension of our analysis is to explore whether there is any Markov perfect equilibrium which implements Myerson's optimal outcome. Although we do not know of any counterexample at this point, it is a far more challenging task to formally prove this. Still, one way to attack this problem is to characterize conditions under which the equilibrium is unique; along with Theorem 2, this necessarily implies that the equilibrium is always efficient. Theorem 3 thus provides a preliminary answer to this issue, showing that asymmetric uniform distributions of the buyers' valuations yield a sufficient condition.

[15] Note that our methodology and analysis can also be generalized to the n-buyer case. With commitment, we can construct a set of functions $\{p_2(p_1), p_2(p_1), \cdots, p_n(p_1)\}$ specifying the relationship among the prices to implement Myerson's optimal outcome. Without commitment, we find a set of functions $\{P_1(x_1, \cdots, x_n), P_2(x_1, \cdots, x_n), \cdots, P_n(x_1, \cdots, x_n)\}$ and $\{x_{2,\omega}(x_1), x_{3,\omega}(x_1), \cdots, x_{n,\omega}(x_1)\}_\omega$ to constitute a Markov perfect equilibrium. We believe that our conclusion still holds in an n-buyer case and will leave such exploration to future research.

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Appendix A: the proofs

Proof of Theorem 1: Given an incentive compatible and individually rational mechanism characterized by $\hat{x}_2(x_1) : [\underline{x}_1, \overline{w}_1] \to [\underline{w}_2, \overline{w}_2]$, let

$$b_1(x_1) \equiv x_1 - \frac{\int_{\underline{x}_1}^{x_1} F_2(\hat{x}_2(x))\,dx}{F_2(\hat{x}_2(x_1))} \text{ for } x_1 \in [\underline{x}_1, \overline{w}_1],$$

$$b_2(x_2) \equiv x_2 - \frac{\int_{\underline{x}_2}^{x_2} F_1(\hat{x}_1(x))\,dx}{F_1(\hat{x}_1(x_2))} \text{ for } x_2 \in [\underline{x}_2, \overline{w}_2],$$

where $\underline{x}_2 \equiv \hat{x}_2(\underline{x}_1)$, $\hat{x}_1(x) \equiv \hat{x}_2^{-1}(x)$. If $F_j(\hat{x}_j(\underline{x}_i)) = 0$, let

$$b_i(\underline{x}_i) = \underline{x}_i = \lim_{x_i \downarrow \underline{x}_i} \left(x_i - \frac{\int_{\underline{x}_i}^{x_i} F_j(\hat{x}_j(x))\,dx}{F_j(\hat{x}_j(x_i))} \right).$$

Note that both $b_1(x)$ and $b_2(x)$ are strictly increasing. Suppose that the seller commits to a price path on which the price to buyer 1, p_1, and the price to buyer 2, p_2, decrease continuously in a relationship whereby $p_2(p_1) = b_2\left(\hat{x}_2\left(b_1^{-1}(p_1)\right)\right)$ for $p_1 \in [\underline{x}_1, b_1(\overline{w}_1)]$. Then if buyer 2 with valuation $x_2 > \hat{x}_2(\underline{x}_1)$ accepts at $b_2(x_2)$, we show that accepting at $b_1(x_1)$ is optimal for buyer 1 with valuation x_1. To see this, note that if the current price for buyer 1 is $p_1 > b_1(x_1)$, the payoff for buyer 1 from accepting now is

$$x_1 - p_1 = x_1 - b_1(x_1'), \text{ where } x_1' = b_1^{-1}(p_1) > x_1,$$

whereas the expected payoff from accepting later at $b_1(x_1)$ is $\frac{F_2(\hat{x}_2(x_1))}{F_2(\hat{x}_2(x_1'))}(x_1 - b_1(x_1))$. Since

$$\begin{aligned} F_2(\hat{x}_2(x_1'))(x_1 - b_1(x_1')) &= (x_1 - x_1') F_2(\hat{x}_2(x_1')) + \int_{\underline{x}_1}^{x_1'} F_2(\hat{x}_2(x))\,dx \\ &< \int_{\underline{x}_1}^{x_1} F_2(\hat{x}_2(x))\,dx = F_2(\hat{x}_2(x_1))(x_1 - b_1(x_1)), \end{aligned}$$

buyer 1 would be better off by waiting and accepting later. If the current price is $b_1(x_1)$, on the other hand, the payoff from accepting now is $x_1 - b_1(x_1)$, whereas the expected payoff from accepting later at $b_1(x_1'')$ is $\frac{F_2(\hat{x}_2(x_1''))}{F_2(\hat{x}_2(x_1))}(x_1 - b_1(x_1))$. Since

$$\begin{aligned} F_2\left(\hat{x}_2\left(x_1''\right)\right)\left(x_1 - b_1\left(x_1''\right)\right) &= \left(x_1 - x_1''\right) F_2\left(\hat{x}_2\left(x_1''\right)\right) + \int_{\underline{x}_1}^{x_1''} F_2\left(\hat{x}_2(x)\right) dx \\ &< \int_{\underline{x}_1}^{x_1} F_2\left(\hat{x}_2(x)\right) dx = F_2\left(\hat{x}_2(x_1)\right)\left(x_1 - b_1(x_1)\right), \end{aligned}$$

it is strictly better for buyer 1 to accept now at $b_1(x_1)$. The same argument applies to buyer 2. This shows that buyer i with valuation x_i accepting at price $b_i(x_i)$ constitutes an equilibrium.

Given that buyer i with valuation x_i accepts $b_i(x_i)$, if the two price offers, p_1 and p_2, satisfy the relationship $p_2(p_1) = b_2\left(\hat{x}_2\left(b_1^{-1}(p_1)\right)\right)$, then the type accepting p_1, $b_1^{-1}(p_1)$, and the type accepting p_2, $b_2^{-1}(p_2)$, follow the relationship $b_2^{-1}(p_2) = \hat{x}_2\left(b_1^{-1}(p_1)\right)$. Therefore, buyer 1 with valuation x_1 accepts earlier than buyer 2 with valuation x_2 if and only if $x_2 < \hat{x}_2(x_1)$. ■

Proof of Lemma 1: We show that the buyers adopt cutoff strategies in equilibrium, that is, a buyer with a higher valuation always accepts earlier. To see this, consider a continuation game starting with price pair (p_1', p_2'). Suppose that the belief is such that buyer 1's and buyer 2's values are in sets $S_1^{p_1'}$ and $S_2^{p_1'}$, respectively.[16] The buyers expect that the seller will offer two continuously decreasing price sequences in a relationship whereby $p_2(p_1)$ for $p_1 \leq p_1'$, and $S_1^{p_1}$ and $S_2^{p_1}$ are the updated beliefs regarding the sets of possible values of the buyers, conditional on that no one has accepted any price before the price pair reaches $(p_1, p_2(p_1))$ (note that $S_i^{p_1} \subset S_i^{p_1'}$).

Now suppose that with price pair (p_1', p_2'), buyer i with value x_i rejects the offer. This implies that buyer i with value x_i obtains a weakly higher payoff if he accepts later with some price pair (p_1'', p_2''), where $p_2'' = p_2(p_1'')$. That is,

$$x_i - p_i' \leq \frac{\Pr\left(x_j \in S_j^{p_1''}\right)}{\Pr\left(x_j \in S_j^{p_1'}\right)}\left(x_i - p_i''\right), \tag{3}$$

where $\frac{\Pr\left(x_j \in S_j^{p_1''}\right)}{\Pr\left(x_j \in S_j^{p_1'}\right)} \leq 1$ is the probability that the object is not taken by buyer j that is yet conditional on the hypothesis that buyer i waits until the price pair reaches (p_1'', p_2''). If

[16]The belief assigns positive support to all the elements in $S_1^{p_1'}$ and $S_2^{p_1'}$.

$\frac{\Pr\left(x_j \in S_j^{p_1''}\right)}{\Pr\left(x_j \in S_j^{p_1'}\right)} < 1$, for $x_i' < x_i$,

$$x_i' - p_i' < \frac{\Pr\left(x_j \in S_j^{p_1''}\right)}{\Pr\left(x_j \in S_j^{p_1'}\right)} \left(x_i' - p_i''\right),$$

so buyer i with value $x_i' < x_i$ will also reject the offer p_i'. If $\frac{\Pr\left(x_j \in S_j^{p_1''}\right)}{\Pr\left(x_j \in S_j^{p_1'}\right)} = 1$, for the case where $p_i'' < p_i'$, buyer i with value x_i' will certainly reject the offer p_i'; for the case where $p_i'' = p_i'$, without loss of generality, we assume that buyer i with value x_i' also rejects the offer p_i'. Therefore, buyer i with a higher value accepts earlier in equilibrium. ■

Proof of Lemma 2: Given that $w_1 > \underline{w}_1$ and $w_2 > \underline{w}_2$, if the seller terminates the negotiation, she obtains a payoff of 0; if he continues the game, there must exist a price path with which the seller's expected payoff is greater than 0, unless buyer 1 with valuation in $[\underline{w}_1, w_1]$ and buyer 2 with valuation in $[\underline{w}_2, w_2]$ only accept at price 0. If that is the case, given that buyer 2 accepts at price 0, buyer 1 must expect that buyer 2's price will be lowered to 0 later than buyer 1's price, so buyer 1 with a valuation close to w_1 is willing to wait and accept at 0. Likewise, buyer 2 must also expect that buyer 1's price will be lowered to 0 later than buyer 2's price. This is a contradiction. Therefore, the situation where the two buyers only accept at price 0 cannot arise in equilibrium, and there always exist price paths which generate a positive payoff for any continuation game. ■

Proof of Theorem 2: We prove that

$$x_{2,\omega}(x_1) = \begin{cases} \underline{w}_2 & \text{for } \underline{w}_1 \le x_1 < \max\{\underline{w}_1 \underline{w}_2\}, \\ x_1 & \text{for } \max\{\underline{w}_1, \underline{w}_2\} \le x_1 \le \min\{w_1, w_2\}, \\ w_2 & \text{for } \min\{w_1, w_2\} < x_1 \le w_1 \end{cases}$$

$$P_1(x_1, x_2) = x_1 - \frac{\int_{\underline{w}_1}^{x_1} F_2(x_{2,\omega}(x))\,dx}{F_2(x_2)},$$

$$P_2(x_2, x_1) = x_2 - \frac{\int_{\underline{w}_2}^{x_2} F_1(x_{1,\omega}(x))\,dx}{F_1(x_1)},$$

are a set of functions satisfying the equilibrium conditions.[17] Note that $x_{2,\omega}(\cdot)$ represents a forty-five degree line for $x_1 \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{w_1, w_2\}]$, which means that buyers with the same valuation in $[\max\{\underline{w}_1, \underline{w}_2\}, \min\{w_1, w_2\}]$ will accept at the same time, and thus implies that the buyer with the higher valuation will accept earlier so that the allocation is efficient and identical to that of the second-price auction.

Given $\{x_{2,\omega}(\cdot)\}_\omega$, $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$ are derived from (1) and (2), so we only need to show that given $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$, $\{x_{2,\omega}(\cdot)\}_\omega$ is the seller's best response. Without loss of generality, assume that $\underline{w}_1 \geq \underline{w}_2$. First notice that given $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$, the price accepted by buyer 2 with valuation $x \in [\underline{w}_2, \underline{w}_1)$ is lower than $\underline{w}_1$, the price accepted by buyer 1 with valuation $\underline{w}_1$; in addition, being paired with buyer 2 with valuation $x \in [\underline{w}_2, \underline{w}_1)$ will not raise the prices accepted by buyer 1.[18] Therefore, given any ω, a path $x_2(x_1)$ going below $\underline{w}_1$ (i.e., $x_2(x) < \underline{w}_1$ for some $x \in [\underline{w}_1, \overline{w}_1]$) is dominated and cannot occur in equilibrium.

Next we show that, given $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$, all the paths that stay above $\underline{w}_1$ yield the same expected payoff to the seller. Let (w_1, w_2) be the initial belief. First consider the case where $w_2 > w_1$ (the other case can be proved in a similar manner). Given $P_1(x_1, x_2)$ and $P_2(x_1, x_2)$, if the seller chooses a function $x_2(x_1)$ such that $x_2(x_1) \geq x_1$ for all $x_1 \in [\underline{w}_1, w_1]$,

[17] If $\underline{w}_2 < \underline{w}_1$, $x_{1,\omega}(x_2) = \underline{w}_1$ for $x_2 < \underline{w}_1$. If $w_2 > w_1$, $x_{1,\omega}(x_2) = w_1$ for $x_2 > w_1$. If $x_j = \underline{w}_j$, let $P_i(x_i, x_j) = x_i - \lim_{x_j \downarrow \underline{w}_j} \frac{\int_{\underline{w}_i}^{x_i} F_j(x_{j,\omega}(x))dx}{F_j(x_j)}$.

[18] Given any x_1, $P_1(x_1, x) \leq P_1(x_1, y)$ for $y \geq \underline{w}_1$.

i.e., $x_2(x_1)$ is above the forty-five degree line, then the expected payoff is

$$\begin{aligned}
&\int_{\underline{w}_1}^{w_1}\left[x_1F_2(x_2(x_1))-\int_{\underline{w}_1}^{x_1}F_2(x)\,dx\right]f_1(x_1)\,dx_1\\
&+\int_{\underline{w}_1}^{w_1}\left[x_1F_1(x_1)-\int_{\underline{w}_1}^{x_1}F_1(x)\,dx\right]f_2(x_2(x_1))\frac{dx_2(x_1)}{dx_1}dx_1\\
&+\left[w_1F_1(w_1)-\int_{\underline{w}_1}^{w_1}F_1(x)\,dx\right][F_2(w_2)-F_2(x_2(w_1))]\\
=&\int_{\underline{w}_1}^{w_1}\{x_1[F_2(x_2(x_1))-F_2(x_1)]+\underline{w}_1F_2(\underline{w}_1)\}\,dF_1(x_1)+\int_{\underline{w}_1}^{w_1}\int_{\underline{w}_1}^{x_1}xdF_2(x)\,dF_1(x_1)\\
&+\int_{\underline{w}_1}^{x_2(w_1)}\int_{\underline{w}_1}^{x_2^{-1}(x_2)}xdF_1(x)\,dF_2(x_2)\\
&+\int_{\underline{w}_1}^{w_1}xdF_1(x)\,[F_2(w_2)-F_2(x_2(w_1))]\\
=&\int_{\underline{w}_1}^{w_1}\int_{\underline{w}_1}^{x_1}xdF_2(x)\,dF_1(x_1)+\int_{\underline{w}_1}^{w_1}\{x_1[F_2(x_2(x_1))-F_2(x_1)]+\underline{w}_1F_2(\underline{w}_1)\}\,dF_1(x_1)\\
&+\int_{\underline{w}_1}^{w_1}x_1[F_2(x_2(w_1))-F_2(x_2(x_1))]\,dF_1(x_1)\\
&+\int_{\underline{w}_1}^{w_1}x_1[F_2(w_2)-F_2(x_2(w_1))]\,dF_1(x_1)\\
=&\int_{\underline{w}_1}^{w_1}\int_{\underline{w}_1}^{x_1}xdF_2(x)\,dF_1(x_1)+\int_{\underline{w}_1}^{w_1}\{x_1[F_2(w_2)-F_2(x_1)]+\underline{w}_1F_2(\underline{w}_1)\}\,dF_1(x_1),
\end{aligned}$$

which is independent of $x_2(x_1)$. The first equation comes from integration by parts: $\int_{\underline{w}_1}^{x_1}F_i(x)\,dx = F_i(x_1)\,x_1 - F_i(\underline{w}_1)\,\underline{w}_1 - \int_{\underline{w}_1}^{x_1}xdF_i(x)$. The second equation comes from changing the order of integration: $\int_{\underline{w}_1}^{x_2(w_1)}\int_{\underline{w}_1}^{x_2^{-1}(x_2)}xdF_1(x)\,dF_2(x_2) = \int_{\underline{w}_1}^{w_1}\int_{x_2(x)}^{x_2(w_1)}xdF_2(x_2)\,dF_1(x)$. Therefore, the seller obtains the same expected payoff for all $x_2(x_1)$ such that $x_2(x_1)\geq x_1$. Similarly, we can prove that the seller receives the same expected payoff for all $x_2(x_1)$ such that $x_2(x_1)\leq x_1$, and can further show that the seller obtains the same expected payoff for any path $x_2(x_1)$.[19] Therefore, $P_1(x_1,x_2)$, $P_2(x_2,x_1)$, and $\{x_{2,\omega}(x_1)\}_\omega$ constitute a Markov perfect equilibrium in which the buyer with the higher valuation obtains the object, and a buyer whose valuation is smaller than or equal to $\max\{\underline{w}_1,\underline{w}_2\}$ receives zero payoff. ■

[19]Note that although the seller receives the same payoff for all $x_2(x_1)$, this does not mean that all of those functions can be sustained in equilibrium. If the seller switches to another function $\tilde{x}_2(x_1)$, $P_1(\cdot)$ and $P_2(\cdot)$, which characterize the buyers' strategies, will change accordingly. Then $\tilde{x}_2(x_1)$ might no longer be optimal for the seller.

Proof of Theorem 3: We prove the uniqueness of the equilibrium under the two restrictions in Assumption 3. The first restriction is on the seller's side and states that, given ω, the path $x_{2,\omega}(x_1)$ is continuous, piecewise differentiable on $[\underline{w}_1, w_1]$, and strictly increasing on the interval $\left(\underline{x}_{1,\omega}, \overline{x}_{1,\omega}\right)$ where $\overline{x}_{1,\omega} \equiv \sup\{x_1 \in [\underline{w}_1, w_1] \mid x_{2,\omega}(x_1) < w_2\}$ and $\underline{x}_{1,\omega} \equiv \inf\{x_1 \in [\underline{w}_1, w_1] \mid x_{2,\omega}(x_1) > \underline{w}_2\}$.[20] Under this restriction, program (P1) can be rewritten as

$$\begin{aligned} x_{2,\omega}(x_1) \quad = \quad & \arg\max_{x_2(x_1)} \int_{\underline{w}_1}^{\overline{x}_1} P_1(x_1, x_2(x_1)) F_2(x_2(x_1)) f_1(x_1) \\ & \qquad + P_2(x_2(x_1), x_1) F_1(x_1) f_2(x_2(x_1)) u(x_1) dx_1 \\ & + \mathbf{1}_{(\overline{x}_1 < w_1)} P_1(\overline{x}_1, x_2(\overline{x}_1)) F_2(x_2(\overline{x}_1)) [F_1(w_1) - F_1(\overline{x}_1)] \\ & + \mathbf{1}_{(x_2(\overline{x}_1) < w_2)} P_2(x_2(\overline{x}_1), \overline{x}_1) F_1(\overline{x}_1) [F_2(w_2) - F_2(x_2(\overline{x}_1))] \qquad (P2) \\ s.t. \quad & \text{either } \overline{x}_1 = w_1 \text{ and } x_2(w_1) \le w_2, \text{ or } \overline{x}_1 \le w_1 \text{ and } x_2(\overline{x}_1) = w_2, \\ & x_2(\underline{w}_1) \ge \underline{w}_2, \\ & \frac{dx_2(x_1)}{dx_1} = u(x_1), \\ & 0 < \frac{dx_2(x_1)}{dx_1} < \infty \text{ for } x_1 \in (\underline{x}_1, \overline{x}_1), \end{aligned}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function and $\overline{x}_1 \equiv \sup\{x_1 \in [\underline{w}_1, w_1] \mid x_2(x_1) < w_2\}$. When $\overline{x}_1 = w_1$ and $x_2(w_1) < w_2$, it means that buyer 2 with valuation between $x_2(w_1)$ and w_2 obtains the object with the same probability, so in equilibrium all those types accept the same price $P_2(x_2(\overline{x}_1), \overline{x}_1)$; this event occurs with probability $F_1(\overline{x}_1)[F_2(w_2) - F_2(x_2(\overline{x}_1))]$ and yields the payoff shown in the forth term of the objective function. Similarly, when $\overline{x}_1 < w_1$ and $x_2(\overline{x}_1) = w_2$, buyer 1 with valuation between $\overline{x}_1$ and w_1 accepts price $P_1(\overline{x}_1, x_2(\overline{x}_1))$; this occurs with probability $F_2(x_2(\overline{x}_1))[F_1(w_1) - F_1(\overline{x}_1)]$ and yields the payoff shown in the third term.

We also need to verify that $P_1(x_1, x_2)$ and $P_2(x_2, x_1)$ satisfy (1) and (2), respectively. Since the expected payoffs of the lowest types of the two buyers are zero, by (1) and (2),

$$P_i(w_i, w_j) = w_i - \frac{C_i(w_1, w_2)}{F_j(w_j)}, \tag{4}$$

where

$$C_i(w_1, w_2) \equiv \int_{\underline{w}_i}^{w_i} F_j(x_{j,\omega}(x)) dx. \tag{5}$$

[20] This class of paths satisfies the equilibrium property described in Lemma 2.

Here, we impose an additional restriction on the buyers' side that $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$ are continuous, and the first derivatives of $C_1(x_1, x_2)$ and $C_2(x_1, x_2)$ with respect to x_1 and x_2 exist.[21]

To prove the uniqueness of the equilibrium, we first use Pontryagin's maximum principle to derive necessary conditions for the seller's equilibrium strategy. Along with (4), we show that given the strategy spaces described in Assumption 3, there is only one set of strategies and beliefs satisfying all the conditions and the requirements.

Consider the seller's optimal control problem in (P2). Define the initial value function as

$$I(\overline{x}_1, x_2(\overline{x}_1)) = \begin{cases} [\overline{x}_1 F_2(x_2(\overline{x}_1)) - C_1(\overline{x}_1, x_2(\overline{x}_1))][F_1(w_1) - F_1(\overline{x}_1)] & \text{if } \overline{x}_1 < w_1, x_2(\overline{x}_1) = w_2, \\ [x_2(\overline{x}_1) F_1(\overline{x}_1) - C_2(\overline{x}_1, x_2(\overline{x}_1))][F_2(w_2) - F_2(x_2(\overline{x}_1))] & \text{if } \overline{x}_1 = w_1, x_2(\overline{x}_1) < w_2, \end{cases}$$

and the Hamiltonian function H as

$$H(x_1, x_2, u, \lambda) = G(x_1, x_2, u) + \lambda g(x_1, x_2, u),$$

where

$$\begin{aligned} G(x_1, x_2, u) &= [x_1 F_2(x_2) - C_1(x_1, x_2)] f_1(x_1) + [x_2 F_1(x_1) - C_2(x_1, x_2)] f_2(x_2) u, \\ g(x_1, x_2, u) &= u. \end{aligned}$$

The following theorem is a restatement of Pontryagin's maximum principle:

Theorem 4 *If a control $u(\cdot)$ with a corresponding state trajectory $x_2(\cdot)$ is optimal, there exists an absolutely continuous function $\lambda : [\underline{w}_1, w_1] \mapsto \mathbb{R}$ such that the maximum condition*

$$H(x_1, x_2(x_1), u(x_1), \lambda(x_1)) = \max\{H(x_1, x_2(x_1), u, \lambda(x_1)) \mid 0 \leq u \leq \infty\},$$

the adjoint equation

$$\lambda'(x_1) = -\frac{\partial H(x_1, x_2(x_1), u(x_1), \lambda(x_1))}{\partial x_2},$$

[21]The optimal control theory only requires that the integrand of the objective function in program (P2) be continuous and that the first derivative of the integrand with respect to x_2 exist. However, in our problem, instead of solving for $x_{2,\omega}(x_1)$, we can change the formulation to solve for $x_{1,\omega}(x_2)$. Therefore, we require that the first derivatives with respect to both x_1 and x_2 exist.

and the transversality conditions

$$\begin{aligned}
& \lambda\left(\underline{w}_1\right) \geq 0; \\
& x_2\left(\underline{w}_1\right) \geq \underline{w}_2; \\
& \left(x_2\left(\underline{w}_1\right) - \underline{w}_2\right) \lambda\left(\underline{w}_1\right) = 0; \\
\textit{if } \overline{x}_1 \;=\; & w_1 \textit{ and } x_2\left(w_1\right) < w_2,\ \lambda\left(w_1\right) = \frac{\partial I}{\partial x_2}; \qquad (6) \\
\textit{if } \overline{x}_1 \;<\; & w_1 \textit{ and } x_2\left(\overline{x}_1\right) = w_2,\ H\left(\overline{x}_1\right) + \frac{\partial I}{\partial \overline{x}_1} = 0. \qquad (7)
\end{aligned}$$

are satisfied.

For later convenience, let $c_1\left(x_1, x_2\right) = \left(\overline{w}_2 - \underline{w}_2\right) C_1\left(x_1, x_2\right)$ and $c_2\left(x_1, x_2\right) = \left(\overline{w}_1 - \underline{w}_1\right) C_2\left(x_1, x_2\right)$. Given this result, we now prove the theorem through a series of lemmas, from lemma 3 to 10.

Lemma 3 $c_1\left(x_1, x_2\right) = \left(x_1 - \underline{w}_1\right)\left(x_2 - \underline{w}_2\right) - c_2\left(x_1, x_2\right).$

Proof: By (5),

$$C_1\left(w_1, w_2\right) = \int_{\underline{w}_1}^{\overline{x}_{1,\omega}} F_2\left(x_{2,\omega}\left(x\right)\right) dx + F_2\left(x_{2,\omega}\left(\overline{x}_{1,\omega}\right)\right)\left[w_1 - \overline{x}_{1,\omega}\right],$$

and

$$C_2\left(w_1, w_2\right) = \int_{\underline{w}_2}^{x_{2,\omega}\left(\overline{x}_{1,\omega}\right)} F_1\left(x_{1,\omega}\left(x\right)\right) dx + F_1\left(\overline{x}_{1,\omega}\right)\left[w_2 - x_{2,\omega}\left(\overline{x}_{1,\omega}\right)\right].$$

Multiplying both sides of the two equations by $\left(\overline{w}_2 - \underline{w}_2\right)$ and $\left(\overline{w}_1 - \underline{w}_1\right)$ respectively, we obtain

$$\begin{aligned}
c_1\left(w_1, w_2\right) &= \int_{\underline{w}_1}^{\overline{x}_{1,\omega}} \left(x_{2,\omega}\left(x\right) - \underline{w}_2\right) dx + \left(x_{2,\omega}\left(\overline{x}_{1,\omega}\right) - \underline{w}_2\right)\left[w_1 - \overline{x}_{1,\omega}\right], \qquad (8) \\
c_2\left(w_1, w_2\right) &= \int_{\underline{w}_2}^{x_{2,\omega}\left(\overline{x}_{1,\omega}\right)} \left(x_{1,\omega}\left(x\right) - \underline{w}_1\right) dx + \left(\overline{x}_{1,\omega} - \underline{w}_1\right)\left[w_2 - x_{2,\omega}\left(\overline{x}_{1,\omega}\right)\right]. \qquad (9)
\end{aligned}$$

Note that either $\overline{x}_{1,\omega} = w_1$ or $x_{2,\omega}\left(\overline{x}_{1,\omega}\right) = w_2$. Therefore, $c_1\left(w_1, w_2\right) + c_2\left(w_1, w_2\right) = \left(w_1 - \underline{w}_1\right)\left(w_2 - \underline{w}_2\right)$. ■

Lemma 4 $c_i\left(\underline{w}_1, x_2\right) = 0$ *and* $c_i\left(x_1, \underline{w}_2\right) = 0.$

Proof: By Lemma 3, $c_1(x_1, \underline{w}_2) + c_2(x_1, \underline{w}_2) = 0$. Since $c_i(\cdot) \geq 0$, $c_1(x_1, \underline{w}_2) = 0$ and $c_2(x_1, \underline{w}_2) = 0$. Similarly, $c_1(\underline{w}_1, x_2) = 0$ and $c_2(\underline{w}_1, x_2) = 0$. ■

Lemma 5 *Let $x_{2,\omega}(x_1)$ be the equilibrium path of the continuation game starting with belief ω. Then*

$$(x_1 - \underline{w}_1)\underline{w}_2 + c_1(x_1, x_{2,\omega}(x_1)) - \int_{\underline{w}_1}^{x_1} x - \frac{\partial c_1(x, x_{2,\omega}(x))}{\partial x_2} + \frac{\partial c_1(x, x_{2,\omega}(x))}{\partial x_2}\frac{dx_{2,\omega}(x)}{dx_1}dx = 0, \tag{10}$$

for all $x_1 \in [\underline{w}_1, \overline{x}_{1,\omega}]$.

Proof: By the maximum condition,

$$u \in (0,\infty) \text{ if } \frac{\partial H}{\partial u} = [x_2 F_1(x_1) - C_2(x_1, x_2)] f_2(x_2) + \lambda = 0. \tag{11}$$

By the adjoint equation,

$$\lambda'(x_1) = -\left\{\left[x_1 f_2(x_{2,\omega}(x_1)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x_1) + \left[F_1(x_1) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_{2,\omega}(x_1))\, u\right\},$$

so $\lambda(x_1) = \lambda(\underline{w}_1) - \int_{\underline{w}_1}^{x_1}\left[x f_2(x_{2,\omega}(x)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_{2,\omega}(x))\, u(x)\, dx$. Plugging into equation (11),

$$\begin{aligned}&[x_{2,\omega}(x_1) F_1(x_1) - C_2(x_1, x_{2,\omega}(x_1))] f_2(x_{2,\omega}(x_1)) + \lambda(\underline{w}_1) \\ - &\int_{\underline{w}_1}^{x_1}\left[x f_2(x_{2,\omega}(x)) - \frac{\partial C_1}{\partial x_2}\right] f_1(x) + \left[F_1(x) - \frac{\partial C_2}{\partial x_2}\right] f_2(x_{2,\omega}(x))\, u(x)\, dx = 0,\end{aligned}$$

which implies that $\lambda(\underline{w}_1) = 0$.[22] Multiplying both sides by $(\overline{w}_1 - \underline{w}_1)(\overline{w}_2 - \underline{w}_2) = \frac{1}{f_1(x_1) f_2(x_{2,\omega}(x_1))}$,

$$[(x_1 - \underline{w}_1) x_{2,\omega}(x_1) - c_2(x_1, x_{2,\omega}(x_1))] - \int_{\underline{w}_1}^{x_1}\left[x - \frac{\partial c_1}{\partial x_2}\right] + \left[(x - \underline{w}_1) - \frac{\partial c_2}{\partial x_2}\right] u(x)\, dx = 0.$$

By Lemma 3, we obtain $(x_1 - \underline{w}_1)\underline{w}_2 + c_1(x_1, x_{2,\omega}(x_1)) - \int_{\underline{w}_1}^{x_1} x_1 - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2}\frac{dx_{2,\omega}}{dx_1}dx_1 = 0$ for all $x_1 \in [\underline{w}_1, \overline{x}_{1,\omega}]$. ■

Lemma 6 *Let $x_{2,\omega}(x_1)$ be the equilibrium path of the continuation game starting with belief ω. Then,*

$$\frac{\partial c_1(x_1, x_{2,\omega}(x_1))}{\partial x_1} + \frac{\partial c_1(x_1, x_{2,\omega}(x_1))}{\partial x_2} = x_1 - \underline{w}_2, \tag{12}$$

for all $x_1 \in [\underline{w}_1, \overline{x}_{1,\omega}]$.

[22] When $x_1 = \underline{w}_1$, the equation holds only if $\lambda(\underline{w}_1) = 0$.

Proof: Since equation (10) holds for all $x_1 \in [\underline{w}_1, \overline{x}_{1,\omega}]$, by taking derivatives with respect to x_1 on both sides, we obtain

$$\frac{\partial c_1}{\partial x_1} + \frac{\partial c_1}{\partial x_2} u = (x_1 - \underline{w}_2) - \frac{\partial c_1}{\partial x_2} + \frac{\partial c_1}{\partial x_2} u.$$

Therefore, $\frac{\partial c_1(x_1, x_{2,\omega}(x_1))}{\partial x_1} + \frac{\partial c_1(x_1, x_{2,\omega}(x_1))}{\partial x_2} = x_1 - \underline{w}_2$. ■

Let $A = \{(x_1, x_{2,\omega}(x_1)) \mid x_1 \in [\underline{w}_1, \overline{x}_{1,\omega}], \underline{w}_1 \leq w_1 \leq \overline{w}_1, \underline{w}_2 \leq w_2 \leq \overline{w}_2\}$, and $\overline{A}$ is the closure of A. A contains all the points on the equilibrium paths of all the continuation games. Note that given $\omega = (x, x)$ where $x \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{\overline{w}_1, \overline{w}_2\}]$,

$$x_{2,\omega}(x_1) = \begin{cases} \underline{w}_2, & \text{for } x_1 \in [\underline{w}_1, \max\{\underline{w}_1, \underline{w}_2\}), \\ x_1, & \text{for } \max\{\underline{w}_1, \underline{w}_2\} \leq x_1 \leq \min\{w_1, w_2\}, \end{cases}$$

so $(x, x) \in A$ for $x \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{\overline{w}_1, \overline{w}_2\}]$.

Lemma 7 *For all $(x_1, x_2) \in \overline{A}$, $c_1(x_1, x_2)$ is of the form $\frac{(x_1 - \underline{w}_2)^2}{2} + \phi(x_2 - x_1)$.*

Proof: If (x_1, x_2) is on the equilibrium path of a continuation game so that it is in A, then the differential equation (12) has to hold. The general solution to the differential equation is $c_1(x_1, x_2) = \frac{(x_1 - \underline{w}_2)^2}{2} + \phi(x_2 - x_1)$. ■

Let $\overline{d} = \max\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$ and $\underline{d} = \min\left\{x_2 - x_1 \mid (x_1, x_2) \in \overline{A}\right\}$.

Lemma 8 $c_1(x_1, x_2) = \frac{(x_1 - \underline{w}_2)^2}{2} - \frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2}$ *for $x_2 \geq x_1$.*

Proof: For $(x_1, x_2) \in \overline{A}$, $c_1(x_1, x_2) = \frac{(x_1 - \underline{w}_2)^2}{2} + \phi(x_2 - x_1)$ by Lemma 7. Suppose that $x_2 \geq x_1$. We first show that for $(x_1, x_2) \in \overline{A}$, $\phi(x_2 - x_1) = -\frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2}$, and then show that for $(x_1, x_2) \notin \overline{A}$, $\phi(x_2 - x_1) = -\frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2}$ as well. Let $UL(x_2) = \min\left\{x_1 \mid (x_1, x_2) \in \overline{A}\right\}$ for $x_2 \in [\underline{w}_2, \overline{w}_2]$. Let $\underline{x} = UL(\max\{\underline{w}_1, \underline{w}_2\})$, $\underline{y} = \inf\{x_2 \in [\underline{w}_2, \overline{w}_2] \mid UL(x_2) > \underline{w}_1\}$,[23] and $\overline{y} = \sup\{x_2 \in [\underline{w}_2, \overline{w}_2] \mid UL(x_2) < \overline{w}_1\}$. $UL(x_2)$ is strictly increasing on $\left[\underline{y}, \overline{y}\right]$,[24] but might not be continuous as shown in Figure 3. For $x_2 \in \left(\max\{\underline{w}_1, \underline{w}_2\}, \underline{y}\right)$, $(\underline{w}_1, x_2) \in \overline{A}$ and $c_1(\underline{w}_1, x_2) = 0$; for $x_1 \in (\underline{x}, \max\{\underline{w}_1, \underline{w}_2\})$,

[23] Note that either $\underline{x} = \underline{w}_1$ or $\underline{y} = \underline{w}_2$ by Assumption 3 (i).

[24] If $UL(\cdot)$ is not strictly increasing, then there exist a and b such that $a > b$ but $UL(a) \leq UL(b)$. However, this implies that for some path $x_{2,\omega}(\cdot)$ passing through $(a, UL(a))$, $x_{2,\omega}(\cdot)$ is not strictly increasing, and cannot be the solution to program (P2).

$(x_1,\underline{w}_2)\in\overline{A}$ and $c_1(x_1,\underline{w}_2)=0$; given $c_1(x_1,x_2)=\frac{(x_1-\underline{w}_2)^2}{2}+\phi(x_2-x_1)$ for $(x_1,x_2)\in\overline{A}$, it is implied that $\phi(d)=-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ for $d\in\left[0,\underline{y}-\underline{x}\right]$. For any $d\in\left(\underline{y}-\underline{x},\overline{d}\right)$, there exists $a\in\left(\underline{y},\overline{w}_2\right)$ such that $a-UL(a)=d$. Since UL is increasing, for any $a_2\in(a,\overline{w}_2]$, $(UL(a),a_2)\notin\overline{A}$. Thus, given belief $w_1=UL(a)$ and $w_2=a_2$, $x_{2,\omega}(UL(a))=a$. By (8), $c_1(UL(a),a_2)=c_1(UL(a),a)$ for all $a_2\in(a,w_2]$, so $\frac{\partial c_1}{\partial x_2}(UL(a),a)=0$. Since $(UL(a),a)\in\overline{A}$, we know $c_1(x_1,x_2)$ is of the form $\frac{(x_1-\underline{w}_2)^2}{2}+\phi(x_2-x_1)$, and hence $\frac{\partial c_1}{\partial x_2}(UL(a),a)=\phi'(a-UL(a))=\phi'(d)=0$. Thus, for $d\in\left(\underline{y}-\underline{x},\overline{d}\right)$, $\phi'(d)=0$. Since $\phi(d)=-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ for $d\in\left[0,\underline{y}-\underline{x}\right]$ and $\phi'(d)=0$ for $d\in\left(\underline{y}-\underline{x},\overline{d}\right)$, $\phi(d)=-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ for $d\in\left[0,\overline{d}\right]$.

[Figure 3 about here]

From the above discussion, we know that, for any $(x_1,x_2)\in\overline{A}$, if $x_2-x_1\in\left[0,\overline{d}\right]$, $c_1(x_1,x_2)=\frac{(x_1-\underline{w}_2)^2}{2}-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$. However, by (8), this implies that $x_{2,\omega}(x_1)=x_1$ for $x_1\in[\max\{\underline{w}_1,\underline{w}_2\},\min\{\overline{w}_1,\overline{w}_2\}]$, so $UL(x_2)=x_2$ for $x_2\in[\max\{\underline{w}_1,\underline{w}_2\},\min\{\overline{w}_1,\overline{w}_2\}]$. Given $UL(x_2)=x_2$, by (8), $c_1(x_1,x_2)=\frac{(x_1-\underline{w}_2)^2}{2}-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ for $x_1\in[\max\{\underline{w}_1,\underline{w}_2\},\min\{\overline{w}_1,\overline{w}_2\}]$ and $x_2\in[x_1,\overline{w}_2]$. ■

Lemma 9 $c_2(x_1,x_2)=\frac{(x_2-\underline{w}_1)^2}{2}-\frac{(\max\{\underline{w}_2-\underline{w}_1,0\})^2}{2}$ *for* $x_1\geq x_2$.

Proof: For $(x_1,x_2)\in\overline{A}$, $c_2(x_1,x_2)=(x_1-\underline{w}_1)(x_2-\underline{w}_2)-\frac{(x_1-\underline{w}_2)^2}{2}-\phi(x_2-x_1)$ by Lemmas 3 and 7. Suppose that $x_1\geq x_2$. We first show that for $(x_1,x_2)\in\overline{A}$, $\phi(x_2-x_1)=-\frac{(x_2-x_1)^2}{2}-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$, and then show that for $(x_1,x_2)\notin\overline{A}$, $\phi(x_2-x_1)=-\frac{(x_2-x_1)^2}{2}-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ as well. Let $LR(x_1)=\min\left\{x_2\mid(x_1,x_2)\in\overline{A}\right\}$ for $x_1\in[\underline{w}_1,\overline{w}_1]$. Let $\underline{y}=LR(\max\{\underline{w}_1,\underline{w}_2\})$, $\underline{x}=\inf\{x_1\in[\underline{w}_1,\overline{w}_1]\mid LR(x_1)>\underline{w}_2\}$,[25] and $\overline{x}=\sup\{x_1\in[\underline{w}_1,\overline{w}_1]\mid LR(x_1)<\overline{w}_2\}$. $LR(x_1)$ is strictly increasing on $[\underline{x},\overline{x}]$, but might not be continuous as shown in Figure 4. For $x_1\in(\max\{\underline{w}_1,\underline{w}_2\},\underline{x})$, $(x_1,\underline{w}_2)\in\overline{A}$ and $c_2(x_1,\underline{w}_2)=0$; for $x_2\in\left(\underline{y},\max\{\underline{w}_1,\underline{w}_2\}\right)$, $(\underline{w}_1,x_2)\in\overline{A}$ and $c_2(\underline{w}_1,x_2)=0$; given $c_2(x_1,x_2)=(x_1-\underline{w}_1)(x_2-\underline{w}_2)-\frac{(x_1-\underline{w}_2)^2}{2}-\phi(x_2-x_1)$ for $(x_1,x_2)\in\overline{A}$, it is implied that $\phi(d)=-\frac{d^2}{2}-\frac{(\max\{\underline{w}_1-\underline{w}_2,0\})^2}{2}$ for $d\in\left[\underline{y}-\underline{x},0\right]$. For any $d\in\left(\underline{d},\underline{y}-\underline{x}\right)$, there exists $a\in(\underline{x},\overline{w}_1)$ such that $LR(a)-a=d$. Since LR is increasing, for any $a_1\in(a,\overline{w}_1]$,

[25] Note that either $\underline{x}=\underline{w}_1$ or $\underline{y}=\underline{w}_2$ by Assumption 3 (i).

$(a_1, LR(a)) \notin \overline{A}$. Thus, given belief $w_1 = a_1$ and $w_2 = LR(a)$, $x_{1,\omega}(LR(a)) = a$. By (9), $c_2(a_1, LR(a)) = c_2(a, LR(a))$ for all $a_1 \in (a, \overline{w}_1]$, so $\frac{\partial c_2}{\partial x_1}(a, LR(a)) = 0$. Since $(a, LR(a)) \in \overline{A}$, we know $c_2(x_1, x_2)$ is of the form $(x_1 - \underline{w}_1)(x_2 - \underline{w}_2) - \frac{(x_1 - \underline{w}_2)^2}{2} - \phi(x_2 - x_1)$, and hence $\frac{\partial c_2}{\partial x_1}(a, LR(a)) = LR(a) - a + \phi'(LR(a) - a) = d + \phi'(d) = 0$. Thus, for $d \in (\underline{d}, \underline{y} - \underline{x})$, $\phi'(d) = -d$. Since $\phi(d) = -\frac{d^2}{2} - \frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2}$ for $d \in [\underline{y} - \underline{x}, 0]$ and $\phi'(d) = -d$ for $d \in (\underline{d}, \underline{y} - \underline{x})$, $\phi(d) = -\frac{d^2}{2} - \frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2}$ for $d \in [\underline{d}, 0]$.

[Figure 4 about here]

From the above discussion, we know that, for any $(x_1, x_2) \in \overline{A}$, if $x_2 - x_1 \in [\underline{d}, 0]$, $c_2(x_1, x_2) = \frac{(x_2 - \underline{w}_1)^2}{2} - \frac{(\max\{\underline{w}_2 - \underline{w}_1, 0\})^2}{2}$. However, by (9), this implies that $x_{1,\omega}(x_2) = x_2$ for $x_2 \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{\overline{w}_1, \overline{w}_2\}]$, so $LR(x_1) = x_1$ for $x_1 \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{\overline{w}_1, \overline{w}_2\}]$. Given $LR(x_1) = x_1$, by (9), $c_2(x_1, x_2) = \frac{(x_2 - \underline{w}_1)^2}{2} - \frac{(\max\{\underline{w}_2 - \underline{w}_1, 0\})^2}{2}$ for $x_2 \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{\overline{w}_1, \overline{w}_2\}]$ and $x_1 \in [x_2, \overline{w}_1]$. ■

Finally, the following lemma completes the proof.

Lemma 10 *In equilibrium,*

$$C_1(x_1, x_2) = \begin{cases} \frac{1}{(\overline{w}_2 - \underline{w}_2)} \left[\frac{(x_1 - \underline{w}_2)^2}{2} - \frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2} \right] & \text{if } x_1 \leq x_2, \\ \frac{1}{(\overline{w}_2 - \underline{w}_2)} \left[(x_1 - \underline{w}_1)(x_2 - \underline{w}_2) - (\overline{w}_1 - \underline{w}_1) C_2(x_1, x_2) \right] & \text{if } x_1 \geq x_2, \end{cases}$$

$$C_2(x_1, x_2) = \begin{cases} \frac{1}{(\overline{w}_1 - \underline{w}_1)} \left[(x_1 - \underline{w}_1)(x_2 - \underline{w}_2) - (\overline{w}_2 - \underline{w}_2) C_1(x_1, x_2) \right] & \text{if } x_1 \leq x_2, \\ \frac{1}{(\overline{w}_1 - \underline{w}_1)} \left[\frac{(x_2 - \underline{w}_1)^2}{2} - \frac{(\max\{\underline{w}_2 - \underline{w}_1, 0\})^2}{2} \right] & \text{if } x_1 \geq x_2, \end{cases}$$

$$A = \{(x_1, x_2) \mid x_1 = x_2, \max\{\underline{w}_1, \underline{w}_2\} \leq x_1 \leq \min\{\overline{w}_1, \overline{w}_2\}\}.$$

Proof: From Lemmas 8, 9, and the discussion in their proofs, we know that under Assumption 3, the only possible c_1 is that

$$c_1(x_1, x_2) = \begin{cases} \frac{(x_1 - \underline{w}_2)^2}{2} - \frac{(\max\{\underline{w}_1 - \underline{w}_2, 0\})^2}{2} & \text{if } x_1 \leq x_2, \\ (x_1 - \underline{w}_1)(x_2 - \underline{w}_2) - \left[\frac{(x_2 - \underline{w}_1)^2}{2} - \frac{(\max\{\underline{w}_2 - \underline{w}_1, 0\})^2}{2} \right] & \text{if } x_1 \geq x_2, \end{cases}$$

which implies that the seller's strategy is

$$x_{2,\omega}(x_1) = \begin{cases} \underline{w}_2 & \text{if } x_1 \in [\underline{w}_1, \max\{\underline{w}_1, \underline{w}_2\}), \\ x_1 & \text{if } x_1 \in [\max\{\underline{w}_1, \underline{w}_2\}, \min\{w_1, w_2\}], \\ w_2 & \text{if } x_1 \in (\min\{w_1, w_2\}, w_1], \end{cases}$$

given the belief ω about the buyers' values at the beginning of a continuation game. ■

Appendix B: the derivation of the price path

With solution $x_{2,\omega}(x_1)$, let

$$Gr(x_{2,\omega}) = \left\{(x_1, x_2) \in [\underline{w}_1, w_1] \times [\underline{w}_2, w_2] \mid x_2 \in \left[\lim_{x \to x_1 -} x_{2,\omega}(x), \lim_{x \to x_1 +} x_{2,\omega}(x)\right]\right\}$$

be the graph of $x_{2,\omega}(x_1)$ with lines connecting $(x_1, \lim_{x \to x_1 -} x_{2,\omega}(x))$ and $(x_1, \lim_{x \to x_1 +} x_{2,\omega}(x))$ when $x_{2,\omega}(\cdot)$ is discontinuous at x_1. Then

$$P_\omega \equiv \{(P_1(x_1, x_2), P_2(x_2, x_1)) \mid (x_1, x_2) \in Gr(x_{2,\omega})\}$$

is the price path submitted by the seller.

Figures

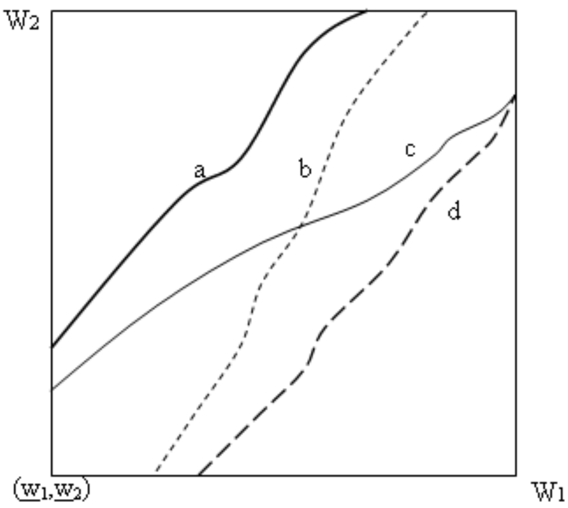

Figure 1: some examples which satisfy Assumption 3

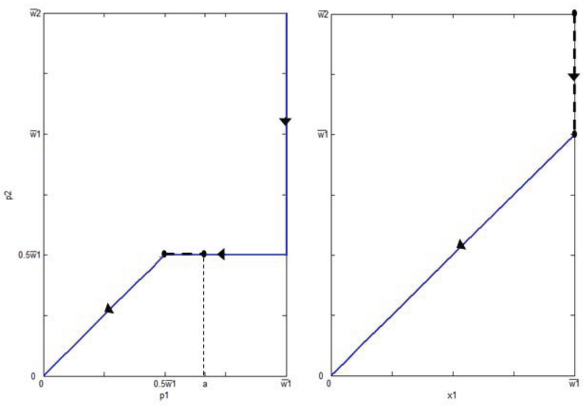

Figure 2: the actual price path and the cutoff type

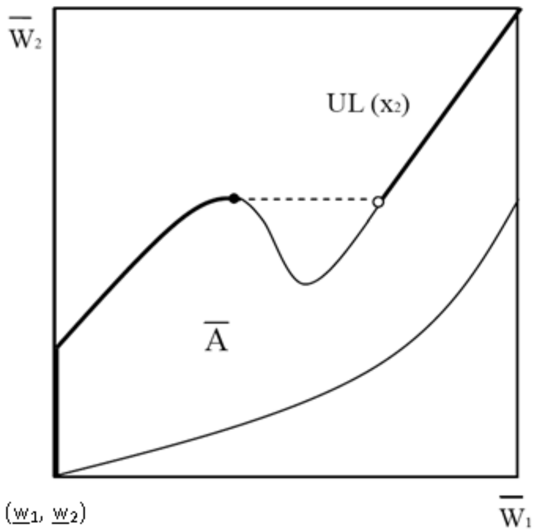

Figure 3

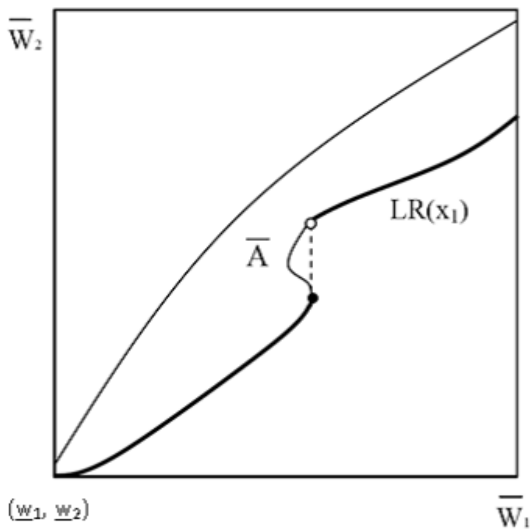

Figure 4