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MULTISTAGE INFORMATION TRANSMISSION WITH VOLUNTARY MONETARY TRANSFER

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Multistage Information Transmission with Voluntary Monetary Transfer*

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Abstract

I analyze a cheap talk model in which an informed sender and an uninformed receiver engage in finite-period communication before the receiver chooses a project. During the communication phase, the sender can gradually convey information through multistage cheap talk communication and the receiver can pay money to the sender voluntarily whenever she receives a message. My results show that under some conditions, (i) the receiver can extract more detailed information from the sender than that in the model of one-shot cheap talk communication and (ii) there exists an equilibrium whose outcome Pareto-dominates all the equilibrium outcomes in the model of one-shot cheap talk communication. Moreover, I find an upper bound of the receiver's equilibrium payoff and provide a sufficient condition for it to be approximated by the receiver's payoff under a certain equilibrium. This result shows that multistage information transmission with voluntary monetary transfer can be more beneficial for the receiver than a wide class of other communication protocols (e.g., mediation and arbitration).

JEL Classification: C72; C73; D82; D83
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1. Introduction

A lack of information typically leads to inefficient decisions. Therefore, in many economic situations, decision makers need to gather the relevant information before making their decisions. One canonical way of gathering information is consulting informed experts. For example, CEOs consult management consultants; politicians seek advice from strategic planners; and law enforcement officers hire informants. In the above-mentioned examples, the individuals who supply information are often paid for doing so.

Contract theory indicates that a properly designed contract containing information-contingent payments helps the decision maker to screen the information possessed by the informed expert. However, if information is transmitted through ordinary and informal talk, or equivalently, through "cheap talk," contractibility does not always exist. In such situations, the decision maker cannot commit to information-contingent payments. Hence, it seems that allowing the decision maker to make "voluntary" payments does not affect information transmission. Nevertheless, the information transmitted via cheap talk is often bought and sold without signing a contract.

Can voluntary payments by the decision maker facilitate cheap talk communication? If they can, how should the decision maker pay for cheap talk messages? To address this question, I enrich the canonical cheap talk model originally provided by Crawford and Sobel (1982) (hereafter, CS). Specifically, I analyze a sender–receiver game in which an informed expert (sender or he) and an uninformed decision maker (receiver or she) engage in finite-period communication. During the communication phase, in each period, the sender sends a cheap talk message to the receiver, and then the receiver pays money to the sender voluntarily. Once the communication phase is over, the receiver chooses a project.

In the CS model, the project choice and underlying asymmetric information are one-dimensional. Moreover, the sender's most desirable project is always higher than that of the receiver to a certain degree. Hence, the sender has an incentive to cheat the receiver into choosing a higher project than the receiver's most profitable one. This fact prevents detailed information transmission. By contrast, if the receiver can make message-contingent payments, by paying more money for the messages inducing the lower projects, the receiver can weaken the sender's exaggeration incentive. However, when the information transmission is one shot, the receiver never pays since making payments after receiving a message is a waste of money. In the present study, I consider a scenario in which information is conveyed in a gradual fashion and show that by combining multistage information transmission with the receiver's voluntary payments, a message-contingent payment scheme can be self-enforcing.[1] As a result, information transmission can be improved even in situations in which there is no contractibility.

[1]Without monetary transfer, allowing multiple rounds of unilateral (one-sided) communication in the CS model does not affect the set of equilibria identified by the original model. Krishna and Morgan (2004) show that allowing multiple rounds of "bilateral" (face-to-face or two-sided) communication in the CS model leads to Pareto improvements.

I find that under some conditions (i) the receiver can obtain more detailed information from the sender than in the CS model[2] and (ii) an equilibrium whose outcome Pareto-dominates all the equilibrium outcomes in the CS model can exist.[3] I also show that no fully separating equilibrium exists in my model. This result implies that information transmission is still limited even in my communication procedure. By considering the well-known uniform-quadratic model, i.e., with quadratic preferences regarding the project and a uniform type distribution, I find an upper bound of the receiver's equilibrium payoff and provide a sufficient condition for it to be approximated by the receiver's payoff under an equilibrium.

To demonstrate the benefit of multistage information transmission with voluntary transfer payments, I construct an interval partition equilibrium in which information about the state of the world is conveyed in order from the right-most interval on the state space. Specifically, in the first period, if the sender sends a message that means that the true state belongs to the right-most interval, the receiver will neither pay money nor obtain additional information in the future. Otherwise, the receiver pays a certain amount of money to the sender. After this payment, in the second period, the sender conveys whether the true state belongs to the second right-most interval that is the neighbor to the left of the first one. If the receiver learns that the true state belongs to the second right-most interval, she will neither pay money nor obtain additional information in the future. Otherwise, the receiver pays money to the sender and then the sender conveys additional information in the next period. This information elicitation is repeated in the communication phase. If the receiver deviates in terms of payment in a period, the sender conveys no information thereafter. Once communication is over, the receiver chooses her best project based on the information she has.

The logic underlying this equilibrium is as follows. First, under the information elicitation explained above, the receiver pays money to the sender whenever the information opposite to the sender's bias is conveyed. As a result, the receiver makes message-contingent payments on the equilibrium path: a higher payment for information inducing a lower project. As noted earlier, this payment scheme weakens the sender's exaggeration incentive. Second, since the sender can gradually convey his information, he can punish the receiver for not paying by babbling. Thus, the receiver makes a payment in the current period to prevent the sender's babbling in the future. Roughly speaking, similar to Benoit and Krishna (1985), the dependence of the selection of the future equilibrium on players' past behavior constructs punishments for their deviation. This fact enables the receiver to make message-contingent payments

[2]This result means that there exists an equilibrium whose partition has a greater number of elements than that achieved in any equilibrium in the CS model.

[3]In my model, there always exists an equilibrium in which the receiver never pays money to the sender. For instance, irrespective of the number of periods in the communication phase, there exists an equilibrium in which the sender sends an informative message to the receiver only in the first period and the receiver never pays. The equilibrium partition achieved in such an equilibrium is achievable in the CS model. Obviously, players waste time on pointless communication; in other words, the receiver does not use long-term communication effectively. Therefore, by constructing equilibria inducing Pareto improvements, I show the benefit of multistage information transmission.

to some extent during the communication phase.

The model I describe is potentially applicable for studying the effective use of informants. The Federal Bureau of Investigation (FBI) mentions that the "use of informants to assist in the investigation of criminal activity may involve an element of deception, ... or cooperation with persons whose reliability and motivation may be open to question."[4] This statement suggests that informants are often biased and that their information might neither be credible nor certifiable. Alemany (2002) indicates that cooperation agreements between the Drug Enforcement Agency (DEA) and informants are often silent with respect to the compensation of the latter. This fact implies that the parties may not always be able to sign a contract containing information-contingent payments. Indeed, there are numerous cases of oral promises made by DEA agents to informants subsequently being broken.[5] The present study shows that by using multistage information elicitation and voluntary transfer payments, information transmission can be improved even in situations in which there is no contractibility.

My results have important implications for the theory of organizational economics regarding designing communication protocols and organizational structures. I show that multistage information transmission with voluntary transfer payments can be more beneficial for the receiver than a wide range of other communication protocols. It is well known that information transmission can be improved when more general communication protocols (i.e., noisy communication) are available.[6] By considering a mediation model under the uniform-quadratic assumption,[7] Goltsman et al. (2009) characterize the optimal level of noise in the communication. I compare my communication procedure with the optimal mediation that maximizes the receiver's ex ante expected payoff and show that under some conditions, the receiver prefers the former to the latter.

Dessein (2002) studies a simple delegation problem[8] and establishes the remarkable result that the receiver prefers full delegation to communication as long as the incentive conflict is not too large. Since the work of Dessein (2002), designing "who decides what" has been extensively studied. Many works investigate general settings in which the parties can commit to an information-contingent decision rule.[9] Under the uniform-

[4]FBI, Frequently Asked Questions, "What is the FBI's policy on the use of informants?" (https://www.fbi.gov/about/faqs/what-is-the-fbis-policy-on-the-use-of-informants).

[5]For details, see Alemany (2002).

[6]Many studies highlight that noisy communication leads to improved information transmission (e.g., Krishna and Morgan, 2004; Blume et al., 2007; Goltsman et al., 2009; Ivanov, 2010; and Ambrus et al., 2013). Goltsman et al. (2009) characterize the optimal mediation mechanism that controls the noise in communication. Blume et al. (2007) and Krishna and Morgan (2004) show that the optimal mediation mechanism can be implemented under some communication protocols without monetary transfer.

[7]Under mediation analyzed by Goltsman et al. (2009), a neutral third party (mediator) asks the sender for information and advises the receiver who chooses a project.

[8]The receiver chooses whether to communicate with the sender. She decides herself after cheap talk communication or fully delegates the decision-making authority to the sender.

[9]One simple decision rule for the receiver is to delegate authority to the sender, but possibly to constrain the set of available decisions. This class of mechanisms (analyzed by Holmström, 1977; Melumad and Shibano, 1991; and Alonso and Matouschek, 2008) is called *delegation mechanism*. Goltsman et al. (2009) show that the optimal arbitration mechanism is deterministic as a consequence and that the optimal arbitration includes the optimal delegation mechanism.

quadratic assumption, Goltsman et al. (2009) characterize an optimal information-contingent decision rule, the *optimal arbitration*.[10] Although, under arbitration, players benefit from a "formal contract" that forces them to commit to the predetermined decision rule, surprisingly, my results show that the receiver can obtain a higher ex ante expected payoff in my communication procedure than under the optimal arbitration.

Related Literature A seminal analysis of the strategic information transmission between an informed sender and an uninformed receiver was provided by CS. In the CS model, the sender sends a costless and unverifiable[11] message about his private information to the receiver, who then decides on the project that affects the payoffs of both players. CS obtain a complete characterization of the set of equilibria in their model and show that the existence of the incentive conflict prevents the full revelation of information. In the present study, I investigate how information transmission can be improved under multistage information transmission with voluntary monetary transfers.

Krishna and Morgan (2008) study an amendment to the CS model by allowing the parties to write a contract containing message-contingent payments. They show that full information revelation is feasible but not optimal and they characterize the optimal contract. In their model, there is a crucial assumption that the receiver can commit herself to compensate the sender for his message. I show that when the communication phase has multiple periods, the receiver can control the sender's incentive through voluntary payments even though there is no contractibility.

My results are closely related to those of Krishna and Morgan (2004). Both their study and my analysis investigate how information transmission can be improved through the receiver's active participation in the communication process. Krishna and Morgan (2004) add a long communication protocol to the CS model.[12] They show that if bilateral (face-to-face) communication between the receiver and sender is possible before the sender sends a message about his private information to the receiver, there exists an equilibrium whose outcome Pareto-dominates all the equilibrium outcomes in the CS model. The key factor to their results is that after the sender conveys some information in the face-to-face communication, multiple equilibria exist in the remaining game. The outcome of this face-to-face communication, which could be random, determines which of these equilibria is played in the future. This affects what the

[10] Under arbitration, a neutral third party (arbitrator) asks the sender for information and chooses a project according to a predetermined potentially stochastic decision rule.

[11] Seidmann and Winter (1997) and Mathis (2008) study the sender–receiver game in which the message sent by the sender is (partially) verifiable, that is, the set of available messages depends on the sender's type. These authors provide the sufficient conditions (Mathis (2008) provides the necessary and sufficient conditions) for the existence of a fully revealing equilibrium. Forges and Koessler (2008) study a multistage sender–receiver game with certifiable messages and geometrically characterize the set of equilibrium payoffs.

[12] Aumann and Hart (2003) study a finite simultaneous-move (long conversation) game in which there are two players, one being better informed than the other. They provide a complete geometrical characterization of the set of equilibrium payoffs when the state of the world is finite and long communication is possible. In this study, the state space and players' action space must be finite. Therefore, I cannot directly apply the results of Aumann and Hart (2003) to my model.

sender conveys during the face-to-face communication. Therefore, in Krishna and Morgan (2004), the receiver tries to control the sender's incentive by controlling the degree of uncertainty associated with the outcome of the face-to-face communication. By contrast, in my model, the receiver tries to control the sender's incentive directly through voluntary transfer payments.

Spence (1973) shows that costly signaling helps people convey their private information credibly. In the framework of the CS model, Austen-Smith and Banks (2000), Kartik (2007), and Karamychev and Visser (2016) show that information transmission can be improved when the sender can send a costly message (money burning, or equivalently, paying money to the receiver) to signal information.[13] In their settings, a fully separating equilibrium that is optimal from the receiver's perspective can exist. However, in the equilibrium that maximizes the sender's ex ante expected payoff, the sender does not pay money to separate an interval of states. Karamychev and Visser (2016) show that in the sender's optimal equilibrium, he pays to adjust the pooling intervals. In the present study, I focus on the situation in which the sender cannot pay money (or equivalently, cannot send a costly signal) to the receiver and show that the signaling structure can be endogenously generated by the receiver's voluntary payment. Moreover, Section 4.5 shows that under the uniform-quadratic assumption, the receiver can obtain the higher ex ante expected payoff than that under the sender's optimal equilibrium in the model analyzed in Karamychev and Visser (2016). This result suggests that in some cases, it might be better for the receiver to generate the signaling structure by herself through voluntary payments rather than to rely on the sender's costly signaling.

In the present study, I focus on information transmission via cheap talk communication and show the benefit of long-term communication with voluntary transfers. By contrast, Hörner and Skrzypacz (2016) study a model of gradual persuasion in which the sender is paid and gradually reveals "certifiable" information. They show that the sequential revelation of partially informative signals can increase payments to the sender who is trying to sell his information to the receiver.

In all the abovementioned studies, once the communication phase is over, the receiver chooses a project; that is, the project choice is once and for all. By contrast, in the studies mentioned hereafter, there are multiple rounds of communication and actions. More precisely, in each period, the sender sends a message and the receiver chooses a project. Hence, these models differ from mine.

Golosov et al. (2014) study strategic information transmission in a finitely repeated cheap talk game. Only the sender knows the state of the world, which remains constant through out the game. They show that the sender can condition his message on the receiver's past actions; in addition, the receiver can choose actions that reward the sender for following a path of messages that eventually leads to the full revelation of information. In contrast to this result, there is no fully revealing equilibrium in my model.

Kolotilin and Li (2017) investigate the optimal relational contracts in an infinitely repeated cheap talk game. In their model, both the sender and receiver can pay

[13] Relatedly, Kartik et al. (2007) and Kartik (2009) study amendments to the CS model with other means of costly signals such as lying costs.

each other. Therefore, there are equilibria in which the sender always reveals his private information completely. They show that full separation can be attained in the equilibrium, whereas partial or complete pooling is optimal if preferences are divergent. In contrast to my study, the sender's private information is not persistent in their model. Hence, gradual information transmission does not appear.

Paper Outline The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 derives the general properties of the perfect Bayesian equilibria in the model. Section 4 analyzes the uniform-quadratic model and shows the benefits of multistage information transmission with voluntary monetary transfers. Section 4.1 shows the two main results by constructing an equilibrium in which information is transmitted within two periods. Section 4.2 shows the benefit of long-term communication. Section 4.3 provides some properties of the optimal equilibria. Section 4.4 discusses the implications for organization design. In Section 4.5, I compare my communication procedure with the sender's optimal signaling. Section 5 generalizes the players' payoff functions and prior probability distribution, and describes two results that correspond to the results in Section 4.1. Section 6 gives some concluding remarks.

2. Model

There are two players, a sender (S) and a receiver (R). R has the authority to choose a project $y \in Y \equiv \mathbb{R}_+$, but the outcome produced by project y depends on S's private information, $\theta \in \Theta \equiv [0, 1]$, which is distributed according to a differentiable distribution function $G(\cdot)$ with density $g(\cdot)$.

Before R chooses a project, R and S engage in T-period communication. Each period consists of two stages, stage 1 and stage 2. At stage 1, S sends a costless and unverifiable message to R. Let $M \equiv [0, 1]$ be S's message space. I denote by m_t a message sent by S at stage 1 in period t. At stage 2, R voluntarily pays money to S. Let $W \equiv \mathbb{R}_+$ be the set of the amount of payment possible for R. I denote by w_t a payment amount, which R pays to S at stage 2 in period t. After T-period communication, the game proceeds to period $T + 1$, in which R chooses a project.

Let $\mathbf{w}$ be a sequence of transfers, $\mathbf{w} \equiv (w_1, \ldots, w_T) \in W^T$. The players' payoff functions $U^R : Y \times \Theta \times W^T \to \mathbb{R}$ and $U^S : Y \times \Theta \times W^T \to \mathbb{R}$ are defined as follows:

$$U^R(y, \theta, \mathbf{w}) \equiv r \cdot u^R(y, \theta) - \sum_{t=1}^{T} w_t$$

$$U^S(y, \theta, \mathbf{w}) \equiv s \cdot u^S(y, \theta, b) + \sum_{t=1}^{T} w_t$$

where r, s, and b are positive constants. The term $\sum_{t=1}^{T} w_t$ represents the total amount of payments.

Here, $r \cdot u^R(y, \theta)$ and $s \cdot u^S(y, \theta, b)$ denote utilities from project y for R and S, respectively. The functions u^R and u^S satisfy CS's assumptions:

- $u^R(y,\theta) \equiv u^S(y,\theta,0)$;
- u^S is twice-continuously differentiable in y, θ, and b for all $y \in \mathbb{R}_+$, $\theta \in \Theta$, and $b \in \mathbb{R}_+$;
- for all $\theta \in \Theta$ and $b \in \mathbb{R}_+$, there exists $y \in \mathbb{R}_+$ such that $u_1^S(y,\theta,b) \equiv \frac{\partial u^S}{\partial y}(y,\theta,b) = 0$; and
- $u_{11}^S(y,\theta,b) \equiv \frac{\partial^2 u^S}{\partial y^2}(y,\theta,b) < 0$, $u_{12}^S(y,\theta,b) \equiv \frac{\partial^2 u^S}{\partial y \partial \theta}(y,\theta,b) > 0$, and $u_{13}^S(y,\theta,b) \equiv \frac{\partial^2 u^S}{\partial y \partial b}(y,\theta,b) > 0$ for all $y \in \mathbb{R}_+$, $\theta \in \Theta$, and $b \in \mathbb{R}_+$.

Under these assumptions, for each given (θ, b), there exists a unique maximizing project: $y^R(\theta) = \arg\max_y u^R(y,\theta)$ and $y^S(\theta,b) = \arg\max_y u^S(y,\theta,b)$. Parameter $b > 0$ represents "bias," which measures how much S's interest differs from R's. Since $u_{13}^S(y,\theta,b) > 0$ and $b > 0$, I obtain $y^R(\theta) < y^S(\theta,b)$. Constants $r > 0$ and $s > 0$ are scalar parameters that measure the relative importance of the project choice versus transfer payments.

The timing of game is summarized as follows:

1. Before the game starts, nature randomly draws a state $\theta \in \Theta$ with common prior $G(\theta)$, and S observes θ privately.
2. R and S engage in T-period communication.
   - At stage 1 in period t, S sends a message m_t to the decision maker,
   - At stage 2 in period t, R voluntarily pays w_t to S.
3. After T-period communication, R chooses a project y and the game ends.

Hereafter, I denote by $\Gamma(b,s,r,T)$ my T-period communication game.

2.1. History and Strategies

A (public) history $h^{(t,j)}$ is defined as a sequence of players' past actions realized until the beginning of stage j in period t.

$$h^{(t,j)} \equiv \begin{cases} (m_1, w_1, \ldots, m_{t-1}, w_{t-1}) & \text{if } j = 1, \\ (m_1, w_1, \ldots, m_{t-1}, w_{t-1}, m_t) & \text{if } j = 2. \end{cases}$$

A (public) history h^{T+1} is defined as a sequence of players' past actions realized until the beginning of period $T+1$, in which R chooses a project.

$$h^{T+1} \equiv (m_1, w_1, \ldots, m_T, w_T).$$

Let $H^{(t,j)}$ and H^{T+1} be the set of $h^{(t,j)}$ and h^{T+1}, respectively. I assume that $H^{(1,1)}$ is a singleton set $\{\phi\}$. I denote the set of all histories at stage j by $\mathcal{H}^j \equiv \bigcup_{t=1}^T H^{(t,j)}$. Let $h_\theta^{(t,1)} \in \Theta \times H^{(t,1)} \equiv H_\Theta^{(t,1)}$ be S's private history at stage 1 in period t. Let $\mathcal{H}_\Theta^1$ be the set of all private histories of S: $\mathcal{H}_\Theta^1 \equiv \Theta \times \mathcal{H}^1$.

S's behavior strategy σ specifies a probability distribution of messages that S of type θ sends at stage 1 in period t: $\sigma : \mathcal{H}^1_\Theta \to \Delta M$.[14] R's pure strategy is a measurable function $\rho : \mathcal{H}^2 \cup H^{T+1} \to \mathbb{R}_+$, which specifies the payment amount and project. Note that $\rho(h^{(t,2)}) \in W$, and $\rho(h^{T+1}) \in Y$.[15] A belief system, $f : \mathcal{H}^2 \cup H^{T+1} \to \Delta\Theta$, specifies R's belief about S's types at history $h \in \mathcal{H}^2 \cup H^{T+1}$.

3. Equilibrium

I analyze (weak) perfect Bayesian equilibria[16]: both players' strategies must maximize their expected payoffs after all histories, and the system of beliefs f must be consistent with the regular conditional probability derived from $((\sigma, \rho), f)$ and G.[17] The formal definition of perfect Bayesian equilibria can be found in Appendix 3.A. Hereafter, I call a perfect Bayesian equilibrium simply *equilibrium*. In this section, I derives the general properties of the equilibria.

3.1. Relationship to the CS Model

I discuss the relationship between the equilibria in the CS model and those in $\Gamma(b, s, r, T)$. Since R cannot obtain additional information about θ after stage 2 in period T, she has no incentive to choose $w^T > 0$. Therefore, w^T must be equal to 0 in any equilibrium. Consequently, $\Gamma(b, s, r, 1)$ is essentially equivalent to the CS model, and I call it *the one-shot cheap talk game*. CS have shown that under the one-shot cheap talk communication, for every $b > 0$, there exists a positive integer $\tilde{n}(b)$ such that, for every $n \in \{1, \ldots, \tilde{n}(b)\}$, there exists at least one equilibrium with an n-element partition: $\{[a_n, a_{n-1}), [a_{n-1}, a_{n-2}), \ldots, [a_1, a_0]\}$. In this equilibrium, S's type $\theta \in [a_{i+1}, a_i)$ conveys that his type belongs to this interval, and after receiving the message that "θ belongs to $[a_{i+1}, a_i)$," R chooses the project $\overline{y}(a_{i+1}, a_i) = \arg\max_y \int_{a_{i+1}}^{a_i} u^R(y, \theta) g(\theta) d\theta$. I define $\overline{y}(a_1, a_0) = y^R(a)$ for $a_1 = a_0 = a$. Since u^R is strictly concave, $\overline{y}(a_{i+1}, a_i)$ is uniquely determined. Moreover, since $u^R_{12}(y, \theta) > 0$, $\overline{y}(a_{i+1}, a_i)$ is strictly increasing in both of its arguments. Since S whose type falls on a boundary between adjacent intervals is indifferent between the associated values of y, the following must be satisfied: for $i = 1, \ldots, n-1$,

$$s \cdot u^S(\overline{y}(a_{i+1}, a_i), a_i, b) - s \cdot u^S(\overline{y}(a_i, a_{i-1}), a_i, b) = 0; \tag{1}$$

$$a_n = 0; \tag{2}$$

$$a_0 = 1. \tag{3}$$

[14] I denote by $\mathbb{B}(X)$ the Borel algebra on a set X. S's behavior strategy is a function $\sigma : \mathbb{B}(M) \times \mathcal{H}^1_\Theta \to [0, 1]$ with the following two properties: (1) for every $\tilde{M} \in \mathbb{B}(M)$, function $\sigma(\tilde{M}, \cdot) : \mathcal{H}^1_\Theta \to [0, 1]$ is measurable, (2) for every $h^{(t,1)}_\theta \in \mathcal{H}^1_\Theta$, function $\sigma(\cdot, h^{(t,1)}_\theta) : \mathbb{B}(M) \to [0, 1]$ is a probability measure. The definition of σ originates from Milgrom and Weber (1985).

[15] Due to the strict concavity of R's preference over projects, she never mixes projects in period $T + 1$.

[16] There always exists an equilibrium that is essentially equivalent to a perfect Bayesian equilibrium in the CS model. Hence, in this study, I do not prove the existence theorem.

[17] Suppose that $((\sigma, \rho), f)$ is an equilibrium. At any payment stage history $h^{(t,2)}$, R does not obtain additional information about S's type from her own action w_t. Therefore, I require that at any $h^{(t,2)}$, any deviation by R from $\rho(h^{(t,2)})$ does not affect the beliefs she uses as the basis for belief updating.

I call a sequence $\mathbf{a} \equiv \{a_0, \ldots, a_n\}$ a (backward) solution of (1) if $\mathbf{a}$ satisfies (1)–(3). I impose the following monotonicity condition on a solution of (1).

Condition M . If $\mathbf{a}'$ and $\mathbf{a}''$ are two solutions of (1) with $a'_0 = a''_0$ and $a'_1 > a''_1$, then $a'_i \geq a''_i$ for all $i \geq 2$.

This condition is met by the uniform-quadratic case: $s \cdot u^S(y, \theta, b) \equiv -s(y - (\theta + b))^2$, $r \cdot u^R(y, \theta) \equiv -r(y - \theta)^2$, and $G(\theta)$ is uniform distribution over $[0, 1]$. CS show that Condition M also holds for more general specifications.

Consider a strategy profile such that S sends an informative message only at stage 1 in period 1 and R pays nothing to S at any payment stage. Obviously, if both S's behavior regarding sending m_1 and R's behavior regarding choosing y depending on m_1 are the same as an equilibrium in the CS model, then this strategy profile constitutes an equilibrium in $\Gamma(b, s, r, T)$. This outcome immediately yields the following Fact 1.

Fact 1. *Any equilibrium partition achieved in the CS model can be achieved under an equilibrium in* $\Gamma(b, s, r, T)$.

3.2. Relationship to Direct Contract

In this subsection, I first characterize the relationship between equilibria in $\Gamma(b, s, r, T)$ and those in a case in which R can sign a contract that specifies the transfer as functions of messages sent by S.

Fix an equilibrium $\xi = ((\sigma, \rho), f)$. Let $\mu_\xi : \Theta \to \Delta(M^T)$ be a probability distribution induced by (σ, ρ) over M^T. When a sequence of messages $\mathbf{m} \in M^T$ is given, a sequence of payments $\mathbf{w} \in W^T$ and a project y are induced from ρ. Let $\omega_\xi : M^T \to W^T$ and $y_\xi : M^T \to Y$ be the functions induced by ρ, respectively.

Now, consider the case in which R can write a indirect contract (M^T, ω_ξ). By the construction of $\omega_\xi : M^T \to W^T$, under this contract, the strategy profile and belief system $((\mu_\xi, y_\xi), f)$ constitutes an equilibrium whose outcome is equivalent to ξ in the sense that both this equilibrium and ξ induce the same probability distribution over $W^T \times Y$ for any θ.

Next, I discuss the relationship between equilibria under this indirect contract (M^T, ω_ξ) and those under a direct contract in which R can sign a contract that specifies the transfer as functions of the direct message $m \in \Theta$ sent by S. Let (Θ, ω) be a direct contract under which S reports $\theta \in \Theta$ and R pays $\omega(\theta)$ for S. Let $y : \Theta \to Y$ be R's strategy under the direct contract (Θ, ω). By the application of the result of Krishna and Morgan (2008),[18] I immediately obtain the following Fact 2.

Fact 2. *Consider an equilibrium under* (M^T, ω_ξ)*. There exists a direct contract* (Θ, ω) *under which there exists a pure strategy equilibrium that is outcome equivalent to the given equilibrium under* (M^T, ω_ξ).

Finally, I characterize the relationship between equilibria in $\Gamma(b, s, r, T)$ and those under a direct contract (Θ, ω). The following Proposition 1 shows that given an equilibrium ξ in $\Gamma(b, s, r, T)$, there exists an equilibrium of a direct contract that is outcome

[18]For details, see Proposition 2 and Appendix B in Krishna and Morgan (2008).

equivalent in the sense that it results in the same projects and transfer as in the original equilibrium ξ for almost every state.

Proposition 1. *Fix an equilibrium ξ in $\Gamma(b, s, r, T)$. There exists a direct contract (Θ, ω) under which there exists a pure strategy equilibrium that is outcome equivalent to ξ.*

Proof. In the indirect contract cases, $\omega_{\xi}(\mathbf{m})$ specifies a sequence of payments, $w_1(\mathbf{m})$, $\ldots$, $w_T(\mathbf{m})$, dependently on $\mathbf{m}$. In the direct contract case, $\omega(\theta)$ specifies the resulting transfer dependently on θ. Fact 2 shows that there exists a direct contract (Θ, ω) such that $\omega(\theta) = \sum_{t=1}^{T} w_t(\mathbf{m}) = \sum_{t=1}^{T} w_t(\mathbf{m}')$ and $y(\theta) = y_{\xi}(\mathbf{m}) = y_{\xi}(\mathbf{m}')$ for almost every θ and for any $\mathbf{m}$, $\mathbf{m}' \in \text{supp}\mu_{\xi}(\cdot|\theta)$. This result means that the outcome of ξ can be replicated by a direct contract (Θ, ω). □

3.3. Partition Equilibrium

As is the case in the CS model, all the equilibria in $\Gamma(b, s, r, T)$ are interval partitional, that is, all the equilibria are partition equilibria.

Definition 1 (Partition Equilibrium)**.** Fix an equilibrium ξ in $\Gamma(b, s, r, T)$. Consider a pure strategy equilibrium, under a direct contract (Θ, ω), which is outcome equivalent to ξ. If there exists a family of sets $\{\mathcal{I}_{\lambda}\}_{\lambda \in \Lambda}$ over Θ such that

1. $\{\mathcal{I}_{\lambda}\}_{\lambda \in \Lambda}$ constitutes an interval partition[19] over Θ;
2. $y(\theta) = y(\theta')$ for all θ, $\theta' \in \mathcal{I}_{\lambda}$; and
3. if $\lambda \neq \lambda'$, $y(\theta) \neq y(\theta')$ for all $\theta \in \mathcal{I}_{\lambda}$ and $\theta' \in \mathcal{I}_{\lambda'}$; then

I call ξ *partition equilibrium*, and $\{\mathcal{I}_{\lambda}\}_{\lambda \in \Lambda}$ *equilibrium partition*.

First, I show the following Proposition 2.

Proposition 2. *Any equilibrium under a direct contract (Θ, ω) is partition equilibrium.*

The proof is in Appendix 3.B. As shown in Subsection 3.2, any equilibrium outcome in $\Gamma(b, s, r, T)$ is also achieved in equilibrium under a corresponding direct contract. Therefore, Proposition 2 means that all equilibria in $\Gamma(b, s, r, T)$ are partition equilibria.

Corollary 1. *All equilibria in $\Gamma(b, s, r, T)$ are partition equilibria.*

The following Proposition 3 shows that there is no fully separating equilibria in $\Gamma(b, s, r, T)$.

Proposition 3. *There exists no fully separating equilibrium in $\Gamma(b, s, r, T)$.*

The proof is in Appendix 3.C. If R can commit herself to compensating for S's message, fully separating equilibria (full revelation contracts) are always feasible. However, in my model, since there is neither commitment nor contractibility, R pays money to S only when paying money is optimal for her. For S's truth telling to be incentive

[19]For all $\lambda \neq \lambda'$, $\mathcal{I}_{\lambda} \cap \mathcal{I}_{\lambda'} = \emptyset$. For all $\lambda \in \Lambda$, $\mathcal{I}_{\lambda}$ is convex, and $\bigcup_{\lambda \in \Lambda} \mathcal{I}_{\lambda} = \Theta$.

compatible, the resulting sum of transfers must be different for each $\theta \in \Theta$. Precisely, $\omega(\theta)$ must be strictly decreasing in $\theta \in \Theta$. This means that if the given R's payment strategy leads to S's truth telling, R almost certainly reaches a history where she pays a certain amount of money to S even though she has already detected the true state. At such a history, R has no incentive to pay. For this reason, there is no fully separating equilibrium.

Whether the cardinality of the equilibrium partition is finite remains an open question. Next, I provide a sufficient condition (Assumptions 1 and 2) for the cardinality of the equilibrium partition to be finite.

Assumption 1. *S's utility function u^S satisfies*

$$u^S(y, \theta, b) = \psi(|y - \theta - b|),$$

where $\psi''(\cdot) < 0$ and $\psi'(0) = 0$.

Assumption 2. *The distribution G and R's utility function u^R jointly satisfy: for any colsed interval $[\underline{a}, \overline{a}]$ with $0 \leq \underline{a} \leq \overline{a} \leq 1$,*

$$\overline{y}(\underline{a}, \overline{a}) = \arg\max_{y \in \mathbb{R}} \int_{\underline{a}}^{\overline{a}} \left[\frac{g(\theta)}{G(\overline{a}) - G(\underline{a})} u^R(y, \theta) \right] d\theta < \frac{\underline{a} + \overline{a}}{2} + b. \tag{4}$$

Assumption 2 is mild. For example, suppose that $u^R(y, \theta, b) = l(|y - \theta|)$, where $l''(\cdot) < 0$ and $l'(0) = 0$, and that G is non increasing. Then, the inequality (4) holds.

Proposition 4. *Under Assumption 1 and 2, in any equilibrium, the equilibrium partition has a finite number of elements.*

The proof is in Appendix 3.D. Proposition 4 shows that under Assumptions 1 and 2, the equilibrium partition is a finite set. In Appendix 4.E, I discuss the fact that an equilibrium which has separating intervals in its partition might exist if Assumption 2 is not satisfied.

Hereafter, $[a_\lambda, a_{\lambda-1})$ denotes $\mathcal{I}_\lambda$, and ω_λ denotes $\omega(\theta)$ for $\theta \in [a_\lambda, a_{\lambda-1})$. In any equilibrium, there must exist $\tilde{\lambda} \in \Lambda$ such that $\omega_{\tilde{\lambda}+1} \leq \omega_{\tilde{\lambda}}$.[20] From S's incentive compatibility condition,

$$\psi(|\overline{y}(a_{\tilde{\lambda}+1}, a_{\tilde{\lambda}}) - a_{\tilde{\lambda}} - b|) \geq \psi(|\overline{y}(a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1}) - a_{\tilde{\lambda}} - b|). \tag{5}$$

Figure 1 illustrates the inequality (5). The blue curve is $\psi(|\overline{y}(a_{\tilde{\lambda}+1}, a_{\tilde{\lambda}}) - \theta - b|)$, and the red curve is $\psi(|\overline{y}(a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1}) - \theta - b|)$. Note that $y_{\tilde{\lambda}+1} = \overline{y}(a_{\tilde{\lambda}+1}, a_{\tilde{\lambda}})$; $y_{\tilde{\lambda}} = \overline{y}(a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1})$; $\psi_{\tilde{\lambda}+1} = \psi(|y_{\tilde{\lambda}+1} - a_{\tilde{\lambda}} - b|)$; and $\psi_{\tilde{\lambda}} = \psi(|y_{\tilde{\lambda}} - a_{\tilde{\lambda}} - b|)$.

Since $\overline{y}(a_{\tilde{\lambda}+1}, a_{\tilde{\lambda}}) < a_{\tilde{\lambda}}$, the left-hand side of the inequality (5) is less than $\psi(b)$. Moreover, from Assumption 2, the right-hand side of the inequality (5) is higher than $\psi([a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}}]/2)$. Therefore, I must have $a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}} > 2b$ irrespective of the length of the communication phase.

[20]Suppose that this condition does not hold. If the true state belongs to the leftmost element of the equilibrium partition, R almost certainly reaches a history where she pays a certain amount to S even though she does not obtain additional information in the future.

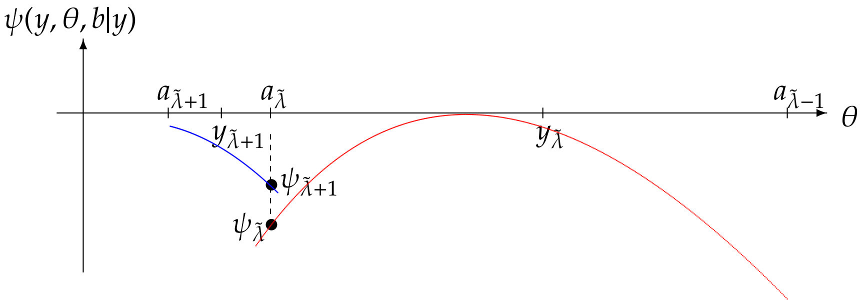

Figure 1: $(\omega_{\tilde\lambda} - \omega_{\tilde\lambda+1})/s = \psi_{\tilde\lambda+1} - \psi_{\tilde\lambda} \geq 0$

This result implies that in any equilibrium, at history h^{T+1} where R believes that $\theta \in [a_{\tilde\lambda}, a_{\tilde\lambda-1})$, R's conditional expected utility from project is strictly less than the optimal:

$$r\int_{a_{\tilde\lambda}}^{a_{\tilde\lambda-1}}\left[\frac{g(\theta)}{G(a_{\tilde\lambda-1})-G(a_{\tilde\lambda})}u^R(\overline{y}(a_{\tilde\lambda},a_{\tilde\lambda-1}),\theta)\right]d\theta < r\int_{a_{\tilde\lambda}}^{a_{\tilde\lambda-1}}\left[\frac{g(\theta)}{G(a_{\tilde\lambda-1})-G(a_{\tilde\lambda})}u^R(y^R(\theta),\theta)\right]d\theta. \tag{6}$$

Moreover, R reaches such a history with probability $G(a_{\tilde\lambda-1}) - G(a_{\tilde\lambda})$. Hence, in any equilibrium, R's expected payoff is strictly less than

$$\begin{aligned}\overline{U} &\equiv r\int_{a_{\tilde\lambda}}^{a_{\tilde\lambda-1}}\left[g(\theta)u^R(\overline{y}(a_{\tilde\lambda},a_{\tilde\lambda-1}),\theta)\right]d\theta + r\int_{\theta\notin[a_{\tilde\lambda},a_{\tilde\lambda-1}]}\left[g(\theta)u^R(y^R(\theta),\theta)\right]d\theta \\ &< r\int_0^1\left[g(\theta)u^R(y^R(\theta),\theta)\right]d\theta.\end{aligned}$$

To make the characterization more specific, I assume the following.

Assumption 3. *R's utility from project u^R satisfies*

$$u^R(y,\theta) = l(|y-\theta|),$$

where $l''(\cdot) < 0$ and $l'(0) = 0$.

Assumption 4. *The distribution G is the uniform distribution.*

Under Assumptions 3 and 4, Assumption 2 is satisfied.

Proposition 5. *Under Assumptions 1, 3, and 4, the upper bound of R's equilibrium payoff is given by*

$$\overline{U}(b,r) = r\int_{\theta\in[0,4b]} l(|2b-\theta|)d\theta.$$

The proof is in Appendix 3.F. One of the main findings in my analysis is that when T is sufficiently high and s/r is small enough, this upper bound $\overline{U}(b,r)$ can be approximated by R's equilibrium payoff. For the details of this result, see Proposition 11 in Section 4.

4. The Uniform-quadratic Case

In this section, I show the benefits of multistage information transmission with voluntary payments, concentrating on the well-known uniform-quadratic case: $r \cdot u^R(y, \theta) = -r(y-\theta)^2$, $s \cdot u^S(y, \theta, b) = -s(y-(\theta+b))^2$, and $G(\theta)$ is a uniform distribution over Θ.

4.1. Two-period Information Elicitation

The key idea on which I build the analysis is that the dependence of future information on past payments ensures that R makes message-contingent payments. To understand the intuition behind this idea, I construct an equilibrium in which information is transmitted within two periods and R pays a positive amount of money to S on the equilibrium path. By constructing such an equilibrium, I show that multistage information transmission with voluntary payments can be more beneficial for both S and R than the one-shot cheap talk communication. In Section 5, I generalize the players' payoff functions and prior probability distribution and show the results that correspond to those in this subsection.

Suppose that $b \in (1/12, 1/4)$. Then, there are two equilibria in the one-shot cheap talk game. One is the uninformative equilibrium: the babbling equilibrium. The other is a partially informative equilibrium: $a_0 = 1$, $a_1 = 1/2 - 2b$ and $a_2 = 0$. CS have shown that both S and R prefer the partially informative equilibrium to the uninformative equilibrium. In the partially informative equilibrium, the ex ante expected payoff of R is $-r(1/48 + b^2)$ whereas that of S is $-s(1/48 + b^2) - sb^2$.

The first result establishes that if $T \geq 2$ and r is large relative to s, there exists an equilibrium whose partition has more steps than the one-shot cheap talk game does.

Proposition 6. *Fix $b \in (1/12, 1/4)$. If $s/r < (1-4b)/(1+12b)$, there is a continuum of 3-element partition equilibria.*

I characterize a class of 3-element partition equilibria in which information is transmitted in order from the rightmost element of the equilibrium partition. In the equilibrium, S gradually conveys his information within the first and second period. If S conveys information contrary to his bias in the first period, then R pays to S in order to extract more precise information in the second period. If R does not pay in the first period, then S never gives additional information. As s becomes smaller, the necessary payment becomes smaller since the effect of the message-contingent payment on S's incentive becomes larger. Furthermore, as r becomes higher, the punishment by babbling message becomes more severe. This is the reason why s/r needs to be small enough.

Proof. Consider a strategy profile under which the information is transmitted in the following steps. At stage 1 in period 1, S of type $\theta < a_1$ randomly sends a message m_1 according to a uniform distribution over $[0, a_1)$, and S of type $\theta \geq a_1$ randomly sends a message m_1 according to a uniform distribution over $[a_1, 1]$. If R receives $m_1 < a_1$ at stage 1 in period 1, then she pays $w_1 = w$ to S. Otherwise, she pays nothing to S at stage 2 in period 1. At stage 1 in period 2, if $m_1 < a_1$ and $w_1 \geq w$, then S of type $\theta < a_2$ randomly sends a message m_2 according to a uniform distribution over $[0, a_2)$, and S of type $\theta \geq a_2$ randomly sends a message m_2 according to a uniform distribution over $[a_2, 1]$. Otherwise, S conveys no information, i.e., any type of S randomly sends a message m_2 according to a uniform distribution over $[0, 1]$. In period $t \geq 2$, R pays nothing to S. In period $t \geq 3$, S conveys no information.

Once communication is over, R chooses her best project based on the information she has. At h^{T+1} such that $m_1 \geq a_1$, since R believes θ is uniformly distributed over $[a_1, 1]$, the optimal project for R is $y_1 = (a_1 + 1)/2$. At h^{T+1} such that $m_1 < a_1$, $w_1 \geq w$, and $m_2 \geq a_2$, since R believes θ is uniformly distributed over $[a_2, a_1)$, the optimal project for R is $y_2 = (a_2 + a_1)/2$. At h^{T+1} such that $m_1 < a_1$, $w_1 \geq w$, and $m_2 < a_2$, since R believes θ is uniformly distributed over $[0, a_2)$, the optimal project for R is $y_3 = a_2/2$. At h^{T+1} such that $m_1 < a_1$ and $w_1 < w$, since R believes θ is uniformly distributed over $[0, a_1)$, the optimal project for R is $\tilde{y} = a_1/2$. Figure 2 illustrates the equilibrium strategy.

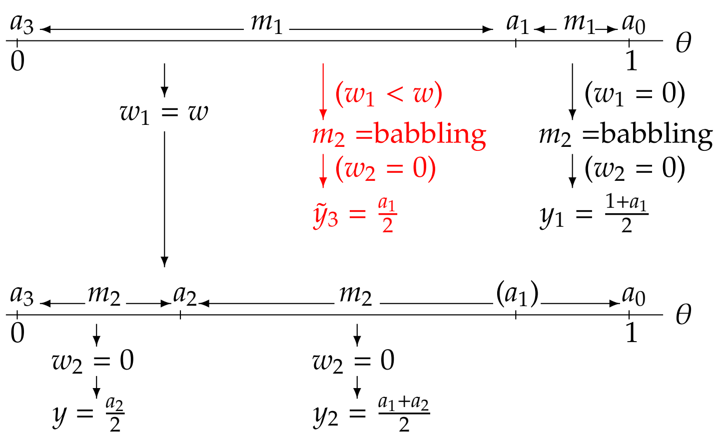

Figure 2: Equilibrium Strategy

In what follows, I ensure that by taking a_1, a_2 and w suitably, I can construct an equilibrium in which S and R follow the abovementioned strategy profile.

In period $t \geq 2$, R always pays nothing to S. Therefore, the partition $\{[0, a_2), [a_2, a_1)\}$ must coincide with the 2-element equilibrium partition achieved in the one-shot cheap talk game in which θ is drawn from the uniform distribution over $[0, a_1)$. By CS, the following must be satisfied:

$$a_2 = a_1/2 - 2b. \tag{7}$$

Since I now focus on a 3-element partition equilibrium, I must have $a_2 > 0$. Hence, $a_1 > 4b$ must be satisfied.

Under the abovementioned strategy profile, S of type $\theta \in (a_i, a_{i-1})$ sends messages so that y_i would be chosen by R. Hence, S's payoff is derived as follows:

$$\begin{aligned}
&-s(y_3 - (\theta + b))^2 + w \;\text{ for }\; \theta \in [0, a_2);\\
&-s(y_2 - (\theta + b))^2 + w \;\text{ for }\; \theta \in [a_2, a_1);\\
&-s(y_1 - (\theta + b))^2 \;\text{ for }\; \theta \in [a_1, 1].
\end{aligned}$$

Since I suppose that $a_2 = a_1/2 - 2b$, I obtain

$$\begin{aligned}
&-s(y_3 - (\theta + b))^2 > -s(y_2 - (\theta + b))^2 \ \text{ for } \ \theta \in [0, a_2);\\
&-s(y_3 - (\theta + b))^2 < -s(y_2 - (\theta + b))^2 \ \text{ for } \ \theta \in (a_2, 1];\\
&-s(y_3 - (\theta + b))^2 = -s(y_2 - (\theta + b))^2 \ \text{ for } \ \theta = a_2.
\end{aligned}$$

Clearly, at stage 1 in period 2 such that $m_1 < a_1$ and $w_1 \geq w$, S has no incentive to deviate from the given strategy. Moreover, if $m_1 < a_1$ and $w_1 < w$, or if $m_1 \geq a_1$, S conveys no information. Therefore, S has no incentive to deviate at such a history. The same can be said in period $t \geq 3$. Hence, I conclude that S has no incentive to deviate in period $t \geq 2$ when $a_2 = a_1/2 - 2b$.

At stage 1 in period 1, if S of type θ sends $m_1 \geq a_1$, then he obtains $-s(y_1 - (\theta + b))^2$. Otherwise, S of type $\theta \geq a_2$ obtains $-s(y_2 - (\theta + b))^2 + w$, and S of type $\theta < a_2$ obtains $-s(y_3 - (\theta + b))^2 + w$. If the following equation (8) holds, then the inequalities (9) and (10) hold.

$$-s(y_1 - (a_1 + b))^2 = -s(y_2 - (a_1 + b))^2 + w; \tag{8}$$

$$-s(y_1 - (\theta + b))^2 \geq \max_{j \in \{1,2\}} \{-s(y_{j+1} - (\theta + b))^2 + w\} \quad \text{for } \theta \geq a_1; \tag{9}$$

$$-s(y_{j+1} - (\theta + b))^2 + w > -s(y_1 - (\theta + b))^2 \quad \text{for } j = \{1, 2\} \text{ and } \theta \in [a_{j+1}, a_j). \tag{10}$$

If (9) and (10) hold, S has no incentive to deviate at stage 1 in period 1.

By equation (8), I obtain

$$w = w(a_1) \equiv s[(2 + 4b - a_1)(-2 + 12b + 3a_1)]/16. \tag{11}$$

Since $w(a_1)$ is strictly increasing in $a_1 \in [4b, 1]$, I have an inverse function of $w(\cdot)$ such that $w^{-1}(w) \equiv a_1(w)$ is strictly increasing in $w \in [w(4b), w(1)]$. Moreover, since I suppose that $b \in (1/12, 1/4)$, R's payment is nonnegative: $w(4b) = s(12b - 1)/4 > 0$. Note that $a_1(w) = \frac{2}{3}\left\{2 - \sqrt{(1 + 6b)^2 - 12w/s}\right\}$ and $a_1(w) \in (4b, 1)$ where $w \in (w(4b), w(1))$.

In summary, I conclude that S has no incentive to deviate from the given strategy when the boundaries of the partition satisfy the following conditions:

$$a_i(w) \equiv \begin{cases} 1 \ \text{ for } \ i = 0, \\ \frac{2}{3}\left\{2 - \sqrt{(1 + 6b)^2 - 12w/s}\right\} \ \text{ for } \ i = 1, \\ \frac{1}{3}\left\{2 - \sqrt{(1 + 6b)^2 - 12w/s}\right\} - 2b \ \text{ for } \ i = 2, \\ 0 \ \text{ for } \ i = 3. \end{cases} \tag{12}$$

where $w \in (w(4b), w(1))$. Figure 3 illustrates S's incentive compatibility conditions.

At any $h^{(t,2)}$, R has no incentive to increase the amount of payment because it would not affect S's behavior. Therefore, I have only to ensure that paying w is optimal for R after receiving $m_1 < a_1$.

If R pays $w_1 \geq w$ after receiving $m_1 < a_1$, then she obtains $u^*(w_1)$:

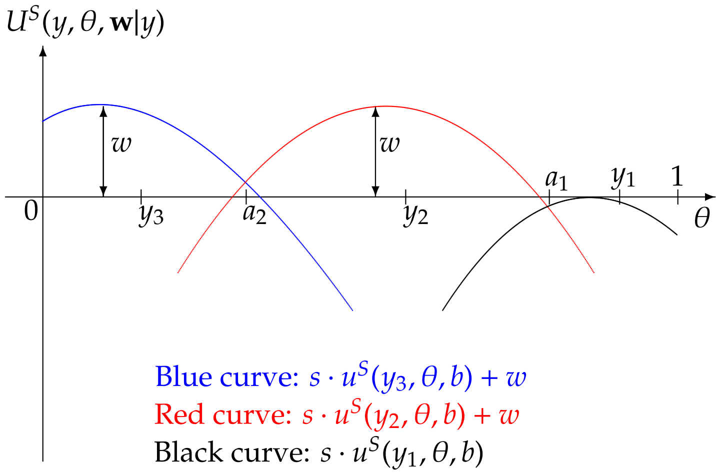

Figure 3: S's payoff on the equilibrium path

$$\begin{aligned} u^*(w_1) &= -w_1 - \frac{1}{a_1}\sum_{i=1}^{2}\int_{a_{i+1}}^{a_i} r\left[\frac{a_{i+1}+a_i}{2}-\theta\right]^2 d\theta \\ &= -w_1 - \frac{r}{12a_1}\sum_{i=1}^{2}(a_i-a_{i+1})^3 \\ &= -w_1 - \frac{r}{12a_1}(a_2^3+(a_1-a_2)^3). \end{aligned}$$

On the other hand, by paying $w_1 < w$, R obtains $\overline{u}(w_1)$:

$$\begin{aligned} \overline{u}(w_1) &= -w_1 + \int_0^{a_1}\frac{1}{a_1}U^R\left(\frac{a_1}{2},\theta\right)d\theta \\ &= -w_1 - \frac{r}{a_1}\int_0^{a_1}\left(\frac{a_1}{2}-\theta\right)^2 d\theta \\ &= -w_1 - \frac{a_1^2}{12}r\,. \end{aligned}$$

The payoffs $u^*(w_1)$ and $\overline{u}(w_1)$ have a unique maximum at $w_1 = w$ and $w_1 = 0$, respectively. Thus, paying w is an optimal decision for R if and only if $u^*(w) \geq \overline{u}(0)$. Using condition (12) yields

$$u^*(w) \geq \overline{u}(0) \iff r\left(\frac{\{a_1(w)\}^2}{16}-b^2\right) \geq w. \tag{13}$$

Since $a_1(w) \equiv \frac{2}{3}\left\{2-\sqrt{(1+6b)^2-12w/s}\right\}$, for any $w \in (w(4b), w(1))$

- $a_1(w)$ is strictly increasing in w;
- $\frac{\{a_1(w)\}^2}{16} - b^2 > 0$ and $\frac{\{a_1(w(4b))\}^2}{16} - b^2 = 0$;

- $\frac{d^2}{dw^2}\{a_1(w)\}^2 > 0$.

Hence, if $r(\{a_1(w(1))\}^2/16 - b^2) > w(1)$, then there exists $\underline{w} \in (w(4b), w(1))$ such that for all $w \in [\underline{w}, w(1))$, the inequality (13) holds.

Since $a_1(w(1)) = 1$ and $w(1) = s(1+4b)(1+12b)/16$, the inequality $r(\{a_1(w(1))\}^2/16 - b^2) > w(1)$ can be simplified into

$$\frac{s}{r} < \frac{1-4b}{1+12b}.$$

Therefore, if $s/r < (1-4b)/(1+12b)$, then the given strategy profile and the system of beliefs constitute an equilibrium when $w \in [\underline{w}, w(1))$ and the boundaries of partition satisfy the condition (12). □

Remark 1. In the equilibrium, meaningful information transmission must occur after R pays w. For this reason, in the equilibrium outlined above, it is necessary that $a_2 = a_1/2 - 2b > 0$. Hence, both $4b < a_1$ and $b < 1/4$ must be satisfied.

There is a possibility of the existence of a 3-element partition equilibrium in which S conveys information in a different order. For example, consider the following strategy profile. In period 1, S reveals whether $\theta \geq a_2$. If $\theta \geq a_2$, then R pays $\tilde{w}$, and then, S reveals whether $\theta < a_1$. Note that $a_2 < a_1$. The following Proposition 7 shows that there is no equilibrium where information is transmitted in such a way.

Proposition 7. *Fix $b \in (1/12, 1/4)$. There exists no 3-element partition equilibrium such that information is transmitted in order from the leftmost element of the equilibrium partition.*

The proof of is in Appendix 4.A. Under the abovementioned strategy profile, R pays $\tilde{w} > 0$ only when she receive the message that means $\theta \geq a_2$. Intuitively, this payment strategy affects S's incentive for misrepresentation negatively, since it strengthens S's exaggeration incentive. Hence, I cannot have equilibria in which information is transmitted in order from the leftmost element of the equilibrium partition.

It can be confirmed that if r is large relative to s, R can obtain the greater expected "revenue from the project" under a 3-element partition equilibrium constructed in Proposition 6 than under the 2-element partition equilibrium in the one-shot cheap talk game. This result is due to the fact that R can obtain more detailed information about S's type. However, since R has to make a payment under the 3-element partition equilibrium, multistage information transmission with voluntary transfer payments is not always beneficial to R. I now show the second result that when r is large relative to s, multistage information transmission with voluntary transfer payments is more beneficial to both R and S than the one-shot cheap talk communication.

In the one-shot cheap talk game, both players always strictly prefer the 2-element partition equilibrium to the babbling equilibrium from the ex-ante perspective. Let $E\hat{U}^{\kappa}$ be the ex ante expected payoff of $\kappa \in \{R, S\}$ under the 2-element partition equilibrium in the one-shot cheap talk game. I denote by $\{[\tilde{a}_2, \tilde{a}_1)[\tilde{a}_1, \tilde{a}_0]$ the equilibrium partition. As noted earlier, $\tilde{a}_1 = 1/2 - 2b$. Let $E\overline{U}^{\kappa}(x)$ be the ex ante expected payoff of $\kappa \in \{R, S\}$ in the 3-element partition equilibrium with $x = a_1$ and $a_2 = x/2 - 2b$.

The following lemma shows that if r is large relative to s, there exists a 3-element partition equilibrium that R prefers to all the equilibria in the one-shot cheap talk game.

Lemma 1. *There exists a positive value $\eta^*(b)$ such that if $s/r < \eta^*(b)$, for some $x \in (\underline{a}, 1)$,*

$$E\overline{U}^R(x) > E\hat{U}^R.$$

Proof. Suppose that $s/r < (1-4b)/(1+12b)$. Fix a 3-element partition equilibrium constructed in the proof of Proposition 6. By the definition of $a_1(w)$, I have

$$w(x) \equiv a_1^{-1}(x) = s[(2+4b-x)(-2+12b+3x)]/16 \quad \text{for } x = a_1 \in [\underline{a}, 1).$$

Hereafter, I denote by $s \cdot \alpha(b,x)$ the function $w(x)$. Recall that $s \cdot \alpha(b,x)$ is strictly greater than zero for $x \in [\underline{a}, 1)$.

R's expected payoff $E\overline{U}^R(x)$ is given by

$$\begin{aligned} E\overline{U}^R(x) &= -\sum_{i=1}^{3} \int_{a_i(x)}^{a_{i-1}(x)} r\left[\frac{a_{i-1}(x)+a_i(x)}{2} - \theta\right]^2 d\theta - xs \cdot \alpha(b,x) \\ &= -r\left\{\frac{x^3}{48} + xb^2\right\} - \frac{r}{12}(1-x)^3 - xs \cdot \alpha(b,x). \end{aligned}$$

CS show that

$$\begin{aligned} E\hat{U}^R &= -\sum_{i=1}^{2} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} r\left[\frac{\tilde{a}_{i-1}+\tilde{a}_i}{2} - \theta\right]^2 d\theta \\ &= -\frac{r}{48} - rb^2. \end{aligned}$$

Let $\delta(b,x) \equiv \frac{1}{r}\{E\overline{U}^R(x) - E\hat{U}^R\}$. I obtain

$$\delta(b,x) = -\frac{1}{16}(1-x^3) + \frac{x}{4}(1-x) + b^2(1-x) - \frac{s}{r}x\alpha(b,x).$$

$\delta(b,x) > 0$ holds if and only if

$$\eta^*(b,x) \equiv \frac{-\frac{1}{16}(1-x^3) + \frac{x}{4}(1-x) + b^2(1-x)}{x\alpha(b,x)} > \frac{s}{r}.$$

$\frac{\partial \eta^*}{\partial x}|_{x=1} < 0$ and $\eta^*(b,1) = 0$. Since $\inf_{x\in[\underline{a},1)} x\alpha(b,x) = \underline{a}\alpha(b,\underline{a}) > 0$, $\eta^*(b,x)$ has a least upper bound $\eta^*(b) = \sup_{x\in[\underline{a},1)} \eta^*(b,x) > 0$. Therefore, if $s/r < \eta^*(b)$, then $\delta(b,x) > 0$ for some $x \in (\underline{a}, 1)$. This completes the proof of Lemma 1. □

Remark 2. Note that x is almost equal to 1. Then, boundaries of the 3-element partition equilibrium almost coincide with boundaries of the 2-element partition equilibrium in the one-shot cheap talk game. Nevertheless, the payment of monetary transfer is strictly higher than 0. Therefore, if $s/r < (1-4b)/(1+12b)$, there always exists a 3-element partition equilibrium that is unfavorable to R: there exists $\hat{\varepsilon} > 0$ such that $\eta^*(b,\underline{x}) < s/r$ for all $\underline{x} \in (1-\hat{\varepsilon}, 1)$.

Next, I show the following lemma.

Lemma 2. $E\overline{U}^R(x) > E\hat{U}^R$ *implies that* $E\overline{U}^S(x) > E\hat{U}^S$.

Proof. Recall that $\overline{E}U^S(x)$ denotes the ex ante expected payoff of S under the 3-element partition equilibrium with $a_1 = x \in (\underline{a}, 1)$. I obtain

$$E\overline{U}^S(x) = -\sum_{i=1}^{3}\int_{a_i(x)}^{a_{i-1}(x)} s\left[\frac{a_{i-1}(x)+a_i(x)}{2}-\theta\right]^2 d\theta - sb^2 + x\cdot w(x)$$
$$= \frac{s}{r}\left\{E\overline{U}^R(x) + x\cdot w(x)\right\} - sb^2 + x\cdot w(x).$$

CS show that

$$E\hat{U}^S = -\sum_{i=1}^{2}\int_{\tilde{a}_i}^{\tilde{a}_{i-1}} s\left[\frac{\tilde{a}_{i-1}+\tilde{a}_i}{2}-\theta\right]^2 d\theta - sb^2$$
$$= \frac{s}{r}E\hat{U}^R - sb^2.$$

Clearly, if $E\overline{U}^R(x) > E\hat{U}^R$, then $E\overline{U}^S(x) > E\hat{U}^S$. □

From Lemma 1 and 2, I immediately have the following result.

Proposition 8. *Fix* $b \in (1/12, 1/4)$. *Then, there exists a positive value* $\eta^*(b)$ *such that if* $s/r < \eta^*(b)$, *there exists a* 3*-element partition equilibrium whose outcome ex ante Pareto-dominates all the equilibrium outcomes in the one-shot cheap talk game.*

It is known that the existence of a non-strategic mediator leads to improved information transmission. Now, I compare my communication procedure with optimal mediation. In the mediation model, S can send a message to an impartial mediator, who then passes on a recommendation to R according to some predetermined stochastic rule. R chooses her best project based on the recommendation from mediator. Goltsman et al. (2009) characterize the optimal mediation under which R's ex ante expected payoff is $-rb(1-b)/3$. The following Proposition 9 shows that in two-period information elicitation with voluntary monetary transfer, R can obtain higher ex ante expected payoff than that under the optimal mediation.

Proposition 9. *Fix* $b \in (1/12, (4+\sqrt{3})/26)$.[21] *Then there exists* $\eta'(b)$ *such that if* $s/r < \eta'(b)$, *for some* $x \in (\underline{a}, 1)$,

$$E\overline{U}^R(x) > -\frac{r}{3}b(1-b).$$

Since this proposition can be proved in the same way as the proof of Lemma 1, the formal proof is omitted. When b is almost equal to $1/4$, boundaries of the 3-element partition equilibrium almost coincide with those of the 2-element partition equilibrium in the one-shot cheap talk game: $a_1 \approx 1$ and $a_2 \approx 1/2 - 2b$. The value of $-rb(1-b)/3$ is always strictly higher than R's equilibrium payoff under the 2-element partition equilibrium in the one-shot cheap talk game. Therefore, the parameter b needs to be strictly less than $1/4$.

[21]Note that $\frac{1}{5} < \frac{1}{26}\left(4+\sqrt{3}\right) < \frac{1}{4}$.

4.2. Effective Long-term Communication

In the previous subsection, I focus on the equilibrium in which information is transmitted within only two periods, regardless of the length of communication. It seems that R does not use T-period communication effectively. In this subsection, I show the benefit of long-term communication.

Recall my earlier discussion of the upper bound of R's equilibrium payoff. Proposition 5 provides it as

$$\overline{U}(b,r) = r\int_{\theta\in[0,4b]} l(|2b-\theta|)d\theta = -\frac{16rb^3}{3}.$$

One of the main findings in my analysis is that when T is long enough, this upper bound $\overline{U}(b,r) = -16rb^3/3$ can be approximated by R's equilibrium payoff.

First, I demonstrate that under a certain condition, there exists an equilibrium in which information is transmitted within the whole T-period in order from the rightmost element of the equilibrium partition. Specifically, I consider the following information elicitation. In period 1, S conveys whether the value of θ is less than a_1. If $\theta < a_1$, then R pays a certain amount of money. After that, in period 2, S conveys whether the value of θ is less than a_2. If $\theta < a_2$, then R pays again. This information elicitation is repeated until the last period in the communication phase. In the last period, S of type $\theta < a_{T-1}$ conveys whether the value of θ is less than a_T. Under this communication process, R eventually learns to which element of a partition $\{[a_{t+1},a_t)\}_{t=1}^{T} \cup [a_1,a_0]$ the state θ belongs. I call this communication process *(monotone) effective T-period communication.*[22] Figure 4 illustrates this information elicitation.

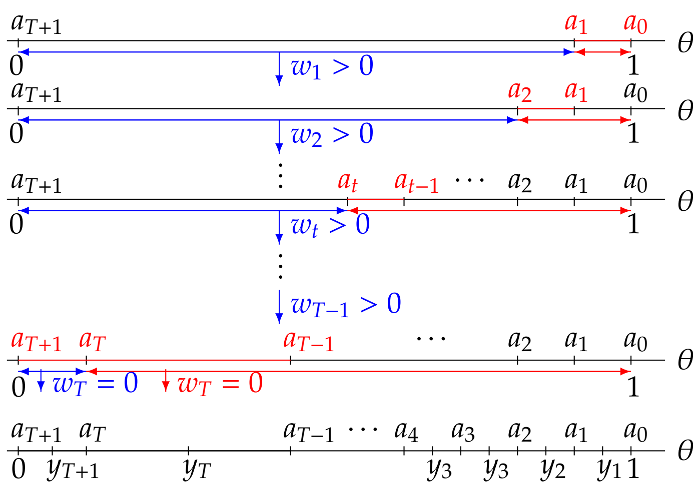

Figure 4: Effective T-period communication

[22] This information elicitation is similar to that in Ivanov (2015) and Hörner and Skrzypacz (2016).

Proposition 10. *Fix $b \in (0, 1/4)$. If $s/r < (1-4b)/(1+12b)$, there exists an equilibrium with effective T-period communication.*

Under the effective T-period communication, the information is transmitted in the following steps. At $h^{(t,1)}$ such that $m_{t'} < a_{t'}$ and $w_{t'} \geq w^*_{t'}$ for all $t' < t$, S of type $\theta < a_t$ randomly sends a message m_t according to a uniform distribution over $[0, a_t)$, and S of type $\theta \geq a_t$ randomly sends a message m_t according to a uniform distribution over $[a_t, 1]$. Otherwise, any type of S randomly sends a message according to the same distribution, a uniform distribution over $[0, 1]$. If S conveys that $\theta < a_t$ at stage 1 in period t, then R pays w^*_t to S at stage 2. Otherwise, he pays nothing.

Let $I(h^{T+1})$ be the closure of $\{\theta \in \Theta : f(\theta|h^{T+1}) > 0\}$. Under the abovementioned strategy profile, for any h^{T+1}, the closed set $I(h^{T+1})$ belongs to $\{[a_t, a_{t-1}]\}_{t=1}^{T+1} \cup \{[a_{T+1}, a_{t-1}]\}_{t=2}^{T}$, and R's posterior belief $f(\theta|h^{T+1})$ is a uniformly distribution on $I(h^{T+1})$. Therefore, R chooses $y = \frac{\min I(h^{T+1}) + \max I(h^{T+1})}{2}$ at h^{T+1}.

At $h^{(T,2)}$ such that $m_{t'} < a_{t'}$ and $w_{t'} \geq w^*_{t'}$ for all $t' < T$, since R does not obtain additional information after making a payment, w^*_T must be equal to 0. Therefore, $\{[a_{T+1}, a_T), [a_T, a_{T-1})\}$ coincides with the 2-element equilibrium partition achieved in the one shot cheap talk game where $\Theta = [0, a_{T-1})$. Hence, I obtain

$$a_T = \frac{a_{T-1}}{2} - 2b.$$

This implies that $a_{T-1} > 4b$. Define a_t and w^*_t as follows:

$$a_t \equiv \begin{cases} 1 - ta & \text{for } t \in \{1, \ldots, T-1\}, \\ \frac{1-(T-1)a}{2} - 2b & \text{for } t = T, \\ 0 & \text{for } i = T+1. \end{cases} \tag{14}$$

$$w^*_t \equiv \begin{cases} 2bsa & \text{for } t \in \{0, \ldots, T-2\}, \\ \frac{s}{16}\{1 + 12b - a(T+1)\}\{1 + 4b - a(T-3)\} & \text{for } t = T-1, \\ 0 & \text{for } t = T. \end{cases} \tag{15}$$

Suppose that $a < (1-4b)/(T-1)$. Then, $4b < a_{T-1}$ and $a_{t-1} - a_t = a > 0$ for $t \in \{1, \ldots, T-1\}$. Note that $w^*_{T-1} > 0$ if $a < \min\{(1+12b)/(T+1), (1+4b)/(T-3)\}$. Since I suppose that $T \geq 3$, I obtain $(1+4b)/(T-3) > (1-4b)/(T-1)$. Therefore, if $a < \min\{(1-4b)/(T-1), (1+12b)/(T+1)\}$, the given boundaries and payments are well-defined. Moreover, for any $t \in \{1, \ldots, T-1\}$, w^*_t becomes a solution to an equation,

$$-s\left(\frac{a_t + a_{t-1}}{2} - (a_t + b)\right)^2 = -s\left(\frac{a_{t+1} + a_t}{2} - (a_t + b)\right)^2 + w^*_t,$$

induced by S's incentive compatibility condition: S whose type falls on the boundaries between adjacent intervals is indifferent between the associated values of y.

The abovementioned strategy profile and system of beliefs, hereafter ξ_T, cannot always be an equilibrium. Whether it is so depends on the value of a. I show that ξ_T can be an equilibrium when a is small enough. R's payment w^*_t in each $t \leq T-2$ goes to 0 as a goes to 0. Consider a history at stage 2 in period $T-1$ such that $m_t < a_t$ for all $t \leq T-1$ and $w_t \geq w_t^*$ for all $t < T-1$. Then, there are two cheap talk equilibria in the remaining game: the babbling equilibrium and the 2-element partition equilibrium. Since I now suppose that $a \approx 0$, if the 2-element partition equilibrium is chosen in period T, R's continuation payoff is approximated by $-r(b^2 - 1/48)$. Otherwise, R's continuation payoff is approximated by $-r/12$. Moreover, $w_{T-1}^* \approx s(1+12b)(1+4b)/16$. Since I suppose that $s/r < (1-4b)/(1+12b)$, I have

$$-r\left(b^2 - \frac{1}{48}\right) - \left(-\frac{r}{12}\right) > \frac{s}{16}(1+12b)(1+4b).$$

Thus, R has an incentive to pay w_{T-1}^* at this history so that the babbling equilibrium would not be chosen in the last period. Furthermore, at $h^{(t,2)}$ where R pays w_t^*, if w_t^* is small enough, R pays to ensure that the babbling equilibrium would not be chosen in the future. Hence, by taking a small enough, I can construct an equilibrium with effective T-period communication. The formal proof is found in Appendix 4.B.

Proposition 10 shows only the possibility of the effective T-period communication. In order for ξ_T to be an equilibrium, it might be necessary for a_{T-1} to be close to 1. If a_{T-1} is close to 1, R reaches a history $h^{(T,1)}$ at which $I(h^{(T,1)}) = [0, a_{T-1}]$ with a high probability on the equilibrium path. Moreover, $\{[a_{T+1}, a_T), [a_T, a_{T-1})\}$ almost coincides with the 2-element equilibrium partition achieved in the one-shot cheap talk game. In such a case, the initial $(T-1)$-period communication does not have much meaning from ex ante perspective. However, as S becomes less concerned with the project, the effects of monetary transfer on S's incentive becomes larger. In other words, the necessary payments for controlling S's incentive goes to 0 as s goes to 0. Hence, if s is small enough, it is not necessary for a_{T-1} to be close to 1. This fact suggests that long-term communication becomes more beneficial for R as s becomes smaller. To see this, I show the following Proposition 11.

Proposition 11. *Fix $b \in (0, 1/4)$. For any $d > 0$, there exists $\overline{T}(b,d)$ and $\underline{\eta}(b,d)$ such that if $T \geq \overline{T}(b,d)$ and $s/r < \underline{\eta}(b,d)$, R can obtain a higher ex ante expected payoff than $-16rb^3/3 - rd$.*

The proof is in Appendix 4.C. I earlier show that an upper bound of R's equilibrium payoff is $-16rb^3/3$. This Proposition 11 shows that if the communication phase has a sufficiently large number of periods and S weighs transfer payments more heavily than the project choice, this upper bound can be approximated by R's equilibrium payoff.

4.3. Some Properties of Optimal Equilibria

Having shown that the upper bound $-16rb^3/3$ can be approximated by R's equilibrium payoff, it remains to identify the characteristics of an optimal equilibrium. Let ξ_d be an equilibrium where R's equilibrium payoff is higher than $-16rb^3/3 - rd$. Proposition 11 shows that the set of parameters that guarantees the existence of ξ_d in $\Gamma(b,s,r,T)$ is open. Let $\Gamma(d)$ be a game in which there exists ξ_d. I denote by Ξ_d the set of ξ_d in $\Gamma(d)$. By definition, all the optimal equilibria in $\Gamma(d)$ belong to Ξ_d. In this subsection, focusing on $\Gamma(d)$ with small $d > 0$, I establish some key properties of optimal equilibria.

Proposition 4 in Section 3.3 shows that all the equilibria in $\Gamma(d)$ are finite partition equilibria. I denote by $\mathcal{I}^d \equiv \{\mathcal{I}^d_\lambda\}_{\lambda\in\{1,\ldots,\Lambda\}}$ the equilibrium partition of ξ_d: each element of the equilibrium partition $\mathcal{I}^d_\lambda$ is an interval $[a^d_\lambda, a^d_{\lambda-1})$ such that $a^d_0 = 1$ and $a^d_\Lambda = 0$. Let ω_λ be the resulting transfer that S of type $\theta \in [a^d_\lambda, a^d_{\lambda-1})$ receives under ξ_d.[23]

Since $\omega_\lambda - \omega_{\lambda+1} = -s((a^d_{\lambda+1} - a^d_\lambda)/2 - b)^2 + s((a^d_{\lambda-1} - a^d_\lambda)/2 - b)^2$. if $\omega_{\lambda+1} \leq \omega_\lambda$, the width of the interval $\mathcal{I}_\lambda$ is strictly higher than $4b$: $a^d_\lambda - a^d_{\lambda+1} + 4b \leq a^d_{\lambda-1} - a^d_\lambda$. Let $\tilde{\Lambda}$ be the number of intervals such that $\omega_{\lambda+1} \leq \omega_\lambda$: $\tilde{\Lambda} \equiv \#\{\lambda \in \{1,\ldots,\Lambda\} : \omega_{\lambda+1} \leq \omega_\lambda\}$. The following proposition establishes the relationship between d and $\tilde{\Lambda}$.

Proposition 12. *Fix $d > 0$ and $\Gamma(d)$. Then, in any $\xi_d \in \Xi_d$, the following must be satisfied:*

$$\frac{3d}{16b^3} + 1 > \tilde{\Lambda}.$$

Proof. Under the given equilibrium ξ_d, the ex ante expected payoff of R is given by

$$EU^R(d) = -r\sum_{\lambda=1}^{\Lambda}\left[\frac{(a^d_\lambda - a^d_{\lambda-1})^3}{12} - (a^d_\lambda - a^d_{\lambda-1})\omega_\lambda\right].$$

Moreover, $a^d_{\tilde{\lambda}-1} - a^d_{\tilde{\lambda}} > 4b$ is satisfied for any $\tilde{\lambda} \in \{\lambda \in \{1,\ldots,\Lambda\} : \omega_{\lambda+1} \leq \omega_\lambda\}$. This finding implies that $-16rb^3\tilde{\Lambda}/3 > EU^R(d)$. By the definition of ξ_d, I must have $EU^R(d) > -16rb^3/3 - rd$. Therefore, $-16rb^3\tilde{\Lambda}/3 > -16rb^3/3 - rd$ must be satisfied. This completes the proof of Proposition 12. □

Corollary 2. *Fix $d \in (0, 16b^3/3)$ and $\Gamma(d)$. Then, in any $\xi_d \in \Xi_d$, $\tilde{\Lambda} = 1$ is satisfied. Moreover, $\exists!\tilde{\lambda} \neq \Lambda$ such that $\omega_\Lambda = \omega_{\tilde{\lambda}}$.*

Proof. Proposition 12 shows that $\tilde{\Lambda} \leq 1$ if $d \in (0, 16b^3/3)$. Now, suppose that $\tilde{\Lambda} = 0$. Then, $\omega_\Lambda > \omega_\lambda$ for any $\lambda < \Lambda$. In this case, if the true state θ belongs to the interval $[a^d_\Lambda, a^d_{\Lambda-1})$, then R certainly reaches a history $h^{(t,2)}$ such that she pays a positive amount of money even though she does not obtain additional information in the future: $\{\theta \in \Theta : f(\theta|h^{(t,2)}) > 0\} = [a^d_\Lambda, a^d_{\Lambda-1}]$ and $\rho(h^{(t,2)}) > 0$. Hence, I conclude that $\tilde{\Lambda} = 1$, and $\omega_\Lambda \leq \omega_{\lambda'}$ for some $\lambda' < \Lambda$. In the same way as this result, the inequality $\omega_{\lambda'} \leq \omega_\Lambda$ holds. This completes the proof of Corollary 2. □

4.4. Comparison with Predetermined Decision Rules

Now, under the uniform-quadratic assumption, I compare my communication procedure with both delegation and arbitration. When R delegates control, her payoff is given by $-rb^2$. As shown by CS, the ex ante expected payoff of R under the one-shot cheap talk communication is given by

$$EU^R_{CS} = -r\left(\frac{1}{12n^2} + \frac{b^2(n^2-1)}{3}\right),$$

[23] Suppose that (Θ, ω_d) is a direct contract under which there exists a pure strategy equilibrium that is outcome equivalent to ξ_d. Then, $\omega_\lambda = \omega_d(\theta)$ is satisfied for $\theta \in [a^d_\lambda, a^d_{\lambda-1})$.

where $n \in \{1, \ldots, \tilde{n}\}$. The maximum number of partition equilibrium outcomes $\tilde{n}$ is given by

$$\tilde{n} \equiv \left\lceil -\frac{1}{2} + \frac{1}{2}\left(1 + \frac{2}{b}\right)^{\frac{1}{2}} \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. Dessein (2002) shows that $EU^R_{CS} < -rb^2$ for $n \geq 2$, and thus, R prefers delegation to the one-shot cheap talk communication whenever informative communication is possible, $b < 1/4$.

By contrast, in my model, if $T \geq \overline{T}(b,d)$ and $s/r < \underline{\eta}(b,d)$, R can obtain the higher equilibrium payoff than $-16rb^3/3 - rd$. If $b < 3/16$ and $d < b^2 - 16b^3/3$, the inequality $-16rb^3/3 - rd > -rb^2$ holds.

Next, consider the situation in which *arbitration* is available. Under arbitration, S sends a message to a neutral third party (arbitrator), and after receiving the messages, the arbitrator announces a project. This announcement serves as a binding recommendation to R. In other words, R cannot choose any action that is different from the recommended one. Goltsman et al. (2009) characterize the optimal arbitration rule and show that R's ex ante expected payoff under optimal arbitration is $-rb^2(1-4b/3)$.[24] I immediately verify that if $d < b^2(1 - 4b/3) - 16b^3/3$ and $b < 3/20$, the inequality $-16rb^3/3 - rd > -rb^2(1 - 4b/3)$ holds.

Therefore, Proposition 11 implies that when the communication phase has a sufficiently large number of periods and R places greater importance on the project than S does, R can obtain higher ex ante expected payoff than under delegation and arbitration.[25]

4.5. Comparison with Sender-optimal Signaling

As noted in Section 1, costly signaling helps people convey their private information credibly. Naturally enough, even in my setting, if S can send a costly message (paying money to R) to signal information, a fully separating equilibrium that is optimal from R's perspective can exist. However, it is known that under general assumptions, the perfect separation is never optimal from S's perspective although it is feasible.[26]

Karamychev and Visser (2016) study an amendment to the CS model by allowing S to use both costless and costly messages. They show that in S's optimal equilibrium, S pays to adjust the pooling intervals. Moreover, under the uniform-quadratic assumption, they characterize Sender-optimal equilibria whose partition has at most $\tilde{n} + 1$ steps.[27] In such equilibria, R's expected payoff is less than $-r/\{12(\tilde{n}+1)^2\}$. Since $\tilde{n} \equiv \left\lceil \left(-1 + \sqrt{1 + 2/b}\right)/2 \right\rceil$, the integer $\tilde{n}$ satisfies that $2\tilde{n}(\tilde{n}+1) \leq b \leq 2\tilde{n}(\tilde{n}-1)$.

[24] Having restricted attention to deterministic mechanism, Melumad and Shibano (1991) provide the optimal arbitration (optimal delegation) rule.

[25] Since the optimal arbitration rule dominates the optimal mediation rule, my communication protocol could strictly dominate the optimal mediation rule.

[26] de Haan et al. (2015) experimentally study the strategic information transmission in a setting where both cheap talk and burning money are available, and they find that the individuals who supply information prefer to communicate through cheap talk.

[27] See Proposition 4 in Karamychev and Visser (2016).

Therefore, if $\tilde{n} > 4$ holds, I obtain

$$-\frac{16rb^3}{3} \geq -\frac{r}{12(\tilde{n}+1)^3} > -\frac{r}{12(\tilde{n}+1)^2}.$$

This inequality and Proposition 11 suggest that in some cases, it might be better for R to generate the signaling structure by herself through voluntary payment rather than to rely on S's costly signaling.

5. Generalization of Proposition 6 and Proposition 8

In this section, under the more general settings where the players' payoff function and the prior probability of the state are kept as is in Section 2, I show two results that correspond to the results in Section 4.1.

Recall that $\tilde{n}$ ($\equiv \tilde{n}(b)$ in Section 3.1) denotes the maximum number of elements of equilibrium partition achievable in the one-shot cheap talk game. As can be observed from the uniform-quadratic case, under my equilibrium construction in Proposition 6, after S conveys some information in period 1, there must be multiple equilibria in the remaining game. Therefore, I assume that $\tilde{n} \geq 2$. In the one-shot cheap talk game, if Condition M holds, then the most informative equilibrium is $\tilde{n}$-element partition equilibrium where $\{[\tilde{a}_{\tilde{n}}, \tilde{a}_{\tilde{n}-1}), \ldots [\tilde{a}_1, \tilde{a}_0]\}$, and $0 = \tilde{a}_{\tilde{n}} < \tilde{a}_{\tilde{n}-1} < \cdots < \tilde{a}_1 < \tilde{a}_0 = 1$.

The following Proposition 13 establishes that an equilibrium whose partition has more steps that that in the one-shot cheap talk game exists.

Proposition 13. *Fix $b > 0$ and suppose that $\tilde{n} \geq 2$. Then, there exists a positive value $\eta(b)$ such that if $s/r < \eta(b)$, there is a continuum of $(\tilde{n}+1)$-element partition equilibria.*

To prove this Proposition, I construct a strategy profile that induces a $(\tilde{n}+1)$-element partition: $\{[\hat{a}_{\tilde{n}+1}, \hat{a}_{\tilde{n}}), \ldots [\hat{a}_1, \hat{a}_0]\}$, and $0 = \hat{a}_{\tilde{n}+1} < \hat{a}_{\tilde{n}} < \cdots < \hat{a}_1 < \hat{a}_0 = 1$. The following strategy profile is an extension of the strategy profile that I construct in Section 4.1.

At stage 1 in period 1, S conveys whether $\theta < \hat{a}_1$. If $\theta < \hat{a}_1$, then R pays a certain amount of money, w^*, to S at stage 2 in period 1. Otherwise, she pays nothing to S. If $\theta < \hat{a}_1$ and $w_1 \geq w^*$, at stage 1 in period 2, S conveys to which element of $\{[\hat{a}_{\tilde{n}+1}, \hat{a}_{\tilde{n}}), \ldots [\hat{a}_2, \hat{a}_1)\}$ the true state θ belongs. Otherwise, S conveys no information regardless of his type. In period $t \geq 2$, R always pays nothing to S. In period $t \geq 3$, S conveys no information. In period $T+1$, R chooses a project $\rho(h^{T+1}) = \arg\max_y \int u^R(y, \theta) f(\theta | h^{T+1}) d\theta$. In the rest of this section, I denote by $(\hat{\sigma}, \hat{\rho})$ the strategy profile defined above, and denote by $\hat{f}$ the belief system derived from $(\hat{\sigma}, \hat{\rho})$.

Under the strategy profile outlined above, I have to take an equilibrium partition whose boundaries $\{\hat{a}_{\tilde{n}+1}, \ldots, \hat{a}_1\}$ coincide with those of the $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game, where the state space is $[0, \hat{a}_1)$. The following inequality must hold for R's payment w^* to be optimal.

$$\frac{r}{G(\hat{a}_1)} \sum_{i=1}^{\tilde{n}} \int_{\hat{a}_{i+1}}^{\hat{a}_i} u^R(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta) g(\theta) d\theta - \frac{r}{G(\hat{a}_1)} \int_0^{\hat{a}_1} u^R\left(\overline{y}(0, \hat{a}_1), \theta\right) g(\theta) d\theta \geq w^*, \quad (16)$$

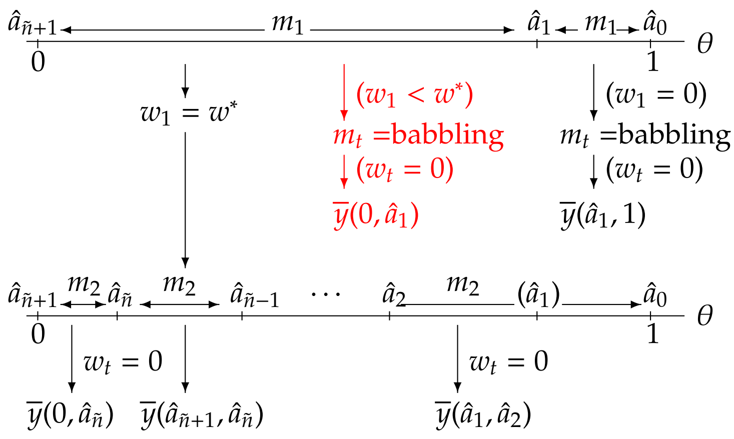

Figure 5: Equilibrium

where $\overline{y}(\hat{a}_{i+1}, \hat{a}_i) = \arg\max_y \int_{\hat{a}_{i+1}}^{\hat{a}_i} u^R(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta) g(\theta) d\theta$. The left-hand side of this inequality represents the value of additional information, that is, the value of the partition $\{[0, \hat{a}_{\tilde{n}}), \ldots, [\hat{a}_2, \hat{a}_1)\}$ that R receives in period 2 by paying w^* after receiving a message that means $\theta < \hat{a}_1$. It is obvious that R always strictly prefers partition $\{[0, \hat{a}_{\tilde{n}}), \ldots, [\hat{a}_2, \hat{a}_1)\}$ to partition $\{[0, \hat{a}_1)\}$, which implies that the left-hand side of the inequality (16) is positive and increasing in r when $\tilde{n} \geq 2$. Since $w^* = s \cdot u^S(\overline{y}(\hat{a}_1, 1), \hat{a}_1, b) - s \cdot u^S(\overline{y}(\hat{a}_2, \hat{a}_1), \hat{a}_1, b)$,[28] the right-hand side of the inequality (16) is decreasing in s and goes to 0 as s goes to 0. Therefore, if r is large enough relative to s, then paying w^* is optimal for R.

In Appendix 5.A, I ensure that there exists $\eta(b) > 0$ such that if $\frac{s}{r} < \eta(b)$, by taking the boundaries of partition $\{[\hat{a}_{\tilde{n}+1}, \hat{a}_{\tilde{n}}), \ldots [\hat{a}_1, \hat{a}_0]\}$ suitably, $((\hat{\sigma}, \hat{\rho}), \hat{f})$ constitutes an equilibrium.

Next, I show that under some conditions, multistage information transmission with voluntary monetary transfer is more beneficial to both R and S than one-shot cheap talk communication. To observe this, I focus on equilibrium, $((\hat{\sigma}, \hat{\rho}), \hat{f})$, which I construct in Proposition 13.

Let $\{[\hat{a}^x_{\tilde{n}+1}, \hat{a}^x_{\tilde{n}}), \ldots, [\hat{a}^x_1, \hat{a}^x_0]\}$ be the partition whose boundaries $\{\hat{a}^x_{\tilde{n}+1}, \ldots, \hat{a}^x_1\}$ coincide with those of the $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game, where the state space is $[0, x)$. I denote by $E\hat{U}^R(x)$ the ex ante expected payoff of R under $((\hat{\sigma}, \hat{\rho}), \hat{f})$ with $(\tilde{n}+1)$-element partition: $\{[\hat{a}^x_{\tilde{n}+1}, \hat{a}^x_{\tilde{n}}), \ldots [\hat{a}^x_1, \hat{a}^x_0]\}$ where $\hat{a}^x_1 \equiv x \in [\underline{a}_1(s/r), 1)$. Let $\underline{a}_1(s/r)$ be the infimum value of z such that (16) holds for all $x \in [z, 1)$.

I obtain

$$E\hat{U}^R(x) = \hat{W}(x) - E[w^*],$$

where $\hat{W}(x)$ denotes R's ex ante expected utility from project:

$$\hat{W}(x) \equiv r \sum_{i=1}^{\tilde{n}+1} \int_{\hat{a}^x_i}^{\hat{a}^x_{i-1}} u^R(\overline{y}(\hat{a}^x_i, \hat{a}^x_{i-1}), \theta) g(\theta) d\theta.$$

[28] Recall Figure 3 in Section 4.1.

CS show that in the one-shot cheap talk game, under Condition M, R always strictly prefers $\tilde{n}$-element partition equilibrium to any other equilibria. I denote by EU^R_{CS} the ex ante expected payoff of R under the $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game. I obtain

$$EU^R_{CS} \quad = \quad r\sum_{i=1}^{\tilde{n}}\int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^R(\overline{y}(\tilde{a}_i,\tilde{a}_{i-1}),\theta)g(\theta)d\theta.$$

The boundaries $\{\hat{a}^x_{\tilde{n}+1},\hat{a}^x_{\tilde{n}},\ldots,\hat{a}^x_1\}$ almost coincide with the boundaries $\{\tilde{a}_{\tilde{n}},\tilde{a}_{\tilde{n}-1},\ldots,\tilde{a}_0\}$ induced by $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game when $x \approx 1$. Therefore, I have $\lim_{x\uparrow 1}\hat{W}(x) = EU^R_{CS}$. This implies that if the following Condition C holds, $\hat{W}(x) > EU^R_{CS}$ for some $x \in (\underline{a}_1(s/r), 1)$.

Condition C . $\left.\frac{d\hat{W}}{dx}\right|_{x=1} < 0.$

Under Condition C, for some $x \in (\underline{a}_1(s/r), 1)$, the partition $\{[\hat{a}^x_{\tilde{n}+1},\hat{a}^x_{\tilde{n}}),\ldots[\hat{a}^x_1,\hat{a}^x_0]\}$ is finer than the partition $\{[\tilde{a}_{\tilde{n}},\tilde{a}_{\tilde{n}-1}),\ldots[\tilde{a}_1,\tilde{a}_0]\}$. Hereafter, I restrict attention to $((u^R,u^S),G)$ under which Condition C holds. Note that there exists a pair of players' payoff functions and the prior distribution of state, $((u^R,u^S),G)$, under which Condition C holds. It is not true that Condition M implies that Condition C. In Remark 6 in Appendix 5.C, I provide an example in which Condition M is satisfied, while Condition C is not.

I now show the following Proposition 14.

Proposition 14. *Fix $b > 0$ and suppose that $\tilde{n} \geq 2$ and Condition C holds. Then, there exists a positive value $\tilde{\eta}(b)$ such that if $s/r < \tilde{\eta}(b)$, there exists a $(\tilde{n}+1)$-element partition equilibrium whose outcome ex ante Pareto-dominates all the equilibrium outcomes in the one-shot cheap talk game.*

I prove Proposition 14 by three steps. Let $((\hat{\sigma},\hat{\rho}),\hat{f})$ be a partition equilibrium constructed in Proposition 13. First, I show that if $s/r < \eta(b)$, the set of $(\tilde{n}+1)$-element partition equilibria that S prefers to all equilibria in $\Gamma(b,s,r,1)$ is nonempty. Second, I show that there exists a positive value $\overline{\eta}(b)$ such that if $s/r < \overline{\eta}(b)$, the set of $(\tilde{n}+1)$-element partition equilibria that R prefers to all the equilibria in $\Gamma(b,s,r,1)$ is nonempty. Finally, I show that there exists a positive value $\tilde{\eta}(b)$ such that if $s/r < \tilde{\eta}(b)$, the intersection of the above two sets is nonempty: The formal proof is in Appendix 5.B.

Finally, I show that Condition C is not necessary for a Prato improvement.

Proposition 15. *Fix $b > 0$ and suppose that $\tilde{n} \geq 3$. Then, there exists a positive value $\ddot{\eta}(b)$ such that if $s/r < \ddot{\eta}(b)$, there exists a $\tilde{n}$-element partition equilibrium whose outcome ex ante Pareto-dominates all the equilibrium outcomes in the one-shot cheap talk game.*

Under the strategy profile on which I focus here, information is elicited in the same way as the previous Proposition 13 and 14, whereas the number of elements of the equilibrium partition is $\tilde{n}$. Let $\{[\tilde{a}^x_{\tilde{n}},\tilde{a}^x_{\tilde{n}-1}),\ldots,[\tilde{a}^x_1,\tilde{a}^x_0]\}$ be the equilibrium partition with $\tilde{a_1}^x = x \in (\tilde{a}_1, 1)$. The boundaries $\{\tilde{a}^x_{\tilde{n}},\ldots,\tilde{a}^x_1\}$ coincide with those of the $(\tilde{n}-1)$-element partition equilibrium in the one-shot cheap talk game, where the state space is $[0,x)$. By the definition, if $x = \tilde{a}_1$, the boundaries $\{\tilde{a}^x_{\tilde{n}},\ldots,\tilde{a}^x_0\}$ coincide with those of the $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game. In Appendix 5.D, I show that the above strategy profile can constitute an equilibrium that leads to a Pareto improvement.

6. Concluding Remarks

In this study, I analyzed a cheap talk game in which an informed sender and an uninformed receiver engage in finite-period communication before the receiver makes a decision. During the communication phase, the sender sends a (cheap talk) message more than once and the receiver can pay money to the sender whenever she receives a message. I have shown that the dependence of future information on past payments creates an incentive for the receiver to pay money. This result ensures that the receiver makes message-contingent payments to some extent even in the situation in which there is no contractibility, and consequently, information transmission can be improved relative to the one-shot cheap talk communication without transfer payments.

Under the assumption of quadratic preferences and a uniform type distribution, I found the upper bound of the receiver's equilibrium payoff, and provided a sufficient condition for it to be approximated by the receiver's payoff under a certain equilibrium. Consequently, when the communication phase has a sufficiently large number of periods and the receiver places greater importance on the project than the sender does, multistage information transmission with voluntary payments can be more beneficial for the receiver than a wide class of other communication protocols (e.g., mediation, arbitration, and the sender's optimal signaling).

In this paper, I focused on the multistage *unilateral* communication. Intuitively, it seems that the sender's punishment by babbling message can create the receiver's payment incentive even in situations in which players engage in more general communication protocols such as multistage *bilateral* communication. Hence, a natural question to ask is whether the receiver's voluntary payment can work jointly with such general communication protocols? Considering such a model remains for further research.

Appendix

Appendix 3.A Perfect Bayesian Equilibria

Let $\mathbb{H} \equiv \Theta \times M_1 \times W_1 \times \cdots \times M_T \times W_T \times Y$ be the set of sequences of the realized state and players' actions, $(\theta, m_1, w_1, \ldots, m_T, w_T, y)$.[29] Let $\mathbb{B}(\mathbb{H})$ be the Borel algebra on $\mathbb{H}$. Given a strategy profile and a prior distribution, $((\sigma, \rho), G)$, a probability measure $\mathbb{P}$ on the measurable space $(\mathbb{H}, \mathbb{B}(\mathbb{H}))$ is uniquely determined. Given $h \in \mathbb{H}$, the values of players' payoffs, both U^R and U^S, are uniquely derived. Moreover, the functions $U^R : \mathbb{H} \to \mathbb{R}$ and $U^S : \mathbb{H} \to \mathbb{R}$ are measurable. Therefore, the players' ex ante expected payoffs $E[U^R(y, \theta, \mathbf{w})|(\sigma, \rho)]$ and $E[U^R(y, \theta, \mathbf{w})|(\sigma, \rho)]$ are well-defined. Let $V^S(\sigma, \rho|h_\theta^{(t,1)}, m_t)$ and $V^R((\sigma, \rho), f|h^{(t,2)}, w_t)$ be the continuation payoff of S after sending m_t at $h_\theta^{(t,1)}$ and the continuation payoff of R after paying w_t at history $h^{(t,2)}$, respectively.

Definition 2. A strategy profile (σ, ρ) and a belief system f constitute a *perfect Bayesian equilibrium* if the following conditions hold. For any $t \in \{1, \ldots, T\}$,

[29] In order to avoid confusion, I add a time operator to the players' action space.

1. for any $h_\theta^{(t,1)} \in H_\Theta^{(t,1)}$ and $m_t \in \text{supp}\{\sigma(\cdot|h_\theta^{(t,1)})\}$,

$$m_t \in \arg\max_{m_t'} V^S(\sigma, \rho|h_\theta^{(t,1)}, m_t'),$$

2. for any $h^{(t,2)} \in H^{(t,2)}$,

$$\rho(h^{(t,2)}) \in \arg\max_{w_t'} \left\{ V^R((\sigma,\rho), f|h^{(t,2)}, w_t') - w_t' \right\},$$

3. for any $h^{T+1} \in H^{T+1}$,

$$\rho(h^{T+1}) \in \arg\max_{y'} r \int u^R(y', \theta) f(d\theta|h^{T+1}),$$

4. the belief system f is consistent with (σ, ρ).

Consistency of the belief system

Given $h^{(t,2)}$, the belief system induces a probability measure $f(\cdot|h^{(t,2)})$ on $(\Theta, \mathbb{B}(\Theta))$. Moreover, since S's behavior strategy $\sigma(\tilde{M}, \cdot|h^{(t,2)}, w_t) : \Theta \to [0,1]$ is measurable for any $\tilde{M} \in \mathbb{B}(M_{t+1})$ and $w_t \in W_t$, I can uniquely define probability measure $\hat{\mathbb{P}}(\cdot|h^{(t,2)}, w_t)$ on $(\Theta \times M_{t+1}, \mathbb{B}(\Theta) \otimes \mathbb{B}(M_{t+1}))$ as follows: for $\tilde{\Theta} \in \mathbb{B}(\Theta)$ and $\tilde{M} \in \mathbb{B}(M_{t+1})$,

$$\hat{\mathbb{P}}(\tilde{\Theta} \times \tilde{M}|h^{(t,2)}, w_t) \equiv \int_{\tilde{\Theta}} \sigma(\tilde{M}, \theta|h^{(t,2)}, w_t) f(d\theta|h^{(t,2)}).$$

Therefore, I calculate the posterior belief: if $\hat{\mathbb{P}}(\Theta \times \tilde{M}|h^{(t,2)}, w_t) > 0$, then

$$f(\tilde{\Theta}|h^{(t,2)}, w_t, \tilde{M}) = \frac{\hat{\mathbb{P}}(\tilde{\Theta} \times \tilde{M}|h^{(t,2)}, w_t)}{\hat{\mathbb{P}}(\Theta \times \tilde{M}|h^{(t,2)}, w_t)}.$$

Moreover, fix $\tilde{\Theta} \in \mathbb{B}(\Theta)$, then $\hat{\mathbb{P}}(\tilde{\Theta}, \cdot|h^{(t,2)}, w_t)/\hat{\mathbb{P}}(\Theta, \cdot|h^{(t,2)}, w_t)$ induce measures $\tilde{\nu}$ on $(M_{t+1}, \mathbb{B}(M_{t+1}))$. Since $\tilde{\nu}$ is absolutely continuous with respect to the Borel measure ν on $(M_{t+1}, \mathbb{B}(M_{t+1}))$ and both measures are σ-finite, there exists a Radon–Nikodym derivative $\zeta(m_{t+1}|\tilde{\Theta}, h^{(t,2)}, w_t)$ such that for any $\tilde{M} \in \mathbb{B}(M_{t+1})$,

$$\tilde{\nu} = \int_{\tilde{M}} \zeta(m_{t+1}|\tilde{\Theta}, h^{(t,2)}, w_t) \nu(dm_{t+1}).$$

Hence, I require that for $m_{t+1} \in \text{supp}(\nu)$,

$$f(\tilde{\Theta}|h^{(t+1,2)}) = \zeta(m_{t+1}|\tilde{\Theta}, h^{(t,2)}, w_t),$$

where $h^{(t+1,2)} = (h^{(t,2)}, w_t, m_{t+1})$.

Appendix 3.B Proof of Proposition 2

Fix a pure strategy equilibrium under a direct contract (Θ, ω). Then, the existence of a partition $\{\mathcal{I}_\lambda\}_{\lambda \in \Lambda}$ that satisfies the conditions 2–3 in Definition 1 is trivial. Hence, I have only to ensure that $\mathcal{I}_\lambda$ is convex for each $\lambda \in \Lambda$. First, I show that R's strategy regarding the project, $y : \Theta \to Y$, satisfies the following property.

Lemma 3. *In a pure strategy equilibrium under a direct contract* (Θ, ω)*, R's strategy regarding the project,* $y(\theta)$*, is nondecreasing.*

Proof of Lemma 3. From S's incentive compatibility condition, for any $\theta, \theta' \in \Theta$,

$$u^S(y(\theta), \theta, b) + \omega(\theta) \geq u^S(y(\theta'), \theta, b) + \omega(\theta'), \text{ and}$$
$$u^S(y(\theta'), \theta', b) + \omega(\theta') \geq u^S(y(\theta), \theta', b) + \omega(\theta).$$

These inequalities can be simplified into

$$u^S(y(\theta), \theta, b) - u^S(y(\theta'), \theta, b) \geq u^S(y(\theta), \theta', b) - u^S(y(\theta'), \theta', b)$$

My assumption $u^S_{12}(y, \theta, b) > 0$ yields $y(\theta) \geq y(\theta')$ for $\theta > \theta'$. □

From Lemma 3, I immediately obtain the following lemma.

Lemma 4. *In a pure strategy equilibrium under a direct contract* (Θ, ω)*, if* $y(\underline{\theta}) = y(\overline{\theta})$ *for* $\underline{\theta} < \overline{\theta}$*, then* $y(\underline{\theta}) = y(\theta) = y(\overline{\theta})$ *for all* $\theta \in [\underline{\theta}, \overline{\theta}]$*. Moreover,* $\omega(\underline{\theta}) = \omega(\theta) = \omega(\overline{\theta})$ *for all* $\theta \in [\underline{\theta}, \overline{\theta}]$.

Lemma 4 implies the convexity of $\mathcal{I}_\lambda$. ◇

Appendix 3.C Proof of Proposition 3

Suppose that a fully separating equilibrium ξ_F exists. Let (Θ, ω_F) be a direct contract under which there exists a pure strategy equilibrium that is outcome equivalent to ξ_F. Let $y_F(\theta)$ be R's equilibrium strategy under (Θ, ω_F). Obviously, $y_F(\theta) = y^R(\theta) = \arg\max_y u^R(y, \theta)$. For truth telling to be incentive compatible, it is necessary to satisfy the following condition:

$$s \cdot u^S(y^R(\theta), \theta, b) + \omega_F(\theta) \geq s \cdot u^S(y^R(\theta'), \theta, b) + \omega_F(\theta') \quad \text{for all } \theta' \neq \theta.$$

From the first-order condition, I obtain the differential equation

$$\frac{d}{d\theta}\omega_F(\theta) = -s \cdot u^S_1(y^R(\theta), \theta, b)\frac{d}{d\theta}y^R(\theta).$$

Since $u^S_1(y^R(\theta), \theta, b) > 0$ and $y'^R(\theta) \equiv \frac{d}{d\theta}y^R(\theta) > 0$, S's incentive compatibility condition requires that

$$\omega_F(\theta) = \omega_F(1) + \int_\theta^1 s \cdot u^S_1(y^R(z), z, b)y'^R(z)dz. \tag{17}$$

From condition (17), the compensation schedule that induces full revelation is strictly decreasing in θ.

Finally, I show that R's payment strategy satisfying condition (17) never satisfies the equilibrium condition. Let $H(\theta)$ be $\mathrm{supp}\{\tilde{\mathbb{P}}(\cdot|\theta)\}$ where $\tilde{\mathbb{P}}(\cdot|\theta)$ is the probability measure on $(H^{T+1}, \mathbb{B}(H^{T+1}))$ induced by (σ, ρ) given θ: the set of h^{T+1} that has a positive probability under the given ξ_F when the true state is θ.

Step 1: Fix $\overline{\theta} \in (0,1)$ and $(\overline{m}_1, \overline{w}_1, \ldots, \overline{m}_T, \overline{w}_T) \in H(\overline{\theta})$. Then, there exists $\overline{t} < T$ such that $\overline{w}_{\overline{t}} > 0$ and $\overline{w}_t = 0$ for any $t > \overline{t}$. Moreover, $\sum_{t=1}^{\overline{t}} \overline{w}_t = \omega_F(\overline{\theta})$ holds. If $\mathrm{supp}\{f(\cdot|(\overline{m}_1, \overline{w}_1, \ldots, \overline{m}_{\overline{t}})\} = \{\overline{\theta}\}$, R has no incentive to pay $\overline{w}_{\overline{t}}$ at this history. Therefore, there exists a $\underline{\theta} \neq \overline{\theta}$ such that $\underline{\theta} \in \mathrm{supp}\{f(\cdot|(\overline{m}_1, \overline{w}_1, \ldots, \overline{m}_{\overline{t}})\}$. Furthermore, since $\omega_F(\theta) < \omega_F(\overline{\theta})$ for $\theta > \overline{\theta}$, I must have $\underline{\theta} < \overline{\theta}$. This implies that there must exist $(\underline{m}_1, \underline{w}_1, \ldots, \underline{m}_T, \underline{w}_T) \in H(\underline{\theta})$ such that $(\underline{m}_1, \underline{w}_1, \ldots, \underline{w}_{\overline{t}}) = (\overline{m}_1, \overline{w}_1, \ldots, \overline{w}_{\overline{t}})$, and $\underline{w}_t > 0$ for some $t \in \{\overline{t}+1, T-1\} \equiv \mathbb{T}_1$.

Step 2: Let $\underline{t}$ be the maximum number that satisfies $\underline{w}_t > 0$. From the definition of $\underline{t}$, $\sum_{t=1}^{\underline{t}} \underline{w}_t = \omega_F(\underline{\theta})$ is satisfied. Similar to Step 1, there exists a $\tilde{\theta} < \underline{\theta}$ such that $\tilde{\theta} \in \mathrm{supp}\{f(\cdot|(\underline{m}_1, \underline{w}_1, \ldots, \underline{m}_{\underline{t}})\}$. Furthermore, there must exist $(\tilde{m}_1, \tilde{w}_1, \ldots, \tilde{m}_T, \tilde{w}_T) \in H(\tilde{\theta})$ such that $(\tilde{m}_1, \tilde{w}_1, \ldots, \tilde{w}_{\underline{t}}) = (\underline{m}_1, \underline{w}_1, \ldots, \underline{w}_{\underline{t}})$, and $\tilde{w}_t > 0$ for some $t \in \{\underline{t}+1, T-1\} \equiv \mathbb{T}_2$.

For ξ_F to be an equilibrium, the above operation must be repeated infinitely regardless of its start point $\overline{\theta}$. However, this is impossible in the set of finite numbers. Hence, I conclude that there exists no fully separating equilibrium. ◇

Appendix 3.D Proof of Proposition 4

Fix an equilibrium ξ. Let (Θ, ω) be a direct contract under which there exists a pure strategy equilibrium that is outcome equivalent to ξ. Let $y(\theta)$ be R's equilibrium strategy under (Θ, ω). Proposition 2 shows that ξ is a partition equilibrium.

Let $[a', a'']$ be an element of the equilibrium partition[30] such that $a' < a''$. Then, I have

- $\lim_{\theta \downarrow a'} s \cdot u^S(y(\theta), \theta, b) + \omega(\theta) = s \cdot \psi(|\overline{y}(a', a'') - \theta' - b|) + \omega(a')$, and
- $s \cdot \psi(|\overline{y}(a', a'') - a'' - b|) + \omega(a'') = \lim_{\theta \uparrow a''} s \cdot u^S(y(\theta), \theta, b) + \omega(\theta)$,

where $\omega(a) = \omega(a') = \omega(a'')$ and $y(\theta) = \overline{y}(a', a'')$ for any $\theta \in [a', a'']$. Moreover, since I assume that $\overline{y}(a', a'') < (a' + a'')/2 + b$, I obtain

$$s \cdot \psi(|\overline{y}(a', a'') - a' - b|) > s \cdot \psi(|\overline{y}(a', a'') - a'' - b|). \tag{18}$$

Let $\hat{\Theta}$ be the set of all boundaries of equilibrium partition. First, I show the following Claim 1.

Claim 1. *If there exists closed intervals $[\theta_{k+1}, \theta_k]$ and $[\theta_j, \theta_{j-1}]$ such that $\theta_{k+1} < \theta_k < \theta_j < \theta_{j-1}$ and $[\theta_{k+1}, \theta_k]$, $[\theta_j, \theta_{j-1}] \subset \hat{\Theta}$,[31] then*

[30] The same argument holds for the cases of $[a', a'')$, $(a', a'']$, and (a', a'')

[31] If $(\theta_{k+1}, \theta_k) \subset \hat{\Theta}$ is satisfied, the $[\theta_{k+1}, \theta_k] \subset \hat{\Theta}$ is also satisfied since $\hat{\Theta}$ is the set of the boundaries of equilibrium partition.

- *ω is strictly decreasing in θ over $[\theta_{k+1}, \theta_k]$ and $[\theta_j, \theta_{j-1}]$, and*
- $\lim_{\theta \uparrow \theta_k} \omega(\theta) \equiv \underline{\omega} > \overline{\omega} \equiv \lim_{\theta \downarrow \theta_j} \omega(\theta)$.

Proof of Claim 1. First, I show that ω is strictly decreasing in θ over $[\theta_{k+1}, \theta_k]$ and $[\theta_j, \theta_{j-1}]$. For all $\theta \in [\theta_{k+1}, \theta_k]$, the truth telling to be a best response requires that

$$s \cdot u^S(y^R(\theta), \theta, b) + \omega(\theta) \geq s \cdot u^S(y^R(\theta'), \theta, b) + \omega(\theta') \quad \text{for all } \theta', \theta \in [\theta_{k+1}, \theta_k].$$

The first-order condition for S results in the differential equation

$$\frac{d}{d\theta}\omega(\theta) = -s \cdot u_1^S(y^R(\theta), \theta, b)\frac{d}{d\theta}y^R(\theta).$$

Since $u_1^S(y^R(\theta), \theta, b) > 0$, and $y'^R(\theta) \equiv \frac{d}{d\theta}y^R(\theta) > 0$, S's incentive compatibility condition requires that

$$\omega(\theta) = \omega(\theta_k) + \int_\theta^{\theta_k} s \cdot u_1^S(y^R(z), z, b)y'^R(z)dz. \tag{19}$$

The same argument holds for interval $[\theta_j, \theta_{j-1}]$. Hence, I obtain

$$\omega(\theta) = \omega(\theta_{j-1}) + \int_\theta^{\theta_{j-1}} s \cdot u_1^S(y^R(z), z, b)y'^R(z)dz. \tag{20}$$

From conditions (19) and (20), the given compensation schedule is strictly decreasing in θ over $[\theta_{k+1}, \theta_k]$ and $[\theta_j, \theta_{j-1}]$. To simplify the proof, I now suppose that there exists no closed interval $[\underline{\theta}, \overline{\theta}] \subset (\theta_k, \theta_j)$ such that $[\underline{\theta}, \overline{\theta}] \subset \hat{\Theta}$. The equilibrium payoffs of S of type θ_k and θ_j are $s \cdot \psi(b) + \underline{\omega}$ and $s \cdot \psi(b) + \overline{\omega}$, respectively.

From condition (18), I conclude that $s \cdot u^S(y(\theta), \theta, b) + \omega(\theta)$ is strictly decreasing in θ over $[\theta_k, \theta_j] \cap \hat{\Theta}$. Therefore, the following must be satisfied

$$\begin{aligned}\lim_{\theta \uparrow \theta_k} s \cdot u^S(y(\theta), \theta, b) + \omega(\theta) &= s \cdot \psi(b) + \underline{\omega} \\ &> s \cdot \psi(b) + \overline{\omega} = \lim_{\theta \downarrow \theta_j} s \cdot u^S(y(\theta), \theta, b).\end{aligned}$$

This outcome completes the proof of Claim 1. □

Now, I suppose that there exists an interval which is subset of $\hat{\Theta}$. Let $[\theta_{k+1}, \theta_k]$ be the leftmost interval such that $[\theta_{k+1}, \theta_k] \subset \hat{\Theta}$ and $\theta_{k+1} < \theta_k$. By Claim 1, for almost every $\theta \in [\theta_{k+1}, \theta_k]$, there is no $\tilde{\theta} \in \Theta \setminus [\theta_{k+1}, \theta_k]$ such that $\omega(\tilde{\theta}) = \omega(\theta)$. In the same way as the proof of Appendix 3.C, I can prove that this result contradicts the fact that the given strategy profile is an equilibrium.[32] Therefore, the equilibrium partition does not include any separating interval.

Next, I show that the cardinality of $\hat{\Theta}$ is finite. I prove this by contradiction. Suppose that the cardinality of $\hat{\Theta}$ is countably infinite: $\{\mathcal{I}_\lambda\}_{\lambda \in \mathbb{N}}$. Let $[a_{n+1}, a_n)$ and $[a_n, a_{n-1})$ be adjacent elements of equilibrium partition. I denote by ω_j the payment amount S of type $\theta \in [a_j, a_{j-1})$ receives. I have the following Claim 2.

[32] The proof is a straightforward application of each step I take in Appendix 3.C. Therefore, it is omitted.

Claim 2. $\#\{[a_n, a_{n-1}) \in \{\mathcal{I}_\lambda\}_{\lambda\in\mathbb{N}} : \omega_n \geq \omega_{n+1}\} < +\infty$.

Proof of Claim 2. Since S of type $\theta = a_n$ is indifferent between $[a_{n+1}, a_n)$ and $[a_n, a_{n-1})$, the following must be satisfied:

$$s \cdot \psi(|\overline{y}(a_{n+1}, a_n) - a_n - b|) + \omega_{n+1} = s \cdot \psi(|\overline{y}(a_n, a_{n-1}) - a_n - b|) + \omega_n.$$

Hence, if $\omega_n \geq \omega_{n+1}$ holds, I have $s \cdot \psi(|\overline{y}(a_{n+1}, a_n) - a_n - b|) \geq s \cdot \psi(|\overline{y}(a_n, a_{n-1}) - a_n - b|)$. Since $s \cdot \psi(|\overline{y}(a_{n+1}, a_n) - a_n - b|)$ is increasing in $a_{n+1} \in [0, a_n]$, if $\omega_n \geq \omega_{n+1}$ holds, I must have

$$s \cdot \psi(b) \geq s \cdot \psi(|\overline{y}(a_n, a_{n-1}) - a_n - b|).$$

$\psi(|y - a_n - b|)$ is strictly increasing in $y \in [0, a_n + b]$ and strictly decreasing in $y \in [a_n + b, \infty)$, and $a_n < \overline{y}(a_n, a_{n-1}) < (a_n + a_{n-1})/2 + b$. Therefore, if $a_{n-1} - a_n \leq b$ is satisfied, $s \cdot \psi(b) < s \cdot \psi(|\overline{y}(a_n, a_{n-1}) - a_n - b|)$. This means that if $\omega_n \geq \omega_{n+1}$ holds, I must have $a_{n-1} - a_n > b$. Therefore, it is satisfied that $\#\{[a_n, a_{n-1}) \in \{\mathcal{I}_\lambda\}_{\lambda\in\mathbb{N}} : \omega_n \geq \omega_{n+1}\} < 1/b$. This completes the proof of Claim 2. □

Claim 2 implies that if the cardinality of $\hat{\Theta}$ is countably infinite, there exists an infinite sequence $\{[a_j, a_{j-1})\}_{j\in\mathbb{N}} \subset \{\mathcal{I}_\lambda\}_{\lambda\in\mathbb{N}}$ such that $\omega_j < \omega_{j+1}$, and $\omega_j \neq \omega(\theta)$ for $\theta \in [0,1] \setminus \{[a_j, a_{j-1})\}_{j\in\mathbb{N}}$. In the same way as the proof of Appendix 3.C, I can prove that this result contradicts the fact that the given strategy profile is an equilibrium.[33] Therefore, the cardinality of $\hat{\Theta}$ must be finite. Claim 1 and Claim 2 conclude that all equilibria are finite partition equilibria. ◇

Appendix 3.E Discussion of Assumption 2

Assumption 2 guarantees Claim 1 that plays a critical role to prove the finiteness of equilibrium partition. To see this, suppose that Assumption 2 does not hold. Then, if there is a pair of $\underline{\theta}$, θ_1, and $\overline{\theta}$ such that

1. $0 < \underline{\theta} < \theta_1 < \overline{\theta} < 1$, and $1 - \overline{\theta} = \underline{\theta}$;
2. $\omega(\theta) = \omega_1$ for $\theta \in (\underline{\theta}, \overline{\theta})$, and

$$\omega(\theta) = \begin{cases} \hat{\omega} + \int_{\theta}^{1} s \cdot u_1^S(y^R(z), z, b) y'^R(z) dz & \text{for } \theta \in [\overline{\theta}, 1], \\ \hat{\omega} + \int_{\theta}^{\overline{\theta}} s \cdot u_1^S(y^R(z), z, b) y'^R(z) dz & \text{for } \theta \in [0, \underline{\theta}], \end{cases}$$

where $u^S = \psi$;[34]

3. $s \cdot \psi(|\overline{y}(\underline{\theta}, \theta_1) - \theta - b|) = s \cdot \psi(|\overline{y}(\theta_1, \overline{\theta}) - \theta_1 - b|)$, $s \cdot \psi(b) + \hat{\omega} = s \cdot \psi(|\overline{y}(\underline{\theta}, \theta_1) - \underline{\theta} - b|) + \omega_1$, and $s \cdot \psi(|\overline{y}(\theta_1, \overline{\theta}) - \overline{\theta} - b|) + \omega_1 = s \cdot \psi(b) + \hat{\omega} + \int_{\overline{\theta}}^{1} s \cdot u_1^S(y^R(z), z, b) y'^R(z) dz$; then

the following strategy profile can be constitute an equilibrium, which has separating intervals.

[33] The proof is a straightforward application of each step I take in Appendix 3.C. Therefore, it is omitted.

[34] Clearly, the given $\omega(\theta)$ does not hold Claim 1.

In the first period, S reveals whether θ belongs to $(\underline{\theta}, \overline{\theta})$. If $\theta \in (\underline{\theta}, \overline{\theta})$, R pays $\hat{w}$, and then, S reveals whether $\theta < \theta_1$. If $\theta \notin (\underline{\theta}, \overline{\theta})$, R pays ω_1 in period 2. After this payment, S's types $\{\epsilon, \epsilon + \overline{\theta}\}$ pool together and send message m_ϵ. After receiving m_ϵ, R pays $\omega(\epsilon) = \int_\epsilon^{\underline{\theta}} s \cdot u_1^S(y^R(z), z, b) y'^R(z) dz$ in period 2. Note that $\omega(\epsilon) = \omega(\overline{\theta} + \epsilon)$ since $1 - \overline{\theta} = \underline{\theta}$. After receiving $\omega(\epsilon)$, S reveals whether $\theta = \epsilon$ or $\epsilon + \overline{\theta}$. If R deviates in terms of payment in a period, S conveys no information thereafter. S's incentive compatibility condition is met by the second and third condition of the abovementioned requirements. Moreover, if s/r is small enough, R makes a payment to prevent S's babbling. However, even if the Assumption 2 is not satisfied, the existence of the pair $(\underline{\theta}, \theta_1, \overline{\theta})$ is not guaranteed. It remains an open question.

Appendix 3.F Proof of Proposition 5

Since G is the uniform distribution and $u^R(y, \theta, b) = l(|y - \theta|)$, the optimal project for R is given by $\overline{y}(a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1}) = (a_{\tilde{\lambda}} + a_{\tilde{\lambda}-1})/2$ for any $[a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1}) \subset [0, 1]$. Recall that $\omega_{\tilde{\lambda}+1} \leq \omega_{\tilde{\lambda}}$. Therefore, I obtain

$$\psi(|(a_{\tilde{\lambda}+1} - a_{\tilde{\lambda}})/2 - b|) - \psi(|(a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}})/2 - b|) = (\omega_{\tilde{\lambda}} - \omega_{\tilde{\lambda}+1})/s \geq 0. \tag{21}$$

Since $a_{\tilde{\lambda}+1} < a_{\tilde{\lambda}}$, the inequality (21) can be simplified into

$$\psi(b) \geq \psi(|(a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}})/2 - b|). \tag{22}$$

Moreover, since $a_{\tilde{\lambda}} < a_{\tilde{\lambda}-1}$, I obtain

$$b \leq (a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}})/2 - b \quad \Leftrightarrow \quad a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}} \geq 4b. \tag{23}$$

Therefore, I obtain

$$\begin{aligned} r \int_{a_{\tilde{\lambda}}}^{a_{\tilde{\lambda}-1}} g(\theta) u^R(\overline{y}(a_{\tilde{\lambda}}, a_{\tilde{\lambda}-1}), \theta) d\theta &= r \int_{a_{\tilde{\lambda}}}^{a_{\tilde{\lambda}-1}} l(|(a_{\tilde{\lambda}-1} - a_{\tilde{\lambda}})/2 - \theta|) \\ &< r \int_0^{4b} l(|2b - \theta|). \end{aligned}$$

Appendix 4.A Proof of Proposition 7

For S's incentive compatibility condition to be satisfied, the partition $\{[a_2, a_1), [a_1, 1]\}$ must coincide with the 2-element equilibrium partition achieved in the one-shot cheap talk game where θ is drawn from the uniform distribution over $[0, a_1)$. By CS, the boundary a_1 satisfies that

$$-s\left(\frac{a_1}{2} - a_1 - b\right)^2 = -s\left(\frac{a_1 + a_2}{2} - a_1 - b\right)^2.$$

This equation implies that

$$1 - a_1 = a_1 - a_2 + 4b. \tag{24}$$

Moreover, similar to the condition (8), the indifference condition for S of type $\theta = a_2$ induces the following equation:

$$\tilde{w} = s\{(a_2 + a_1)/2 - (a_2 + b)\}^2 - s(a_2/2 - (a_2 + b))^2.$$

The value of $\tilde{w}$ is positive if and only if $a_1 - a_2 > a_2 + 4b$. This means that $a_1 - a_2 > 4b$. Hence, I obtain

$$\begin{aligned}(a_2 - 0) + (a_1 - a_2) + (1 - a_1) &= 2(a_1 - a_2) + 4b + a_2 \\ &> 12b + 3a_2.\end{aligned}$$

Since I now suppose that $b \in (1/12, 1/4)$, I obtain $12b + 3a_2 > 1$. Therefore, boundaries of the partition and the payment $\tilde{w}$ are not well defined. This outcome implies that I cannot construct a 3-element partition equilibrium described in Proposition 7. ◇

Appendix 4.B Proof of Proposition 10

First, I now ensure of the optimality of S's strategy. At history $h^{(t,1)}$ such that $w_{t'} < w^*_{t'}$ or $m_{t'} \geq a_{t'}$ for some $t' < t$, any type of S randomly sends a message according to the same distribution, a uniform distribution over $[0, 1]$. Therefore, there is no profitable deviation for S at such a history.

At history $h^{(1,1)}$ or $h^{(t,1)}$ such that $m_{t'} < a_{t'}$ and $w_{t'} \geq w^*_{t'}$ for all $t' < t$, if S of type θ sends $m_t \geq a_t$, then he will obtain $-s\left((a_t + a_{t-1})/2 - (\theta + b)\right)^2$ in the future. Otherwise, the continuation payoff of S can be $-s\left((a_{\tilde{t}+1} + a_{\tilde{t}})/2 - (\theta + b)\right)^2 + \sum_{l=t}^{\tilde{t}} w^*_l$ for some $\tilde{t} \in \{t, \ldots, T\}$. Since a_t and w^*_t satisfy (14)–(15), it is easy to verify that for any $\tilde{t} \in \{t, \ldots, T\}$,

$$-s\left(\frac{a_t + a_{t-1}}{2} - (\theta + b)\right)^2 > -s\left(\frac{a_{\tilde{t}+1} + a_{\tilde{t}}}{2} - (\theta + b)\right)^2 + \sum_{l=t}^{\tilde{t}} w^*_l \qquad \text{for any } \theta > a_t, \quad (25)$$

$$-s\left(\frac{a_t + a_{t-1}}{2} - (\theta + b)\right)^2 < -s\left(\frac{a_{\tilde{t}+1} + a_{\tilde{t}}}{2} - (\theta + b)\right)^2 + \sum_{l=t}^{\tilde{t}} w^*_l \quad \text{for any } \theta \in [a_{\tilde{t}+1}, a_{\tilde{t}}), \quad (26)$$

$$-s\left(\frac{a_t + a_{t-1}}{2} - (\theta + b)\right)^2 = -s\left(\frac{a_{t+1} + a_t}{2} - (\theta + b)\right)^2 + w^*_t \qquad \text{for } \theta = a_t. \quad (27)$$

Moreover, take $\theta = a_t$, then t solves

$$\max_{\tilde{t} \in \{t, \ldots, T\}} \left\{-s\left(\frac{a_{\tilde{t}+1} + a_{\tilde{t}}}{2} - (\theta + b)\right)^2 + \sum_{l=t}^{\tilde{t}} w^*_l\right\}.$$

Hence, (25)–(27) imply that there is no profitable deviation for S from ξ_T.

Next, I ensure of the optimality of R's strategy. At any history $h^{T+1} \in H^{T+1}$, the posterior belief $f(\theta | h^{T+1} \equiv (h^{(T,2)}, w_T)) = f(\theta | h^{(T,2)})$ is a uniform distribution supported on an interval whose mid-point is equal to $\frac{\min I(h^{T+1}) + \max I(h^{T+1})}{2}$. Therefore, $y = \frac{\min I(h^{T+1}) + \max I(h^{T+1})}{2}$ is an optimal project for R at any $h^{T+1} \in H^{T+1}$.

Consider a history $h^{(t,2)}$ for $t \in \{1, \ldots, T-1\}$. If $w_{t'} < w^*_{t'}$ or $m_{t'} \geq a_{t'}$ for some $t' < t$, then R has no chance to obtain additional information in the future. Therefore, she must pay nothing to S at such a history. If $m_{t'} < a_{t'}$ and $w_{t'} \geq w^*_{t'}$ for all $t' < t$ and $m_t < a_t$,[35] by paying $w_t \geq w^*_t$, R obtains $u^*_t(w_t)$:

$$\begin{aligned} u^*_t(w_t) &= -w_t - \sum_{i=t+1}^{T} w^*_i \frac{a_i}{a_t} - r \sum_{i=t}^{T} \int_{a_{i+1}}^{a_i} \frac{1}{a_t} \left(\frac{a_{i+1} + a_i}{2} - \theta \right)^2 d\theta \\ &= -w_t - \sum_{i=t+1}^{T} w^*_i \frac{a_i}{a_t} - r \left(\frac{a_{T-1} b^2}{a_t} + \frac{(a_{T-1})^3}{48 a_t} + (T - 1 - t) \frac{(a)^3}{48 a_t} \right). \end{aligned}$$

On the other hand, by paying $w_t < w^*_t$, she obtains $\overline{u}_t(w_t)$:

$$\begin{aligned} \overline{u}_t(w_t) &= -w_t - r \int_0^{a_t} \frac{1}{a_t} \left(\frac{a_t}{2} - \theta \right)^2 d\theta \\ &= -w_t - r \frac{(a_t)^2}{12}. \end{aligned}$$

Clearly, $u^*_t(w_t)$ and $\overline{u}_t(w_t)$ have a maximum at $w_t = w^*_t$ and $w_t = 0$, respectively. Therefore, paying w^*_t is optimal for R if and only if $u^*_t(w^*_t) \geq \overline{u}_t(0)$

$$\iff r \left(-\frac{a_{T-1} b^2}{a_t} - \frac{(a_{T-1})^3}{48 a_t} - (T - 1 - t) \frac{(a)^3}{48 a_t} + \frac{(a_t)^2}{12} \right) \geq \sum_{i=t}^{T} w^*_i \frac{a_i}{a_t}. \tag{28}$$

By making a sufficiently close to 0, the left-hand side of this inequality can be made as close to $r(1/16 - b^2)$ as desired and the right-hand side of this inequality can be made as close to $s(1 + 12)(1 + 4b)/16$ as desired. It is obvious that if $s/r < (1 - 4b)/(1 + 12b)$, there exists $\tilde{a}(b, T) > 0$ such that if $a < \tilde{a}(b, T)$, then $u^*_t > \overline{u}_t$ for any $t \in \{1, \ldots, T - 1\}$. Take $a < \min \left\{ \frac{1+12b}{T+1}, \frac{1-4b}{T-1}, \tilde{a}(b, T) \right\}$. Then, ξ_T constitutes an equilibrium. ◇

Appendix 4.C Proof of Proposition 11

I now impose a condition, $a = \{1 - (4b + \varepsilon)\}/(T - 1)$, on ξ_T. Since $a_{T-1} = 4b + \varepsilon \in (4b, 1)$, $\varepsilon \in (0, 1 - 4b)$ must be satisfied. Moreover, $a = \{1 - (4b + \varepsilon)\}/(T - 1) < (1 - 4b)/(T - 1) < (1 + 4b)/(T - 3)$ holds. Therefore, if $a < (1 + 12b)/(T + 1)$, a_i and w^*_t are well defined. I now suppose that $T > \tilde{T}(b) \equiv 1/8b + 1/2$, and then $a < (1 + 12b)/(T + 1)$ for any $\varepsilon \in (0, 1 - 4b)$. Let ξ_ε be this modified strategy profile and system of beliefs. The following lemma shows that if r is large relative to s, then ξ_ε can be an equilibrium.

Lemma 5. *Fix $b \in (0, 1/4)$, and $T \geq \tilde{T}(b)$. Then, for any $\varepsilon \in (0, 1 - 4b)$, there exists $\underline{\eta}(b, T, \varepsilon)$ such that if $s/r < \underline{\eta}(b, T, \varepsilon)$, then ξ_ε can be an equilibrium.*

Proof of Lemma 5. It is obvious that the restriction $a = \{1 - (4b + \varepsilon)\}/(T - 1)$ affects only R's optimal decision at $h^{(t,2)}$ such that $m_{t'} < a_{t'}$ and $w_{t'} \geq w^*_{t'}$ for all $t' < t - 1$ and $m_t < a_t$. Therefore, I have only to ensure whether inequality (28) holds.

The left-hand side of the inequality (28) can be simplified into

$$r \frac{a_{T-1}}{a_t} \left\{ \frac{4(a_t)^3 - (a_{T-1})^3 - (T - 1 - t) a^3}{48 a_{T-1}} - \frac{a_{T-1}}{a_t} b^2 \right\}.$$

[35] R learns $\theta < a_t$ at the immediately preceding stage.

Since $a_t = a_{T-1} + (T-1-t)a$, I obtain

$$\frac{4(a_t)^3 - (a_{T-1})^3 - (T-1-t)a^3}{48a_{T-1}} > \frac{(a_t)^3}{16a_{T-1}} > \frac{(a_t)^2}{16} > b^2 > \frac{a_{T-1}}{a_t}b^2.$$

This implies that

$$\frac{a_{T-1}}{a_t}\left\{\frac{4(a_t)^3 - (a_{T-1})^3 - (T-1-t)a^3}{48a_{T-1}} - \frac{a_{T-1}}{a_t}b^2\right\} > 0.$$

Moreover, since $w_t^* > 0$, the right-hand side of the inequality (28) is higher than 0. Therefore, I obtain

$$\begin{aligned} u_t^*(w_t^*) \geq \overline{u}_t(0) \qquad &\Longleftrightarrow \\ &\frac{s}{r} < \frac{\frac{a_{T-1}}{a_t}\left\{\frac{4(a_t)^3-(a_{T-1})^3-(T-1-t)a^3}{48a_{T-1}} - \frac{a_{T-1}}{a_t}b^2\right\}}{\frac{1}{s}\sum_{i=t}^{T} w_i^* \frac{a_i}{a_t}}. \end{aligned} \tag{29}$$

Note that the value of w_i^*/s does not depend on s. Now, I conclude that there exists $\underline{\eta}(b, T, \varepsilon)$ such that if $s/r < \underline{\eta}(b, T, \varepsilon)$, then the inequality (29) holds and ξ_ε constitutes an equilibrium. □

I denote by $EU^R(\varepsilon)$ the ex ante expected payoff of R under a strategy profile ξ_ε.

$$EU^R(\varepsilon) = rW(\varepsilon) - \sum_{i=1}^{T} w_i^* a_i.$$

$rW(\varepsilon)$ denotes the expected revenue from the project under ξ_ε:

$$\begin{aligned} rW(\varepsilon) &= -r\sum_{i=1}^{T+1}\int_{a_i}^{a_{i-1}}\left(\frac{a_i + a_{i-1}}{2} - \theta\right)^2 d\theta \\ &= r\left[-(4b+\varepsilon)b^2 - \frac{(4b+\varepsilon)^3}{48} - \frac{1}{48}\frac{\{1-(4b+\varepsilon)\}^3}{(T-1)^2}\right]. \end{aligned}$$

There exists $\varepsilon(b, d) > 0$ such that if $\varepsilon \in (0, \varepsilon(b, d))$, then

$$r\left[-(4b+\varepsilon)b^2 - \frac{(4b+\varepsilon)^3}{48}\right] > -\frac{16}{3}rb^3 - rd.$$

This implies that for any $\varepsilon \in (0, \varepsilon(b, d))$, there exists $\overline{T}(b, \varepsilon, d)$ such that for any $T \geq \overline{T}(b, \varepsilon, d)$,

$$rW(\varepsilon) > -\frac{16}{3}rb^3 - rd. \tag{30}$$

Recall that w_i^* is linearly increasing in s for all $i \in \{1, \ldots, T\}$. Suppose that $T \geq \overline{T}(b, \varepsilon, d)$. Then, for any $\varepsilon \in (0, \varepsilon(b, d))$, there exists $\hat{\eta}(b, T, \varepsilon, d)$ such that

$$s/r < \hat{\eta}(b, T, \varepsilon, d) \equiv \frac{W(\varepsilon) + 16b^3/3 + d}{\frac{1}{s}\sum_{i=1}^{T} w_i^* a_i} \quad \Longrightarrow \quad EU^R(\varepsilon) > -16rb^3/3 - rd.$$

By Lemma 5, if $s/r < \eta(b,T,\varepsilon)$, then ξ_ε constitutes an equilibrium. Therefore, if $s/r < \tilde{\eta}(b,T,\varepsilon,d) \equiv \min\{\overline{\eta}(b,T,\varepsilon,d), \underline{\eta}(b,T,\varepsilon)\}$, the strategy profile ξ_ε constitutes an equilibrium under which $EU^R(\varepsilon) > -16rb^3/3 - rd$.

I define $\overline{T}(b,d)$ and $\mathbb{E}(b,d)$ as follows.

$$\overline{T}(b,d) \equiv \max\left\{\min_{\varepsilon\in(0,\varepsilon(b,d))} \overline{T}(b,\varepsilon,d), \tilde{T}(b)\right\}, \text{ and}$$
$$\mathbb{E}(b,d) \equiv \left\{\varepsilon \in (0,\varepsilon(b,d)) : \overline{T}(b,\varepsilon,d) = \overline{T}(b,d)\right\}.$$

Define $\underline{\eta}(b,T,d)$ as follows:

$$\underline{\eta}(b,T,d) \equiv \sup_{\varepsilon\in\mathbb{E}(b,d)} \tilde{\eta}(b,T,\varepsilon,d).$$

Suppose that ξ_ε constitutes an equilibrium of $\Gamma(b,s,r,T)$ where R obtains $EU^R(\varepsilon) > -16rb^3/3 - rd$. Consider $\Gamma(b,s,r,T')$ where $T' > T$. Now, construct a strategy profile ξ'_ε by modifying ξ_ε. In particular, under ξ'_ε, players follow ξ_ε until period T, and then S conveys no information and R never pays money to S in the future. It is obvious that ξ'_ε constitutes an equilibrium of $\Gamma(b,s,r,T')$ and R's equilibrium payoff is equal to $EU^R(\varepsilon) > -16rb^3/3 - rd$. Hence, by taking $\underline{\eta}(b,d)$ as $\underline{\eta}(b,\overline{T}(b,d),d)$, I complete the proof. $\diamond$

Appendix 5.A Proof of Proposition 13

Formally, the strategy profile $(\hat{\sigma},\hat{\rho})$ is defined as follows. At stage 1 in period 1, S of type $\theta \geq \hat{a}_1$ sends a message m_1 randomly according to a uniform distribution over $[\hat{a}_1, 1]$, and S of type $\theta < \hat{a}_1$ sends a message m_1 randomly according to a uniform distribution over $[0,\hat{a}_1)$. If $m_1 < \hat{a}_1$, then R pays a certain amount of money, w^*, to S at stage 2 in period 1. Otherwise, she pays nothing to S. If $m_1 < \hat{a}_1$ and $w_1 \geq w^*$, then, at stage 1 in period 2, S of type $\theta \geq \hat{a}_2$ randomly sends a message m_2 according to a uniform distribution over $[\hat{a}_2, 1]$, and S of type $\theta \in [\hat{a}_{i+1},\hat{a}_i)$, for $i \in \{2,\ldots,\tilde{n}\}$, randomly sends a message m_2 according to a uniform distribution over $[\hat{a}_{i+1},\hat{a}_i)$. Otherwise, S randomly sends a message m_2 according to uniform distribution over $[0,1]$ regardless of his type. In period $t \geq 2$, R always pays nothing to S. In period $t \geq 3$, S always sends babbling message. In period $T+1$, R chooses a project $\rho(h^{T+1}) \equiv \arg\max_y \int u^R(y,\theta) f(\theta|h^{T+1})d\theta$.

Let $\mathcal{H}$ be the set of all histories where R makes a decision, $\mathcal{H} \equiv \{\bigcup_{t=1}^{T} H^{(t,2)}\} \cup H^{T+1}$. I denote by $I(h)$ the closure of the set $\{\theta \in \Theta : f(\theta|h \in \mathcal{H}) > 0\}$. Under the belief system $\hat{f}$, I obtain $I(h^{T+1}) \in \{[\hat{a}_{\tilde{n}+1},\hat{a}_{\tilde{n}}],\ldots[\hat{a}_1,\hat{a}_0],[\hat{a}_{\tilde{n}+1},\hat{a}_1]\}$ for any $h^{T+1} \in H^{T+1}$. Therefore, at h^{T+1} such that $I(h^{T+1}) = [\hat{a}_{i+1},\hat{a}_i]$ for $i \in \{0,\ldots\tilde{n}\}$, the optimal project for R is $\overline{y}(\hat{a}_{i+1},\hat{a}_i) = \arg\max_y \int_{\hat{a}_{i+1}}^{\hat{a}_i} u^R(y,\theta)g(\theta)d\theta$, and at h^{T+1} such that $I(h^{T+1}) = [\hat{a}_{\tilde{n}+1},\hat{a}_1]$, the optimal project for R is $\overline{y}(\hat{a}_{\tilde{n}+1},\hat{a}_1) = \arg\max_y \int_{\hat{a}_{\tilde{n}+1}}^{\hat{a}_1} u^R(y,\theta)g(\theta)d\theta$. Hence, $\hat{\rho}(h^{T+1})$ becomes $\overline{y}(\hat{a}_{i+1},\hat{a}_i)$ at h^{T+1} such that $I(h^{T+1}) = [\hat{a}_{i+1},\hat{a}_i]$ for $i \in \{0,\ldots\tilde{n}\}$, and $\hat{\rho}(h^{T+1})$ becomes $\overline{y}(\hat{a}_{\tilde{n}+1},\hat{a}_1)$ at h^{T+1} such that $I(h^{T+1}) = [\hat{a}_{\tilde{n}+1},\hat{a}_1]$.

In period $t \geq 2$, R always pays nothing to S, which implies that $\{[\hat{a}_{i+1},\hat{a}_i)\}_{i=1}^{\tilde{n}}$ must coincide with the $\tilde{n}$-element equilibrium partition achieved in a model with one-shot information transmission where θ is drawn from a distribution with density $\{g(\theta)/G(\hat{a}_1)\} \cdot \mathbb{1}_{[0,\hat{a}_1)}(\theta)$. Therefore, the boundaries of this partition, $\{[\hat{a}_{i+1}, \hat{a}_i)\}_{i=1}^{\tilde{n}}$, must be solutions to the following non-linear difference equation whose initial and terminal conditions are $a_1 = \hat{a}_1$ and $a_{\tilde{n}+1} = 0$: for $i = 2, \ldots, \tilde{n}$,

$$s \cdot u^S(\overline{y}(a_{i+1}, a_i), a_i, b) - s \cdot u^S(\overline{y}(a_i, a_{i-1}), a_i, b) = 0. \tag{31}$$

When $\hat{a}_1 = 1$, the solution to (31) induces a partition that coincides with $\tilde{n}$-element equilibrium partition in the one-shot cheap talk game.[36] Moreover, the solution to (31) varies continuously with respect to initial condition $a_1 = \hat{a}_1$. Recall that a solution to (1)–(3) in Section 3.1 induces a partition: $0 = \tilde{a}_{\tilde{n}} < \cdots < \tilde{a}_1 < \tilde{a}_0 = 1$. Therefore, there exists $\underline{x} \in (\tilde{a}_1, 1)$ such that (31) is well defined for all $\hat{a}_1 \in (\underline{x}, 1)$. Let $\underline{a}_1$ be the minimum value of $\underline{x}$ such that for all $\hat{a}_1 \in (\underline{x}, 1)$, the solution to (31) induces an $\tilde{n}$-element partition: $0 = \hat{a}_{\tilde{n}+1} < \hat{a}_{\tilde{n}} < \cdots < \hat{a}_1 = \hat{a}_1$. Since the solution to (31) does not depends on both s and r, the value of $\underline{a}_1$ also does not depends on both s and r.

Suppose that $\{\hat{a}_2, \ldots, \hat{a}_{\tilde{n}+1}\}$ is a solution to (31) where $\hat{a}_1 \in (\underline{a}_1, 1)$. Then, there is no profitable deviation for S from $\hat{\sigma}$ at any $h_\theta^{(2,1)}$ such that $m_1 < \hat{a}_1$ and $w_1 \geq w^*$. Moreover, S conveys no information at any $h_\theta^{(2,1)}$ such that $m_1 \geq \hat{a}_1$, or $m_1 < a_1$ and $w_1 < w^*$. The same can be said at any $h_\theta^{(t,1)}$ for $t \geq 3$. This implies that if $\{\hat{a}_1, \ldots, \hat{a}_{\tilde{n}+1}\}$ is a solution to (31) where $\hat{a}_1 \in (\underline{a}_1, 1)$, then $\hat{\sigma}$ is optimal for S at any $h_\theta^{(t,1)}$ for $t \geq 2$.

At stage 1 in period 1, if S of type θ sends $m_1 \geq \hat{a}_1$, then he obtains $s \cdot u^S(\overline{y}(\hat{a}_1, 1), \theta, b)$. Otherwise, S of type $\theta \geq \hat{a}_2$ obtains $s \cdot u^S(\overline{y}(\hat{a}_2, \hat{a}_1), \theta, b) + w^*$, and S of type $\theta \in [\hat{a}_{i+1}, \hat{a}_i)$, for $i \geq 2$, obtains $s \cdot u^S(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta, b) + w^*$. I assume that $u_{11}^S(y, \theta, b) < 0$ and $u_{12}^S(y, \theta, b) > 0$. Moreover, $\overline{y}(\hat{a}_{i+1}, \hat{a}_i) > \overline{y}(\hat{a}_i, \hat{a}_{i-1})$ holds. Therefore, if the following is satisfied

$$s \cdot u^S(\overline{y}(\hat{a}_1, 1), \hat{a}_1, b) - s \cdot u^S(\overline{y}(\hat{a}_2, \hat{a}_1), \hat{a}_1, b) = w^*, \quad \text{then} \tag{32}$$

$$s \cdot u^S(\overline{y}(\hat{a}_1, 1), \theta, b) \geq \max_{j \in \{1,\ldots,\tilde{n}\}} s \cdot u^S\left(\overline{y}(\hat{a}_{j+1}, \hat{a}_j), \theta, b\right) + w^* \quad \text{for } \theta \geq \hat{a}_1, \quad \text{and} \tag{33}$$

$$s \cdot u^S(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta, b) + w^* > s \cdot u^S(\overline{y}(\hat{a}_1, 1), \theta, b) \quad \text{for } i \geq 1 \text{ and } \theta \in [\hat{a}_{i+1}, \hat{a}_i). \tag{34}$$

When (33) and (34) hold, S has no incentive to deviate from $\hat{\sigma}$ at stage 1 in period 1. Since I assume that R's payment must be non-negative, w^* must be non-negative. If $w^*(\hat{a}_1) = 0$ for some $\hat{a}_1 \in (\underline{a}_1, 1)$, then (1)–(3) has a solution: $0 = \hat{a}_{\tilde{n}+1} < \hat{a}_{\tilde{n}} < \cdots < \hat{a}_0 = 1$. This is incompatible with the definition of $\tilde{n}$. Hence, R's payment, $w^*(\hat{a}_1) \equiv s \cdot u^S(\overline{y}(\hat{a}_1, 1), \hat{a}_1, b) - s \cdot u^S(\overline{y}(\hat{a}_2, \hat{a}_1), \hat{a}_1, b)$, which holds for equation (32), is positive for any $\hat{a}_1 \in (\underline{a}_1, 1)$. Since $\overline{y}(\hat{a}_1, 1)$, $\overline{y}(\hat{a}_2, \hat{a}_1)$ and $\hat{a}_2$ is continuous in $\hat{a}_1 \in (\underline{a}_1, 1]$, $w^*(\hat{a}_1)$ is continuous in $\hat{a}_1 \in (\underline{a}_1, 1]$. Moreover, since $\hat{a}_2 = \tilde{a}_1$ when $\hat{a}_1 = 1$, I obtain $w^*(1) = s \cdot u^S(y^R(1), 1, b) - s \cdot u^S(\overline{y}(\tilde{a}_1, 1), 1, b)$. Note that $w^*(1) > 0$ since $u_{11}^S(y, \theta, b) < 0$ and $\overline{y}(\tilde{a}_1, 1) < y^R(1) < y^S(1, b)$. Therefore, $w^*(\hat{a}_1) > 0$ for any $\hat{a}_1 \in (\underline{a}_1, 1]$.

At any $h^{(t,2)}$, R has no incentive to increase the amount of payment because that does not affect S's behavior. Therefore, I have only to ensure the optimality of ρ at $h^{(1,2)}$ such that $m_1 < \hat{a}_1$. At this history, if R pays $w_1 < w^*$, then she obtains $\overline{u}(w_1)$:

$$\overline{u}(w_1) \quad = \quad -w_1 + \frac{r}{G(\hat{a}_1)} \int_0^{\hat{a}_1} u^R(\overline{y}(0, \hat{a}_1), \theta)\, g(\theta) d\theta.$$

[36] Condition M ensures that the above difference equation has at most one solution for the given $\tilde{n}$. See pages 1444–1445 of CS (1982).

On the other hand, by paying $w_1 \geq w^*$ at history $h^{(1,2)}$ such that $m_1 < a_1$, she obtains $u^*(w_1)$:

$$u^*(w_1) = -w_1 + \frac{r}{G(\hat{a}_1)} \sum_{i=1}^{\tilde{n}} \int_{\hat{a}_{i+1}}^{\hat{a}_i} u^R(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta) g(\theta) d\theta.$$

Therefore, paying w^* is an optimal decision for R at $h^{(1,2)}$ such that $m_1 < \hat{a}_1$ if and only if $u^*(w^*) \geq \overline{u}(0) \Longleftrightarrow$

$$\frac{r}{G(\hat{a}_1)} \sum_{i=1}^{\tilde{n}} \int_{\hat{a}_{i+1}}^{\hat{a}_i} u^R(\overline{y}(\hat{a}_{i+1}, \hat{a}_i), \theta) g(\theta) d\theta - \frac{r}{G(\hat{a}_1)} \int_0^{\hat{a}_1} u^R\left(\overline{y}(0, \hat{a}_1), \theta\right) g(\theta) d\theta \geq w^*. \quad (35)$$

I denote by $r \cdot V(\hat{a}_1)$ the left-hand side of inequality (35). $V(\hat{a}_1)$ is continuous in $\hat{a}_1 \in (\underline{a}_1, 1]$, and $V(\hat{a}_1) > 0$ for $\hat{a}_1 \in (\underline{a}_1, 1]$. Moreover, $V(1) = EU^R_{CS,\tilde{n}} - EU^R_{CS,ui}$ where $EU^R_{CS,\tilde{n}} \equiv \sum_{i=1}^{\tilde{n}} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^R(\overline{y}(\tilde{a}_i, \tilde{a}_{i-1}), \theta) g(\theta) d\theta$ and $EU^R_{CS,ui} \equiv \int_0^1 u^R(\overline{y}(0,1), \theta) g(\theta) d\theta$. Let $\alpha(\hat{a}_1)$ be $u^S\left(\overline{y}(\hat{a}_1, 1), \hat{a}_1, b\right) - u^S\left(\overline{y}(\hat{a}_2, \hat{a}_1), \hat{a}_1, b\right)$. In the following part, $s \cdot \alpha(\hat{a}_1)$ denotes R's payment, $w^*(\hat{a}_1)$, which holds for equation (32). Inequality (35) can be simplified into $s/r \leq V(\hat{a}_1)/\alpha(\hat{a}_1)$. It is obvious that $V(\hat{a}_1)/\alpha(\hat{a}_1)$ is continuous in $\hat{a}_1 \in (\underline{a}_1, 1]$, and

$$\frac{V(1)}{\alpha(1)} = \frac{EU^R_{CS,\tilde{n}} - EU^R_{CS,ui}}{u^S(y^*(1), 1, b) - u^S(\overline{y}(\tilde{a}_1, 1), 1, b)} > 0.$$

Therefore, if $s/r < \eta(b) \equiv V(1)/\alpha(1)$, then $\{\hat{a}_1 \in (\underline{a}_1, 1) : s/r \leq V(\hat{a}_1)/\alpha(\hat{a}_1)\} \neq \emptyset$. This outcome implies that if $s/r < \eta(b)$, there exists a non-empty set $\{\hat{a}_1 \in (\underline{a}_1, 1) : s/r \leq V(\hat{a}_1)/\alpha(\hat{a}_1)\}$ such that $((\hat{\sigma}, \hat{\rho}), \hat{f})$ constitutes an $(\tilde{n}+1)$-element partition equilibrium when $\hat{a}_1 \in \{\hat{a}_1 \in (\underline{a}_1, 1) : s/r \leq V(\hat{a}_1)/\alpha(\hat{a}_1)\}$. $\diamond$

Remark 3. Since $V(\hat{a}_1)/\alpha(\hat{a}_1) > 0$ for $\hat{a}_1 \in (\underline{a}_1, 1]$ and $V(\hat{a}_1)/\alpha(\hat{a}_1)$ is continuous in $\hat{a}_1 \in (\underline{a}_1, 1]$, there exists $z \in (\underline{a}_1, 1)$ such that $s/r \leq V(\hat{a}_1)/\alpha(\hat{a}_1)$ holds for any $\hat{a}_1 \in (z, 1)$. Let $\underline{a}_1(s/r)$ be the infimum value of z. Then, the value of $\underline{a}_1(s/r)$ is strictly decreasing and goes to $\underline{a}_1$ as s/r goes to 0.

Appendix 5.B Proof of Proposition 14

First, I show the following Lemma 6.

Lemma 6. *Fix $b > 0$ and suppose that $\tilde{n} \geq 2$. If $s/r < \eta(b)$, there exists a $(\tilde{n}+1)$-element partition equilibrium $((\hat{\sigma}, \hat{\rho}), \hat{f})$ such that S always strictly prefers $((\hat{\sigma}, \hat{\rho}), \hat{f})$ to any equilibrium in the one-shot cheap talk game.*

Proof of Lemma 6. Now, I denote by EU^S_{CS} the ex ante expected payoff of S under the $\tilde{n}$-element partition equilibrium with $\{\tilde{a}_{\tilde{n}}, \ldots, \tilde{a}_0\}$ in the one-shot cheap talk game. I denote by $E\hat{U}^S(x)$ the ex ante expected payoff of S under the $(\tilde{n}+1)$-element partition equilibrium $((\hat{\sigma}, \hat{\rho}), \hat{f})$ with $(\tilde{n}+1)$-element partition: $\{[\hat{a}^x_{\tilde{n}+1}, \hat{a}^x_{\tilde{n}}), \ldots [\hat{a}^x_1, \hat{a}^x_0]\}$ where $\hat{a}^x_1 \equiv x \in (\underline{a}_1(s/r), 1)$.

In the one-shot cheap talk game, under Condition M, S always strictly prefers ex ante $\tilde{n}$-element partition equilibrium to any other equilibria. I have

$$EU^S_{CS} = s \sum_{i=1}^{\tilde{n}} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^S(\overline{y}(\tilde{a}_i, \tilde{a}_{i-1}), \theta, b) g(\theta) d\theta.$$

By Proposition 13, it must be satisfied that $s/r < \eta(b)$ in order for an equilibrium $((\hat{\sigma}, \hat{\rho}), \hat{f})$ to exist. Therefore, in what follows, I suppose that $s/r < \eta(b)$.

The ex ante expected payoff of S under $((\hat{\sigma}, \hat{\rho}), \hat{f})$ is

$$E\hat{U}^S(x) = s\left[\sum_{i=1}^{\tilde{n}+1} \int_{\hat{a}_i^x}^{\hat{a}_{i-1}^x} u^S(\overline{y}(\hat{a}_i^x, \hat{a}_{i-1}^x), \theta, b) g(\theta) d\theta + G(x) \cdot \alpha(x)\right].$$

Recall that $s \cdot \alpha(x) \equiv w^*(x) = s \cdot u^S(\overline{y}(x, 1), x, b) - s \cdot u^S\left(\overline{y}(\hat{a}_2^x, x), x, b\right)$ is positive for $x > \underline{a}_1$, and $s \cdot \alpha(x)$ is continuous in $x > \underline{a}_1$.

Let $\Delta(x)$ denote $E\hat{U}^S(x) - EU_{CS}^S$. Since $\lim_{x \uparrow 1} \Delta(x) = \alpha(1) > 0$ and $\Delta(x)$ is continuous in $x \in (\underline{a}_1, 1]$, there exists $d < 1$ such that $d \geq \underline{a}_1(s/r)$ and

$$\Delta(x) > 0 \text{ for all } x \in (d, 1).$$

This completes the proof of Lemma 6. □

Remark 4. Define $\underline{d}(s/r) \equiv \inf\{d : d \geq \underline{a}_1(s/r) \text{ and } \Delta(x) > 0 \text{ for all } x \in (d, 1)\}$. Since $\underline{a}_1(s/r)$ is decreasing as s/r is decreasing and $\Delta(x)$ does not depend on both s and r, the following is satisfied: $\underline{d}(s/r)$ is decreasing (but not always strictly decreasing) as s/r is decreasing.

Next, I show the following Lemma 7.

Lemma 7. *Fix $b > 0$ and suppose that $\tilde{n} \geq 2$. Then, there exists a positive value $\overline{\eta}(b)$ such that if $s/r < \overline{\eta}(b)$, there exists $\overline{x} \in (\underline{a}_1(s/r), 1)$ such that*

$$E\hat{U}^R(\overline{x}) > EU_{CS}^R.$$

Intuitively, R seems to prefer the $(\tilde{n} + 1)$-element partition with $\{\hat{a}_{\tilde{n}+1}^x, \ldots, \hat{a}_0^x\}$ to the $\tilde{n}$-element partition with $\{\tilde{a}_{\tilde{n}}, \ldots, \tilde{a}_0\}$ since the former has more steps than the latter. As I earlier show, if Condition C holds, then there exists $x < 1$ such that $\hat{W}(x) > EU_{CS}^R$. Fix x, then $\hat{W}(x) - EU_{CS}^R$ is increasing in r. Moreover, since w^* is decreasing and goes to 0 as s goes to 0, the expected payment $E[w^*]$ is also decreasing and goes to 0 as s goes to 0. Thus, if r is large enough relative to s, then there exists $\overline{x}$ such that $E\hat{U}^R(\overline{x}) > EU_{CS}^R$.

Proof of Lemma 7. In common with the proof of Lemma 6, I suppose that $s/r < \eta(b)$.

Let $\delta(x, s, r)$ denote $\{E\hat{U}^R(x) - EU_{CS}^R\}/r$. I obtain

$$\delta(x, s, r) = \hat{W}(x) - \frac{s}{r} G(x) \cdot \alpha(x) - \sum_{i=1}^{\tilde{n}} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^R(\overline{y}(\tilde{a}_i, \tilde{a}_{i-1}), \theta) g(\theta) d\theta.$$

$\delta(x, s, r) > 0$ holds if and only if

$$\overline{\eta}(b, x) \equiv \frac{\hat{W}(x) - \sum_{i=1}^{\tilde{n}} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^R(\overline{y}(\tilde{a}_i, \tilde{a}_{i-1}), \theta) g(\theta) d\theta}{G(x) \cdot \alpha(x)} > \frac{s}{r}.$$

Since x belongs to $(\underline{a}_1(s/r), 1)$ and $\inf_{x \in (\underline{a}_1(s/r), 1)} G(x)\alpha(x) > 0$, $\overline{\eta}(b, x)$ has a least upper bound $\overline{\eta}(b|s/r) = \sup_{x \in (\underline{a}_1(s/r), 1)} \overline{\eta}(b, x)$. Under Condition C, $\overline{\eta}(b, x) > 0$ for some $x \in (\underline{a}_1(s/r), 1)$. This implies that $\overline{\eta}(b|s/r) > 0$. Moreover, since $\underline{a}_1(s/r)$ is not increasing as

s/r is decreasing, $\overline{\eta}(b|s/r)$ is not decreasing as s/r is decreasing. Therefore, I can take a supremum of the value of s/r, which satisfies $\overline{\eta}(b|s/r) > s/r$. I denote this supremum by $\overline{\eta}(b)$. Note that $\overline{\eta}(b) < +\infty$, since $\hat{W}(x) - \sum_{i=1}^{\tilde{n}} \int_{\tilde{a}_i}^{\tilde{a}_{i-1}} u^R(\overline{y}(\tilde{a}_i, \tilde{a}_{i-1}), \theta) g(\theta) d\theta < +\infty$ for any $x \in (\underline{a}_1, 1)$. This completes the proof of Lemma 7. □

Remark 5. Suppose that x almost equal to 1. Then, the partition under the $(\tilde{n} + 1)$-element partition equilibrium almost coincides with the partition under the $\tilde{n}$-element partition equilibrium in the one-shot cheap talk game. Nevertheless, the expected payment of monetary transfer is high (almost coincides with $s\alpha(1)$). Therefore, if $s/r < \eta(b)$, there always exists a $(\tilde{n}+1)$-element partition equilibrium that is unfavorable to R. This means that there always exists $\underline{x} \approx 1$ such that $E\hat{U}^R(\underline{x}) < EU^R_{CS}$.

Finally, I complete the proof of Proposition 14 by demonstrating that if r is large enough relative to s, then I can take $x \in (\underline{a}_1(s/r), 1)$ such that

$$E\hat{U}^R(x) > EU^R_{CS} \text{ and } E\hat{U}^S(x) > EU^S_{CS}.$$

Pproof of Proposition 14 Suppose that $s''/r'' < s'/r' < \overline{\eta}(b)$. In the proof of Lemma 7, I show that $\{x \in (\underline{a}_1(s'/r'), 1) : \delta(x, s', r') > 0\} \neq \emptyset$ and $\{x \in (\underline{a}_1(s''/r''), 1) : \delta(x, s'', r'') > 0\} \neq \emptyset$. Since $\underline{a}_1(s/r)$ is decreasing as s/r is decreasing,

$$\{x \in (\underline{a}_1(s'/r'), 1) : \delta(x, s', r') > 0\} \subset \{x \in (\underline{a}_1(s''/r''), 1) : \delta(x, s'', r'') > 0\}.$$

Moreover, since $\frac{d\hat{W}}{dx}\Big|_{x=1} < 0$, $\frac{d\overline{\eta}}{dx}\Big|_{x=1} < 0$. Furthermore, $\overline{\eta}(b, 1) = 0$ and $\overline{\eta}(b, x)$ is continuous in $x \in (\underline{a}_1, 1)$. Therefore, I obtain

$$\lim_{s/r \downarrow 0} \sup\{x \in (\underline{a}_1(s/r), 1) : \delta(x, s, r) > 0\} = 1.$$

Since $\Delta(x) > 0$ for $x \in (\underline{d}(s/r), 1)$ and $\underline{d}(s/r)$ is not increasing as s/r is decreasing, there exists $\tilde{\eta}(b)$ such that if $s/r < \tilde{\eta}(b)$, then $\{x \in (\underline{a}_1(s/r), 1) : \delta(x, s, r) > 0\} \cap (\underline{d}(s/r), 1) \neq \emptyset$. This completes the proof of Proposition 14. ◇

Appendix 5.C Condition C

Suppose that $s \cdot u^S(y, \theta, b) \equiv -s(y - (\theta + b))^2$, $r \cdot u^R(y, \theta) \equiv -r(y - \theta)^2$, and $G(\theta)$ is uniform distribution over $[0, 1]$. In this case, the boundaries of the partition induced from $((\hat{\sigma}, \hat{\rho}), \hat{f})$ are given by

$$\hat{a}_i^x = \begin{cases} 1 & \text{for } i = 0, \\ x & \text{for } i = 1, \\ \frac{\tilde{n}+1-i}{\tilde{n}} x - 2b(\tilde{n} + 1 - i)(i - 1) & \text{for } i = 2, \dots, \tilde{n}, \\ 0 & \text{for } i = \tilde{n} + 1. \end{cases}$$

Proposition 13 shows that for $\tilde{n} \geq 2$, there exists $\eta(b)$ such that if $s/r < \eta(b)$, then $((\hat{\sigma}, \hat{\rho}), \hat{f})$ constitutes an equilibrium whose partition is induced by $\hat{a}_i^x$ where $x \in (\underline{a}_1(s/r), 1)$. Note that $\overline{y}(\hat{a}_{i+1}^x, \hat{a}_i^x) = (\hat{a}_{i+1}^x + \hat{a}_i^x)/2$ for $i = 0, \dots, \tilde{n}$.

The envelope theorem yields

$$\frac{d}{dx}\hat{W}(x)=\sum_{i=1}^{\tilde{n}} g(\hat{a}_i^x)\frac{d\hat{a}_i^x}{dx}[u^R(\overline{y}(\hat{a}_{i+1}^x,\hat{a}_i^x),\hat{a}_i^x)-u^R(\overline{y}(\hat{a}_i^x,\hat{a}_{i-1}^x),\hat{a}_i^x)].$$

Since $\lim_{x\uparrow 1}\hat{a}_i^x=\tilde{a}_{i-1}$, I obtain

$$\begin{aligned}\left.\frac{d\hat{W}}{dx}\right|_{x=1} &= \sum_{j=1}^{\tilde{n}-1} g(\tilde{a}_j)[u^R(\overline{y}(\tilde{a}_{j+1},\tilde{a}_j),\tilde{a}_j)-u^R(\overline{y}(\tilde{a}_j,\tilde{a}_{j-1}),\tilde{a}_j)]\left.\frac{d\hat{a}_{j+1}^x}{dx}\right|_{x=1}\\ &\quad +g(\tilde{a}_0)[u^R(\overline{y}(\tilde{a}_1,\tilde{a}_0),\tilde{a}_0)-u^R(y^R(\tilde{a}_0),\tilde{a}_0)]\left.\frac{d\hat{a}_1^x}{dx}\right|_{x=1}.\end{aligned}$$

Therefore, I obtain

$$\left.\frac{d\hat{W}}{dx}\right|_{x=1}=\sum_{j=1}^{\tilde{n}-1}\left[-\left(\frac{\tilde{a}_{j+1}-\tilde{a}_j}{2}\right)^2+\left(\frac{-\tilde{a}_j+\tilde{a}_{j-1}}{2}\right)^2\right]\frac{\tilde{n}-j}{\tilde{n}}-\left(\frac{1-\tilde{a}_1}{2}\right)^2.$$

Since $\tilde{a}_j=\frac{\tilde{n}-j}{\tilde{n}}-2bj(\tilde{n}-j)$,

$$-\left(\frac{\tilde{a}_{j+1}-\tilde{a}_j}{2}\right)^2+\left(\frac{-\tilde{a}_j+\tilde{a}_{j-1}}{2}\right)^2>0\quad\text{for } j=1,\dots,\tilde{n}-1.$$

Moreover,

$$\begin{aligned}\sum_{j=1}^{\tilde{n}-1}\left[-\left(\frac{\tilde{a}_{j+1}-\tilde{a}_j}{2}\right)^2+\left(\frac{-\tilde{a}_j+\tilde{a}_{j-1}}{2}\right)^2\right]\frac{\tilde{n}-j}{\tilde{n}} &< \sum_{j=1}^{\tilde{n}-1}\left[-\left(\frac{\tilde{a}_{j+1}-\tilde{a}_j}{2}\right)^2+\left(\frac{-\tilde{a}_j+\tilde{a}_{j-1}}{2}\right)^2\right]\\ &< \left(\frac{1-\tilde{a}_1}{2}\right)^2.\end{aligned}$$

This establishes $\left.\frac{d\hat{W}}{dx}\right|_{x=1}<0$.

Remark 6. Suppose that $s\cdot u^S(y,\theta,b)\equiv -s(y-(\theta+b))^2$, $r\cdot u^R(y,\theta)\equiv -r(y-\theta)^2$, and $G(\theta)$ is a distribution over $[0,1]$ with a density $g(\theta)=-2\theta+2$. By Theorem 2 in CS, any solution to (1) satisfies Condition M. By Condition M and $u_{13}^S(y,\theta,b)>0$, I obtain $d\hat{a}_i^x/dx>0$ and

$$u^R(\overline{y}(\tilde{a}_{j+1},\tilde{a}_j),\tilde{a}_j)-u^R(\overline{y}(\tilde{a}_j,\tilde{a}_{j-1}),\tilde{a}_j)\geq u^S(\overline{y}(\tilde{a}_{j+1},\tilde{a}_j),\tilde{a}_j,b)-u^S(\overline{y}(\tilde{a}_j,\tilde{a}_{j-1}),\tilde{a}_j,b)\geq 0.$$

Since $g(1)=0$, this means that $\left.\frac{d\hat{W}}{dx}\right|_{x=1}>0$.

The implication of this result is as follows. Let $\{[0,a),[a,1)\}$ be a 2-element equilibrium partition under this setting. Take $x\in(a,1)$ and $w^*(x)$ such that the 3-element partition $\{[0,a^x),[a^x,x),[x,1]\}$ holds S's incentive compatibility condition. Then, I obtain $x-a^x>1-a$ for $x\approx 1$. Namely, adding new interval $[x,1]$ distorts S's incentive significantly. Furthermore, the prior probability with which $\theta\in[x,1]$ is almost equal to 0. Hence, intuitively, the negative effect of the fact that the interval $[x-a^x)$ becomes wider dominates the positive effect of adding a new interval.

Appendix 5.D Proof of Proposition 15

Under the given strategy profile, If S reveals that $\theta < x$ in the first period, R pays $\tilde{w}(x) = s \cdot u^S(\overline{y}(x,1),x,b) - s \cdot u^S(\overline{y}(\tilde{a}_2^x,x),x,b)$. Since each element of the boundaries $\{\tilde{a}_{\tilde{n}}^x,\dots,\tilde{a}_0^x\}$ is continuous in x and converges to a corresponding element of $\{\tilde{a}_{\tilde{n}},\dots,\tilde{a}_0\}$ as x goes to 0, I must have $\tilde{w}(\tilde{a}_1) = 0$.

Let $\left\{[\tilde{a}_{\tilde{n}-1}^{\tilde{n}-1},\tilde{a}_{\tilde{n}-2}^{\tilde{n}-1}),\dots,[\tilde{a}_1^{\tilde{n}-1},\tilde{a}_0^{\tilde{n}-1})\right\}$ be the equilibrium partition of $(\tilde{n}-1)$-element partition equilibrium in the one-shot cheap talk game. I obtain $\tilde{w}(1) = s \cdot u^S(y^R(1),1,b) - s \cdot u^S(\overline{y}(\tilde{a}_1^{\tilde{n}-1},1),1,b)$. Note that $\tilde{w}(1) > 0$ since $u_{11}^S(y,\theta,b) < 0$ and $\overline{y}(\tilde{a}_1,1) < y^R(1) < y^S(1,b)$. Condition M requires the uniqueness of the solution to the deference equation in Section 3.1. The necessary payment $\tilde{w}(x)$ is strictly positive in $x \in (\tilde{a}_1,1)$.

The following inequality must hold for R's payment $\tilde{w}(x)$ to be optimal.

$$\frac{r}{G(x)}\sum_{i=1}^{\tilde{n}-1}\int_{\tilde{a}_{i+1}^x}^{\tilde{a}_i^x} u^R(\overline{y}(\tilde{a}_{i+1},\tilde{a}_i),\theta)g(\theta)d\theta - \frac{r}{G(x)}\int_0^x u^R\left(\overline{y}(0,x),\theta\right)g(\theta)d\theta \geq \tilde{w}(x) \tag{36}$$

The left-hand side of (36) is strictly positive when $\tilde{n} \geq 3$ and $x > \tilde{a}_1$. Recall that $\tilde{w}(x) \equiv s\cdot\tilde{\alpha}(x,b) = s\cdot u^S(\overline{y}(x,1),x,b) - s\cdot u^S(\overline{y}(\tilde{a}_2^x,x),x,b) > 0$ for any $x \in (\tilde{a}_1,1)$. Summarizing the above, I conclude that for any $x \in (\tilde{a}_1,1)$, there exists $\ddot{\eta}(b,x)$ such that if $s/r < \ddot{\eta}(b,x)$, the given strategy profile constitutes a $\tilde{n}$-element partition equilibrium.

The players' equilibrium payoffs are given as follows:

$$E\tilde{U}^R(x) = r\tilde{W}^R(x) - s \cdot \tilde{\alpha}(x,b),$$

where $\tilde{W}^R(x) = \sum_{i=1}^{\tilde{n}}\int_{\tilde{a}_i^x}^{\tilde{a}_{i-1}^x} u^R(\overline{y}(\tilde{a}_i^x,\tilde{a}_{i-1}^x),\theta)g(\theta)d\theta$, and

$$E\tilde{U}^S(x) = s\tilde{W}^S(x) + s \cdot \tilde{\alpha}(x,b),$$

where $\tilde{W}^S(x) = s\sum_{i=1}^{\tilde{n}}\int_{\tilde{a}_i^x}^{\tilde{a}_{i-1}^x} u^S(\overline{y}(\tilde{a}_i^x,\tilde{a}_{i-1}^x),\theta)g(\theta)d\theta$. By the definition of $\tilde{W}^R(x)$ and $\tilde{W}^S(x)$, $\tilde{W}^R(\tilde{a}_1) = EU_{CS}^R$ and $\tilde{W}^S(\tilde{a}_1) = EU_{CS}^S$, respectively.

Now I ensure that if Condition M holds, $\left.\frac{d\tilde{W}^\kappa}{dx}\right|_{x=\tilde{a}_1} > 0$ for $\kappa \in \{R,S\}$. The envelope theorem yields

$$\frac{d}{dx}\tilde{W}^R(x) = \sum_{i=1}^{\tilde{n}-1} g(\tilde{a}_i^x)\frac{d\tilde{a}_i^x}{dx}[u^R(\overline{y}(\tilde{a}_{i+1}^x,\tilde{a}_i^x),\tilde{a}_i^x) - u^R(\overline{y}(\tilde{a}_i^x,\tilde{a}_{i-1}^x),\tilde{a}_i^x)],$$

$$\left.\frac{d\tilde{W}^R}{dx}\right|_{x=\tilde{a}_1} = \sum_{i=1}^{\tilde{n}-1} g(\tilde{a}_i)[u^R(\overline{y}(\tilde{a}_{i+1},\tilde{a}_i),\tilde{a}_i) - u^R(\overline{y}(\tilde{a}_i,\tilde{a}_{i-1}),\tilde{a}_i)]\left.\frac{d\tilde{a}_i^x}{dx}\right|_{x=\tilde{a}_1}.$$

Condition M guarantees that $da_i^x/dx > 0$ for $i \in \{1,\dots,\tilde{n}-1\}$, and

$$\begin{aligned}&u^R(\overline{y}(\tilde{a}_{i+1},\tilde{a}_i),\tilde{a}_i) - u^R(\overline{y}(\tilde{a}_i,\tilde{a}_{i-1}),\tilde{a}_i)\\ &\quad > u^S(\overline{y}(\tilde{a}_{i+1},\tilde{a}_i),\tilde{a}_i) - u^S(\overline{y}(\tilde{a}_i,\tilde{a}_{i-1}),\tilde{a}_i) = 0\end{aligned}$$

The first inequality holds since $\overline{y}(\tilde{a}_{i+1}, \tilde{a}_i) < \overline{y}(\tilde{a}_i, \tilde{a}_{i-1})$, $\frac{\partial^2 u^S}{\partial y \partial b}\Big|_{b=0} > 0$, and $u^R(y,\theta) = u^S(y,\theta,0)$. The second equality holds since $\tilde{a}_i$ is a solution of the deference equation in Section 3.1.

The total derivative of $\tilde{W}^S(x)$ is

$$\frac{d}{dx}\tilde{W}^R(x) = \sum_{i=1}^{\tilde{n}-1} g(\tilde{a}_i^x)\frac{d\tilde{a}_i^x}{dx}[u^s(\overline{y}(\tilde{a}_{i+1}^x, \tilde{a}_i^x), \tilde{a}_i^x) - u^s(\overline{y}(\tilde{a}_i^x, \tilde{a}_{i-1}^x), \tilde{a}_i^x)]$$
$$+ \sum_{i=0}^{\tilde{n}-1} \frac{d\overline{y}(\tilde{a}_{i+1}^x, \tilde{a}_i^x)}{dx} \int_{\tilde{a}_{i+1}^x}^{\tilde{a}_i^x} \left[u_1^S(\overline{y}(\tilde{a}_{i+1}^x, \tilde{a}_i^x), \theta, b) g(\theta) d\theta\right].$$

Since $\tilde{a}_i$ is a solution of the deference equation in Section 3.1, the first term is equal to 0 at $x = \tilde{a}_1$. Hence,

$$\frac{d\tilde{W}^S}{dx}\bigg|_{x=\tilde{a}_1} = +\sum_{i=0}^{\tilde{n}-1} \int_{\tilde{a}_{i+1}}^{\tilde{a}_i} \left[u_1^S(\overline{y}(\tilde{a}_{i+1}, \tilde{a}_i), \theta, b) g(\theta) d\theta \; \frac{d\overline{y}(\tilde{a}_{i+1}^x, \tilde{a}_i^x)}{dx}\bigg|_{x=\tilde{a}_1} \right].$$

Condition M guarantees that $\frac{d\overline{y}(\tilde{a}_{i+1}^x, \tilde{a}_i^x)}{dx}\Big|_{x=\tilde{a}_1} > 0$. Since, moreover, I assume that $u_{13}^S(y,\theta,b) > 0$, I obtain $\frac{d\tilde{W}^S}{dx}\Big|_{x=\tilde{a}_1} > 0$.

I have already shown that $s \cdot \tilde{\alpha}(x,b) > 0$ for $x \in (\tilde{a}_1, 1)$. Hence, these results conclude that there exists $\ddot{\eta}(b)$ such that for $\kappa \in \{R, S\}$, $E\tilde{U}^\kappa(x) > EU_{CS}^\kappa$ holds for some $x \in (\tilde{a}_1, 1)$. ◇

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