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Instrumental Variable Estimation of Dynamic Linear Panel Data Models with Defactored Regressors and a Multifactor Error Structure

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Abstract

This paper develops an instrumental variable (IV) estimator for consistent estimation of dynamic panel data models with a multifactor error structure when both N and T, the cross-sectional and time series dimensions respectively, are large. Our approach projects out the common factors from observed variables, the exogenous regressors of the model, using principal components analysis and then uses the defactored regressors as instruments to estimate the unknown parameters, as in a standard 2SLS procedure. The approach requires estimating solely the common factors contained in the regressors, leaving those that only influence the dependent variable into the errors. Hence our approach is computationally attractive. Since our estimator is based on instrumental variables, it is not subject to the Nickell bias that arises with least squares type estimators in dynamic panel data models. The finite sample performance of the proposed estimator is investigated using simulated data. The results show that the estimator performs well in terms of bias, RMSE and size. The performance of an overidentifying restrictions test is also explored and the evidence suggests that it has high power when the key assumption, strong exogeneity of (a subset of) the regressors, is violated.

Key Words: method of moments; dynamic panel data; cross-sectional dependence

JEL Classification: C13, C15, C23.

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1 Introduction

The rapid increase in the availability of panel data during the last few decades has invoked a large interest in developing ways to model and analyse them effectively. In particular, the issue of how to characterise 'between group' or cross-sectional dependence, and then creating consistent estimation methods and making asymptotically valid inferences, has proven both popular and challenging. The factor structure approach has been widely used to model cross-sectional dependence. It escapes from the curse of dimensionality by asserting that there exists a common component, which is a linear combination of a finite number of time-varying common factors with individual-specific factor loadings. One can provide different interpretations of this approach, depending on the application in mind. In macroeconomic panels the unobserved factors are frequently viewed as economy-wide shocks, affecting all individuals albeit with different intensities; see e.g. Favero et al. (2005). In microeconomic panels the factor error structure may be thought to reflect distinct sources of unobserved individual-specific heterogeneity, the impact of which varies over time. For instance, in a model of wage determination the factor loadings may represent several unmeasured skills, specific to each individual, while the factors may capture the price of these skills, which changes intertemporally in an arbitrary way; see e.g. Carneiro et al. (2003) and Heckman et al. (2006).

A large body of the literature has focused on developing statistical inference for models with an error factor structure. For large panels, two estimation methods have been popular: Pesaran (2006) proposed the Common Correlated Effects (CCE) estimator, that consists of approximating the unobserved factors by the linear combinations of cross section averages of the dependent and explanatory variables. Bai (2009) proposed an iterative least squares estimator with bias correction, approximating the unobserved factors by principal components (PC).[1] For both estimators it is assumed that the regressors are strictly exogenous with respect to the idiosyncratic error component, whereas possible correlation between the regressors and the error factor component is permitted. Under somewhat weaker assumptions, Moon and Weidner (2015) show that the estimator of Bai (2009) is interpretable as a quasi maximum likelihood estimator (QMLE), the consistency of which is maintained even when the number of factors is not specified correctly, so long as it is larger than or equal to the true number of factors.

In this paper we consider estimation of linear dynamic panel data models with an error factor structure in large panels.[2] Recently, the CCE and the PC estimators have been shown to remain consistent in such models. In particular, Chudik and Pesaran (2015a) propose mean group CCE estimation for panel autoregressive distributed lag models. Notably they allow cross-sectionally heterogenous slope coefficients, and they propose to alleviate the small sample bias using jackknife bias correction. The cost of allowing this generality is twofold. First, when the

[1] See Westerlund and Urbain (2015) for comparison analysis of the CCE and PC estimation. Chudik and Pesaran (2015b), Sarafidis and Wansbeek (2012) and Bai and Wang (2016) also provide excellent surveys on the related literature.

[2] Estimation of such models for short panel is considered by Ahn et al. (2013) and Robertson and Sarafidis (2015).

number of unobserved factors is larger than the number of right-hand side variables plus one, a set of external variables, which are not in the original model of interest but form a part of the dynamic system with the dependent variable, should be found. In practice, this may not be a trivial exercise. Second, in order to mitigate the effects of weak exogeneity, the CCE approach potentially requires augmenting the model by several lags of weakly exogenous variables.[3] This can result in a large loss of degrees of freedom.

On the other hand, Moon and Weidner (2017) propose a bias-corrected PC (or QMLE) estimator and put forward three classical likelihood based test statistics. However, the statistical properties of the estimator are shown to be sensitive to the quality of the estimate of the number of factors. In particular, There can be a considerable loss of efficiency of the PC estimator when the number of factors specified is larger than the true number.[4] Finally, there is evidence suggesting that the bias-corrected PC estimator can still exhibit some finite sample bias for the model with exogenous regressors.[5]

In this paper we propose an instrumental variable (IV) estimator for dynamic linear panel data models with error factor structure when both cross section and time dimensions are large. Our estimator is potentially robust to the above problems and computationally attractive[6]. Our approach asymptotically projects out the common component from the regressors using principal components analysis at first stage and then uses the defactored regressors as instruments to estimate the structural parameters. Our methodology can be regarded as an extension of the approach taken by Sarafidis et al. (2009). The required assumption underlying our approach is that endogeneity of the covariates arises due to the non-zero correlation between the common components in the covariates and in the disturbance. Importantly, this assumption can be tested using an overidentifying restrictions test.

Although both our approach and the QMLE approach of Moon and Weidner (2017) are based on principal components, there are important differences in practice; firstly, our method estimates the factors from observed data (the covariates), rather than the disturbances. In addition, our procedure requires estimating solely the common factors included in the regressors. Due to these differences, it is expected that our approach will be less sensitive to possible overestimation of the number of factors. Moreover, since our estimator is an instrumental variable estimator, it is not subject to the Nickell bias that arises with least squares type estimators in dynamic panel data models. Finally, we employ the PC approach rather than the CCE type approach for defactoring the exogenous regressors, since with the former approach it is not necessary to seek external variables to approximate the factors when the number of unobserved factors is larger than the number of regressors plus one.

Our approach can be regarded as the opposite one employed by Bai and Ng (2010) and Kapetanios and Marcellino (2010). In specific, in their model the

[3]See equation (24) and the discussion around it in Chudik and Pesaran (2015a).

[4]See, for example, Table 2 in Moon and Weidner (2017).

[5]See Table V in Appendix E in the supplement to Bai (2009).

[6]We only consider the models with cross-sectionally homogeneous slopes. See Chudik and Pesaran (2015a) for the estimation of such models.

idiosyncratic errors of the reduced form regression of the endogenous variable cause endogeneity, therefore, no error factor structure is considered in the structural model of interest. They propose finding instruments for the endogenous regressors by extracting the common components from external variables and the endogenous regressors in the model. Our approach essentially complements theirs.

Using simulated data, the finite sample performance of the proposed IV estimator and the associated t-test is investigated, along with the QMLE estimator of Moon and Weidner (2017). The results show that the proposed estimator performs well under a variety of designs both in terms of bias and size of the t-test. Furthermore, the overidentifying restrictions test appears to have high power when the key assumption, strong exogeneity, is violated.

The paper is organised as follows. Section 2 sets out the model and assumptions, and puts forward the proposed estimation approach. Section 3 extends the results to the more general case. Section 4 studies the performance of the estimator in small samples using simulated data. Section 5 contains some concluding remarks. Proofs of propositions, theorems and corollaries, together with necessary lemmas, are contained in Appendix A. The proofs of the lemmas are available in Supplemental Material.

2 Model and Estimation Method

Consider the following autoregressive distributed lag, ARDL(1,0), panel data model with a multi-factor error structure[7]:

$$y_{it} = \alpha + \rho y_{i,t-1} + \boldsymbol{\beta}'\mathbf{x}_{it} + u_{it}; \quad i = 1, 2, ..., N; \quad t = 1, 2, ..., T, \tag{1}$$

with

$$u_{it} = \boldsymbol{\gamma}_i'\mathbf{f}_{x,t} + \boldsymbol{\lambda}_i'\mathbf{f}_{y,t} + \varepsilon_{it}, \tag{2}$$

where $|\rho| < 1$, $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_k)'$ with at least one of $\{\beta_\ell\}_{\ell=1}^k$ being non-zero, $\mathbf{x}_{it} = (x_{1it}, x_{2it}, ..., x_{kit})'$ is a $k \times 1$ vector of regressors, $\mathbf{f}_{x,t} = (f_{x,1t}, f_{x,2t}, ..., f_{x,m_x t})'$ and $\mathbf{f}_{y,t} = (f_{y,1t}, f_{y,2t}, ..., f_{y,m_y t})'$ denote $m_x \times 1$ and $m_y \times 1$ vectors of unobservable factors, respectively. The $m_x \times 1$ vector $\boldsymbol{\gamma}_i$ and the $m_y \times 1$ vector $\boldsymbol{\lambda}_i$ contain factor loadings associated with $\mathbf{f}_{x,t}$ and $\mathbf{f}_{y,t}$, respectively, whereas ε_{it} is an idiosyncratic error. $\mathbf{x}_{it}$ is subject to the following process:

$$\mathbf{x}_{it} = \boldsymbol{\Gamma}_{xi}'\mathbf{f}_{x,t} + \mathbf{v}_{it}, \tag{3}$$

where $\boldsymbol{\Gamma}_{xi} = (\boldsymbol{\gamma}_{1i}, \boldsymbol{\gamma}_{2i}, ..., \boldsymbol{\gamma}_{ki})$ denotes an $m_x \times k$ factor loading matrix and $\mathbf{v}_{it} = (v_{1it}, v_{2it}, ..., v_{kit})'$ is an idiosyncratic error term.[8]

Remark 1 *When time invariant individual effects exist in u_{it} and $\mathbf{x}_{it}$, one can transform the variables, by taking first differences, applying the within transformation, or orthogonal deviations; this does not alter the discussion below.*

[7]The main results of this paper naturally extend to models with higher order lags, i.e. ARDL(p,q) for $p > 0$ and $q \geq 0$.

[8]We do not explicitly discuss the case in which u_{it} contains a subset of factor components in $\mathbf{x}_{it}$ only, since it is easily seen that all the results in this paper will still hold.

Stacking the T observations for each i yields

$$\mathbf{y}_i = \rho \mathbf{y}_{i,-1} + \mathbf{X}_i \boldsymbol{\beta} + \mathbf{u}_i, \tag{4}$$

with

$$\mathbf{u}_i = \mathbf{F}_x \boldsymbol{\gamma}_i + \mathbf{F}_y \boldsymbol{\lambda}_i + \boldsymbol{\varepsilon}_i, \tag{5}$$

where $\mathbf{y}_i = (y_{i1}, y_{i2}, ..., y_{iT})'$, $\mathbf{y}_{i,-1} = L^1 \mathbf{y}_i = (y_{i0}, y_{i1}, ..., y_{iT-1})'$ with L^j being the j^{th} lag operator, $\mathbf{X}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, ..., \mathbf{x}_{iT})'$, $\mathbf{u}_i = (u_{i1}, u_{i2}, ..., u_{iT})'$, $\mathbf{F}_x = (\mathbf{f}_{x,1}, \mathbf{f}_{x,2}, ..., \mathbf{f}_{x,T})'$, $\mathbf{F}_y = (\mathbf{f}_{y,1}, \mathbf{f}_{y,2}, ..., \mathbf{f}_{y,T})'$ and $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2}, ..., \varepsilon_{iT})'$. Similarly,

$$\mathbf{X}_i = \mathbf{F}_x \boldsymbol{\Gamma}_{xi} + \mathbf{V}_i, \tag{6}$$

where $\mathbf{V}_i = (\mathbf{v}_{i1}, \mathbf{v}_{i2}, ..., \mathbf{v}_{iT})'$ and $\mathbf{F}_x$ is defined above.

Let $\mathbf{W}_i = (\mathbf{y}_{i,-1}, \mathbf{X}_i)$ and $\boldsymbol{\theta} = (\rho, \boldsymbol{\beta}')'$. The model in (4) can be written more concisely as follows:

$$\mathbf{y}_i = \mathbf{W}_i \boldsymbol{\theta} + \mathbf{u}_i. \tag{7}$$

$\mathbf{W}_i$ is heterogeneously cross sectionally correlated because the factor loadings vary across i. Also the composite error, $\mathbf{u}_i$, is allowed to be serially correlated through serial correlation in the factors, $\mathbf{f}_{x,t}$ and $\mathbf{f}_{y,t}$.

Our proposed approach involves asymptotically eliminating at first stage the common factors in $\mathbf{X}_i$ by projecting them out, and then using the defactored regressors as instruments to estimate the structural parameters of the model. To see the main idea, consider the following projection matrices:

$$\mathbf{M}_{F_x} = \mathbf{I}_T - \mathbf{F}_x \left(\mathbf{F}_x' \mathbf{F}_x\right)^{-1} \mathbf{F}_x'; \; \mathbf{M}_{F_{x,-1}} = \mathbf{I}_T - \mathbf{F}_{x,-1} \left(\mathbf{F}_{x,-1}' \mathbf{F}_{x,-1}\right)^{-1} \mathbf{F}_{x,-1}', \tag{8}$$

where $\mathbf{F}_{x,-1} = L^1 \boldsymbol{F}_x$. If $\mathbf{F}_x$ were observed, premultiplying $\mathbf{X}_i$ by $\mathbf{M}_{F_x}$ would yield $\mathbf{M}_{F_x} \mathbf{X}_i = \mathbf{M}_{F_x} \mathbf{V}_i$. Assuming $\mathbf{V}_i$ is independent from $\boldsymbol{\varepsilon}_i, \mathbf{F}_x, \mathbf{F}_y$, it is easily seen that $E(\mathbf{X}_i' \mathbf{M}_{F_x} \mathbf{u}_i) = E(\mathbf{V}_i' \mathbf{M}_{F_x} \mathbf{u}_i) = \mathbf{0}$.

Furthermore, let

$$\mathbf{X}_{i,-j} = L^j \mathbf{X}_i. \tag{9}$$

So long as $\{y_{it}, \mathbf{x}_{it}'\}$, $t = 0, 1, ..., T$ is observed, the $T \times k$ matrix $\mathbf{X}_{i,-1}$ is also observed. Using similar assumptions, one can show that $E(\mathbf{X}_{i,-1}' \mathbf{M}_{F_{x,-1}} \mathbf{M}_{F_x} \mathbf{u}_i) = E(\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}} \mathbf{M}_{F_x} \mathbf{u}_i) = \mathbf{0}$.

Define

$$\mathbf{Z}_i \equiv \left[\mathbf{X}_i, \mathbf{M}_{F_{x,-1}} \mathbf{X}_{i,-1}\right] \; (T \times 2k). \tag{10}$$

Given the model in equation (7) it is clear that the defactored regressors satisfy instrument relevance, i.e. $E(\mathbf{Z}_i' \mathbf{M}_{F_x} \mathbf{W}_i) \neq \mathbf{0}$. Therefore, it is straightforward to apply instrumental variable (IV) estimation using $\mathbf{M}_{F_x} \mathbf{Z}_i$ as an instrument vector for $\mathbf{W}_i$.

Remark 2 *Since our approach makes use of transformed x's as instruments, identification of ρ requires that* at least one *element in $\boldsymbol{\beta}$ is not equal to zero. We believe this is a mild restriction, especially compared to imposing $\boldsymbol{\beta} \neq \mathbf{0}$. Specifically, identification of the autoregressive parameter can be achieved based on the covariate(s) and lagged value(s) corresponding to the non-zero slope coefficient(s). Notably, it is not necessary to know which covariates have non-zero coefficients since by construction the 2SLS procedure does not require that all instruments are relevant to all endogenous regressors.*

More instruments become available when further lags of $\mathbf{x}_{it}$ are observed. In particular, given model (3), when $\{\mathbf{x}_{it}\}_{t=0-j}^{T}$ for $j \geq 0$ are observable, $(j+1)k$ instruments, $\{\mathbf{X}_{i,-(r-1)}\}_{r=1}^{j+1}$, become available. Furthermore, as $\mathbf{V}_i$ is strictly exogenous, we could exploit $(j+1)kT^2$ moment conditions for asymptotically efficient estimation. However, in such case it is well known that the estimator will be subject to the overfitting bias that arises in GMM estimation with a large number of instruments; for further analysis see Alvarez and Arellano (2003) among others. To avoid this issue we stick to the set as in (10), such that the number of instruments is fixed and does not depend on T. The analysis of overfitting bias with a large number of instruments is beyond the scope of this paper.

The assumption that $\mathbf{V}_i$ is independent of $\boldsymbol{\varepsilon}_i$, $\mathbf{F}_x$ and $\mathbf{F}_y$ implies that the covariates are strongly exogenous with respect to the idiosyncratic error component (i.e. $E(\boldsymbol{\varepsilon}_i'\mathbf{X}_i) = \mathbf{0}$). Dynamic panel data models with strongly exogenous regressors is a widely used framework in the economics literature; some examples include partial adjustment models for labour supply (e.g. Bover, 1991), household consumption models with habits (e.g. Becker et al., 1994) and production functions with adjustment costs (e.g. Blundell and Bond, 2000). In these applications the autoregressive parameter captures consumption inertia due to habits, or costs of adjustment, so it has a structural significance; see e.g. Arellano (2003, Ch. 7). Notwithstanding strong exogeneity with respect to the idiosyncratic disturbance, it is reasonable to expect that the regressors may be correlated with the unobserved common factors and are, therefore, endogenous. For instance, in a production function the input decisions of the firm are likely to be correlated with its individual-specific unobserved characteristics, $\boldsymbol{\gamma}_i$, that may or may not vary over time. Likewise, determinants of labour supply, such as the level of wage offered to an individual, are likely to be correlated with the common factors influencing supply itself. Essentially, this is the standard fixed effects assumption employed in panel data models, extended to the factor structure. However, notice that under the current, more general, framework, first-differencing does not remove endogeneity since the factor component remains in the residuals. The strong exogeneity assumption of the covariates with respect to the purely idiosyncratic error component can be tested using an overidentifying restrictions test, as shown below.

Note that our model can be extended to allow for additional regressors which are weakly exogenous or endogenous with respect to the idiosyncratic disturbance, provided that there are appropriate instruments available. For example, such sets of instruments can be formed based on lagged values of the endogenous regressors, if these are not correlated with the common factor component. This case is analysed in detail by Sarafidis, Yamagata and Robertson (2009). External instruments may be used in (10) if one wishes to allow for weakly exogenous regressors that are correlated with the common factor component, as in a standard two-stage least squares procedure. This case is analysed by Harding and Lamarche (2011).

In practice, $\mathbf{M}_{F_x}$ is not known because the factors $\mathbf{F}_x$ are not observed. As a result, we propose estimating $\mathbf{F}_x$ using the principal components approach, as advanced in Bai (2003) and Bai (2009).[9] To obtain our results it is sufficient to

[9] We could also adopt Pesaran's (2006) approach to estimate the common factors in the regressors.

make the following assumptions, where $\text{tr}[\mathbf{A}]$ and $||\mathbf{A}|| = \sqrt{\text{tr}[\mathbf{A}'\mathbf{A}]}$ denote the trace and Frobenius (Euclidean) norm of matrix $\mathbf{A}$, respectively, and Δ is a finite positive constant.

Assumption 1 (idiosyncratic error in y): ε_{it} is independently distributed across i and t, with mean zero, $E(\varepsilon_{it}^2) = \sigma_{\varepsilon,it}^2$, and $E|\varepsilon_{it}|^{8+\delta} \leq \Delta < \infty$ for small positive constant δ.

Assumption 2 (idiosyncratic error in x): (i) $v_{\ell it}$ is independently distributed across i and group-wise independent from ε_{it}; (ii) $E(v_{\ell it}) = 0$ and $E|v_{\ell it}|^{8+\delta} \leq \Delta < \infty$; (iii) $T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T} E|v_{\ell is}v_{\ell it}|^{1+\delta} \leq \Delta < \infty$; (iv) $E\left|N^{-1/2}\sum_{i=1}^{N}[v_{\ell is}v_{\ell it} - E(v_{\ell is}v_{\ell it})]\right|^4 \leq \Delta < \infty$ for every ℓ, t and s; (v) $N^{-1}T^{-2}\sum_{i=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{r=1}^{T}\sum_{w=1}^{T}|\text{cov}(v_{\ell is}v_{\ell it}, v_{\ell ir}v_{\ell iw})| \leq \Delta < \infty$; (vi) the largest eigenvalue of $E(\mathbf{v}_{\ell i}\mathbf{v}_{\ell i}')$ is bounded uniformly for every ℓ, i and T.

Assumption 3 (stationary factors): $\mathbf{f}_{x,t} = \boldsymbol{C}_x(L)\mathbf{e}_{f_x,t}$ and $\mathbf{f}_{y,t} = \boldsymbol{C}_y(L)\mathbf{e}_{f_y,t}$, where $\boldsymbol{C}_x(L)$ and $\boldsymbol{C}_y(L)$ are absolutely summable, $\mathbf{e}_{f_x,t} \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_{fx})$ and $\mathbf{e}_{f_y,t} \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_{fy})$, where $\boldsymbol{\Sigma}_{fx}$ and $\boldsymbol{\Sigma}_{fy}$ are positive definite matrices. Each element of $\mathbf{e}_{f_x,t}$ and $\mathbf{e}_{f_y,t}$ has finite fourth order moments and are group-wise independent from $\mathbf{v}_{it}$ and ε_{it}.

Assumption 4 (random factor loadings): (i) $\boldsymbol{\Gamma}_{xi} \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_{\Gamma x})$, $\boldsymbol{\gamma}_i \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_{\gamma})$, $\boldsymbol{\lambda}_i \sim iid(\mathbf{0}, \boldsymbol{\Sigma}_{\lambda})$, where $\boldsymbol{\Sigma}_{\Gamma x}$ is positive definite and $\boldsymbol{\Sigma}_{\gamma}$ and $\boldsymbol{\Sigma}_{\lambda}$ are positive semi-definite, and each element of $\boldsymbol{\Gamma}_{xi}$, $\boldsymbol{\gamma}_i$ and $\boldsymbol{\lambda}_i$ has finite fourth order moments. $\boldsymbol{\Gamma}_{xi}$, $\boldsymbol{\gamma}_i$ and $\boldsymbol{\lambda}_i$ are independent groups from ε_{it}, $\mathbf{v}_{it}$, $\mathbf{e}_{f_x,t}$ and $\mathbf{e}_{f_y,t}$; (ii) $\boldsymbol{\Gamma}_{xi}$ and $\boldsymbol{\lambda}_i$ are independent of each other.

Assumption 5 (identification of $\boldsymbol{\theta}$): (i) $\mathbf{A}_{i,T} = T^{-1}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{W}_i$ and $\mathbf{B}_{i,T} = T^{-1}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{Z}_i$ have full column rank for all i and T; (ii) $E\|\mathbf{A}_{i,T}\|^{2+2\delta} \leq \Delta < \infty$ and $E\|\mathbf{B}_{i,T}\|^{2+2\delta} \leq \Delta < \infty$ for all i and T; (iii) $E\left\|\boldsymbol{\varphi}_{FiT,t}\right\|^{2+\delta} \leq \Delta < \infty$ for all i and T, where $\boldsymbol{\varphi}_{FiT} = T^{-1/2}\mathbf{Z}_i{}'\mathbf{M}_{F_x}\mathbf{u}_i$, and $E(\boldsymbol{\varphi}_{FiT}\boldsymbol{\varphi}_{FiT}')$ is a positive definite matrix for any i, T. In addition, $\lim_{N,T\to\infty} N^{-1}\sum_{i=1}^{N} E(\boldsymbol{\varphi}_{FiT}\boldsymbol{\varphi}_{FiT}') = \boldsymbol{\Omega}$, which is a fixed positive definite matrix.

The assumptions above require some discussion. First of all, notice that Assumption 1 allows non-normality and (unconditional) times-series and cross-sectional heteroskedasticity in the idiosyncratic errors in the equation for y. Assumptions 2 and 3 allow for serial correlation in the idiosyncratic errors in the equation for x and the factors. Assumption 2 is in line with Bai (2003) but assumes independence across i, which can be relaxed such that the factors and $(\varepsilon_{it}, \mathbf{v}_{it})$ and/or ε_{jt} and ε_{is} are weakly dependent, provided that there exist higher order moments; see Assumptions D-F in Bai (2003)[10]. Assumptions 3 and 4 are standard in the principal components literature; see e.g. Bai (2003) among others. Notice that the zero-mean restriction on the factor loadings is not binding because for large N one can always remove the non-zero mean by transforming

[10]This includes conditional heteroskedasticity, such as ARCH or GARCH processes.

the variables in terms of deviations from time-specific averages or by adding time dummies into the model (4). The resulting correlation between the factor loadings is clearly $O_p(1/N)$, thus the results we obtain below are not affected by this transformation; see Sarafidis et al. (2009) for more details. Assumption 4 allows for possible non-zero correlations between the loadings associated with the same factors in the y and x equations, i.e. $E(\boldsymbol{\gamma}_i \boldsymbol{\gamma}'_{\ell i}) \neq \mathbf{0}$ for $\ell = 1, 2, ..., k$. Since the variables y_{it} and $\mathbf{x}_{it}$ of the same individual unit i can be affected in a related manner by the same common shocks, allowing for this possibility is potentially important in practice. Meanwhile Assumption 4(ii) implies that $E(\boldsymbol{\lambda}_i \boldsymbol{\gamma}'_{\ell i}) = \mathbf{0}$ for $\ell = 1, 2, ..., k$, i.e. the loadings of the factors entering only the process for y are uncorrelated with those in x. This can be seen admissible in some empirical applications, where different common shocks are thought to have associated effects on cross-section units in unrelated ways. However, to pursue more general results, we will relax this assumption in Section 3.

Finally, Assumption 5(i)-(ii) is common in overidentified instrumental variable (IV) estimation; for example, see Wooldridge (2002, Ch5). Assumption 5(iii) is required for identification of the estimator, the consistency property of the variance-covariance estimator and the asymptotic normality of the estimator as N and T tend to infinity jointly.

The first step of our approach is to consistently estimate the number of factors in $\mathbf{X}_i$ using, for example, the method proposed by Bai and Ng (2002), as T and N tend jointly to infinity. Since these estimators are consistent, our discussion below treats the number of factors, m_x, as known. Given m_x, the factors are extracted using principal components from $\{\mathbf{X}_i\}_{i=1}^N$. Define $\hat{\mathbf{F}}_x$ as $\sqrt{T}$ times the eigenvectors corresponding to the m_x largest eigenvalues of the $T \times T$ matrix $\frac{1}{NT}\sum_{i=1}^N \mathbf{X}_i \mathbf{X}'_i$. $\hat{\mathbf{F}}_{x,-1}$ is defined in the same way, but this time based on $\frac{1}{NT}\sum_{i=1}^N \mathbf{X}_{i,-1}\mathbf{X}'_{i,-1}$. Note that $\mathbf{F}_x$ and $\mathbf{\Gamma}_{xi}$ are estimated up to an invertible $m_x \times m_x$ matrix transformation. Since our aim is to marginalise out the unobservable common components, the principal components estimator $\hat{\mathbf{F}}_x$ can be treated as consistent, without loss of generality.

The empirical counterpart of the projection matrices defined in (8) is given by

$$\mathbf{M}_{\hat{F}_x} = \mathbf{I}_T - \hat{\mathbf{F}}_x \left(\hat{\mathbf{F}}'_x \hat{\mathbf{F}}_x\right)^{-1} \hat{\mathbf{F}}'_x; \ \mathbf{M}_{\hat{F}_{x,-1}} = \mathbf{I}_T - \hat{\mathbf{F}}_{x,-1} \left(\hat{\mathbf{F}}'_{x,-1} \hat{\mathbf{F}}_{x,-1}\right)^{-1} \hat{\mathbf{F}}'_{x,-1}. \quad (11)$$

The associated transformed instrument matrix discussed above is

$$\mathbf{M}_{\hat{F}_x} \hat{\mathbf{Z}}_i, \text{ where } \hat{\mathbf{Z}}_i = \left(\mathbf{X}_i, \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{X}_{i,-1}\right). \quad (12)$$

We propose the following instrumental variable (IV) or two-stage least squares estimator of $\boldsymbol{\theta}$:

$$\hat{\boldsymbol{\theta}}_{IV} = \left(\hat{\mathbf{A}}'_{NT} \hat{\mathbf{B}}^{-1}_{NT} \hat{\mathbf{A}}_{NT}\right)^{-1} \hat{\mathbf{A}}'_{NT} \hat{\mathbf{B}}^{-1}_{NT} \hat{\mathbf{g}}_{NT}, \quad (13)$$

where

$$\hat{\mathbf{A}}_{NT} = \frac{1}{NT}\sum_{i=1}^N \hat{\mathbf{Z}}'_i \mathbf{M}_{\hat{F}_x} \mathbf{W}_i, \ \hat{\mathbf{B}}_{NT} = \frac{1}{NT}\sum_{i=1}^N \hat{\mathbf{Z}}'_i \mathbf{M}_{\hat{F}_x} \hat{\mathbf{Z}}_i, \ \hat{\mathbf{g}}_{NT} = \frac{1}{NT}\sum_{i=1}^N \hat{\mathbf{Z}}'_i \mathbf{M}_{\hat{F}_x} \mathbf{y}_i. \quad (14)$$

Firstly we show consistency of the above estimator. To begin with, from (7) and (13) we obtain

$$\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV}-\boldsymbol{\theta}\right)=\left(\hat{\mathbf{A}}_{NT}'\hat{\mathbf{B}}_{NT}^{-1}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}_{NT}'\hat{\mathbf{B}}_{NT}^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\right). \quad (15)$$

Since the asymptotic properties of the estimator are primarily determined by those of $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i$, we focus on this term. It turns out that the asymptotic effect of the replacement of $\mathbf{F}_x$ with $\hat{\mathbf{F}}_x$ is $O_p\left(\frac{\sqrt{NT}}{\min\{N,T\}}\right)$, which is either $O_p\left(\sqrt{\frac{T}{N}}\right)$ or $O_p\left(\sqrt{\frac{N}{T}}\right)$. The result of formal analysis is provided as a proposition below, where $(N,T)\overset{j}{\rightarrow}\infty$ signifies that N and T tend to infinity jointly.

Proposition 1 *Consider the model in equations (1)-(3). Under Assumptions 1-4(i)(ii), as* $(N,T)\overset{j}{\rightarrow}\infty$ *such that* $N/T\rightarrow c$ *with* $0<c<\infty$*,*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{u}_i+\sqrt{\frac{T}{N}}\mathbf{b}_{1NT}+o_p(1),$$

where $\hat{\mathbf{Z}}_i$*,* $\mathbf{M}_{\hat{F}_x}$*,* $\mathbf{Z}_i$ *and* $\mathbf{M}_{F_x}$ *are defined in (12), (11), (10) and (8), respectively,* $\mathbf{b}_{1NT}=[\mathbf{b}_{11NT}',\mathbf{b}_{12NT}']'$ *with*

$$\mathbf{b}_{11NT}=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_i'\mathbf{V}_j}{T}\boldsymbol{\Gamma}_{xj}'\boldsymbol{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T};$$

$$\mathbf{b}_{12NT}=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{V}_j}{T}\boldsymbol{\Gamma}_{xj}'\boldsymbol{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T},$$

where $\tilde{\mathbf{V}}_i=\mathbf{V}_i-\frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_n\boldsymbol{\Gamma}_{xn}'\boldsymbol{\Upsilon}_{kN}^{-1}\boldsymbol{\Gamma}_{xi}$*,* $\tilde{\mathbf{V}}_{i,-1}=\mathbf{V}_{i,-1}-\frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_{n,-1}\boldsymbol{\Gamma}_{xn}'\boldsymbol{\Upsilon}_{kN}^{-1}\boldsymbol{\Gamma}_{xi}$*, and* $\boldsymbol{\Upsilon}_{kN}=\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\boldsymbol{\gamma}_{\ell i}\boldsymbol{\gamma}_{\ell i}'$*.*

Remark 3 *The source of the bias term in Proposition 1 is different than the bias terms reported in Bai (2009) and Moon and Weidner (2017). In particular, the bias term of our estimator arises primarily due to the correlation between the factor loadings associated with* $\mathbf{F}_x$ *in* x *and the error term in the equation of* y*,* $\mathbf{u}_i$*. On the other hand, the two bias terms in Bai (2009) and Moon and Weidner (2017) arise from error serial dependence and weak cross-sectional dependence. In our case, error serial correlation in the idiosyncratic part of the* x *process,* $v_{\ell it}$*, does not result in bias because* $v_{\ell it}$ *is not correlated with the error term in the* y *equation,* ε_{it}*. Also note that Moon and Weidner (2017) report additional bias term that generalizes the Nickell bias which typically occurs in dynamic panel models with fixed effects. Our estimator is not subject to incidental parameter problem as it is based on instrumental variables, therefore such a bias term does not arise in our case.*

Remark 4 *Our expression of the bias estimator involves the composite error u_{it}, rather than the idiosyncratic error, ε_{it}. This underlines the simplicity and robustness of our approach, which does not require estimation of the factor components in the error term for bias correction or statistical inference of the estimator.*

From the result stated in Proposition 1 it is easily seen that $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i$ is $O_p(1)$ and tends to a multivariate normal random variable. In addition, $\sqrt{\frac{T}{N}}\mathbf{b}_{1NT}$ is $O_p(1)$ as T/N tends to a finite positive constant c $(0 < c < \infty)$ when N and $T \to \infty$ jointly. Therefore, in such situation the IV estimator is $\sqrt{NT}$-consistent.

The above discussion is summarised in the following theorem:

Theorem 1 *Consider model (1)-(3) and suppose that Assumptions 1-5(i)(ii)(iii) hold true. Then,*

$$\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta} \xrightarrow{p} \mathbf{0}$$

as N and $T \to \infty$ jointly in such a way that $T/N \to c$ with $0 < c < \infty$, where $\hat{\boldsymbol{\theta}}_{IV}$ is defined in (13).

Now we turn our attention to the asymptotic normality properties of the estimator. To this end, we propose a bias corrected estimator; otherwise the limiting distribution of $\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right)$ will not be centered at zero. Based on the result in Proposition 1 the bias corrected estimator is defined as

$$\hat{\hat{\boldsymbol{\theta}}}_{IV} = \hat{\boldsymbol{\theta}}_{IV} - \left(\hat{\mathbf{A}}_{NT}'\hat{\mathbf{B}}_{NT}^{-1}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}_{NT}'\hat{\mathbf{B}}_{NT}^{-1}\frac{1}{\sqrt{NT}}\widehat{\mathbf{b}}_{NT}, \tag{16}$$

where $\widehat{\mathbf{b}}_{NT} = \sqrt{\frac{T}{N}}\widehat{\mathbf{b}}_{1NT}$ with $\widehat{\mathbf{b}}_{1NT} = [\widehat{\mathbf{b}}_{11NT}', \widehat{\mathbf{b}}_{12NT}']'$, and

$$\widehat{\mathbf{b}}_{11NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\hat{\hat{\mathbf{V}}}_i'\hat{\mathbf{V}}_j}{T}\hat{\boldsymbol{\Gamma}}_{xj}'\hat{\boldsymbol{\Upsilon}}_{kN}^{-1}\frac{\hat{\mathbf{F}}_x'\hat{\mathbf{u}}_i}{T};$$

$$\widehat{\mathbf{b}}_{12NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\hat{\hat{\mathbf{V}}}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\hat{\mathbf{V}}_j}{T}\hat{\boldsymbol{\Gamma}}_{xj}'\hat{\boldsymbol{\Upsilon}}_{kN}^{-1}\frac{\hat{\mathbf{F}}_x'\hat{\mathbf{u}}_i}{T};$$

$$\hat{\hat{\mathbf{V}}}_i = \hat{\mathbf{V}}_i - \frac{1}{N}\sum_{n=1}^{N}\hat{\mathbf{V}}_n\hat{\boldsymbol{\Gamma}}_{xn}'\hat{\boldsymbol{\Upsilon}}_{kN}^{-1}\hat{\boldsymbol{\Gamma}}_{xi};\ \hat{\hat{\mathbf{V}}}_{i,-1} = \hat{\mathbf{V}}_{i,-1} - \frac{1}{N}\sum_{n=1}^{N}\hat{\mathbf{V}}_{n,-1}\hat{\boldsymbol{\Gamma}}_{xn}'\hat{\boldsymbol{\Upsilon}}_{kN}^{-1}\hat{\boldsymbol{\Gamma}}_{xi};$$

$$\hat{\boldsymbol{\Upsilon}}_{kN} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\hat{\boldsymbol{\gamma}}_{\ell i}\hat{\boldsymbol{\gamma}}_{\ell i}';\ \hat{\boldsymbol{\gamma}}_{\ell i} = T^{-1}\hat{\mathbf{F}}_x'\mathbf{x}_{\ell i};\ \hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{W}_i\hat{\boldsymbol{\theta}}_{IV};\ \hat{\mathbf{v}}_{\ell i} = \mathbf{M}_{\hat{F}_x}\mathbf{x}_{\ell i}.$$

The following theorem proves asymptotic normality of the distribution of the bias adjusted estimator, based on Hansen's (2007) law of large numbers and central limit theorem, which are restated as Lemmas 1 and 2 in Appendix A.

Theorem 2 *Suppose that Assumptions 1-5(i)(ii)(iii) hold true under model (1)-(3). Then, assuming that $plim_{N,T\to\infty}\mathbf{b}_{NT} = \mathbf{b}$ exists,* *(i) as N and $T \to \infty$ jointly in such a way that $T/N \to c$ with $0 < c < \infty$*

$$\sqrt{NT}\left(\widehat{\widehat{\boldsymbol{\theta}}}_{IV} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \boldsymbol{\Psi}\right),$$

where $\widehat{\widehat{\boldsymbol{\theta}}}_{IV}$ is defined by (16) and

$$\boldsymbol{\Psi} = \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\boldsymbol{\Omega}\mathbf{B}^{-1}\mathbf{A}\left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}$$

is a positive definite matrix, $\mathbf{A} = \text{plim}_{N,T\to\infty}\hat{\mathbf{A}}_{NT}$ and $\mathbf{B} = \text{plim}_{N,T\to\infty}\hat{\mathbf{B}}_{NT}$ with $\hat{\mathbf{A}}_{NT}$ and $\hat{\mathbf{B}}_{NT}$ defined in (14), and $\boldsymbol{\Omega}$ is defined in Assumption 5.

(ii) $\widehat{\boldsymbol{\Psi}}_{NT} - \boldsymbol{\Psi} \xrightarrow{p} \mathbf{0}$ as N and $T \to \infty$ jointly in such a way that $T/N \to c$ with $0 < c < \infty$, where

$$\widehat{\boldsymbol{\Psi}}_{NT} = \left(\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}^{-1}_{NT}\hat{\boldsymbol{\Omega}}_{NT}\hat{\mathbf{B}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\left(\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}, \quad (17)$$

with

$$\hat{\boldsymbol{\Omega}}_{NT} = \frac{1}{NT}\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{u}}_i\hat{\mathbf{u}}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i \quad (18)$$

and $\hat{\mathbf{u}}_i = \mathbf{y}_i - \mathbf{W}_i\hat{\boldsymbol{\theta}}_{IV}$.

Define the two-step bias corrected IV estimator as

$$\widehat{\widehat{\boldsymbol{\theta}}}_{IV2} = \hat{\boldsymbol{\theta}}_{IV2} - \left(\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\frac{1}{\sqrt{NT}}\hat{\mathbf{b}}_{NT} \quad (19)$$

with

$$\hat{\boldsymbol{\theta}}_{IV2} = \left(\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\hat{\mathbf{g}}_{NT}. \quad (20)$$

The following corollary describes the asymptotic properties of the estimator:

Corollary 1 *Suppose that Assumptions 1-5(i)(ii)(iii) hold true under model (1)-(3). Then, as N and $T \to \infty$ jointly in such a way that $T/N \to c$ with $0 < c < \infty$,*

$$\sqrt{NT}\left(\widehat{\widehat{\boldsymbol{\theta}}}_{IV2} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \left(\mathbf{A}'\boldsymbol{\Omega}^{-1}\mathbf{A}\right)^{-1}\right),$$

where $\widehat{\widehat{\boldsymbol{\theta}}}_{IV2}$ is defined by (19), $\mathbf{A} = \text{plim}_{N,T\to\infty}\hat{\mathbf{A}}_{NT}$ and $\boldsymbol{\Omega}$ is defined in Assumption 5.

The associated overidentifying restrictions test statistic is given by

$$S_{NT} = \frac{1}{NT}\left(\sum_{i=1}^{N}\widehat{\widehat{\mathbf{u}}}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i\right)\hat{\boldsymbol{\Omega}}^{-1}_{NT}\left(\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\widehat{\widehat{\mathbf{u}}}_i\right), \quad (21)$$

where $\widehat{\widehat{\mathbf{u}}}_i = \mathbf{y}_i - \mathbf{W}_i\widehat{\widehat{\boldsymbol{\theta}}}_{IV2}$, and $\hat{\boldsymbol{\Omega}}$ is defined by (18). Hansen (2007) shows in the context of a standard panel fixed effects estimation that the t-test based on the variance estimator (17) is asymptotically valid even when T and N tend jointly to infinity. Using similar arguments, the asymptotic validity of the two-step IV estimator and the associated overidentifying restrictions test can be verified. The result is summarised in the following theorem:

Theorem 3 *Suppose that Assumptions 1-5(i)(ii)(iii) hold true under model (1)-(3). Then, as N and $T \to \infty$ jointly in such a way that $T/N \to c$ with $0 < c < \infty$*

$$S_{NT} \xrightarrow{d} \chi^2_{k-1}, \tag{22}$$

for $k > 1$, under the null hypothesis of strong exogeneity of the covariates, where S_{NT} is defined in (21).

The overidentifying restrictions test is particularly useful in our approach in order to test the assumption of strong exogeneity of the idiosyncratic error in the equation for x.

3 The Case of Correlated Factor Loadings

In the previous section, we placed Assumption 4(ii) which leads to zero correlation between the loadings of the factors that enter only y and those in x, i.e. $E(\boldsymbol{\lambda}_i \boldsymbol{\gamma}'_{\ell i}) = \mathbf{0}$ for $\ell = 1, 2, ..., k$. In this section, we drop this assumption. This is an important extension to consider, since, in our approach, we only estimate the factors in the regressors, leaving the factor components exclusively in u unestimated.

Note that if $\mathbf{F}_x$ were observed, even with such correlated loadings, the factor component would be projected out from $\mathbf{X}_i$ completely, and so the defactored regressors, $\mathbf{M}_{F_x}\mathbf{X}_i$, would be free from $\mathbf{F}_y\boldsymbol{\lambda}_i$. However, since in practice the factors are unobserved, in the absence of Assumption 4(ii) estimation of $\mathbf{F}_x$ can induces additional non-zero correlations, which in turn imply extra asymptotic bias terms.

First, for a purely theoretical derivation purpose, we introduce the following:

$$\tilde{\mathbf{Z}}_i = \left[\tilde{\mathbf{X}}_i, \mathbf{M}_{F_{x,-1}}\tilde{\mathbf{X}}_{i,-1}\right] \tag{23}$$

where $\tilde{\mathbf{X}}_i = \boldsymbol{X}_i - \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{X}_n \boldsymbol{\Gamma}'_{xn} \boldsymbol{\Upsilon}^{-1}_{kN} \boldsymbol{\Gamma}_{xi}$, $\tilde{\mathbf{X}}_{i,-1} = \boldsymbol{X}_{i,-1} - \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{X}_{n,-1} \boldsymbol{\Gamma}'_{xn} \boldsymbol{\Upsilon}^{-1}_{kN} \boldsymbol{\Gamma}_{xi}$. Note that we will not make use of $\tilde{\mathbf{Z}}_i$ or its estimated version to compute our estimates. Then, we replace Assumption 5(iii) with a version appropriate for $\tilde{\mathbf{Z}}_i$:

Assumption 5(iv) $E\left\|\tilde{\boldsymbol{\varphi}}_{FiT,t}\right\|^{2+\delta} \leq \Delta < \infty$ for all i and T, where $\tilde{\boldsymbol{\varphi}}_{FiT} = T^{-1/2}\tilde{\mathbf{Z}}'_i \mathbf{M}_{F_x}\mathbf{u}_i$ and $E(\tilde{\boldsymbol{\varphi}}_{FiT}\tilde{\boldsymbol{\varphi}}'_{FiT})$ are positive definite for any i, T. In addition, $\lim_{N,T\to\infty} N^{-1}\sum_{i=1}^{N} E(\tilde{\boldsymbol{\varphi}}_{FiT}\tilde{\boldsymbol{\varphi}}'_{FiT}) = \tilde{\boldsymbol{\Omega}}$, which is a fixed positive definite matrix.[11]

The asymptotic expansion of $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \hat{\mathbf{Z}}'_i \mathbf{M}_{\hat{F}_x}\mathbf{u}_i$ is summarised in the following proposition.

Proposition 2 *Under Assumptions 1-3,4(i),5(i)(ii)(iv), as $(N, T) \xrightarrow{j} \infty$ such that $N/T \to c$ with $0 < c < \infty$,*

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \hat{\mathbf{Z}}'_i \mathbf{M}_{\hat{F}_x}\mathbf{u}_i = \frac{1}{\sqrt{NT}}\sum_{i=1}^{N} \tilde{\mathbf{Z}}'_i \mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{1NT} + \sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{2NT} + o_p(1),$$

[11] Assumption 5(iv) is in line with Assumption F in Bai (2003).

where $\hat{\mathbf{Z}}_i$, $\mathbf{M}_{\hat{F}_x}$, $\tilde{\mathbf{Z}}_i$ *and* $\mathbf{M}_{F_x}$ *are defined by (12), (11), (23) and (8), respectively,* $\tilde{\mathbf{b}}_{1NT} = [\tilde{\mathbf{b}}'_{11NT}, \tilde{\mathbf{b}}'_{12NT}]'$, $\tilde{\mathbf{b}}_{2NT} = [\tilde{\mathbf{b}}'_{21NT}, \tilde{\mathbf{b}}'_{22NT}]'$, *with*

$$\tilde{\mathbf{b}}_{11NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_i\mathbf{V}_j}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}^{-1}_{kN}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}'_x\mathbf{u}_i}{T};$$

$$\tilde{\mathbf{b}}_{12NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_{i,-1}\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}^{-1}_{kN}\left(\frac{\mathbf{F}'_{x,-1}\mathbf{F}_{x,-1}}{T}\right)^{-1}\frac{\mathbf{F}'_{x,-1}\mathbf{M}_{F_x}\mathbf{u}_i}{T}$$
$$+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_{i,-1}\mathbf{M}_{F_{x,-1}}\mathbf{V}_j}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}^{-1}_{kN}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}'_x\mathbf{u}_i}{T};$$

$$\tilde{\mathbf{b}}_{21NT} = -\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}'_{xi}\mathbf{\Upsilon}^{-1}_{kN}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\mathbf{F}'_x\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_x}\mathbf{u}_i;$$

$$\tilde{\mathbf{b}}_{22NT} = -\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}'_{xi}\mathbf{\Upsilon}^{-1}_{kN}\left(\frac{\mathbf{F}'_{x,-1}\mathbf{F}_{x,-1}}{T}\right)^{-1}\mathbf{F}'_{x,-1}\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_x}\mathbf{u}_i,$$

where $\bar{\mathbf{\Sigma}}_{kNT} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}E\left(\mathbf{v}_{\ell j}\mathbf{v}'_{\ell j}\right)$ *and* $\bar{\mathbf{\Sigma}}_{kNT,-1} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}E\left(\mathbf{v}_{\ell j,-1}\mathbf{v}'_{\ell j,-1}\right)$.

Remark 5 *In comparison with the asymptotic bias term* $\mathbf{b}_{1NT}$ *arising under Assumption 4(ii) in Proposition 1, dropping this assumption results in additional asymptotic bias terms:* $\tilde{\mathbf{b}}_{21NT}$, $\tilde{\mathbf{b}}_{22NT}$ *and the first term of* $\tilde{\mathbf{b}}_{12NT}$. *These terms arise due to that fact that pre-multiplying* $\mathbf{u}_i$ *by* $\mathbf{M}_{F_x}$ *does not eliminate the factor component* $\mathbf{F}_y\boldsymbol{\lambda}_i$, *which will be correlated with* $\mathbf{F}_x\mathbf{\Gamma}_{xi}$ *in the absence of Assumption 4(ii).*

Since $\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{u}_i$ is $O_p(1)$, the following theorem verifies consistency of the estimator:

Theorem 4 *Consider model (1)-(3) and suppose that Assumptions 1-3,4(i),5(i)(ii)(iv) hold true. Then,*

$$\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta} \xrightarrow{p} \mathbf{0}$$

as N *and* $T \to \infty$ *jointly such that* $T/N \to c$ *with* $0 < c < \infty$, *where* $\hat{\boldsymbol{\theta}}_{IV}$ *is defined in (13).*

Based on the result of Proposition 2, the bias corrected estimator is defined as

$$\hat{\hat{\tilde{\boldsymbol{\theta}}}}_{IV} = \hat{\boldsymbol{\theta}}_{IV} - \left(\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}^{-1}_{NT}\frac{1}{\sqrt{NT}}\hat{\tilde{\mathbf{b}}}_{NT}, \tag{24}$$

where

$$\hat{\tilde{\mathbf{b}}}_{NT} = \sqrt{\frac{T}{N}}\hat{\tilde{\mathbf{b}}}_{1NT} + \sqrt{\frac{N}{T}}\hat{\tilde{\mathbf{b}}}_{2NT}, \tag{25}$$

with $\hat{\hat{\mathbf{b}}}_{1NT} = [\hat{\hat{\mathbf{b}}}'_{11NT}, \hat{\hat{\mathbf{b}}}'_{12NT}]'$, $\hat{\hat{\mathbf{b}}}_{2NT} = [\hat{\hat{\mathbf{b}}}'_{21NT}, \hat{\hat{\mathbf{b}}}'_{22NT}]'$, and

$$
\begin{aligned}
\hat{\hat{\mathbf{b}}}_{11NT} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\hat{\hat{\mathbf{V}}}'_i \hat{\mathbf{V}}_j}{T} \hat{\boldsymbol{\Gamma}}'_{xj} \hat{\boldsymbol{\Upsilon}}^{-1}_{kN} \frac{\hat{\mathbf{F}}'_x \hat{\mathbf{u}}_i}{T}; \\
\hat{\hat{\mathbf{b}}}_{12NT} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\hat{\hat{\mathbf{V}}}'_{i,-1} \hat{\mathbf{V}}_{j,-1}}{T} \hat{\boldsymbol{\Gamma}}'_{xj} \hat{\boldsymbol{\Upsilon}}^{-1}_{kN} \frac{\hat{\mathbf{F}}'_{x,-1} \mathbf{M}_{\hat{F}_x} \hat{\mathbf{u}}_i}{T} \\
&\quad + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\hat{\hat{\mathbf{V}}}'_{i,-1} \mathbf{M}_{\hat{F}_{x,-1}} \hat{\mathbf{V}}_j}{T} \hat{\boldsymbol{\Gamma}}'_{xj} \hat{\boldsymbol{\Upsilon}}^{-1}_{kN} \frac{\hat{\mathbf{F}}'_x \hat{\mathbf{u}}_i}{T}; \\
\hat{\hat{\mathbf{b}}}_{21NT} &= -\frac{1}{N}\sum_{i=1}^{N} \hat{\boldsymbol{\Gamma}}'_{xi} \hat{\boldsymbol{\Upsilon}}^{-1}_{kN} \left(\frac{\widehat{\hat{\mathbf{F}}'_x \boldsymbol{\Sigma}_{kNT} \hat{\mathbf{u}}_i}}{T} - \frac{\widehat{\hat{\mathbf{F}}'_x \boldsymbol{\Sigma}_{kNT} \hat{\mathbf{F}}_x}}{T} \frac{\hat{\mathbf{F}}'_x \hat{\mathbf{u}}_i}{T} \right); \\
\hat{\hat{\mathbf{b}}}_{22NT} &= -\frac{1}{N}\sum_{i=1}^{N} \hat{\boldsymbol{\Gamma}}'_{xi} \hat{\boldsymbol{\Upsilon}}^{-1}_{kN} \left(\frac{\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{u}}_i}}{T} - \frac{\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{F}}_x}}{T} \frac{\hat{\mathbf{F}}'_x \hat{\mathbf{u}}_i}{T} \right. \\
&\quad \left. - \frac{\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{F}}_{x,-1}}}{T} \frac{\hat{\mathbf{F}}'_{x,-1} \mathbf{M}_{\hat{F}_x} \hat{\mathbf{u}}_i}{T} \right).
\end{aligned}
$$

We have chosen a Newey-West type estimator:

$$
\widehat{\hat{\mathbf{F}}'_x \boldsymbol{\Sigma}_{kNT} \hat{\mathbf{u}}_i} = \sum_{\ell=1}^{k} \frac{1}{N} \sum_{j=1}^{N} \frac{1}{T} \left[\hat{\mathbf{f}}_{x,t} \hat{v}^2_{\ell,jt} \hat{u}_{it} + \sum_{s=1}^{S} \sum_{t=s+1}^{T} \left(1 - \frac{s}{S+1}\right) \hat{v}_{\ell,jt} \hat{v}_{\ell,jt-s} \left(\hat{\mathbf{f}}_{x,t} \hat{u}'_{it-s} + \hat{\mathbf{f}}_{x,t-s} \hat{u}'_{it} \right) \right],
$$

where $\widehat{\hat{\mathbf{F}}'_x \boldsymbol{\Sigma}_{kNT} \hat{\mathbf{F}}_x}$, $\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{u}}_i}$, $\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{F}}_{x,-1}}$ and $\widehat{\hat{\mathbf{F}}'_{x,-1} \boldsymbol{\Sigma}_{kNT,-1} \hat{\mathbf{F}}_x}$ are defined in an analogous manner. We set $S = \lfloor T^{1/4} \rfloor$.

We introduce the following assumption:[12]

Assumption 6:

$$
\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \tilde{\mathbf{Z}}'_i \mathbf{M}_{F_x} \mathbf{u}_i \xrightarrow{d} N\left(\mathbf{0}, \tilde{\boldsymbol{\Omega}}\right). \tag{26}
$$

The asymptotic normality of the IV estimators is ready to be shown:

Theorem 5 *Suppose that Assumptions 1-3,4(i),5(i)(ii)(iv) and 6 hold true under model (1)-(3). Then, assuming that* $plim_{N,T\to\infty} \hat{\hat{\mathbf{b}}}_{NT} = \tilde{\mathbf{b}}$ *exists, as* N *and* $T \to \infty$ *jointly in such a way that* $T/N \to c$ *with* $0 < c < \infty$

(i)

$$
\sqrt{NT}\left(\hat{\hat{\tilde{\boldsymbol{\theta}}}}_{IV} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \tilde{\boldsymbol{\Psi}}\right), \tag{27}
$$

where $\hat{\hat{\tilde{\boldsymbol{\theta}}}}_{IV}$ *is defined by (24), and*

$$
\tilde{\boldsymbol{\Psi}} = \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1} \mathbf{A}'\mathbf{B}^{-1} \widetilde{\boldsymbol{\Omega}} \mathbf{B}^{-1}\mathbf{A} \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1} \tag{28}
$$

[12]This assumption is in line with Assumption E of Bai (2009).

is a positive definite matrix, where $\widetilde{\boldsymbol{\Omega}}$ *is defined in Assumption 5(iv);*

(ii) $\hat{\boldsymbol{\Psi}}_{NT} - \tilde{\boldsymbol{\Psi}} = o_p(1)$*, where* $\hat{\boldsymbol{\Psi}}_{NT}$ *is defined by (17);*

(iii)

$$\sqrt{NT}\left(\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV2} - \boldsymbol{\theta}\right) \stackrel{d}{\to} N\left(\mathbf{0}, \left(\mathbf{A}'\widetilde{\boldsymbol{\Omega}}^{-1}\mathbf{A}\right)^{-1}\right), \tag{29}$$

where

$$\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV2} = \hat{\boldsymbol{\theta}}_{IV2} - \left(\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\boldsymbol{\Omega}}^{-1}_{NT}\frac{1}{\sqrt{NT}}\hat{\tilde{\mathbf{b}}}_{NT} \tag{30}$$

with $\hat{\boldsymbol{\theta}}_{IV2}$ *defined by (20).*

The asymptotic properties of the overidentifying restrictions test statistic is given in the following theorem.

Theorem 6 *Suppose that Assumptions 1-3,4(i),5(i)(ii)(iv) and 6 hold true under model (1)-(3). Then, as* N *and* $T \to \infty$ *jointly in such a way that* $T/N \to c$ *with* $0 < c < \infty$

$$\tilde{S}_{NT} \stackrel{d}{\to} \chi^2_{k-1}, \tag{31}$$

for $k > 1$*, under the null hypothesis of strong exogeneity of the covariates, where*

$$\tilde{S}_{NT} = \frac{1}{NT}\left(\sum_{i=1}^{N}\hat{\tilde{\hat{\mathbf{u}}}}{}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i\right)\hat{\boldsymbol{\Omega}}^{-1}_{NT}\left(\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\hat{\tilde{\hat{\mathbf{u}}}}_i\right), \tag{32}$$

with $\hat{\tilde{\hat{\mathbf{u}}}}_i = \mathbf{y}_i - \mathbf{W}_i\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV2}$.

4 Monte Carlo Experiments

This section investigates the finite sample behaviour of the proposed estimator by means of Monte Carlo experiments. In particular, we focus on bias, standard deviation, root mean square error (RMSE), empirical size of the t-test of the bias-corrected two-step estimator $\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV2}$, which is defined by (30), as well as size and power of the overidentifying restrictions test, where the test statistic is given by (32). The small sample performance of our estimator is also compared to the performance of the bias-corrected quasi maximum likelihood estimator (QMLE) recently proposed by Moon and Weidner (2017), as defined in Corollary 3.7 in their paper.[13]

4.1 Design

We consider the following panel data model with two covariates and three factors:

$$y_{it} = \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2}\beta_\ell x_{\ell it} + u_{it};\;\; u_{it} = \lambda_i f_{y,t} + \sum_{s=1}^{2}\gamma_{si}f_{x,st} + \varepsilon_{it};$$
$$i = 1, 2, ..., N;\;\; t = -49, -48, ..., T, \tag{33}$$

[13] We are grateful to Martin Weidner for providing us the computational algorithm for the QMLE estimator.

where $\alpha_i \sim i.i.d.N(0,(1-\rho)^2)$, $\lambda_i \sim i.i.d.N(0,1)$, $\gamma_{si} \sim i.i.d.N(0,1)$ for $s=1,2$, and

$$f_{y,t} = \rho_{fy} f_{y,t-1} + (1-\rho_{fy}^2)^{1/2}\zeta_{y,t};$$
$$f_{x,st} = \rho_{fx,s} f_{x,st-1} + (1-\rho_{fx,s}^2)^{1/2}\zeta_{x,st},$$

with $\zeta_{y,t} \sim i.i.d.N(0,1)$ and $\zeta_{x,st} \sim i.i.d.N(0,1)$ for $s=1,2$.

The idiosyncratic error, ε_{it}, is non-normal and heteroskedastic across both i and t, such that $\varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it}-1)/\sqrt{2}$, $\epsilon_{it} \sim i.i.d.\chi_1^2$, with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim iid\chi_2^2/2$, and $\varphi_t = t/T$ for $t = 0,1,...,T$ and unity otherwise.

The process for the covariates is given by

$$x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si} f_{x,st} + v_{\ell it};\; i = 1,2,...,N; t = -49,-48,...,T,$$

for $\ell = 1,2$, which means that only a subset of the factors in y enter the process for x_1 and x_2.

The individual-specific effects in x_1 and x_2 are allowed to be correlated with those in the equation for y in the following way:

$$\mu_{\ell i} = \rho_{\mu,\ell}\alpha_i + (1-\rho_{\mu,\ell}^2)^{1/2}\omega_{\ell i},\; \omega_{\ell i} \sim i.i.d.N(0,(1-\rho)^2)$$

for $\ell = 1,2$. Furthermore, the factor loadings of x_1 are drawn as

$$\gamma_{1si} = \rho_{\gamma,1s}\gamma_{si} + (1-\rho_{\gamma,1s}^2)^{1/2}\xi_{1si};\; \xi_{1si} \sim i.i.d.N(0,1),$$

and the factor loadings of x_2 are generated by

$$\gamma_{2si} = \rho_{\gamma,2s}\lambda_i + (1-\rho_{\gamma,2s}^2)^{1/2}\xi_{2si};\; \xi_{2si} \sim i.i.d.N(0,1),$$

for $s = 1,2$. This allows for the case where Assumption 4(ii) is violated, that is, the factor loadings in the process for x_2 are both correlated with the loadings corresponding to the non-overlapping factor in y, namely λ_i.

The idiosyncratic errors of the process for the covariates are serially correlated, such that

$$v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_{v,\ell}^2)^{1/2}\varpi_{\ell it},\; \varpi_{\ell it} \sim i.i.d.N(0,\varsigma_v^2\sigma_{\varpi_{\ell,i}}^2),\; \sigma_{\varpi_{\ell,i}}^2 \sim i.i.d.U[0.5,1.5],$$

for $\ell = 1,2$.

We consider $\rho \in \{0.5, 0.8\}$, whereas we set $\beta_1 = 3$ and $\beta_2 = 1$ as a benchmark case following Bai(2009). In order to investigate the properties of the estimator when one of the slope coefficients is equal to zero, we specify $\beta_1 = 3$ and $\beta_2 = 0$. Moreover, we set $\rho_{\mu,\ell} = 0.5$, $\rho_{\gamma,\ell} = 0.5$, $\rho_{fx,s} = 0.5$, $\rho_{fy} = 0.5$, $\rho_{v,\ell} = 0.5$ for $\ell = 1,2$, $s = 1,2$.

It is straightforward to see that overall average of $\text{var}(\varpi_{\ell it})$ over i and t is ς_v^2, for $\ell = 1,2$, since $\sigma_{\varpi_{\ell,i}}^2$ merely allows for cross-sectional heteroskedasticity and $E\left(\sigma_{\varpi_{\ell,i}}^2\right) = 1$. Let π_x denote the proportion of the variance of $x_{\ell it}$ that is due to $v_{\ell it}$ for all ℓ. That is, we define $\pi_x := \varsigma_v^2/\left(m_x + \varsigma_v^2 + (1-\rho_v)^2\right)$. Solving in terms of ς_v^2 yields

$$\varsigma_v^2 = \frac{\pi_x}{(1-\pi_x)}\left[1 + (1-\rho_v)^2\right].$$

Thus, for example, $\pi_x = 3/4$ means that, for each of the covariates, the variance of the idiosyncratic error accounts for 75% of the total variance in x. In this case most of the variation in the covariates is due to the idiosyncratic component and the factor structure has relatively minor significance. We set ς_v^2 such that $\pi_x \in \{1/4, 3/4\}$. These values are motivated by the results in Sargent and Sims (1977), who show that two common factors explain a large proportion of the variation in many macroeconomic series.[14]

It is easily seen that the DGP ensures that the overall average of $\sigma_{it}^2 = E(\varepsilon_{it}^2)$ over i and t is ς_ε^2. Denoting $\rho_v = \rho_{v,\ell}$, $\ell = 1, 2$, we define the signal-to-noise ratio of the model, conditional on the factor structure and the individual-specific effects, as follows:

$$SNR := \frac{\text{var}\left[(y_{it} - \varepsilon_{it})\,|\mathcal{L}\right]}{\overline{\text{var}}\,(\varepsilon_{it})} = \frac{\left(\frac{\beta_1^2+\beta_2^2}{1-\rho_v^2}\right)\varsigma_v^2 + \frac{\varsigma_\varepsilon^2}{1-\rho_v^2} - \varsigma_\varepsilon^2}{\varsigma_\varepsilon^2},$$

where $\mathcal{L}$ is the information set that contains the factor structure and the individual-specific effects,[15] and $\overline{\text{var}}\,(\varepsilon_{it})$ is the overall average of $E(\varepsilon_{it}^2)$ over i and t. Solving for ς_ε^2 yields

$$\varsigma_\varepsilon^2 = \left(\frac{\beta_1^2 + \beta_2^2}{1-\rho_v^2}\right)\varsigma_v^2 \left[SNR - \frac{\rho_v^2}{1-\rho_v^2}\right]^{-1}.$$

We set ς_ε^2 such that $SNR = 2$. We consider all the combinations of (T, N), for $T \in \{25, 50, 100, 200\}$ and $N \in \{25, 50, 100, 200\}$.

In order to investigate the power of the overidentifying restrictions test, which is defined in (21), we change the DGP such that $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_\ell^2)^{1/2}\varpi_{\ell it}$, $\varpi_{\ell it} = \tau_\ell \varepsilon_{it} + (1-\tau_\ell^2)^{1/2}\varrho_{\ell it}$ with $\varrho_{\ell it} \sim i.i.d.N(0,1)$, $\ell = 1, 2$. We set $\tau_1 = 0.5$ and $\tau_2 = 0$ so that the idiosyncratic error of x_{1it} is correlated with ε_{it}.

All results are obtained based on 2,000 replications, and all tests are conducted at the 5% significance level.

4.2 Results

Table 1 reports the bias, standard deviation, RMSE and size of the t-test based on the IV and QMLE estimators for the panel dynamic model with $\rho = 0.5$, $\beta_1 = 3$, $\beta_2 = 1$.[16] Panel A reports the results of the estimators of ρ, and Panel B those for β_2. The results for β_1 are not reported here as they are qualitatively similar to those for β_2 (which are available upon request from the authors). IV refers to the bias-corrected two-step instrumental variables estimator defined in (30), whereas QMLE stands for the bias-corrected quasi maximum likelihood estimator proposed by Moon and Weidner (2017; Corollary 3.7)

When using the IV estimator, an estimate of m_x, i.e. $\hat{m}_x$, is obtained in each replication, which is based on the information criteria IC_1 proposed by Bai

[14]Indicatively, they find that two common factors explain about 93% of the variation in real GNP, 86% of the variation in unemployment rate and 26% of the variation in residential construction.

[15]The reason we condition on these variables is that they influence both the composite error in the equation for the dependent variable and the covariates.

[16]The quantities reported for bias, standard deviation and RMSE are scaled by a factor of 10 in order to make the results easier to discern.

and Ng (2002). We set the maximum number of factors equal to three.[17] The estimate of common factors is denoted by $\hat{\mathbf{F}}_x$, which is a $T \times \hat{m}_x$ matrix obtained by extracting the principal components from $\sum_{i=1}^{N} \left(\mathbf{X}_i - \overline{\mathbf{X}}_i\right)\left(\mathbf{X}_i - \overline{\mathbf{X}}_i\right)'$, where $\overline{\mathbf{X}}_i = \boldsymbol{\iota}_T \times T^{-1} \sum_t \mathbf{x}'_{it}$ and $\boldsymbol{\iota}_T$ is a $T \times 1$ column vector of ones. In order to deal with individual specific effects, we use the matrix that projects out the common factors including a column of ones, such that $\widehat{\mathbf{H}} = \left[\boldsymbol{\iota}_T;\ \widehat{\mathbf{F}_x}\right]$ with $\mathbf{M}_{\widehat{H}} = \mathbf{I}_T - \widehat{\mathbf{H}}(\widehat{\mathbf{H}}'\widehat{\mathbf{H}})^{-1}\widehat{\mathbf{H}}'$, rather than $\mathbf{M}_{\widehat{F_x}}$. By doing so, we swipe away the individual effects in $\mathbf{x}_{it}$. As for the QMLE, $(y_{it}, \mathbf{x}'_{it})$ are transformed prior to estimation by taking deviations from individual-specific averages, while $m_x + m_y$ is estimated from the residuals of the model using the same criteria of Bai and Ng (2002), IC_1, where the maximum number of factors is set equal to four.

First let us discuss the small sample properties of the estimators of ρ, where the results are presented in Panel A, Table 1. We see that the IV estimator appears to have virtually no bias. The largest absolute bias reported in Table 1 is 0.0011 for $N = T = 25$, and it decreases in magnitude as N or T increases. There is little evidence that the value of π_x affects the bias of IV estimator. Absolute bias of the QMLE reported in Table 1 is always much larger than that of the IV estimator, and it seems to be sensitive to the values taken by T, N and π_x. When $\pi_x = 1/4$, the bias of the QMLE is negative in most of the cases considered, and the bias decreases in absolute value as T increases. For instance, when $N = 50$, for $T = 25, 50, 100, 200$, the biases of QMLE are -0.0060, -0.0026, -0.0012 and -0.0005, respectively. On the other hand, when $\pi_x = 3/4$, the bias does not necessarily decrease monotonically as T increases, unless N is sufficiently large.

The standard deviation of both the IV estimator and QMLE becomes smaller as the values of either T and/or N increases. Even though the standard deviation of IV estimator is comparable to that of QMLE, the standard deviation of the QMLE is smaller than that of IV estimator in most of the cases under consideration. This is expected because the IV estimator is based on estimating the common factors in x only, whereas the factors in the error term of y are also estimated when using the QMLE. This difference in dispersion of two estimators is, however, smaller when $\pi_x = 3/4$, since the IV estimator gains efficiency when π_x gets larger, as it increases the correlation between the instruments and the endogenous variables.

The performance in terms of RMSE reflects the insights drawn from the results of bias and standard deviation discussed above. When $\pi_x = 1/4$, the smaller standard deviation overwhelms the larger bias of the QMLE, so that the RMSE of QMLE is smaller for all the combinations of N and T considered. When $\pi_x = 3/4$, the larger bias of QMLE and improved relative efficiency of the IV estimator make the RMSEs of IV estimator and QMLE very similar in magnitude. Indeed, in seven cases out of 16 combinations of N and T, the RMSE of IV estimator is smaller than that of QMLE. Notwithstanding, the size of t-test based on the QMLE appears to be quite severely distorted. Even when $N = T = 200$ and $\pi_x = 1/4$, which can

[17] Simulations in Bai and Ng (2002) show that the performance of this information criterion is robust provided $\min\{N, T\} > 40$. Our results show that notwithstanding that $\min\{N, T\} < 40$ our IV estimator appears to perform very well. Intuitively, this is because in our experiment the results suggest that for small values of either N or T, the information criterion tends to overshoot the true number of factors, which however does not affect consistency for our estimator.

be seen as the most favorable case for QMLE, the size of the t-test is 13.4% at the 5% nominal level. In contrast, the size of the t-test based on the IV estimator is very close to nominal size of 5% for all the combinations of N and T and for both $\pi_x = 1/4$ and $3/4$.

Now let us turn our attention to Panel B in Table 1, which summarizes the results for the estimators of β_2. Surprisingly, the QMLE of β_2 exhibits much larger bias than that of ρ, and it gets larger in magnitude when the value of π_x increases. For example, when $N = T = 50$ and $\pi_x = 1/4$, the bias of QMLE is 0.0208, but it becomes 0.106 with $\pi_x = 3/4$, which is a positive bias of 10.6%. On the other hand, the results reported in Panel B in Table 1 suggest that the bias of IV estimator is very small in absolute value. Furthermore, the performance of IV estimator in terms of standard deviation dominates that of QMLE. In fact, in seven (fifteen) cases out of 16 combinations of N and T, the standard deviation of IV estimator is smaller than that of QMLE when $\pi_x = 1/4$ ($\pi_x = 3/4$). As for the RMSE, the superior relative performance of IV estimator is even more pronounced. When $\pi_x = 3/4$, the value of RMSE of IV estimator is twice as small as that of QMLE in a vast majority of cases. Similarly as results reported in Panel A, the empirical size of the t-test based on the QMLE presented in Panel B exhibits considerable upward distortions, whereas very small or no size distortions are seen for the t-test based on the IV estimator.

Table 2 provides the results obtained using the IV estimator and QMLE to estimate the dynamic panel model with $\rho = 0.5$, $\beta_1 = 3$, $\beta_2 = 0$. Similar conclusions are drawn based on these results, so we do not discuss them in detail to save space. In order to see how the small sample performance of the two estimators is affected when the DGP exhibits higher degree of persistency, further experiments with $\rho = 0.8$, $\beta_1 = 3$, $\beta_2 = 1$ are implemented and the results are summarized in Table 3. The relative performance of the IV estimator and QMLE is qualitatively similar to that under $\rho = 0.5$, but the differences in results described above are more apparent. In particular, the bias of QMLE of β_2 is very large in magnitude when $\rho = 0.8$ (see Panel B, Table 3) and much more severe than that under $\rho = 0.5$. For example, when $\rho = 0.5$, $\pi_x = 3/4$ and $N = T = 50$, the bias of QMLE of β_2 is 0.106 (see Panel B, Table 1), while it is 0.206 when $\rho = 0.8$ (see Panel B, Table 3). More surprisingly, when $\pi_x = 1/4$ with $\rho = 0.8$ and $N = T = 50$, the bias of QMLE is even larger in absolute value compared to the previous two cases and it is equal to 0.401. The corresponding bias of the IV estimator is smaller in magnitude and takes values of -0.0022, -0.0080 and -0.0103, respectively. The large values of absolute bias of the QMLE result in bigger RMSE and more severe size distortions of the t-test based on this estimator.

Finally, Table 4 reports the empirical size and power of the overidentifying restrictions test. The size of test is very close to the nominal value in most of combinations of N and T, unless N is very small, in which case it is slightly distorted downwards. Notably, the test has high power when the idiosyncratic error in x equation is correlated with the idiosyncratic error in y equation. Thus, it appears that this test can be a reliable statistical tool to check the key assumption within our approach.

5 Concluding Remarks

This paper has proposed a computationally attractive instrumental variables procedure for consistent estimation of dynamic linear panel data models with error cross-sectional dependence when both N and T are large. Our approach involves projecting out the common factors from the regressors at first stage, and then using the defactored regressors as instruments for the endogenous variables. The estimated number of factors and the factors themselves are obtained from observed variables rather than residuals. Since our procedure is based on instrumental variables, there is no need to correct for Nickell bias induced due to predeterminedness.

Aside from computational simplicity the method has the feature that it does not require estimating possible distinct factors that enter directly only into the y process, thus leaving these factors in the residuals. Therefore, full specification of the error term of the model for y is not required.

The finite sample evidence reported in the paper suggests that the proposed estimator performs reasonably well under all circumstances examined, and therefore it presents a good alternative way of estimation to existing approaches. In particular, the estimator appears to have little bias, and small dispersion unless either N or T is small. Furthermore, the empirical size of the t-test appears to be close to nominal one in most cases, which makes our estimator particularly suitable for inferential purposes. The results of the overidentifying restrictions test statistic suggest that the test statistic has good power to detect violations from basic assumptions employed within our approach.

In practice, it is also possible that (a subset of) the factors that hit the covariates are orthogonal to the composite disturbance of the y process. In this case the proposed approach in this paper is asymptotically valid, though, full defactoring is not necessary for consistency of the IV estimator. Empirically, this issue can be addressed using a sequential testing method based on the overidentifying restrictions test that we have explored in this paper. In particular, one may start by testing whether the untransformed covariates are strongly exogenous with respect to the composite disturbance. Notice that the null hypothesis will also be satisfied if the covariates do not have a factor structure at all. If the null is rejected, one may project out one factor (based on the largest eigenvalue) and test whether the defactored regressors yield valid instruments using the same statistic. If the null is rejected, one may project out two factors and so on. Naturally, the significance level used for this sequential method needs to be appropriately adjusted. The interested reader is recommended to refer to Proposition 2 of Ahn et al. (2013).

Finally, notice that although the proofs of our results require N and T both large, under certain restrictions imposed in the covariates $-$ in particular, asymptotic homoskedasticity and serial uncorrelatedness $-$ it is possible to derive consistency and asymptotic normality of our estimator even for T fixed; see Bai (2003). On the other hand, the simulation evidence we have presented suggests that even if these conditions are not met in practice, the bias of the estimator appears to be small and the size of the t-test satisfactory unless *both* N and T are small. Therefore, we hope that our approach provides a computationally attractive way to estimate dynamic panel data models with multi-factor residual structures, even in cases where either T or N are moderately small.

Table 1: Bias, Standard Deviation, RMSE and Size of the t-test of IV and QMEL estimators, for the Panel Dynamic Model with $\rho = .5$, $\beta_1 = 3$ and $\beta_2 = 1$.

PANEL A: Bais, standard deviation, RMSE and size of t-test of the estimates of ρ																
	IV								**QMLE**							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
	bias×10															
25	-.011	.000	-.004	.000	-.011	-.001	-.003	.001	-.060	-.070	-.081	-.085	-.055	-.079	-.114	-.140
50	.004	.006	.000	.000	.005	.006	.000	-.001	-.026	-.039	.040	.040	.140	-.007	-.041	-.068
100	.001	.005	.001	.001	.001	.004	.001	.001	-.012	-.019	-.019	-.012	.043	.019	-.011	-.038
200	-.002	.001	-.002	-.001	-.001	.001	-.001	-.001	-.005	-.010	-.009	-.010	.062	.035	.001	-.002
	st.dev.×10															
25	.371	.257	.177	.125	.325	.233	.153	.109	.149	.100	.081	.053	.203	.162	.134	.117
50	.216	.154	.109	.078	.183	.133	.094	.066	.081	.057	.039	.027	.132	.104	.078	.061
100	.145	.102	.070	.050	.123	.086	.060	.042	.053	.037	.024	.024	.087	.067	.052	.036
200	.101	.070	.048	.035	.085	.059	.042	.030	.037	.025	.017	.011	.066	.050	.036	.023
	RMSE×10															
25	.371	.257	.177	.125	.325	.233	.153	.109	.150	.124	.108	.100	.210	.180	.176	.183
50	.216	.154	.109	.078	.183	.134	.094	.066	.085	.069	.056	.055	.132	.104	.088	.091
100	.145	.102	.070	.050	.123	.086	.060	.042	.055	.041	.031	.026	.097	.070	.053	.052
200	.101	.070	.048	.035	.085	.059	.042	.030	.037	.026	.019	.015	.090	.060	.036	.032
	size of t-test															
25	.069	.061	.047	.045	.078	.063	.049	.045	.208	.136	.107	.090	.210	.251	.409	.603
50	.059	.056	.050	.058	.070	.060	.060	.065	.259	.172	.115	.093	.170	.173	.240	.449
100	.061	.060	.041	.044	.075	.064	.053	.056	.380	.245	.145	.088	.171	.164	.150	.319
200	.073	.050	.049	.049	.081	.059	.059	.058	.593	.395	.222	.134	.301	.230	.131	.221

Table 1, continued.

PANEL B: Bais, standard deviation, RMSE and size of t-test of the estimates of β_2																
	IV								**QMLE**							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
	bias×10															
25	.013	-.011	.004	.017	.207	.101	.094	.091	1.13	.786	.359	.151	1.12	1.19	1.08	.966
50	.007	-.022	-.009	.009	.045	.006	.015	.027	.687	.208	.081	.035	1.15	1.06	.911	.667
100	-.005	.005	-.005	-.002	.016	.016	.005	.005	.435	.117	.026	.013	1.11	.942	.667	.311
200	-.001	.008	-.001	.005	.007	.010	.004	.006	.298	.044	.011	.010	1.06	.837	.435	.112
	st.dev.×10															
25	1.32	.896	.637	.453	1.05	.728	.523	.362	1.65	1.19	.742	.453	1.00	.744	.576	.445
50	.869	.594	.426	.289	.693	.466	.333	.227	1.28	.696	.368	.233	.781	.588	.467	.369
100	.588	.412	.282	.198	.455	.318	.221	.157	.998	.426	.233	.158	.676	.486	.411	.288
200	.413	.295	.202	.146	.322	.222	.153	.113	.786	.263	.156	.107	.578	.451	.355	.178
	RMSE×10															
25	1.32	.896	.637	.453	1.07	.735	.531	.373	2.06	1.42	.824	.477	1.59	1.40	1.22	1.06
50	.869	.594	.427	.289	.694	.466	.333	.229	1.46	.727	.376	.235	1.39	1.21	1.02	.762
100	.588	.412	.283	.198	.455	.318	.221	.157	1.09	.442	.235	.159	1.32	1.06	.782	.424
200	.413	.295	.202	.146	.322	.222	.153	.113	.841	.267	.157	.107	1.21	.951	.562	.207
	size of t-test															
25	.066	.064	.062	.060	.076	.072	.068	.063	.497	.384	.242	.151	.517	.643	.740	.823
50	.078	.057	.055	.051	.084	.058	.058	.052	.390	.192	.097	.078	.643	.756	.810	.774
100	.075	.068	.049	.049	.077	.060	.045	.055	.342	.138	.088	.076	.785	.824	.767	.485
200	.072	.073	.064	.063	.078	.062	.047	.059	.346	.115	.078	.061	.853	.862	.637	.218

Notes: The DGP follows $y_{it} = \alpha_i + \rho y_{it-1} + \sum_{\ell=1}^{2} \beta_\ell x_{\ell it} + \lambda_i f_{y,t} + \sum_{s=1}^{2} \gamma_{si} f_{x,st} + \varepsilon_{it}$, where $\alpha_i \sim i.i.d.N(0,(1-\rho)^2), \lambda_i \sim i.i.d.N(0,1), \gamma_{si} \sim i.i.d.N(0,1)$ for $s = 1,2, \varepsilon_{it} = \varsigma_\varepsilon \sigma_{it}(\epsilon_{it} - 1)/2^{1/2}, \epsilon_{it} \sim i.i.d.\chi_1^2$, with $\sigma_{it}^2 = \eta_i \varphi_t$, $\eta_i \sim i.i.d.\chi_2^2/2$, $\varphi_t = t/T$ for $t = 0, ..., T$, otherwise unity. $x_{\ell it} = \mu_{\ell i} + \sum_{s=1}^{2} \gamma_{\ell si} f_{st} + v_{\ell it}$ for all ℓ, where $\mu_{\ell i} = \rho_{\mu,\ell}\alpha_i + (1-\rho_{\mu,\ell}^2)^{1/2}\omega_{\ell i}$, $\omega_{\ell i} \sim i.i.d.N(0,(1-\rho)^2)$. The factor loadings, $\gamma_{\ell si}$, in $x_{\ell it}$ are drawn as $\gamma_{\ell si} = \rho_{\gamma,\ell s}\gamma_{si} + (1-\rho_{\gamma,\ell s}^2)^{1/2}\xi_{\ell si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$ for $\ell = 1$ and $s = 1,2$, whereas $\gamma_{\ell si} = \rho_{\gamma,\ell s}\lambda_i + (1-\rho_{\gamma,\ell s}^2)^{1/2}\xi_{\ell si}$, $\xi_{\ell si} \sim i.i.d.N(0,1)$ for $\ell = 2$ and $s = 1,2$. Moreover, $f_{x,st} = \rho_{fx,s} f_{x,st-1} + (1-\rho_{fx,s}^2)^{1/2}\zeta_{x,st}$, and $f_{y,t} = \rho_{fy} f_{y,t-1} + (1-\rho_{fy}^2)^{1/2}\zeta_{y,t}$ with $\zeta_{x,st} \sim i.i.d.N(0,1), s = 1,2$ and $\zeta_{y,t} \sim i.i.d.N(0,1)$. $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1-\rho_{v,\ell}^2)^{1/2}\varpi_{\ell it}$, $\varpi_{\ell it} \sim i.i.d.N(0, \varsigma_v^2 \sigma_{\varpi_{\ell,i}}^2)$. We set $\rho_{\mu,\ell} = 0.5$, $\rho_{\gamma,\ell s} = 0.5$, $\rho_{f,xs} = \rho_{f,y} = 0.5$, $\rho_{v,\ell} = 0.5$ for all ℓ and s. ς_ε^2 is set such that $SNR = 2$, while ς_v^2 is determined by π_x, the proportion of the total variance in x due to the idiosyncratic component. $\pi_x \in \{1/4, 3/4\}$, $\rho = \{0.5, 0.8\}$ with $\beta_1 = 3$, whereas $\beta_2 = \{1, 0\}$. IV refers to the bias-corrected two-step instrumental variables estimator defined in (30). QMLE refers to the bias-corrected quasi maximum likelihood estimator put forward by Moon and Weidner (2017; Corollary 3.7). For IV and QMLE, the number of factors are estimated using IC_1 proposed by Bai and Ng (2002). All experiments are based on 2,000 replications and nominal level of the test is set to 5%. The results for the estimates of β_1 are very similar and not reported (available upon request from the authors).

Table 2: Bias, Standard Deviation, RMSE and Size of the t-test of IV and QMEL estimators, for the Panel Dynamic Model with $\rho = .5$, $\beta_1 = 3$ and $\beta_2 = 0$.

PANEL A: Bais, standard deviation, RMSE and size of t-test of the estimates of ρ																
	IV								**QMLE**							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
bias×10																
25	-.011	.001	-.004	.001	-.007	.000	-.002	.002	-.064	-.074	-.014	-.085	-.053	-.082	-.120	-.146
50	.005	.006	.001	-.001	.006	.005	.000	.000	-.026	-.039	-.019	.040	.015	-.008	-.044	-.072
100	.001	.004	.002	.001	.001	.003	.002	.000	-.014	-.019	-.019	-.019	.045	.018	-.014	-.040
200	.000	.002	-.002	-.001	.000	.002	-.002	-.001	-.005	-.010	-.009	-.010	.063	.032	-.002	-.024
st.dev.×10																
25	.379	.259	.177	.125	.329	.225	.152	.109	.143	.082	.054	.052	.206	.167	.136	.117
50	.223	.156	.111	.079	.186	.134	.094	.067	.081	.057	.036	.027	.134	.105	.078	.060
100	.148	.103	.071	.051	.125	.086	.060	.042	.054	.036	.024	.018	.090	.069	.052	.035
200	.102	.070	.050	.035	.085	.059	.042	.029	.037	.025	.017	.011	.068	.051	.037	.022
RMSE×10																
25	.379	.259	.177	.125	.329	.225	.152	.109	.156	.086	.055	.100	.213	.186	.181	.187
50	.223	.156	.111	.079	.186	.134	.094	.067	.085	.069	.041	.055	.135	.105	.089	.094
100	.148	.103	.071	.051	.125	.086	.060	.042	.055	.041	.031	.026	.101	.071	.054	.054
200	.102	.070	.050	.035	.085	.059	.042	.029	.037	.027	.019	.015	.092	.060	.037	.033
size of t-test																
25	.067	.055	.047	.044	.081	.063	.049	.050	.207	.251	.371	.584	.212	.264	.413	.612
50	.064	.052	.050	.054	.066	.067	.059	.064	.138	.174	.245	.398	.167	.166	.244	.473
100	.059	.061	.046	.042	.078	.064	.056	.055	.111	.109	.145	.209	.185	.153	.154	.337
200	.074	.046	.047	.048	.083	.056	.058	.052	.096	.091	.092	.063	.303	.241	.137	.237

Table 2, continued.

PANEL B: Bais, standard deviation, RMSE and size of t-test of the estimates of β_2																
	IV								QMLE							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
bias×10																
25	.010	-.009	.003	.017	.200	.102	.092	.091	1.07	.622	.266	.089	1.16	1.10	.965	.835
50	.007	-.021	-.009	.008	.047	.009	.015	.027	.554	.140	.046	.012	1.10	.984	.802	.530
100	-.004	.007	-.004	-.001	.018	.019	.006	.005	.343	.082	.011	.001	1.09	.868	.555	.212
200	-.003	.008	-.002	.005	.006	.011	.003	.005	.221	.023	.004	.004	1.03	.763	.337	.065
st.dev.×10																
25	1.29	.870	.617	.437	1.01	.694	.498	.343	1.60	1.10	.664	.406	.979	.729	.568	.433
50	.837	.575	.412	.280	.652	.441	.314	.215	1.19	.626	.331	.219	.767	.583	.463	.361
100	.571	.400	.273	.191	.433	.300	.208	.147	.904	.375	.219	.149	.672	.485	.406	.258
200	.400	.286	.195	.142	.304	.209	.145	.107	.686	.236	.147	.101	.578	.452	.334	.144
RMSE×10																
25	1.29	.870	.617	.438	1.02	.701	.506	.354	1.92	1.27	.711	.415	1.52	1.32	1.12	.941
50	.837	.576	.412	.280	.654	.441	.315	.217	1.31	.642	.334	.220	1.34	1.14	.926	.642
100	.571	.400	.273	.191	.433	.301	.208	.147	.967	.384	.219	.159	1.28	.994	.688	.334
200	.400	.286	.195	.142	.305	.209	.145	.107	.721	.238	.147	.101	1.18	.887	.475	.158
size of t-test																
25	.067	.065	.065	.060	.079	.072	.071	.065	.465	.339	.203	.155	.518	.617	.691	.755
50	.076	.055	.056	.053	.082	.060	.054	.052	.357	.161	.088	.075	.638	.741	.738	.659
100	.076	.067	.049	.048	.080	.064	.049	.054	.300	.122	.086	.072	.774	.796	.674	.363
200	.074	.075	.065	.062	.079	.065	.053	.061	.300	.105	.076	.063	.843	.828	.537	.147

Notes: See notes to Table 1.

Table 3: Bias, Standard Deviation, RMSE and Size of the t-test of IV and QMEL estimators, for the Panel Dynamic Model with $\rho = .8$, $\beta_1 = 3$ and $\beta_2 = 1$.

PANEL A: Bais, standard deviation, RMSE and size of t-test of the estimates of ρ																
	IV								**QMLE**							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
bias$\times 10$																
25	-.053	-.005	-.012	.006	-.053	-.010	-.014	.005	-.219	-.300	-.490	-.980	-.528	-.596	-.795	-1.37
50	.012	.016	-.001	-.004	.015	.015	-.002	-.004	-.049	-.074	-.097	-.132	-.209	-.225	-.234	-.255
100	.002	.012	.004	.002	.002	.011	.004	.002	.013	-.002	-.025	-.058	-.080	-.085	-.094	-.098
200	-.004	.004	-.005	-.003	-.004	.004	-.005	-.003	.044	.030	.006	-.023	-.023	-.029	-.030	-.033
st.dev.$\times 10$																
25	1.28	.852	.584	.405	1.33	.875	.597	.419	.388	.447	.732	1.11	.581	.521	.689	1.12
50	.639	.466	.325	.227	.637	.465	.324	.225	.152	.121	.105	.098	.225	.174	.140	.130
100	.407	.283	.198	.139	.405	.279	.197	.136	.092	.068	.060	.046	.138	.102	.075	.057
200	.272	.189	.134	.096	.267	.186	.132	.094	.060	.051	.037	.028	.087	.068	.048	.036
RMSE$\times 10$																
25	1.28	.258	.584	.405	1.33	.875	.597	.419	.446	.539	.881	1.48	.785	.792	1.05	1.77
50	.639	.466	.325	.227	.637	.465	.324	.225	.160	.142	.142	.164	.306	.284	.273	.286
100	.407	.283	.198	.139	.405	.279	.197	.136	.092	.068	.065	.075	.160	.133	.120	.114
200	.277	.189	.134	.096	.267	.186	.132	.094	.074	.059	.038	.037	.090	.074	.057	.049
size of t-test																
25	.057	.049	.045	.045	.070	.056	.046	.045	.367	.522	.701	.887	.502	.688	.875	.974
50	.064	.060	.055	.062	.069	.063	.053	.065	.212	.298	.461	.706	.311	.472	.676	.861
100	.071	.064	.046	.055	.078	.062	.051	.053	.152	.139	.217	.524	.182	.237	.366	.580
200	.077	.060	.063	.059	.079	.062	.064	.060	.203	.245	.145	.279	.082	.123	.141	.226

Table 3, continued.

PANEL B: Bais, standard deviation, RMSE and size of t-test of the estimates of β_2																
	IV								QMLE							
	$\pi_x = 1/4$				$\pi_x = 3/4$				$\pi_x = 1/4$				$\pi_x = 3/4$			
T,N	25	50	100	200	25	50	100	200	25	50	100	200	25	50	100	200
	bias$\times 10$															
25	.034	-.078	.003	.038	.266	.029	.078	.100	4.05	3.85	3.75	3.18	2.01	2.08	1.93	1.79
50	-.041	-.103	-.035	.028	-.032	-.080	-.017	.042	3.93	4.01	3.89	3.78	1.92	2.06	2.03	2.05
100	-.015	.008	-.007	-.002	.004	.002	.011	.004	3.99	3.91	3.79	3.42	1.89	1.98	2.02	2.01
200	-.004	.004	-.005	-.003	-.018	.015	.011	.015	3.86	3.87	3.65	2.73	1.91	1.95	1.93	1.92
	st.dev.$\times 10$															
25	4.41	2.92	2.04	1.41	4.08	2.80	1.99	1.37	2.03	1.59	1.33	1.51	2.82	2.02	1.60	1.30
50	2.71	1.83	1.31	.882	2.61	1.78	1.26	.863	1.70	1.14	.906	.703	1.86	1.42	1.04	.733
100	1.77	1.23	.858	.611	1.73	1.19	.833	.599	1.51	1.05	.816	.784	1.36	.944	.715	.518
200	1.24	.859	.596	.442	1.20	.829	.574	.427	1.38	1.04	.865	1.05	1.01	.723	.522	.371
	RMSE$\times 10$															
25	4.41	2.92	2.04	1.41	4.09	2.80	1.99	1.37	4.54	4.17	3.98	3.52	3.47	2.90	2.51	2.21
50	2.71	1.83	1.31	.882	2.61	1.78	1.26	.863	4.28	4.17	4.00	3.85	2.67	2.51	2.28	2.18
100	1.77	1.23	.858	.611	1.73	1.19	.833	.599	4.26	4.04	3.87	3.51	2.33	2.20	2.15	2.07
200	1.24	.859	.596	.442	1.20	.829	.574	.427	4.10	4.00	3.75	2.93	2.17	2.08	2.00	1.96
	size of t-test															
25	.051	.057	.055	.057	.058	.068	.063	.056	.367	.891	.935	.887	.273	.331	.428	.541
50	.068	.056	.052	.046	.081	.064	.056	.049	.899	.990	.994	.999	.324	.495	.651	.885
100	.073	.062	.044	.055	.081	.060	.045	.055	.960	.996	.998	.994	.459	.677	.903	.982
200	.071	.064	.046	.060	.078	.059	.048	.060	.980	.991	.994	.965	.660	.874	.987	1.00

Notes: See notes to Table 1.

Table 4: Estimated Size and Power of the Overidentifying Restrictions Test, for the panel dynamic model with $\rho = .5$, $\beta_1 = 3$ and $\beta_2 = 1$

	Size				**Power**			
T,N	25	50	100	200	25	50	100	200
				$\pi_x = 1/4$				
25	.042	.044	.045	.049	.389	.636	.888	.992
50	.034	.050	.049	.050	.678	.993	.994	1.00
100	.049	.049	.050	.047	.912	.995	1.00	1.00
200	.039	.049	.058	.046	.995	1.00	1.00	1.00
				$\pi_x = 3/4$				
25	.039	.044	.047	.053	.643	.932	.996	1.00
50	.038	.046	.050	.053	.934	1.00	1.00	1.00
100	.054	.050	.048	.049	.998	1.00	1.00	1.00
200	.038	.050	.049	.045	1.00	1.00	1.00	1.00

Notes: The DGP is the same as that for Table 1, except that follows $v_{\ell it} = \rho_{v,\ell} v_{\ell it-1} + (1 - \rho_\ell^2)^{1/2} \varpi_{\ell it}$, $\varpi_{\ell it} = \tau_\ell \varepsilon_{it} + (1 - \tau_\ell^2)^{1/2} \varrho_{\ell it}$ with $\varrho_{\ell it} \sim i.i.d.N(0,1)$, $\ell = 1, 2$. We set $\tau_1 = 0.5$ and $\tau_2 = 0$ so that the idiosyncratic error of x_{1it} is correlated with ε_{it}. The overidentifying restrictions test statistic is defined by (21), and the 5% critical value from χ_1^2 distribution is used for the test. All the experiments are based on 2,000 replications.

Appendix A: Proofs of Main Results

We rely on the law of large numbers and central limit theorem results, which are stated in Lemmas 1 and 2; see Hansen (2007) for more details. The proofs of all the Lemmas are provided in the Appendix B in the Supplemental Material.

Lemma 1 *Suppose $\{X_{i,T}\}$ are independent across $i = 1, 2, ..., N$ for all T with $E(X_{i,T}) = \mu_{i,T}$ and $E|X_{i,T}|^{1+\delta} < \Delta < \infty$ for some $\delta > 0$ and all i, T. Then $N^{-1}\sum_{i=1}^{N}(X_{i,T} - \mu_{i,T}) \xrightarrow{p} 0$ as $(N,T) \xrightarrow{j} \infty$.*

Lemma 2 *Suppose $\{\mathbf{x}_{i,T}\}$, $h \times 1$ random vectors, are independent across $i = 1, 2, ..., N$ for all T with $E(\mathbf{x}_{i,T}) = \mathbf{0}$, $E(\mathbf{x}_{i,T}\mathbf{x}'_{i,T}) = \mathbf{\Sigma}_{i,T}$ and $E\|\mathbf{x}_{i,T}\|^{2+\delta} < \Delta < \infty$ for some $\delta > 0$ and all i, T. Assume $\mathbf{\Sigma} = \lim_{N,T\to\infty} N^{-1}\sum_{i=1}^{N}\mathbf{\Sigma}_{i,T}$ is positive definite and the smallest eigenvalue of $\mathbf{\Sigma}$ is strictly positive. Then, $N^{-1/2}\sum_{i=1}^{N}\mathbf{x}_{i,T} \xrightarrow{d} N(\mathbf{0}, \mathbf{\Sigma})$ as $(N,T) \xrightarrow{j} \infty$.*

Consistency of factor estimators and other related results are in line with the discussion in Bai (2009). The proof of some elementary results are very similar to Bai (2009) and are therefore omitted; readers are invited to refer to these papers for this purpose. In what follows, we define $\delta_{NT}^{-1} = 1/\min\left\{\sqrt{N}, \sqrt{T}\right\}$, $\delta_{NT}^{-2} = 1/\min\{N, T\}$.

Since our aim is to marginalize out the unobservable common components, we assume the principal component estimator $\hat{\mathbf{F}}_x$ ($\hat{\mathbf{F}}_{x,-1}$) is consistent for $\mathbf{F}_x$ ($\mathbf{F}_{x,-1}$) without loss of generality. This is valid because the factors and factor loadings in the model can always be redefined as $\mathbf{F}_x\mathbf{G}$ ($\mathbf{F}^*_{x,-1}\mathbf{G}^*$) and $\mathbf{G}^{-1}\mathbf{\Gamma}_{xi}$ ($\mathbf{G}^{*-1}\mathbf{\Gamma}^*_{xi}$), respectively, for some invertible matrix $\mathbf{G}$ ($\mathbf{G}^*$). However, we clarify this difference when necessary.

Next, since our instruments are $\mathbf{M}_{\hat{F}_x}\mathbf{X}_i$ and $\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}$, we consider first

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{X}}'_{i,-1}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i,$$

followed by

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}'_i\mathbf{M}_{\hat{F}_x}\mathbf{u}_i,$$

where $\hat{\mathbf{X}}_{i,-1} = \mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}$.

Let $\mathbf{\Xi}_{kNT}$ and $\mathbf{\Xi}_{kNT,-1}$ be $m_x \times m_x$ diagonal matrices that consist of the first m_x largest eigenvalues of $\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_i\mathbf{X}'_i$ and $\frac{1}{NT}\sum_{i=1}^{N}\mathbf{X}_{i,-1}\mathbf{X}'_{i,-1}$, respectively. As it is well known, the factor estimator is up to the rotation, which is sufficient for our purposes. Denote the $T \times m_x$ matrix of true factors $\mathbf{F}^0_x$ and $m_x \times k$ matrix of true factor loadings as $\mathbf{\Gamma}^0_{xi}$. In a similar way, denote by $\mathbf{F}^0_y$, $\boldsymbol{\gamma}^0_i$ and $\boldsymbol{\lambda}^0_i$ the true factors and factor loadings in y equation, which are $T \times m_y$, $m_x \times 1$ and $m_y \times 1$ matrices, correspondingly.

For any invertible $m_x \times m_x$ matrices $\mathbf{G}$ and $\mathbf{G}^*$, now define

$$\mathbf{F}_x = \mathbf{F}^0_x\mathbf{G},\ \mathbf{\Gamma}_{xi} = \mathbf{G}^{-1}\mathbf{\Gamma}^0_{xi} \text{ and } \mathbf{F}^*_{x,-1} = \mathbf{F}^0_{x,-1}\mathbf{G}^*,\ \mathbf{\Gamma}^*_{xi} = \mathbf{G}^{*-1}\mathbf{\Gamma}^0_{xi}.$$

Then, $\mathbf{M}_{F_x} = \mathbf{I}_T - \mathbf{F}^0_x\mathbf{G}\left(\mathbf{G}'\mathbf{F}^{0\prime}_x\mathbf{F}^0_x\mathbf{G}\right)^{-1}\mathbf{G}'\mathbf{F}^{0\prime}_x = \mathbf{I}_T - \mathbf{F}^0_x\mathbf{G}\left(\mathbf{G}\right)^{-1}\left(\mathbf{F}^{0\prime}_x\mathbf{F}^0_x\right)^{-1}\left(\mathbf{G}'\right)^{-1}\mathbf{G}'\mathbf{F}^{0\prime}_x = \mathbf{M}_{F^0_x}$, so that $\mathbf{M}_{F_x}\mathbf{F}^0_x = \mathbf{M}_{F^0_x}\mathbf{F}^0_x = \mathbf{0}$. In the same way, we have $\mathbf{M}_{F^*_{x,-1}}\mathbf{F}^0_{x,-1} = \mathbf{M}_{F^0_{x,-1}}\mathbf{F}^0_{x,-1} = \mathbf{0}$. This implies that the consistent estimators $\hat{\mathbf{F}}_x$ of $\mathbf{F}_x$ and $\hat{\mathbf{F}}_{x,-1}$ of $\mathbf{F}^*_{x,-1}$ serve the purpose of marginalizing out the effect of the factor components $\mathbf{F}^0_x$ and $\mathbf{F}^0_{x,-1}$, respectively. The following restrictions are imposed

$$T^{-1}\mathbf{F}'_x\mathbf{F}_x = \mathbf{I}_{m_x},\ \sum_{\ell=1}^{k}\sum_{i=1}^{N}\boldsymbol{\gamma}_{\ell i}\boldsymbol{\gamma}'_{\ell i} \text{ is diagonal},$$

and

$$T^{-1}\mathbf{F}^{*\prime}_{x,-1}\mathbf{F}^{*}_{x,-1} = \mathbf{I}_{m_x}, \sum_{\ell=1}^{k}\sum_{i=1}^{N}\boldsymbol{\gamma}^{*}_{\ell i}\boldsymbol{\gamma}^{*\prime}_{\ell i} \text{ is diagonal,}$$

so we have

$$T^{-1}\hat{\mathbf{F}}'_x\hat{\mathbf{F}}_x = \mathbf{I}_{m_x}, \text{ and } T^{-1}\hat{\mathbf{F}}'_{x,-1}\hat{\mathbf{F}}_{x,-1} = \mathbf{I}_{m_x}.$$

Following the discussion in Bai (2009), p.1266, we write

$$\begin{aligned}\hat{\mathbf{F}}_x\boldsymbol{\Xi}_{kNT} &= \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{F}^0_x\boldsymbol{\gamma}^0_{\ell i}\mathbf{v}'_{\ell i}\hat{\mathbf{F}}_x + \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i}\boldsymbol{\gamma}^{0\prime}_{\ell i}\mathbf{F}^{0\prime}_x\hat{\mathbf{F}}_x \\ &+ \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i}\mathbf{v}'_{\ell i}\hat{\mathbf{F}}_x + \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{F}^0_x\boldsymbol{\gamma}^0_{\ell i}\boldsymbol{\gamma}^{0\prime}_{\ell i}\mathbf{F}^{0\prime}_x\hat{\mathbf{F}}_x \\ &= \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4. \end{aligned} \tag{A.1}$$

Define

$$\boldsymbol{\Upsilon}^0_{kN} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\boldsymbol{\gamma}^0_{\ell i}\boldsymbol{\gamma}^{0\prime}_{\ell i}, \ \boldsymbol{\Lambda}_{0\hat{F}_x} = T^{-1}\mathbf{F}^{0\prime}_x\hat{\mathbf{F}}_x. \tag{A.2}$$

Observing that

$$\mathbf{E}_4 = \mathbf{F}^0_x\boldsymbol{\Upsilon}^0_{kN}\boldsymbol{\Lambda}_{0\hat{F}_x}, \tag{A.3}$$

we have

$$\hat{\mathbf{F}}_x\boldsymbol{\Xi}_{kNT} - \mathbf{F}^0_x\boldsymbol{\Upsilon}^0_{kN}\boldsymbol{\Lambda}_{0\hat{F}_x} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3. \tag{A.4}$$

Post-multiplying the above equation by $\hat{\mathbf{Q}} = \left(\boldsymbol{\Upsilon}^0_{kN}\boldsymbol{\Lambda}_{0\hat{F}_x}\right)^{-1}$ yields

$$\hat{\mathbf{F}}_x\boldsymbol{\Xi}_{kNT}\hat{\mathbf{Q}} - \mathbf{F}^0_x = \left(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3\right)\hat{\mathbf{Q}}. \tag{A.5}$$

Let

$$\mathbf{G} = \left(\boldsymbol{\Xi}_{kNT}\hat{\mathbf{Q}}\right)^{-1}, \tag{A.6}$$

where $\boldsymbol{\Xi}_{kNT}$ is assumed to be invertible (the invertibility of $\boldsymbol{\Xi}_{kNT}$ is proved in Bai 2009, p.1267) so that

$$\begin{aligned}\hat{\mathbf{F}}_x\mathbf{G}^{-1} - \mathbf{F}^0_x &= \left(\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3\right)\hat{\mathbf{Q}} \\ &= \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{F}^0_x\boldsymbol{\gamma}^0_{\ell i}\mathbf{v}'_{\ell i}\hat{\mathbf{F}}_x\hat{\mathbf{Q}} + \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i}\boldsymbol{\gamma}^{0\prime}_{\ell i}\mathbf{F}^{0\prime}_x\hat{\mathbf{F}}_x\hat{\mathbf{Q}} \\ &+ \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i}\mathbf{v}'_{\ell i}\hat{\mathbf{F}}_x\hat{\mathbf{Q}}. \end{aligned} \tag{A.7}$$

Similarly, we have

$$\begin{aligned}\hat{\mathbf{F}}_{x,-1}\mathbf{G}^{*-1} - \mathbf{F}^0_{x,-1} &= \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{F}^0_{x,-1}\boldsymbol{\gamma}^0_{\ell i}\mathbf{v}'_{\ell i,-1}\hat{\mathbf{F}}_{x,-1}\hat{\mathbf{Q}}_{-1} + \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i,-1}\boldsymbol{\gamma}^{0\prime}_{\ell i}\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}\hat{\mathbf{Q}}_{-1} \\ &+ \frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{v}_{\ell i,-1}\mathbf{v}'_{\ell i,-1}\hat{\mathbf{F}}_{x,-1}\hat{\mathbf{Q}}_{-1}, \end{aligned} \tag{A.8}$$

where $\mathbf{G}^* = \left(\boldsymbol{\Xi}_{kNT,-1}\hat{\mathbf{Q}}_{-1}\right)^{-1}$, with $\boldsymbol{\Xi}_{kNT,-1}$ being assumed to be invertible, $\hat{\mathbf{Q}}_{-1} = \left(\boldsymbol{\Upsilon}^0_{kN}\boldsymbol{\Lambda}_{0\hat{F}_{x,-1}}\right)^{-1}$ and $\boldsymbol{\Lambda}_{0\hat{F}_{x,-1}} = T^{-1}\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}$.

Lemma 3 *Under Assumptions 2-3, the following statements hold for $\ell = 1, 2, ..., k$ and $s = 1, 2, ..., T$:*

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{f}_{y,t}^{0}\left[v_{\ell i,t}v_{\ell i,s}-E\left(v_{\ell i,t}v_{\ell i,s}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.9}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{f}_{x,t}^{0}\left[v_{\ell i,t}v_{\ell i,s}-E\left(v_{\ell i,t}v_{\ell i,s}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.10}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{v}_{it}\left[v_{\ell i,t}v_{\ell i,s}-E\left(v_{\ell i,t}v_{\ell i,s}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.11}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{f}_{x,t-1}^{0}\left[v_{\ell i,t}v_{\ell i,s}-E\left(v_{\ell i,t}v_{\ell i,s}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.12}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{v}_{it-1}\left[v_{\ell i,t}v_{\ell i,s}-E\left(v_{\ell i,t}v_{\ell i,s}\right)\right]\right\|^{2}\leq\Delta<\infty. \tag{A.13}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{f}_{y,t}^{0}\left[v_{\ell i,t-1}v_{\ell i,s-1}-E\left(v_{\ell i,t-1}v_{\ell i,s-1}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.14}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{f}_{x,t}^{0}\left[v_{\ell i,t-1}v_{\ell i,s-1}-E\left(v_{\ell i,t-1}v_{\ell i,s-1}\right)\right]\right\|^{2}\leq\Delta<\infty, \tag{A.15}$$

$$E\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbf{v}_{it}\left[v_{\ell i,t-1}v_{\ell i,s-1}-E\left(v_{\ell i,t-1}v_{\ell i,s-1}\right)\right]\right\|^{2}\leq\Delta<\infty. \tag{A.16}$$

Lemma 4 *Under Assumptions 1-3,4(i), as $(N,T)\xrightarrow{j}\infty$ such that $N/T\to c$ with $0<c<\infty$,*

$$T^{-r/2}\left\|\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right\|^{r}=T^{-r/2}\sum_{t=1}^{T}\left\|\hat{\mathbf{f}}_{x,t}-\mathbf{G}'\mathbf{f}_{x,t}^{0}\right\|^{r}=O_{p}\left(\delta_{NT}^{-r}\right),\ r=1,2, \tag{A.17}$$

$$T^{-r/2}\left\|\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right\|^{r}=T^{-r/2}\sum_{t=1}^{T}\left\|\hat{\mathbf{f}}_{x,t-1}-\mathbf{G}^{*\prime}\mathbf{f}_{x,t-1}^{0}\right\|^{r}=O_{p}\left(\delta_{NT}^{-r}\right),\ r=1,2, \tag{A.18}$$

$$\frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\hat{\mathbf{F}}_{x}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\hat{\mathbf{F}}_{x}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\hat{\mathbf{F}}_{x,-1}}{T}=O_{p}\left(\delta_{NT}^{-2}\right), \tag{A.19}$$

$$\frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\mathbf{F}_{x}^{0}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\mathbf{F}_{x}^{0}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\mathbf{F}_{x,-1}^{0}}{T}=O_{p}\left(\delta_{NT}^{-2}\right), \tag{A.20}$$

$$\frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\mathbf{F}_{y}^{0}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\mathbf{F}_{y}^{0}}{T}=O_{p}\left(\delta_{NT}^{-2}\right), \tag{A.21}$$

$$\frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\boldsymbol{\varepsilon}_{i}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\boldsymbol{\varepsilon}_{i}}{T}=O_{p}\left(\delta_{NT}^{-2}\right)\ \text{for } i=1,2,..,N, \tag{A.22}$$

$$\frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\mathbf{v}_{\ell i}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\mathbf{v}_{\ell i}}{T}=O_{p}\left(\delta_{NT}^{-2}\right);\ \frac{\left(\hat{\mathbf{F}}_{x}-\mathbf{F}_{x}^{0}\mathbf{G}\right)'\mathbf{v}_{\ell i,-1}}{T}=O_{p}\left(\delta_{NT}^{-2}\right), \tag{A.23}$$

for $i = 1, 2, .., N$ *and* $\ell = 1, 2, .., k$,

$$\frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{W}_i}{T} = O_p\left(\delta_{NT}^{-2}\right); \quad \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^0 \mathbf{G}^*\right)' \mathbf{W}_i}{T} = O_p\left(\delta_{NT}^{-2}\right) \text{ for } i = 1, 2, .., N, \tag{A.24}$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{v}_{\ell i}}{T} \gamma_{\ell i}^{0\prime} = O\left(N^{-1/2}\right) + O_p\left(\delta_{NT}^{-2}\right) \text{ for } \ell = 1, 2, .., k, \tag{A.25}$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^0 \mathbf{G}^*\right)' \mathbf{v}_{\ell i}}{T} \gamma_{\ell i}^{0\prime} = O\left(N^{-1/2}\right) + O_p\left(\delta_{NT}^{-2}\right) \text{ for } \ell = 1, 2, .., k, \tag{A.26}$$

$$\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{v}_{\ell i,-1}}{T} \gamma_{\ell i}^{0\prime} = O\left(N^{-1/2}\right) + O_p\left(\delta_{NT}^{-2}\right) \text{ for } \ell = 1, 2, .., k, \tag{A.27}$$

$$\mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1} = O_p\left(\delta_{NT}^{-2}\right); \ \mathbf{G}^*\mathbf{G}^{*\prime} - \left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1} = O_p\left(\delta_{NT}^{-2}\right), \tag{A.28}$$

$$\frac{\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x}{T} \xrightarrow{p} \mathbf{\Lambda} \text{ and } \frac{\mathbf{F}_{x,-1}^{0\prime}\hat{\mathbf{F}}_{x,-1}}{T} \xrightarrow{p} \mathbf{\Lambda}_{-1} \text{ as } (N,T) \xrightarrow{j} \infty, \text{ where } \mathbf{\Lambda} \text{ and } \mathbf{\Lambda}_{-1} \text{ are invertible} \tag{A.29}$$

$m_x \times m_x$ *matrices.*

Lemma 5 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\left\|\mathbf{P}_{\hat{F}_x} - \mathbf{P}_{F_x^0}\right\| = O_p(\delta_{NT}^{-1})$ *and* $\left\|\mathbf{P}_{\hat{F}_{x,-1}} - \mathbf{P}_{F_{x,-1}^0}\right\| = O_p(\delta_{NT}^{-1})$.

Lemma 6 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\frac{1}{NT} \sum_{i=1}^{N} \hat{\mathbf{Z}}_i' \mathbf{M}_{\hat{F}_x} \hat{\mathbf{Z}}_i - \mathbf{Z}_i' \mathbf{M}_{F_x^0} \mathbf{Z}_i = o_p(1), \tag{A.30}$$

$$\frac{1}{NT} \sum_{i=1}^{N} \left(\hat{\mathbf{Z}}_i' \mathbf{M}_{\hat{F}_x} - \mathbf{Z}_i' \mathbf{M}_{F_x^0}\right) \mathbf{W}_i = o_p(1). \tag{A.31}$$

Lemma 7 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\begin{aligned} &\frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^0\right)^{-1} \left(\frac{\hat{\mathbf{F}}_x' \mathbf{F}_x^0}{T}\right)^{-1} \hat{\mathbf{F}}_x' \left(\mathbf{\Sigma}_{kNT} - \bar{\mathbf{\Sigma}}_{kNT}\right) \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\ &= O_p\left(T^{-1/2}\right) + O_p\left(\delta_{NT}^{-1}\right) + \sqrt{T} O_p\left(\delta_{NT}^{-2}\right), \end{aligned} \tag{A.32}$$

and

$$\begin{aligned} &\frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^0\right)^{-1} \left(\frac{\hat{\mathbf{F}}_{x,-1}' \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \hat{\mathbf{F}}_{x,-1}' \left(\mathbf{\Sigma}_{kNT,-1} - \bar{\mathbf{\Sigma}}_{kNT,-1}\right) \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\ &= O_p\left(T^{-1/2}\right) + O_p\left(\delta_{NT}^{-1}\right) + \sqrt{T} O_p\left(\delta_{NT}^{-2}\right), \end{aligned} \tag{A.33}$$

where $\mathbf{\Sigma}_{kNT} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N} \mathbf{v}_{\ell j}\mathbf{v}_{\ell j}'$, $\mathbf{\Sigma}_{kNT,-1} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N} \mathbf{v}_{\ell j,-1}\mathbf{v}_{\ell j,-1}'$ *and* $\bar{\mathbf{\Sigma}}_{kNT} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N} E\left(\mathbf{v}_{\ell j}\mathbf{v}_{\ell j}'\right)$, $\bar{\mathbf{\Sigma}}_{kNT,-1} = \frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N} E\left(\mathbf{v}_{\ell j,-1}\mathbf{v}_{\ell j,-1}'\right)$.

Lemma 8 *Under Assumptions 1-3,4(i), as $(N,T) \xrightarrow{j} \infty$ such that $N/T \to c$ with $0 < c < \infty$,*

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \mathbf{F}_x^{0\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&= -\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_j^{\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&- \frac{1}{\sqrt{N} T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\hat{\mathbf{F}}_x^{\prime} \mathbf{F}_x^{0}}{T}\right)^{-1} \hat{\mathbf{F}}_x^{\prime} \mathbf{\Sigma}_{kNT} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&+ o_p(1), \qquad \text{(A.34)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&= -\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_{j,-1}^{\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&- \frac{1}{\sqrt{N} T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{\Sigma}_{kNT,-1} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&+ o_p(1). \qquad \text{(A.35)}
\end{aligned}$$

Lemma 9 *Under Assumptions 1-3,4(i), as $(N,T) \xrightarrow{j} \infty$ such that $N/T \to c$ with $0 < c < \infty$,*

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_j^{\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_j^{\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_n^{\prime} \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^{0}}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} \\
&+ o_p(1), \qquad \text{(A.36)}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_{j,-1}^{\prime} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_{j,-1}^{\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}^{\prime} \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}^{\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^{0}}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} \\
&+ o_p(1). \qquad \text{(A.37)}
\end{aligned}$$

Lemma 10 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\hat{\mathbf{F}}_x^{\prime} \mathbf{F}_x^0}{T}\right)^{-1} \hat{\mathbf{F}}_x^{\prime} \bar{\mathbf{\Sigma}}_{kNT} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i$$
$$= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \mathbf{F}_x^{0\prime} \bar{\mathbf{\Sigma}}_{kNT} \mathbf{M}_{F_x^0} \mathbf{u}_i + o_p(1), \tag{A.38}$$

and

$$\frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \hat{\mathbf{F}}_{x,-1}^{\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i$$
$$= \frac{1}{\sqrt{N}T^{3/2}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_i$$
$$+ o_p(1). \tag{A.39}$$

Lemma 11 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i^{\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i$$
$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_i^{\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i$$
$$+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{V}_i^{\prime} \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} + o_p(1), \tag{A.40}$$

and

$$\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i$$
$$= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_i$$
$$+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}^{\prime} \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T}$$
$$+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} + o_p(1). \tag{A.41}$$

Lemma 12 *Under Assumptions 1-3,4(i)(ii), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_j^{\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i = o_p(1), \tag{A.42}$$

$$\frac{1}{NT} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \mathbf{F}_x^{0\prime} \bar{\mathbf{\Sigma}}_{kNT} \mathbf{M}_{F_x^0} \mathbf{u}_i = o_p(1), \tag{A.43}$$

$$\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_{j,-1}^{\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_i = o_p(1), \tag{A.44}$$

$$\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T} = o_p(1), \tag{A.45}$$

$$\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}=o_{p}(1). \tag{A.46}$$

Proof of Proposition 1. Consider

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_{i}^{\prime}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}, \tag{A.47}$$

where $\hat{\mathbf{Z}}_i=\left[\mathbf{X}_i,\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}\right]$. We start with the second component of $\hat{\mathbf{Z}}_i$, which is $\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}$. By making use of the result in equation (A.57) in Proposition 2 obtained under Assumptions 1-3 and 4(i) as $(N,T)\xrightarrow{j}\infty$ such that $N/T\to c$ with $0<c<\infty$, and imposing further Assumption 4(ii) yields:

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}\\
&-\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}\\
&-\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{u}_{i}+o_{p}\left(1\right)\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}\\
&+o_{p}\left(1\right),
\end{aligned} \tag{A.48}$$

where the second equality is by Lemma 12. By dropping the superscript "0" without loss of generality and making use of $\mathbf{M}_{F_{x,-1}}\mathbf{F}_{x,-1}=\mathbf{0}$ and , we get

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left(\mathbf{V}_{i,-1}+\mathbf{F}_{x,-1}\mathbf{\Gamma}_{xi}\right)^{\prime}\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_{x}}\mathbf{u}_{i},\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{\prime}\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_{x}^{\prime}\mathbf{F}_{x}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{\prime}\mathbf{u}_{i}}{T}+o_{p}\left(1\right)\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_{x}}\mathbf{u}_{i}+\sqrt{\frac{T}{N}}\mathbf{b}_{12NT}+o_{p}\left(1\right),
\end{aligned} \tag{A.49}$$

where

$$\tilde{\mathbf{V}}_{i,-1} = \mathbf{V}_{i,-1} - \frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_{n,-1}\mathbf{\Gamma}'_{xn}\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi}$$

$$\mathbf{b}_{12NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_{i,-1}\mathbf{M}_{F_{x,-1}}\mathbf{V}_j}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}'_x\mathbf{u}_i}{T}.$$

Next consider the first component of $\hat{\mathbf{Z}}_i$, which is $\mathbf{X}_i$. By following the same steps as before and using the result in equation (A.59) in Proposition 2 which is obtained under Assumptions 1, 2, 3 and 4(i) as $(N,T) \xrightarrow{j} \infty$ such that $N/T \to c$ with $0 < c < \infty$ and imposing again Assumption 4(ii), we get:

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}'_i\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}'_i\mathbf{M}_{F_x^0}\mathbf{u}_i \\
&- \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\mathbf{\Gamma}_{xj}^0\mathbf{V}'_j\mathbf{M}_{F_x^0}\mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_i\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&- \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_x^0}\mathbf{u}_i + o_p(1) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}'_i\mathbf{M}_{F_x^0}\mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_i\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} + o_p(1).
\end{aligned} \tag{A.50}$$

where the second equality is obtained by using Lemma 12. Now, by getting rid of the superscript "0" and making use of $\mathbf{M}_{F_x}\mathbf{F}_x = \mathbf{0}$, we obtain

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}'_i\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left(\mathbf{V}_i + \mathbf{F}_x\mathbf{\Gamma}_{xi}\right)'\mathbf{M}_{F_x}\mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_i\mathbf{V}_j}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}'_x\mathbf{u}_i}{T} + o_p(1) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}'_i\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}\mathbf{b}_{11NT} + o_p(1),
\end{aligned} \tag{A.51}$$

where

$$\tilde{\mathbf{V}}_i = \mathbf{V}_i - \frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_n\mathbf{\Gamma}'_{xn}\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi}$$

$$\mathbf{b}_{11NT} = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}'_i\mathbf{V}_j}{T}\mathbf{\Gamma}'_{xj}\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}'_x\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}'_x\mathbf{u}_i}{T}.$$

By putting all the results together, we therefore have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\mathbf{X}_i, \mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}\right]'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\mathbf{X}_i, \mathbf{M}_{F_{x,-1}}\mathbf{X}_{i,-1}\right]'\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}[\mathbf{b}_{11NT}', \mathbf{b}_{12NT}']' + o_p(1) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}\mathbf{b}_{1NT} + o_p(1),
\end{aligned}
$$

where $\mathbf{Z}_i = \left[\mathbf{X}_i, \mathbf{M}_{F_{x,-1}}\mathbf{X}_{i,-1}\right]$ and $\mathbf{b}_{1NT} = [\mathbf{b}_{11NT}', \mathbf{b}_{12NT}']'$, which provides the required expression stated in Proposition 1. ■

Proof of Proposition 2. Consider

$$
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i, \tag{A.52}
$$

where $\hat{\mathbf{Z}}_i = \left[\mathbf{X}_i, \mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}\right]$. We begin with the second component of $\hat{\mathbf{Z}}_i$, which is $\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}$. Firstly, note that

$$
\begin{aligned}
\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&\quad + \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i.
\end{aligned} \tag{A.53}
$$

By using the results of Lemmas 7 and 8, as $(N,T)\xrightarrow{j}\infty$ such that $N/T \to c$ with $0 < c < \infty$, the first term in (A.53) is given by

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= -\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}'\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&+ o_p(1).
\end{aligned} \tag{A.54}
$$

Then, by Lemmas 9 and 10, we have

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}$$

$$= -\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}$$

$$-\sqrt{\frac{T}{N}}\frac{1}{N^{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xn}^{0}\frac{\mathbf{V}_{n,-1}^{\prime}\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}$$

$$-\sqrt{\frac{T}{N}}\frac{1}{N^{2}}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xn}^{0}\frac{\mathbf{V}_{n,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}$$

$$-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}+o_{p}\left(1\right). \tag{A.55}$$

By making use of Lemma 11, the second term in (A.53) is given by

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}$$

$$= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}^{\prime}\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}+o_{p}\left(1\right). \tag{A.56}$$

So, by adding (A.55) and (A.56) together and rearranging the terms, we get

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}$$

$$= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}$$

$$-\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}$$

$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}^{\prime}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}$$

$$-\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}$$

$$+o_{p}\left(1\right) \tag{A.57}$$

By further using $\mathbf{M}_{F_{x,-1}}\mathbf{F}_{x,-1}=\mathbf{0}$ and dropping the superscript "0" without loss of generality, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left(\mathbf{V}_{i,-1}+\mathbf{F}_{x,-1}\mathbf{\Gamma}_{xi}\right)'\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_x}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xj}\left(\mathbf{V}_{j,-1}+\mathbf{F}_{x,-1}\mathbf{\Gamma}_{xi}\right)'\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_x}\mathbf{u}_i\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_{x,-1}'\mathbf{F}_{x,-1}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}'\mathbf{M}_{F_x}\mathbf{u}_i}{T}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T}\\
&-\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_{x,-1}'\mathbf{F}_{x,-1}}{T}\right)^{-1}\mathbf{F}_{x,-1}'\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_x}\mathbf{u}_i+o_p(1)\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\tilde{\mathbf{X}}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{M}_{F_x}\mathbf{u}_i+\sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{12NT}+\sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{22NT}+o_p(1),
\end{aligned}
\tag{A.58}
$$

where

$$
\begin{aligned}
\tilde{\mathbf{X}}_{i,-1}&=\mathbf{X}_{i,-1}-\frac{1}{N}\sum_{n=1}^{N}\mathbf{X}_{n,-1}\mathbf{\Gamma}_{xn}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi},\\
\tilde{\mathbf{V}}_{i,-1}&=\mathbf{V}_{i,-1}-\frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_{n,-1}\mathbf{\Gamma}_{xn}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbf{b}}_{12NT}&=\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_{x,-1}'\mathbf{F}_{x,-1}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}'\mathbf{M}_{F_x}\mathbf{u}_i}{T}\\
&\quad+\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{V}_{j,}}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T},\\
\tilde{\mathbf{b}}_{22NT}&=-\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_{x,-1}'\mathbf{F}_{x,-1}}{T}\right)^{-1}\mathbf{F}_{x,-1}'\bar{\mathbf{\Sigma}}_{kNT,-1}\mathbf{M}_{F_{x,-1}}\mathbf{u}_i.
\end{aligned}
$$

As for the first component of $\hat{\mathbf{Z}}_i$, which is $\mathbf{X}_i$, by following the same steps as before and using again Lemmas 7, 8, 9, 10 and 11, we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_i'\mathbf{M}_{F_x^0}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_j'\mathbf{M}_{F_x^0}\mathbf{u}_i\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_i'\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\\
&-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_x^0}\mathbf{u}_i+o_p(1).
\end{aligned}
\tag{A.59}
$$

Next, by using $\mathbf{M}_{F_x}\mathbf{F}_x = \mathbf{0}$ and suspending the superscript "0", we get

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{X}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left(\mathbf{V}_i + \mathbf{F}_x\mathbf{\Gamma}_{xi}\right)'\mathbf{M}_{F_x}\mathbf{u}_i \\
&- \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xj}\left(\mathbf{V}_j + \mathbf{F}_x\mathbf{\Gamma}_{xj}\right)'\mathbf{M}_{F_x}\mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_i'\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T} \\
&- \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\mathbf{F}_x'\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_x}\mathbf{u}_i + o_p(1) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\tilde{\mathbf{X}}_i'\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{11NT} + \sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{21NT} + o_p(1),
\end{aligned} \tag{A.60}$$

where

$$\begin{aligned}
\tilde{\mathbf{X}}_i &= \mathbf{X}_i - \frac{1}{N}\sum_{n=1}^{N}\mathbf{X}_n\mathbf{\Gamma}_{xn}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi}, \\
\tilde{\mathbf{V}}_i &= \mathbf{V}_i - \frac{1}{N}\sum_{n=1}^{N}\mathbf{V}_n\mathbf{\Gamma}_{xn}'\mathbf{\Upsilon}_{kN}^{-1}\mathbf{\Gamma}_{xi}, \\
\tilde{\mathbf{b}}_{11NT} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\tilde{\mathbf{V}}_i'\mathbf{V}_j}{T}\mathbf{\Gamma}_{xj}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\frac{\mathbf{F}_x'\mathbf{u}_i}{T}, \\
\tilde{\mathbf{b}}_{21NT} &= -\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}'\mathbf{\Upsilon}_{kN}^{-1}\left(\frac{\mathbf{F}_x'\mathbf{F}_x}{T}\right)^{-1}\mathbf{F}_x'\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_x}\mathbf{u}_i.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\mathbf{X}_i, \mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}\right]'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\left[\tilde{\mathbf{X}}_i, \mathbf{M}_{F_{x,-1}}\tilde{\mathbf{X}}_{i,-1}\right]'\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}[\tilde{\mathbf{b}}_{11NT}', \tilde{\mathbf{b}}_{12NT}']' \\
&+ \sqrt{\frac{N}{T}}[\tilde{\mathbf{b}}_{21NT}', \tilde{\mathbf{b}}_{22NT}']' + o_p(1) \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\tilde{\mathbf{Z}}_i'\mathbf{M}_{F_x}\mathbf{u}_i + \sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{1NT} + \sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{2NT} + o_p(1),
\end{aligned}$$

where $\tilde{\mathbf{Z}}_i = \left[\tilde{\mathbf{X}}_i, \mathbf{M}_{F_{x,-1}}\tilde{\mathbf{X}}_{i,-1}\right]$, $\tilde{\mathbf{b}}_{1NT} = [\tilde{\mathbf{b}}_{11NT}', \tilde{\mathbf{b}}_{12NT}']'$ and $\tilde{\mathbf{b}}_{2NT} = [\tilde{\mathbf{b}}_{21NT}', \tilde{\mathbf{b}}_{22NT}']'$, which provides the expression given in Proposition 2. ■

Lemma 13 *Under Assumptions 1-3,4(i)(ii), as $(N,T) \xrightarrow{j} \infty$ such that $N/T \to c$ with $0 < c < \infty$, $\sqrt{\frac{T}{N}}\hat{\mathbf{b}}_{1NT} - \sqrt{\frac{T}{N}}\mathbf{b}_{1NT} = o_p(1)$.*

Lemma 14 *Under Assumptions 1-3,4(i), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\sqrt{\frac{T}{N}}\hat{\tilde{\mathbf{b}}}_{1NT} - \sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{1NT} = o_p(1)$ *and* $\sqrt{\frac{T}{N}}\hat{\tilde{\mathbf{b}}}_{2NT} - \sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{2NT} = o_p(1)$.

For notational conciseness define

$$\boldsymbol{\xi}_{\hat{F}iT} = \hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i - \frac{\mathbf{b}}{\sqrt{NT}}, \tag{A.61}$$

$$\hat{\boldsymbol{\xi}}_{\hat{F}iT} = \hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\hat{\mathbf{u}}_i - \frac{\mathbf{b}}{\sqrt{NT}}, \tag{A.62}$$

$$\tilde{\boldsymbol{\xi}}_{\hat{F}iT} = \hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{u}_i - \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}, \tag{A.63}$$

$$\hat{\tilde{\boldsymbol{\xi}}}_{\hat{F}iT} = \hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\hat{\mathbf{u}}_i - \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}, \tag{A.64}$$

which are centred, where

$$\mathbf{b} = \mathrm{plim}_{N,T\to\infty}\sqrt{\frac{T}{N}}\mathbf{b}_{1NT}, \tag{A.65}$$

$$\tilde{\mathbf{b}} = \mathrm{plim}_{N,T\to\infty}\left(\sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{1NT} + \sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{2NT}\right), \tag{A.66}$$

which are assumed to exist.

Lemma 15 *Under Assumptions 1-3, 4(i)(ii) and 5(i)(ii)(iii), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\frac{1}{NT}\sum_{i=1}^{N}\hat{\boldsymbol{\xi}}_{\hat{F}iT}\hat{\boldsymbol{\xi}}_{\hat{F}iT}' = \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}_{\hat{F}iT}' + o_p(1)$, *where* $\boldsymbol{\xi}_{\hat{F}iT}$ *and* $\hat{\boldsymbol{\xi}}_{\hat{F}iT}$, *are defined by (A.61) and (A.62), respectively.*

Lemma 16 *Under Assumptions 1-3, 4(i) and 5(i)(ii)(iv), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\frac{1}{NT}\sum_{i=1}^{N}\hat{\tilde{\boldsymbol{\xi}}}_{\hat{F}iT}\hat{\tilde{\boldsymbol{\xi}}}_{\hat{F}iT}' = \frac{1}{NT}\sum_{i=1}^{N}\tilde{\boldsymbol{\xi}}_{\hat{F}iT}\tilde{\boldsymbol{\xi}}_{\hat{F}iT}' + o_p(1)$, *where* $\tilde{\boldsymbol{\xi}}_{\hat{F}iT}$ *and* $\hat{\tilde{\boldsymbol{\xi}}}_{\hat{F}iT}$, *are defined by (A.63) and (A.64), respectively.*

Lemma 17 *Under Assumptions 1-5(i)(ii)(iii), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}_{\hat{F}iT}' - \boldsymbol{\Omega} = o_p(1)$, *where* $\boldsymbol{\Omega} = \mathrm{plim}_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}E\left(T^{-1}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{u}_i\mathbf{u}_i'\mathbf{M}_{F_x}\mathbf{Z}_i\right)$. *Also* $\frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\xi}_{\hat{F}iT} + \frac{\mathbf{b}}{\sqrt{NT}}\right)\left(\boldsymbol{\xi}_{\hat{F}iT} + \frac{\mathbf{b}}{\sqrt{NT}}\right)' - \boldsymbol{\Omega} = o_p(1)$ *for any* $\mathbf{b}$ *such that* $\|\mathbf{b}\| \leq \Delta < \infty$.

Lemma 18 *Under Assumptions 1-3, 4(i), 5(i)(ii)(iv) and 6, as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\frac{1}{NT}\sum_{i=1}^{N}\left(\tilde{\boldsymbol{\xi}}_{\hat{F}iT} + \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}\right)\left(\tilde{\boldsymbol{\xi}}_{\hat{F}iT} + \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}\right)' - \tilde{\boldsymbol{\Omega}} = o_p(1)$ *for any* $\tilde{\mathbf{b}}$ *such that* $\left\|\tilde{\mathbf{b}}\right\| \leq \Delta < \infty$, *where* $\tilde{\boldsymbol{\Omega}}$ *is defined in Assumption 5(iv).*

Proposition 3 *Under Assumptions 1-3, 4(i)(ii) and 5(i)(ii)(iii), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$,

$$\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT} \xrightarrow{d} N(\mathbf{0},\boldsymbol{\Omega}).$$

Proof. Proposition 1 and Lemma 17, together with Lemma 2, yield the required result. ■

Lemma 19 *Under Assumptions 1-3,4(i)(ii),5(i)(ii)(iv), as* $(N,T) \xrightarrow{j} \infty$ *such that* $N/T \to c$ *with* $0 < c < \infty$, $\hat{\mathbf{A}}_{NT} \xrightarrow{p} \mathbf{A}$, $\mathbf{B}_{NT} \xrightarrow{p} \mathbf{B}$, *where* $\hat{\mathbf{A}}_{NT} = \frac{1}{N}\sum_{i=1}^{N}T^{-1}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\mathbf{W}_i$, $\hat{\mathbf{B}}_{NT} = \frac{1}{N}\sum_{i=1}^{N}T^{-1}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i$ *and* $\mathbf{A} = \lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}E(\mathbf{A}_{i,T})$, $\mathbf{B} = \lim_{N,T\to\infty}\frac{1}{N}\sum_{i=1}^{N}E(\mathbf{B}_{i,T})$, $\mathbf{A}_{i,T} = T^{-1}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{W}_i$, $\mathbf{B}_{i,T} = T^{-1}\mathbf{Z}_i'\mathbf{M}_{F_x}\mathbf{Z}_i$.

Proof of Theorem 1. By using the expression in (15) and the result of Proposition 1, which states that $\sqrt{\frac{T}{N}}\mathbf{b}_{1NT}$ is $O_p(1)$ when $N/T \to c$ with $0 < c < \infty$, we have $\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) = O_p(1)$, which implies the required result. ■

Proof of Theorem 2. (i)$\sqrt{NT}\left(\widehat{\hat{\boldsymbol{\theta}}}_{IV} - \boldsymbol{\theta}\right) = \sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}_{NT}^{-1}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}_{NT}^{-1}\hat{\mathbf{b}}_{NT} = \sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\mathbf{b} + o_p(1)$ by Lemmas 19 and 13, assuming $\operatorname{plim}_{N,T\to\infty}\mathbf{b}_{NT} = \mathbf{b}$ exists. Next, using Lemma 15 we have $\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\mathbf{b} = \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\right) + o_p(1)$. Thus, by the result of Proposition 3, we have $\sqrt{NT}\left(\widehat{\hat{\boldsymbol{\theta}}}_{IV} - \boldsymbol{\theta}\right) \to N\left(\mathbf{0}, \boldsymbol{\Psi}\right)$, as required. (ii) $\hat{\boldsymbol{\Psi}} - \boldsymbol{\Psi} = o_p(1)$ follows immediately from Lemmas 15, 17 and 19. ■

Proof of Theorem 3. Under Assumptions 1-3, 4(i)(ii) and 5(i)(ii)(iii), together with the $\sqrt{NT}$-consistency result of $\hat{\boldsymbol{\theta}}_{IV}$, shown in Theorem 1, and the LLN, we have by Lemma 15 $\hat{\boldsymbol{\Omega}}_{NT} - \boldsymbol{\Omega} \xrightarrow{p} \mathbf{0}$ as $N \to \infty$ and $T \to \infty$ jointly in such way that T/N tends to a finite positive constant. The consistency of $\hat{\boldsymbol{\Omega}}_{NT}$ leads to the $\sqrt{NT}$-consistency of $\widehat{\hat{\boldsymbol{\theta}}}_{IV2}$, and therefore under the null hypothesis a similar discussion for Theorem 2 yields $\frac{1}{\sqrt{NT}}\hat{\boldsymbol{\Omega}}_{NT}^{-1/2}\sum_{i=1}^{N}\mathbf{Z}_i'\mathbf{M}_{\hat{F}_x}\widehat{\hat{\mathbf{u}}}_i \xrightarrow{d} N(\mathbf{0}, \mathbf{I}_{2k})$. Finally, applying a standard proof for the asymptotic distribution of the overidentifying restrictions test under the null hypothesis, such as in Arellano (2003), yields the desired result. ■

Proof of Theorem 4. The proof is obtained making use of the expression in (15) as well as Proposition 2, which states that $\sqrt{\frac{T}{N}}\tilde{\mathbf{b}}_{1NT}$ and $\sqrt{\frac{N}{T}}\tilde{\mathbf{b}}_{2NT}$ are $O_p(1)$ when $N/T \to c$ with $0 < c < \infty$, we have $\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) = O_p(1)$. This provides the required result. ■

Proof of Theorem 5. (i)$\sqrt{NT}\left(\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV} - \boldsymbol{\theta}\right) = \sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}_{NT}^{-1}\hat{\mathbf{A}}_{NT}\right)^{-1}\hat{\mathbf{A}}'_{NT}\hat{\mathbf{B}}_{NT}^{-1}\hat{\tilde{\mathbf{b}}}_{NT} = \sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\tilde{\mathbf{b}} + o_p(1)$ by Lemmas 19 and 14, assuming $\operatorname{plim}_{N,T\to\infty}\tilde{\mathbf{b}}_{NT} = \tilde{\mathbf{b}}$ exists. Next, using Lemma 16 we have $\sqrt{NT}\left(\hat{\boldsymbol{\theta}}_{IV} - \boldsymbol{\theta}\right) - \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\tilde{\mathbf{b}} = \left(\mathbf{A}'\mathbf{B}^{-1}\mathbf{A}\right)^{-1}\mathbf{A}'\mathbf{B}^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\tilde{\boldsymbol{\xi}}_{\hat{F}iT}\right) + o_p(1)$. Then, by the Assumption 6, we have $\sqrt{NT}\left(\hat{\tilde{\hat{\boldsymbol{\theta}}}}_{IV} - \boldsymbol{\theta}\right) \to N\left(\mathbf{0}, \tilde{\boldsymbol{\Psi}}\right)$, as required. (ii) $\hat{\boldsymbol{\Psi}} - \tilde{\boldsymbol{\Psi}} = o_p(1)$ are obtained by using Lemmas 16, 18 and 19. ■

Proof of Theorem 6. The proof is analogous to that of Theorem 3 and it is therefore omitted. ■

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Supplemental Material to

"Instrumental Variable Estimation of Dynamic Linear Panel Data Models with Defactored Regressors and a Multifactor Error Structure"

by Milda Norkutė, Vasilis Sarafidis and Takashi Yamagata

Appendix B: Proofs of Lemmas

Proof of Lemma 1. See Proof of Lemma 1 in Appendix, Hansen (2007). ■

Proof of Lemma 2. See Proof of Lemma 2 in Appendix, Hansen (2007). ■

Proof of Lemma 3. The proof of Lemma 3 can be obtained in a similar manner based on the proof of Lemma A.2 provided in Bai (2009, p.1268). Arellano (2003). ■

Proof of Lemma 4. The proof of (A.17) is given in Bai (2009; Proposition A.1). No modification is required because of our assumption of cross-sectional independence and serial correlation of $v_{it\ell}$, see Assumption 2. A similar point applies to the proofs of (A.19)-(A.28), which are given by Bai (2009) as proofs of corresponding Lemmas A3(ii), A4(i), A4(ii), A3(iv), A4(iii) and A7(i). The result (A.29) is given as part of Proposition 1 in Bai (2003) with its proof therein. ■

Proof of Lemma 5. $\left\|\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\|^2 = tr\left[\left(\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right)^2\right] = tr\left[\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{\hat{F}_x}\mathbf{P}_{F_x^0}-\mathbf{P}_{F_x^0}\mathbf{P}_{\hat{F}_x}+\mathbf{P}_{F_x^0}\right] = tr\left[\mathbf{P}_{\hat{F}_x}\right]-2tr\left[\mathbf{P}_{\hat{F}_x}\mathbf{P}_{F_x^0}\right]+tr\left[\mathbf{P}_{F_x^0}\right] = 2m-2tr\left[T^{-1}\hat{\mathbf{F}}_x'\mathbf{P}_{F_x^0}\hat{\mathbf{F}}_x\right]$, where $T^{-1}\hat{\mathbf{F}}_x'\mathbf{P}_{F_x^0}\hat{\mathbf{F}}_x = T^{-1}\hat{\mathbf{F}}_x'\mathbf{F}_x^0\mathbf{G}\left(\mathbf{G}'\mathbf{F}_x^{0\prime}\mathbf{F}_x^0\mathbf{G}\right)^{-1}\mathbf{G}'\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x = T^{-2}\left(\hat{\mathbf{F}}_x'\mathbf{F}_x^0\mathbf{G}\right)\left(\mathbf{G}'\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x\right)$. By making use of (A.20), we have $T^{-1}\mathbf{G}'\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x = T^{-1}\mathbf{G}'\mathbf{F}_x^{0\prime}\mathbf{F}_x^0\mathbf{G}+T^{-1}\mathbf{G}'\mathbf{F}_x^{0\prime}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right) = \mathbf{I}_{m_x}+O_p\left(\delta_{NT}^{-2}\right)$. Hence, we have $\left\|\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\|^2 = 2m_x-2tr\left[\left(T^{-1}\hat{\mathbf{F}}_x'\mathbf{F}_x^0\mathbf{G}\right)\left(T^{-1}\mathbf{G}'\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x\right)\right] = 2m_x-2tr\left[\mathbf{I}_{m_x}+O_p\left(\delta_{NT}^{-2}\right)\right] = O_p\left(\delta_{NT}^{-2}\right)$. Following similar arguments, it can be shown that $\left\|\mathbf{P}_{\hat{F}_{x,-1}}-\mathbf{P}_{F_{x,-1}^0}\right\|^2 = O_p\left(\delta_{NT}^{-2}\right)$, as required. ■

Proof of Lemma 6. We begin with (A.30). By using Lemma 5, we have $\left\|\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0}\right\| = \left\|\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\| = O_p(\delta_{NT}^{-1})$, $\left\|\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\right\| = ||\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{\hat{F}_x}\mathbf{M}_{F_{x,-1}^0}+\mathbf{M}_{\hat{F}_x}\mathbf{M}_{F_{x,-1}^0}-\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}|| \leq \left\|\mathbf{M}_{\hat{F}_x}\right\| \|\,\mathbf{P}_{\hat{F}_{x,-1}}-\mathbf{P}_{F_{x,-1}^0}\,\|+\|\,\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\|\left\|\mathbf{M}_{F_{x,-1}^0}\right\| = O_p(\delta_{NT}^{-1})$ and $||\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}|| = ||\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}+\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}|| \leq \left\|\mathbf{P}_{\hat{F}_{x,-1}}-\mathbf{P}_{F_{x,-1}^0}\,\|\|\,\mathbf{M}_{\hat{F}_x}\right\|\left\|\mathbf{M}_{\hat{F}_{x,-1}}\right\|+\left\|\mathbf{M}_{F_{x,-1}^0}\right\|\left\|\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\right\| = O_p(\delta_{NT}^{-1})$, since $\left\|\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\right\| = O_p(\delta_{NT}^{-1})$ as shown above. By using this result together with $\frac{\|\mathbf{X}_i\|}{\sqrt{T}} = O_p(1)$ and $\frac{\|\mathbf{X}_{i,-1}\|}{\sqrt{T}} = O_p(1)$ which can be shown by using Assumptions 2-4(i), we have the following: $\frac{1}{NT}\sum_{i=1}^N\left\|\mathbf{X}_i'\mathbf{M}_{\hat{F}_x}\mathbf{X}_i-\mathbf{X}_i\mathbf{M}_{F_x^0}\mathbf{X}_i\right\| \leq \frac{1}{N}\sum_{i=1}^N\frac{\|\mathbf{X}_i\|}{\sqrt{T}}\left\|\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\|\frac{\|\mathbf{X}_i\|}{\sqrt{T}} = O_p(\delta_{NT}^{-1})$, $\frac{1}{NT}\sum_{i=1}^N\left\|\mathbf{X}_i'\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}-\mathbf{X}_i\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\mathbf{X}_{i,-1}\right\| \leq \frac{1}{N}\sum_{i=1}^N\frac{\|\mathbf{X}_i\|}{\sqrt{T}}\left\|\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\right\|\frac{\|\mathbf{X}_{i,-1}\|}{\sqrt{T}} = O_p(\delta_{NT}^{-1})$, and therefore $\frac{1}{NT}\sum_{i=1}^N||\mathbf{X}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{X}_i-\mathbf{X}_{i,-1}\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{F_x^0}\mathbf{X}_i|| = O_p(\delta_{NT}^{-1})$ and $\frac{1}{NT}\sum_{i=1}^N\left\|\mathbf{X}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{X}_{i,-1}-\mathbf{X}_{i,-1}\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\mathbf{X}_{i,-1}\right\| \leq \frac{1}{N}\sum_{i=1}^N\frac{\|\mathbf{X}_{i,-1}\|}{\sqrt{T}}\left\|\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0}\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\right\|\frac{\|\mathbf{X}_{i,-1}\|}{\sqrt{T}} = O_p(\delta_{NT}^{-1})$. Hence, we have

$\left\|\frac{1}{NT}\sum_{i=1}^{N}\hat{\mathbf{Z}}_i'\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i-\mathbf{Z}_i'\mathbf{M}_{F_x^0}\mathbf{Z}_i\right\|\leq O_p(\delta_{NT}^{-1})$ which leads to the result in (A.30). The second equation in (A.31) can be derived in a similar manner. ■

Proof of Lemma 7. We start with (A.33). First note that by using $\mathbf{M}_{\hat{F}_{x,-1}}=\mathbf{I}_T-T^{-1}\hat{\mathbf{F}}_{x,-1}\hat{\mathbf{F}}_{x,-1}'$ the left-hand-side of (A.33) can be written as

$$\begin{aligned}
&\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&=\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&-\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\frac{\hat{\mathbf{F}}_{x,-1}\hat{\mathbf{F}}_{x,-1}'}{T}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&=\mathbf{e}_1+\mathbf{e}_2.
\end{aligned}$$

$$\begin{aligned}
\mathbf{e}_1&=\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&+\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^{0}\mathbf{G}^{*}\right)'\\
&\times\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{M}_{\hat{F}_x}\mathbf{u}_i\\
&=\mathbf{a}_1+\mathbf{a}_2.
\end{aligned}$$

By using now $\mathbf{M}_{\hat{F}_x}=\mathbf{I}_T-T^{-1}\hat{\mathbf{F}}_x\hat{\mathbf{F}}_x'$ and $\mathbf{u}_i=\mathbf{F}_x^0\boldsymbol{\gamma}_i^0+\mathbf{F}_y^0\boldsymbol{\lambda}_i^0+\boldsymbol{\varepsilon}_i$, we have

$$\begin{aligned}
\mathbf{a}_1&=\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{M}_{\hat{F}_x}\mathbf{u}_i}{T}\\
&=\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{F}_x^0\boldsymbol{\gamma}_i^0}{T}\\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{F}_y^0\boldsymbol{\lambda}_i^0}{T}\\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\boldsymbol{\varepsilon}_i}{T}\\
&-\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{G}^{*\prime}\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_x}{T}\frac{\hat{\mathbf{F}}_x'\mathbf{u}_i}{T}\\
&=\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4,
\end{aligned}$$

$$
\begin{aligned}
|\mathbf{b}_1| &\leq \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|\left\|\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\|\left\|\mathbf{G}^{*}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\| \\
&\quad\times\left\|\frac{\mathbf{F}_{x}^{0}}{\sqrt{T}}\right\|\left\|\boldsymbol{\gamma}_{i}^{0}\right\| \\
&= \frac{1}{\sqrt{T}}\left\|\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\|\left\|\mathbf{G}^{*}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}_{x}^{0}}{\sqrt{T}}\right\| \\
&\quad\times\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|\left\|\boldsymbol{\gamma}_{i}^{0}\right\|\right) \\
&= O_p\left(T^{-1/2}\right)
\end{aligned}
$$

as $\left\|\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\right\| = O_p\left(1\right)$ by Assumption 4(i), $\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\| = O_p\left(1\right)$ by (A.29), $\left\|\mathbf{G}^{*}\right\| = O_p\left(1\right)$, $\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\| = O_p\left(1\right)$ by (A.10), $\left\|\frac{\mathbf{F}_{x}^{0}}{\sqrt{T}}\right\| = O_p\left(1\right)$ by Assumption 3, and $\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|\left\|\boldsymbol{\gamma}_{i}^{0}\right\| \leq \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|^{2}}\sqrt{\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\gamma}_{i}^{0}\right\|^{2}} = O_p\left(1\right)$ by Assumption 4(i). Similarly, $\mathbf{b}_2 = O_p\left(T^{-1/2}\right)$ and $\mathbf{b}_3 = O_p\left(T^{-1/2}\right)$.

$$
\begin{aligned}
\mathbf{b}_4 &= -\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x}}{T}\frac{\hat{\mathbf{F}}_{x}^{\prime}\mathbf{F}_{x}^{0}}{T}\boldsymbol{\gamma}_{i}^{0} \\
&\quad-\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x}}{T}\frac{\hat{\mathbf{F}}_{x}^{\prime}\mathbf{F}_{y}^{0}}{T}\boldsymbol{\lambda}_{i}^{0} \\
&\quad-\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x}}{T}\frac{\hat{\mathbf{F}}_{x}^{\prime}\boldsymbol{\varepsilon}_{i}}{T} \\
&= \mathbf{c}_1+\mathbf{c}_2+\mathbf{c}_3.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{c}_1| &\leq \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|\left\|\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\|\left\|\mathbf{G}^{*}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\| \\
&\quad\times\left\|T^{-1/2}\hat{\mathbf{F}}_{x}\right\|^{2}\left\|T^{-1/2}\mathbf{F}_{x}^{0}\right\|\left\|\boldsymbol{\gamma}_{i}^{0}\right\| \\
&= \frac{1}{\sqrt{T}}\left\|\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\|\left\|\mathbf{G}^{*}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT,-1}-\bar{\mathbf{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\| \\
&\quad\times\left\|T^{-1/2}\hat{\mathbf{F}}_{x}\right\|^{2}\left\|T^{-1/2}\mathbf{F}_{x}^{0}\right\|\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\mathbf{\Gamma}_{xi}^{0\prime}\right\|\left\|\boldsymbol{\gamma}_{i}^{0}\right\|\right) \\
&= O_p\left(T^{-1/2}\right),
\end{aligned}
$$

by making the same arguments as above and because $\left\|T^{-1/2}\hat{\mathbf{F}}_{x}\right\|^{2} = \operatorname{tr}\left(T^{-1}\hat{\mathbf{F}}_{x}^{\prime}\hat{\mathbf{F}}_{x}\right) = \operatorname{tr}\left(\mathbf{I}_{m_x}\right) = m_x$. By similar reasoning, we have $\mathbf{c}_2 = O_p\left(T^{-1/2}\right)$, $\mathbf{c}_3 = O_p\left(T^{-1/2}\right)$ and therefore $\mathbf{b}_4 = O_p\left(T^{-1/2}\right)$. Thus, we conclude $\mathbf{a}_1 = O_p\left(T^{-1/2}\right)$.

By using similar arguments as above together with (A.15), (A.14) and (A.17), we have $\mathbf{a}_2 = O_p\left(\delta_{NT}^{-1}\right)$. Thus, $\mathbf{e}_1 = O_p\left(T^{-1/2}\right)+O_p\left(\delta_{NT}^{-1}\right)$. Next, consider $\mathbf{e}_2$ which is

$$
\begin{aligned}
|\mathbf{e}_2| \leq& \sqrt{\frac{N}{T}}\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\|\right)\left\|\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\right\|\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\right\| \\
&\times\left\|\frac{\hat{\mathbf{F}}_{x,-1}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x,-1}}{T}\right\|\left\|\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{M}_{\hat{F}_x}}{\sqrt{T}}\right\| \\
=& \sqrt{\frac{N}{T}}O_p(1)\left\|\frac{\hat{\mathbf{F}}_{x,-1}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x,-1}}{T}\right\|,
\end{aligned}
$$

because $\left\|T^{-1/2}\hat{\mathbf{F}}_{x,-1}'\mathbf{M}_{\hat{F}_x}\right\| \leq \left\|T^{-1/2}\hat{\mathbf{F}}_{x,-1}'\right\|+\left\|T^{-3/2}\hat{\mathbf{F}}_{x,-1}'\hat{\mathbf{F}}_x\hat{\mathbf{F}}_x'\right\| \leq \left\|T^{-1/2}\hat{\mathbf{F}}_{x,-1}\right\|+\left\|T^{-1/2}\hat{\mathbf{F}}_{x,-1}\right\|$ $\left\|T^{-1/2}\hat{\mathbf{F}}_x\right\|^2 = O_p(1)$, $\left\|\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\right\| = O_p(1)$, $\left\|\left(\frac{\hat{\mathbf{F}}_{x,-1}'\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\right\| = O_p(1)$, and

$$
\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \leq& \frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\boldsymbol{\gamma}_i^0\right\|\left\|\frac{\mathbf{F}_x^0}{\sqrt{T}}\right\|+\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\boldsymbol{\lambda}_i^0\right\|\left\|\frac{\mathbf{F}_y^0}{\sqrt{T}}\right\|+\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\| \\
=& O_p(1),
\end{aligned}
$$

by the same arguments as above and Assumptions 1, 3, 4(i). We also have

$$
\begin{aligned}
&\left\|\sqrt{\frac{N}{T}}\frac{1}{T}\hat{\mathbf{F}}_{x,-1}'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\hat{\mathbf{F}}_{x,-1}\right\| \\
&\leq \left\|\sqrt{\frac{N}{T}}\frac{1}{T}\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{F}_{x,-1}^0\mathbf{G}^*\right\| \\
&+\left\|\sqrt{\frac{N}{T}}\frac{1}{T}\mathbf{G}^{*\prime}\mathbf{F}_{x,-1}^{0\prime}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^0\mathbf{G}^*\right)\right\| \\
&+\left\|\sqrt{\frac{N}{T}}\frac{1}{T}\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^0\mathbf{G}^*\right)'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\mathbf{F}_{x,-1}^0\mathbf{G}^*\right\| \\
&+\left\|\sqrt{\frac{N}{T}}\frac{1}{T}\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^0\mathbf{G}^*\right)'\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\left(\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^0\mathbf{G}^*\right)\right\| \\
&= \|\mathbf{L}_1\|+\|\mathbf{L}_2\|+\|\mathbf{L}_3\|+\|\mathbf{L}_4\|.
\end{aligned}
$$

$$
\|\mathbf{L}_1\| \leq \sqrt{\frac{1}{T}}\left\|\mathbf{G}^{*\prime}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}_{x,-1}^0}{\sqrt{T}}\right\|\left\|\mathbf{G}^*\right\| = O_p\left(T^{-1/2}\right),
$$

by (A.10), and Assumption 3

$$
\begin{aligned}
\|\mathbf{L}_2\| = \|\mathbf{L}_3\| \leq& \sqrt{\frac{1}{T}}\left\|\mathbf{G}^{*\prime}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT,-1}-\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)}{\sqrt{T}}\right\|\left\|\frac{\hat{\mathbf{F}}_{x,-1}-\mathbf{F}_{x,-1}^0\mathbf{G}^*}{\sqrt{T}}\right\| \\
=& \sqrt{\frac{1}{T}}O_p\left(\delta_{NT}^{-1}\right),
\end{aligned}
$$

by (A.10) and (A.17).

$$
\|\mathbf{L}_4\| = \left\|\sqrt{\frac{1}{T}}\frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(\hat{\mathbf{f}}_{x,t-1}-\mathbf{G}^{*\prime}\mathbf{f}_{x,t-1}^0\right)\left(\hat{\mathbf{f}}_{x,s-1}-\mathbf{G}^{*\prime}\mathbf{f}_{x,s-1}^0\right)'\frac{1}{\sqrt{N}}\sum_{l=1}^{k}\sum_{i=1}^{N}\left[v_{\ell it-1}v_{\ell is-1}-E\left(v_{\ell it-1}v_{\ell is-1}\right)\right]\right\|,
$$

so that, by the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\|\mathbf{L}_4\| &\leq \sqrt{T}\left(\frac{1}{T}\left\|\hat{\mathbf{F}}_{x,-1}-\mathbf{F}^0_{x,-1}\mathbf{G}^*\right\|^2\right) \\
&\times\left\{\frac{1}{T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[\frac{1}{\sqrt{N}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left[v_{\ell it-1}v_{\ell is-1}-E\left(v_{\ell it-1}v_{\ell is-1}\right)\right]\right]^2\right\}^{1/2} \\
&=\sqrt{T}O_p\left(\delta_{NT}^{-2}\right).
\end{aligned}
$$

Thus, $\mathbf{e}_2 = O_p\left(T^{-1/2}\right)+\sqrt{T}O_p\left(\delta_{NT}^{-2}\right)$. Collecting all the results, the required expression is obtained. The result in (A.32) is proved in a similar way. ■

Proof of Lemma 8. We begin with (A.35). From (A.8) we have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\mathbf{F}^{0\prime}_{x,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&=-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left[\mathbf{G}^{*-1\prime}\hat{\mathbf{F}}'_{x,-1}-\mathbf{F}^{0\prime}_{x,-1}\right]\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&=-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\hat{\mathbf{Q}}'_{-1}\hat{\mathbf{F}}'_{x,-1}\mathbf{v}_{\ell j,-1}\boldsymbol{\gamma}^{0\prime}_{\ell j}\mathbf{F}^{0\prime}_{x-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\hat{\mathbf{Q}}'_{-1}\hat{\mathbf{F}}'_{x,-1}\mathbf{F}^{0}_{x,-1}\boldsymbol{\gamma}^{0}_{\ell j}\mathbf{v}'_{\ell j,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\hat{\mathbf{Q}}'_{-1}\hat{\mathbf{F}}'_{x,-1}\mathbf{v}_{\ell j,-1}\mathbf{v}'_{\ell j,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\
&=-\left(\mathbf{d}_1+\mathbf{d}_2+\mathbf{d}_3\right).
\end{aligned}
$$

Start with $\mathbf{d}_1$, which is given by

$$
\frac{\mathbf{d}_1}{\sqrt{NT}}=\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\hat{\mathbf{Q}}'_{-1}\mathbf{A}_{kNT}\left[\mathbf{G}^{*-1\prime}\hat{\mathbf{F}}'_{x,-1}-\mathbf{F}^{0\prime}_{x,-1}\right]\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i
$$

which is a $k\times 1$ vector, where

$$
\mathbf{A}_{kNT}=\frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\hat{\mathbf{F}}'_{x-1}\mathbf{v}_{\ell j,-1}}{T}\boldsymbol{\gamma}^{0\prime}_{\ell j}.
$$

We have

$$
\begin{aligned}
\frac{1}{N}\sum_{j=1}^{N}\frac{\hat{\mathbf{F}}'_{x,-1}\mathbf{v}_{\ell j,-1}}{T}\boldsymbol{\gamma}^{0\prime}_{\ell j}&=\frac{1}{N}\sum_{j=1}^{N}\mathbf{G}^{*\prime}\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{v}_{\ell j,-1}}{T}\boldsymbol{\gamma}^{0\prime}_{\ell j}+\frac{1}{N}\sum_{j=1}^{N}\frac{\left(\hat{\mathbf{F}}_{x-1}-\mathbf{F}^0_{x,-1}\mathbf{G}^*\right)'\mathbf{v}_{\ell j,-1}}{T}\boldsymbol{\gamma}^{0\prime}_{\ell j} \\
&=O_p\left(T^{-1/2}N^{-1/2}\right)+O_p\left(N^{-1}\right)+N^{-1/2}O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

as the first term is $O_p\left(T^{-1/2}N^{-1/2}\right)$ by independence of $\mathbf{v}_{\ell j,-1}$ and $\boldsymbol{\gamma}^0_{\ell j}$ and the second term is $O_p\left(N^{-1}\right)+N^{-1/2}O_p\left(\delta_{NT}^{-2}\right)$ by (A.27) in Lemma 4. This gives the following

$$
\|\mathbf{A}_{kNT}\|=O_p\left(T^{-1/2}N^{-1/2}\right)+O_p\left(N^{-1}\right)+N^{-1/2}O_p\left(\delta_{NT}^{-2}\right).
$$

Next,

$$
\begin{aligned}
\frac{|\mathbf{d}_1|}{\sqrt{NT}} &\leq \left\| \frac{1}{NT} \sum_{i=1}^{N} \boldsymbol{\Gamma}_{xi}^{0\prime} \hat{\mathbf{Q}}_{-1}^{\prime} \mathbf{A}_{kNT} \left(\hat{\mathbf{F}}_{x-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \mathbf{u}_i \right\| \\
&+ \left\| \frac{1}{NT^2} \sum_{i=1}^{N} \boldsymbol{\Gamma}_{xi}^{0\prime} \hat{\mathbf{Q}}_{-1}^{\prime} \mathbf{A}_{kNT} \left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_x \hat{\mathbf{F}}_x^{\prime} \mathbf{u}_i \right\| \\
&+ \left\| \frac{1}{NT^2} \sum_{i=1}^{N} \boldsymbol{\Gamma}_{xi}^{0\prime} \hat{\mathbf{Q}}_{-1}^{\prime} \mathbf{A}_{kNT} \left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_{x,-1} \hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \right\| \\
&\leq \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\gamma}_i^0 \right\| \right) \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \mathbf{F}_x^0}{T} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\lambda}_i^0 \right\| \right) \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \mathbf{F}_y^0}{T} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \boldsymbol{\varepsilon}_i}{T} \right\| \right) \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\gamma}_i^0 \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_x}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_x^{\prime} \mathbf{F}_x^0}{T} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\lambda}_i^0 \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_x}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_x^{\prime} \mathbf{F}_y^0}{T} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_x}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_x^{\prime}}{\sqrt{T}} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\gamma}_i^0 \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_{x,-1}}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{M}_{\hat{F}_x}}{\sqrt{T}} \right\| \left\| \frac{\mathbf{F}_x^0}{\sqrt{T}} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \boldsymbol{\lambda}_i^0 \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_{x,-1}}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{M}_{\hat{F}_x}}{\sqrt{T}} \right\| \left\| \frac{\mathbf{F}_y^0}{\sqrt{T}} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \right) \left\| \frac{\left(\hat{\mathbf{F}}_{x,-1} - \mathbf{F}_{x,-1}^{0} \mathbf{G}^{*} \right)^{\prime} \hat{\mathbf{F}}_{x,-1}}{T} \right\| \left\| \frac{\hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{M}_{\hat{F}_x}}{\sqrt{T}} \right\| \left\| \hat{\mathbf{Q}}_{-1}^{\prime} \right\| \left\| \mathbf{A}_{kNT} \right\| \\
&= O_p\left(\delta_{NT}^{-2}\right) \left[O_p\left(T^{-1/2} N^{-1/2}\right) + O_p\left(N^{-1}\right) + N^{-1/2} O_p\left(\delta_{NT}^{-2}\right) \right],
\end{aligned}
$$

by (A.19), (A.20), (A.21), (A.22), Assumptions 1, 3, 4(i) and using $\left\| T^{-1} \hat{\mathbf{F}}_x^{\prime} \mathbf{F}_x^0 \right\| = O_p(1)$, $\left\| T^{-1} \hat{\mathbf{F}}_x^{\prime} \mathbf{F}_y^0 \right\| = O_p(1)$, $\left\| T^{-1/2} \hat{\mathbf{F}}_{x,-1}^{\prime} \mathbf{M}_{\hat{F}_x} \right\| = O_p(1)$ and $\left\| \mathbf{A}_{kNT} \right\| = O_p\left(T^{-1/2} N^{-1/2}\right) + O_p\left(N^{-1}\right) + N^{-1/2} O_p\left(\delta_{NT}^{-2}\right)$ as shown above. We therefore have

$$
\mathbf{d}_1 = \sqrt{NT} O_p\left(\delta_{NT}^{-2}\right) \times \left[O_p\left(T^{-1/2} N^{-1/2}\right) + O_p\left(N^{-1}\right) + N^{-1/2} O_p\left(\delta_{NT}^{-2}\right) \right].
$$

Now consider $\mathbf{d}_2$ which can be written as

$$\begin{aligned}\mathbf{d}_2 &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\gamma_{\ell j}^{0}\mathbf{v}_{\ell j,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\sum_{\ell=1}^{k}\gamma_{\ell j}^{0}\mathbf{v}_{\ell j,-1}^{\prime}\right)\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\boldsymbol{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i,\end{aligned}$$

Consider now $\mathbf{d}_3$. Defining $\boldsymbol{\Sigma}_{kNT,-1} = N^{-1}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\mathbf{v}_{\ell j,-1}\mathbf{v}_{\ell j,-1}^{\prime}$, we have

$$\begin{aligned}\mathbf{d}_3 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\hat{\mathbf{Q}}_{-1}^{\prime}\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{v}_{\ell j,-1}\mathbf{v}_{\ell j,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{T}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\hat{\mathbf{Q}}_{-1}^{\prime}\hat{\mathbf{F}}_{x,-1}^{\prime}\left(\frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\mathbf{v}_{\ell j,-1}\mathbf{v}_{\ell j,-1}^{\prime}\right)\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\hat{\mathbf{Q}}_{-1}^{\prime}\hat{\mathbf{F}}_{x,-1}^{\prime}\boldsymbol{\Sigma}_{kNT,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}(\boldsymbol{\Upsilon}_{kN}^{0})^{-1}\boldsymbol{\Lambda}_{0\hat{F}_{x,-1}}^{-1\prime}\hat{\mathbf{F}}_{x,-1}^{\prime}\boldsymbol{\Sigma}_{kNT,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}(\boldsymbol{\Upsilon}_{kN}^{0})^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}\boldsymbol{\Sigma}_{kNT,-1}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i.\end{aligned}$$

where the definitions of $\hat{\mathbf{Q}}_{-1}$ and $\boldsymbol{\Lambda}_{0\hat{F}_{x,-1}}$ are given above. Hence, the expressions for $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_3$ gives the required result in (A.35). The result in (A.34) is obtained in an analogous manner. ■

Proof of Lemma 9. First consider (A.37). The left-hand-side of (A.37) can be written as

$$\begin{aligned}&\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\boldsymbol{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\sum_{\ell=1}^{k}\gamma_{\ell j}^{0}\mathbf{v}_{\ell j,-1}^{\prime}\right)\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\gamma_{\ell j}^{0}\mathbf{v}_{\ell j,-1}^{\prime}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\gamma_{\ell j}^{0}\mathbf{u}_{i}^{\prime}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{v}_{\ell j,-1} \\ &= \frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xj}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\gamma_{\ell i}^{0}\mathbf{u}_{j}^{\prime}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{v}_{\ell i,-1} \\ &= \frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left(\frac{1}{N}\sum_{j=1}^{N}\boldsymbol{\Gamma}_{xj}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\gamma_{\ell i}^{0}\mathbf{u}_{j}^{\prime}\right)\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{v}_{\ell i,-1} \\ &= \frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{H}_{\ell,i}^{\prime}\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{v}_{\ell i,-1},\end{aligned}$$

where $\mathbf{H}_{\ell,i} = \frac{1}{N}\sum_{j=1}^{N} \mathbf{u}_j \gamma_{\ell i}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^0\right)^{-1} \mathbf{\Gamma}_{xj}^0$. By adding and subtracting terms we get

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' \mathbf{M}_{\hat{F}_x} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{v}_{\ell i,-1} \\
&= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' \mathbf{M}_{F_x^0} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&+ \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' (\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0}) \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&+ \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' \mathbf{M}_{\hat{F}_x} (\mathbf{M}_{\hat{F}_{x,-1}} - \mathbf{M}_{F_{x,-1}^0}) \mathbf{v}_{\ell i,-1}. \qquad \text{(A.1)}
\end{aligned}
$$

Now, consider the second term in (A.1). Since $\hat{\mathbf{F}}_x' \hat{\mathbf{F}}_x / T = \mathbf{I}_{m_x}$, we have $\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0} = \mathbf{P}_{F_x^0} - \mathbf{P}_{\hat{F}_x} = -\left(\frac{\hat{\mathbf{F}}_x \hat{\mathbf{F}}_x'}{T} - \mathbf{P}_{F_x^0}\right)$. Using this result and by adding and subtracting terms, we get

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' (\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0}) \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= -\frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \mathbf{H}_{\ell,i}' \left(\frac{\hat{\mathbf{F}}_x \hat{\mathbf{F}}_x'}{T} - \mathbf{P}_{F_x^0}\right) \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= -\frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \mathbf{G}' \mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&- \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&- \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \mathbf{G} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&- \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left[\mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1}\right] \mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= -\left(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\right).
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{e}_1| &\le \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \left\| \frac{\mathbf{H}_{\ell,i}' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \mathbf{G}' \mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \right\| \\
&\le \frac{1}{\sqrt{N}} \frac{1}{N} \sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N} \left\| \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^0\right)^{-1} \gamma_{\ell i}^0 \frac{\mathbf{u}_j' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \mathbf{G}' \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{\sqrt{T}} \right\| \\
&\le \sqrt{N} \left(\frac{1}{N} \sum_{j=1}^{N} \left\|\mathbf{\Gamma}_{xj}^0\right\| \left\| \left(\mathbf{\Upsilon}_{kN}^0\right)^{-1} \right\| \left\| \frac{\mathbf{u}_j' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| \right) \\
&\times \left(\frac{1}{N} \sum_{\ell=1}^{k}\sum_{i=1}^{N} \left\|\gamma_{\ell i}^0\right\| \left\|\mathbf{G}\right\| \left\| \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{\sqrt{T}} \right\| \right) = \sqrt{N} O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

as $\left\| \frac{\mathbf{u}_j' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| \le \left\|\gamma_j^{0\prime}\right\| \left\| \frac{\mathbf{F}_x^{0\prime} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| + \left\|\boldsymbol{\lambda}_j^{0\prime}\right\| \left\| \frac{\mathbf{F}_y^{0\prime} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| + \left\| \frac{\boldsymbol{\varepsilon}_j' \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| = O_p\left(\delta_{NT}^{-2}\right)$ by (A.20), (A.21) and (A.22).

$$
\begin{aligned}
|\mathbf{e}_2| &\leq \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}'_{\ell,i}\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{v}_{\ell i,-1} \right\| \\
&\leq \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \left\| \boldsymbol{\Gamma}_{xj}^{0\prime} \left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1} \gamma_{\ell i}^0 \frac{\mathbf{u}'_j\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \right\| \\
&\leq \sqrt{NT} \left(\frac{1}{N} \sum_{j=1}^{N} \left\| \boldsymbol{\Gamma}_{xj}^0 \right\| \left\| \left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1} \right\| \left\| \frac{\mathbf{u}'_j\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)}{T} \right\| \right) \\
&\times \left(\frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \boldsymbol{\gamma}_{\ell i}^0 \right\| \left\| \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \right\| \right) \\
&= \sqrt{NT} O_p\left(\delta_{NT}^{-4}\right),
\end{aligned}
$$

by using $\left\| \frac{\mathbf{u}'_j(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G})}{T} \right\| = O_p\left(\delta_{NT}^{-2}\right)$ and $\left\| \frac{(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G})' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \right\| \leq \left\| \frac{(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G})' \mathbf{v}_{\ell i,-1}}{T} \right\| + \frac{1}{\sqrt{T}} \left\| \frac{(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G})' \mathbf{F}_x^0}{T} \right\|$ $\left\| \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_x^{0\prime} \mathbf{v}_{\ell i,-1}}{\sqrt{T}} \right\| = O_p\left(\delta_{NT}^{-2}\right)$ due to (A.20) and (A.23).

$$
\begin{aligned}
\mathbf{e}_3 &= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \mathbf{G} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G}\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \mathbf{G}\mathbf{G}' \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&+ \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \left[\mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right] \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0\right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= \mathbf{a}_1 + \mathbf{a}_2.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{e}_4| &\leq \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \left[\mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right] \mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \right\| \\
&\leq \sqrt{N} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \right\| \left\| \mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{\sqrt{T}} \right\| \\
&= \sqrt{N} O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

by (A.28) and because

$$
\left\| \frac{\mathbf{H}'_{\ell,i} \mathbf{F}_x^0}{T} \right\| \leq \frac{1}{N} \sum_{j=1}^{N} \left\| \boldsymbol{\Gamma}_{xj}^{0\prime} \right\| \left\| \left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1} \right\| \left\| \boldsymbol{\gamma}_{\ell i}^0 \right\| \left\| \frac{\mathbf{u}'_j}{\sqrt{T}} \right\| \left\| \frac{\mathbf{F}_x^0}{\sqrt{T}} \right\| = O_p(1),
$$

by Assumptions 1, 2, 3, 4(i). Next, consider $\mathbf{a}_1$ and $\mathbf{a}_2$ in the expression of $\mathbf{e}_3$. Start with $\mathbf{a}_2$ which is as follows

$$
\begin{aligned}
|\mathbf{a}_2| &\leq \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left[\mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right] \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \right\| \left\| \mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \mathbf{G}^{-1} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \right\| \left\| \mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \mathbf{G}^{-1} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)' \mathbf{v}_{\ell i,-1}}{T} \right\| \\
&+ \sqrt{N} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \right\| \left\| \mathbf{G}\mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \mathbf{G}^{-1} \right\| \left\| \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)' \mathbf{F}_{x,-1}^0}{T} \right\| \\
&\times \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{v}_{\ell i,-1}}{\sqrt{T}} \right\| \\
&= \sqrt{NT} O_p\left(\delta_{NT}^{-4} \right),
\end{aligned}
$$

by (A.20), (A.23), (A.28) and $\left\| \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \right\| = O_p(1)$, which is shown above. As for $\mathbf{a}_1$, we have

$$
\begin{aligned}
\mathbf{a}_1 &= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= \frac{1}{\sqrt{NT}} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \left(\mathbf{E}_1' + \mathbf{E}_2' + \mathbf{E}_3' \right) \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&= \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \sum_{h=1}^{k} \frac{\hat{\mathbf{F}}_x' \mathbf{v}_{hj}}{T} \boldsymbol{\gamma}_{hj}^{0\prime} \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \frac{\hat{\mathbf{F}}_x' \mathbf{F}_x^0}{T} \sum_{h=1}^{k} \boldsymbol{\gamma}_{hj}^{0} \frac{\mathbf{v}_{hj}' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \sum_{h=1}^{k} \frac{\hat{\mathbf{F}}_x' \mathbf{v}_{hj}}{T} \frac{\mathbf{v}_{hj}' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&= \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_1 &= \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \sum_{h=1}^{k} \frac{\hat{\mathbf{F}}_x' \mathbf{v}_{hj}}{T} \boldsymbol{\gamma}_{hj}^{0\prime} \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&= \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \mathbf{G}' \sum_{h=1}^{k} \frac{\mathbf{F}_x^{0\prime} \mathbf{v}_{hj}}{T} \boldsymbol{\gamma}_{hj}^{0\prime} \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\mathbf{H}_{\ell,i}' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \hat{\mathbf{Q}}' \sum_{h=1}^{k} \frac{\left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)' \mathbf{v}_{hj}}{T} \boldsymbol{\gamma}_{hj}^{0\prime} \frac{\mathbf{F}_x^{0\prime} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1}}{T} \\
&= \mathbf{d}_1 + \mathbf{d}_2
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{d}_1| &\leq \frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\left(\sum_{h=1}^{k}\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\frac{\mathbf{F}_x^{0\prime}\mathbf{v}_{hj}}{\sqrt{T}}\gamma_{hj}^{0\prime}\right)\frac{\mathbf{F}_x^{0\prime}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\right\| \\
&\leq T^{-1/2}\left(\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\hat{\mathbf{Q}}'\right\|\left\|\mathbf{G}'\right\|\left\|\frac{\mathbf{F}_x^{0\prime}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{\sqrt{T}}\right\|\right) \\
&\times\left\|\frac{1}{\sqrt{N}}\sum_{h=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{F}_x^{0\prime}\mathbf{v}_{hj}}{\sqrt{T}}\gamma_{hj}^{0\prime}\right\| \\
&= T^{-1/2}O_p(1) = O_p\left(T^{-1/2}\right).
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{d}_2| &\leq \sqrt{N}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\left(\frac{1}{N}\sum_{h=1}^{k}\sum_{j=1}^{N}\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{v}_{hj}}{T}\gamma_{hj}^{0\prime}\right)\frac{\mathbf{F}_x^{0\prime}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{\sqrt{T}}\right\| \\
&\leq \sqrt{N}\left(\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\hat{\mathbf{Q}}'\right\|\left\|\frac{\mathbf{F}_x^{0\prime}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{\sqrt{T}}\right\|\right) \\
&\times\left(\frac{1}{N}\sum_{h=1}^{k}\sum_{j=1}^{N}\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{v}_{hj}}{T}\right\|\left\|\gamma_{hj}^{0\prime}\right\|\right) \\
&= \sqrt{N}O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

by (A.23).

$$
\begin{aligned}
\mathbf{c}_2 &= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\frac{\hat{\mathbf{F}}'_x\mathbf{F}_x^0}{T}\sum_{h=1}^{k}\gamma_{hj}^0\frac{\mathbf{v}'_{hj}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\sum_{h=1}^{k}\gamma_{hj}^0\frac{\mathbf{v}'_{hj}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_3 &= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\sum_{h=1}^{k}\frac{\hat{\mathbf{F}}'_x\mathbf{v}_{hj}}{T}\frac{\mathbf{v}'_{hj}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\sum_{h=1}^{k}\frac{\mathbf{F}_x^{0\prime}\mathbf{v}_{hj}}{T}\frac{\mathbf{v}'_{hj}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}'_{\ell,i}\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\sum_{h=1}^{k}\frac{\left(\hat{\mathbf{F}}_x\mathbf{G}^{-1}-\mathbf{F}_x^0\right)'\mathbf{v}_{hj}}{T}\frac{\mathbf{v}'_{hj}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&= \mathbf{d}_1 + \mathbf{d}_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{d}_1 &= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\sum_{h=1}^{k}\frac{\mathbf{F}_x^{0\prime}\mathbf{v}_{hj}}{T}\frac{\mathbf{v}_{hj}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&= \sqrt{\frac{T}{N}}\frac{1}{T}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\frac{\mathbf{F}_x^{0\prime}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}}\frac{1}{T}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\frac{\mathbf{F}_x^{0\prime}\left(\mathbf{\Sigma}_{kNT}-\bar{\mathbf{\Sigma}}_{kNT}\right)\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T} \\
&= \mathbf{b}_1+\mathbf{b}_2.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{b}_1| &\leq \sqrt{\frac{T}{N}}\frac{1}{T}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\frac{\mathbf{F}_x^{0\prime}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\right\| \\
&= \sqrt{\frac{T}{N}}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\gamma_{\ell i}^0\frac{\mathbf{u}_j'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\frac{\mathbf{F}_x^{0\prime}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\right\| \\
&= \sqrt{\frac{N}{T}}\frac{1}{N^2}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\left\|\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\gamma_{\ell i}^0\frac{\mathbf{v}_{\ell i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{F}_x^0}{T}\mathbf{G}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_j}{T}\right\| \\
&\leq \sqrt{\frac{N}{T}}\frac{1}{\sqrt{N}}\left(\frac{1}{N}\sum_{j=1}^{N}\left\|\mathbf{\Gamma}_{xj}^0\right\|\left\|\frac{\mathbf{u}_j}{\sqrt{T}}\right\|\right)\left\|\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\gamma_{\ell i}^0\mathbf{v}_{\ell i,-1}'\mathbf{M}_{F_{x,-1}^0}\right\| \\
&\times \lambda_{\max}\left(\bar{\mathbf{\Sigma}}_{kNT,-1}\right)\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^0}{\sqrt{T}}\right\|^2 = O_p\left(N^{-1/2}\right),
\end{aligned}
$$

where $\lambda_{\max}\left(\bar{\mathbf{\Sigma}}_{kNT,-1}\right)$ is the largest eigenvalue of $\bar{\mathbf{\Sigma}}_{kNT,-1}$, which is $O_p(1)$ by Assumption 2. Moreover,

$$
\begin{aligned}
|\mathbf{b}_2| &\leq \sqrt{\frac{T}{N}}\frac{1}{T}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\frac{\mathbf{F}_x^{0\prime}\left(\mathbf{\Sigma}_{kNT}-\bar{\mathbf{\Sigma}}_{kNT}\right)\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\right\| \\
&\leq \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\hat{\mathbf{Q}}'\right\|\|\mathbf{G}'\|\left\|\frac{\mathbf{F}_x^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT}-\bar{\mathbf{\Sigma}}_{kNT}\right)}{\sqrt{T}}\right\|\left\|\frac{\mathbf{v}_{\ell i,-1}}{\sqrt{T}}\right\| \\
&+ \frac{1}{T}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\|\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\hat{\mathbf{Q}}'\right\|\|\mathbf{G}'\|\left\|\frac{\mathbf{F}_x^{0\prime}\sqrt{N}\left(\mathbf{\Sigma}_{kNT}-\bar{\mathbf{\Sigma}}_{kNT}\right)}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}}{\sqrt{T}}\right\| \\
&\times \left\|\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{v}_{\ell i,-1}}{\sqrt{T}}\right\| \\
&= O_p(T^{-1/2}),
\end{aligned}
$$

by (A.10). Also, we have

$$
\begin{aligned}
\mathbf{d}_2 &= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\sum_{h=1}^{k}\frac{\left(\hat{\mathbf{F}}_x\mathbf{G}^{-1}-\mathbf{F}_x^0\right)'\mathbf{v}_{hj}}{T}\frac{\mathbf{v}_{hj}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\\
&= \sqrt{\frac{T}{N}}\frac{1}{N^2}\sum_{\ell=1}^{k}\sum_{h=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\mathbf{\Gamma}_{xn}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\gamma_{\ell i}^0\frac{\mathbf{u}_n'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\hat{\mathbf{Q}}'\mathbf{G}'\\
&\times\frac{\left(\hat{\mathbf{F}}_x\mathbf{G}^{-1}-\mathbf{F}_x^0\right)'\mathbf{v}_{hj}}{T}\frac{\mathbf{v}_{hj}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\\
&= \sqrt{\frac{T}{N}}\frac{1}{N^2}\sum_{\ell=1}^{k}\sum_{h=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\mathbf{\Gamma}_{xn}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\gamma_{\ell i}^0\frac{\mathbf{v}_{\ell i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{hj}}{T}\frac{\mathbf{v}_{hj}'\left(\hat{\mathbf{F}}_x\mathbf{G}^{-1}-\mathbf{F}_x^0\right)}{T}\\
&\times\mathbf{G}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_n}{T},
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{d}_2| &\leq \sqrt{N}\sqrt{\frac{T}{N}}\left(\frac{1}{N}\sum_{n=1}^{N}\left\|\mathbf{\Gamma}_{xn}^{0\prime}\right\|\left\|\frac{\mathbf{u}_n}{\sqrt{T}}\right\|\right)\left\|\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\right\|\left\|\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\gamma_{\ell i}^0\mathbf{v}_{\ell i,-1}'\mathbf{M}_{F_{x,-1}^0}\right\|\\
&\times\left(\frac{1}{N}\sum_{h=1}^{k}\sum_{j=1}^{N}\left\|\frac{\mathbf{v}_{hj}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{v}_{hj}'\left(\hat{\mathbf{F}}_x'\mathbf{G}^{-1}-\mathbf{F}_x^0\right)}{T}\right\|\right)\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\\
&= \sqrt{N}O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

by (A.23). By putting the results together, we therefore get

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{H}_{\ell,i}'(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}\\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\sum_{h=1}^{k}\gamma_{hj}^0\frac{\mathbf{v}_{hj}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\\
&+\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right)\\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\left(\mathbf{\Upsilon}_{kN}^0\right)^{-1}\mathbf{\Gamma}_{xj}^0\frac{\mathbf{V}_j'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}}{T}\\
&+\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right).
\end{aligned}
$$

Consider now the third term in (A.1). By following the same steps as in the discussion above and by replacing $\mathbf{H}_{\ell,i}$ with $\mathbf{M}_{\hat{F}_x}\mathbf{H}_{\ell,i}$, $(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})$ with $(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0})$, and $\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}$ with $\mathbf{v}_{\ell i,-1}$, we get

$$\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{H}_{\ell,i}'\mathbf{M}_{\hat{F}_x}(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0})\mathbf{v}_{\ell i,-1}$$
$$=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{M}_{\hat{F}_x}\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}$$
$$+\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right)$$
$$=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{M}_{F_x^0}\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}$$
$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}$$
$$+\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right)$$
$$=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{M}_{F_x^0}\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}$$
$$=\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right),$$

since

$$\left\|\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}\right\|$$
$$=\left\|\frac{1}{N^2}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\boldsymbol{\Gamma}_{xn}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\gamma}_{\ell i}^0\frac{\mathbf{u}_n'(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_j'\mathbf{v}_{\ell i,-1}}{T}\right\|$$
$$=\left\|\frac{1}{N^2}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{n=1}^{N}\boldsymbol{\Gamma}_{xn}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\gamma}_{\ell i}^0\frac{\mathbf{v}_{\ell i,-1}'\mathbf{V}_j}{T}\boldsymbol{\Gamma}_{xj}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{u}_n}{T}\right\|$$
$$\leq\frac{1}{N}\sum_{n=1}^{N}\left\|\boldsymbol{\Gamma}_{xn}^{0\prime}\right\|\left\|\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\right\|\left\|\frac{1}{\sqrt{N}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\frac{\boldsymbol{\gamma}_{\ell i}^0\mathbf{v}_{\ell i,-1}'}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{N}}\sum_{h=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{v}_{hj}\boldsymbol{\gamma}_{hj}^{0\prime}}{\sqrt{T}}\right\|$$
$$\times\left\|\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\mathbf{P}_{\hat{F}_x}-\mathbf{P}_{F_x^0}\right\|\left\|\frac{\mathbf{u}_n}{\sqrt{T}}\right\|$$
$$=O_p\left(\delta_{NT}^{-1}\right),$$

by Lemma 5. By putting the results together we have

$$\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{H}_{\ell,i}'\mathbf{M}_{\hat{F}_x}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{v}_{\ell i,-1}$$
$$=\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{H}_{\ell,i}'\mathbf{M}_{F_x^0}\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell i,-1}$$
$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{F}_x^0}{T}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_j'\mathbf{v}_{\ell i,-1}}{T}$$
$$+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{H}_{\ell,i}'\mathbf{M}_{F_x^0}\mathbf{F}_{x,-1}^0}{T}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\boldsymbol{\Gamma}_{xj}^0\frac{\mathbf{V}_{j,-1}'\mathbf{v}_{\ell i,-1}}{T}$$
$$+\sqrt{N}O_p\left(\delta_{NT}^{-2}\right)+O_p\left(\delta_{NT}^{-1}\right)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \mathbf{u}_j' \mathbf{M}_{F_x^0} \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell i,-1} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xn}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \frac{\mathbf{u}_n' \mathbf{F}_x^0}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \frac{\mathbf{V}_j' \mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xn}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \frac{\mathbf{u}_n' \mathbf{M}_{F_x^0} \mathbf{F}_{x,-1}^0}{T} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \frac{\mathbf{V}_{j,-1}' \mathbf{v}_{\ell i,-1}}{T} \\
&+ \sqrt{N} O_p\left(\delta_{NT}^{-2}\right) + O_p\left(\delta_{NT}^{-1}\right) \\
&= \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \mathbf{v}_{\ell i,-1}' \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_j \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xn}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \frac{\mathbf{v}_{\ell i,-1}' \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_n}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xn}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \gamma_{\ell i}^{0} \frac{\mathbf{v}_{\ell i,-1}' \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_n}{T} \\
&+ \sqrt{N} O_p\left(\delta_{NT}^{-2}\right) + O_p\left(\delta_{NT}^{-1}\right) \\
&= \frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \mathbf{V}_{n,-1}' \mathbf{M}_{F_{x,-1}^0} \mathbf{M}_{F_x^0} \mathbf{u}_i \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{V}_j}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T}\right)^{-1} \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} \\
&+ \sqrt{\frac{T}{N}} \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T}\right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T} \\
&+ \sqrt{N} O_p\left(\delta_{NT}^{-2}\right) + O_p\left(\delta_{NT}^{-1}\right), \qquad \text{(A.2)}
\end{aligned}
$$

which provides the required result in (A.37). The result in (A.36) is proved in an analogous way.
■

Proof of Lemma 10. We first prove (A.39). By noting that $\hat{\mathbf{A}}\hat{\mathbf{B}} = \mathbf{AB} + \left(\hat{\mathbf{A}} - \mathbf{A}\right)\hat{\mathbf{B}} + \mathbf{A}\left(\hat{\mathbf{B}} - \mathbf{B}\right)$, the left-hand-side of (A.39) can be written as

$$
\begin{aligned}
&\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&+\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}\bar{\boldsymbol{\Sigma}}_{kNT}\left(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^{0}}\right)\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&+\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left[\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}-\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\right]\\
&\times\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&+\sqrt{\frac{N}{T}}\left(\mathbf{a}_{1}+\mathbf{a}_{2}\right).
\end{aligned}
$$

$$
\begin{aligned}
\left|\mathbf{a}_{1}\right|&\leq\frac{1}{NT}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}\bar{\boldsymbol{\Sigma}}_{kNT}\left(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^{0}}\right)\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\right\|\\
&\leq\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\frac{\mathbf{u}_{i}}{\sqrt{T}}\right\|\right)\left\|\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\left\|\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right\|\left\|\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}}{\sqrt{T}}\right\|\\
&\times\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)\left\|\mathbf{P}_{\hat{F}_{x,-1}}-\mathbf{P}_{F_{x,-1}^{0}}\right\|\left\|\mathbf{M}_{\hat{F}_{x}}\right\|\\
&=O_{p}\left(\delta_{NT}^{-1}\right),
\end{aligned}
$$

by Lemma 5 and $T^{-1/2}\left\|\bar{\boldsymbol{\Sigma}}_{kNT,-1}\hat{\mathbf{F}}_{x,-1}\right\|\leq\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right)T^{-1/2}\left\|\hat{\mathbf{F}}_{x,-1}\right\|=O_{p}(1)$ by Assumption 2 and 3. Next, we have

$$
\begin{aligned}
&\left|\mathbf{a}_{2}\right|\\
&\leq\frac{1}{NT}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\left[\left(\frac{\hat{\mathbf{F}}_{x,-1}^{\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\hat{\mathbf{F}}_{x,-1}^{\prime}-\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\mathbf{F}_{x,-1}^{0\prime}\right]\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\right\|\\
&\leq\frac{1}{\sqrt{T}}\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\boldsymbol{\Gamma}_{xi}^{0\prime}\right\|\left\|\frac{\mathbf{u}_{i}}{\sqrt{T}}\right\|\right)\left\|\left[\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}-\mathbf{F}_{x,-1}^{0}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right]\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\right\|\\
&\times\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left\|\mathbf{M}_{\hat{F}_{x,-1}}\right\|\left\|\mathbf{M}_{\hat{F}_{x}}\right\|\\
&=O_{p}(1)\frac{1}{\sqrt{T}}\left\|\left[\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}-\mathbf{F}_{x,-1}^{0}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\right]\left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1}\right\|,
\end{aligned}
$$

by Assumptions 1, 2, 3, 4(i), and because

$$\frac{1}{\sqrt{T}}\left\|\left[\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}-\mathbf{F}^{0}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\right]\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|$$

$$=\frac{1}{\sqrt{T}}\left\|\left[\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}-\mathbf{F}^{0}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}\right]\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|$$

$$=\frac{1}{\sqrt{T}}\left\|\left[\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}-\mathbf{P}_{F^{0}_{x,-1}}\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}\right]\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|$$

$$=\frac{1}{\sqrt{T}}\left\|\mathbf{M}_{F^{0}_{x,-1}}\hat{\mathbf{F}}_{x,-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|\leq\frac{1}{\sqrt{T}}\left\|\mathbf{M}_{F^{0}_{x,-1}}\hat{\mathbf{F}}_{x,-1}\right\|\left\|\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|$$

$$=O_p\left(\delta_{NT}^{-1}\right),$$

since $T^{-1/2}\left\|\mathbf{M}_{F^{0}_{x,-1}}\hat{\mathbf{F}}_{x,-1}\right\|=T^{-1/2}\left\|(\mathbf{M}_{F^{0}_{x,-1}}-\mathbf{M}_{\hat{F}_{x,-1}})\hat{\mathbf{F}}_{x,-1}\right\|\leq\left\|\mathbf{M}_{F^{0}_{x,-1}}-\mathbf{M}_{\hat{F}_{x,-1}}\right\|T^{-1/2}\left\|\hat{\mathbf{F}}_{x,-1}\right\|=\left\|\mathbf{P}_{F^{0}_{x,-1}}-\mathbf{P}_{\hat{F}_{x,-1}}\right\|T^{-1/2}\left\|\hat{\mathbf{F}}_{x,-1}\right\|=O_p\left(\delta_{NT}^{-1}\right)$ by Lemma 5, $\left\|\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\hat{\mathbf{F}}_{x,-1}}{T}\right)^{-1}\right\|=O_p(1)$ and $\left\|\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\right\|=O_p(1)$.

Hence, by putting the results together we get the required result in (A.39)

$$\frac{1}{\sqrt{N}T^{3/2}}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\left(\frac{\hat{\mathbf{F}}'_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\hat{\mathbf{F}}'_{x,-1}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i$$

$$=\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\mathbf{F}^{0\prime}_{x,-1}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F^{0}_{x,-1}}\mathbf{M}_{\hat{F}_x}\mathbf{u}_i+o_p(1)$$

$$=\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\mathbf{F}^{0\prime}_{x,-1}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F^{0}_{x,-1}}\mathbf{M}_{F^{0}_x}\mathbf{u}_i$$

$$+\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\mathbf{F}^{0\prime}_{x,-1}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F^{0}_{x,-1}}(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F^{0}_x})\mathbf{u}_i+o_p(1)$$

$$=\sqrt{\frac{N}{T}}\frac{1}{NT}\sum_{i=1}^{N}\mathbf{\Gamma}^{0\prime}_{xi}\left(\mathbf{\Upsilon}^{0}_{kN}\right)^{-1}\left(\frac{\mathbf{F}^{0\prime}_{x,-1}\mathbf{F}^{0}_{x,-1}}{T}\right)^{-1}\mathbf{F}^{0\prime}_{x,-1}\bar{\mathbf{\Sigma}}_{kNT}\mathbf{M}_{F^{0}_{x,-1}}\mathbf{M}_{F^{0}_x}\mathbf{u}_i+o_p(1),$$

where the second term in the second equality is

$$\begin{aligned}
&\left\| \frac{1}{NT} \sum_{i=1}^{N} \boldsymbol{\Gamma}_{xi}^{0\prime} \left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\boldsymbol{\Sigma}}_{kNT} \mathbf{M}_{F_{x,-1}^{0}} (\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0}) \mathbf{u}_i \right\| \\
&\leq \frac{1}{NT} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\boldsymbol{\Sigma}}_{kNT} \mathbf{M}_{F_{x,-1}^{0}} (\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0}) \mathbf{u}_i \right\| \\
&\leq \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \gamma_i^0 \right\| \right) \left\| \left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime}}{\sqrt{T}} \right\| \\
&\times \lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right) \left\| \mathbf{M}_{F_{x,-1}^{0}} \right\| \left\| \mathbf{P}_{\hat{F}_x} - \mathbf{P}_{F_x^0} \right\| \left\| \frac{\mathbf{F}_y^0}{\sqrt{T}} \right\| \\
&+ \left(\frac{1}{N} \sum_{i=1}^{N} \left\| \boldsymbol{\Gamma}_{xi}^{0\prime} \right\| \left\| \frac{\boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \right) \left\| \left(\boldsymbol{\Upsilon}_{kN}^{0}\right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime}}{\sqrt{T}} \right\| \\
&\times \lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT,-1}\right) \left\| \mathbf{M}_{F_{x,-1}^{0}} \right\| \left\| \mathbf{P}_{\hat{F}_x} - \mathbf{P}_{F_x^0} \right\| \\
&= O_p\left(\delta_{NT}^{-1}\right),
\end{aligned}$$

by Lemma 5 and $T^{-1/2} \left\| \bar{\boldsymbol{\Sigma}}_{kNT,-1} \hat{\mathbf{F}}_{x,-1} \right\| = O_p(1)$. The result in (A.38) is proved by following the same steps. ■

Proof of Lemma 11. We start with (A.41), which is given by

$$\begin{aligned}
&\frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{\hat{F}_{x,-1}} \mathbf{M}_{\hat{F}_x} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} \mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \mathbf{M}_{F_{x,-1}^{0}} \left(\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x^0} \right) \mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \mathbf{V}_{i,-1}^{\prime} \left(\mathbf{M}_{\hat{F}_{x,-1}} - \mathbf{M}_{F_{x,-1}^{0}} \right) \mathbf{M}_{\hat{F}_x} \mathbf{u}_i. \qquad \text{(A.3)}
\end{aligned}$$

Now consider the second term in (A.3). By using $\mathbf{M}_{\hat{F}_x} - \mathbf{M}_{F_x} = -\left(\frac{\hat{\mathbf{F}}_x \hat{\mathbf{F}}_x^{\prime}}{T} - \mathbf{P}_{F_x^0} \right)$, and adding and subtracting terms, we get

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}}\left(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x}\right)\mathbf{u}_i\\
&=-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\left(\frac{\hat{\mathbf{F}}_x\hat{\mathbf{F}}_x'}{T}-\mathbf{P}_{F_x^0}\right)\mathbf{u}_i\\
&=-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\mathbf{G}'\mathbf{F}_x^{0\prime}\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{F}_x^0}{T}\mathbf{G}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i\\
&-\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{F}_x^0}{T}\left[\mathbf{G}\mathbf{G}'-\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right]\mathbf{F}_x^{0\prime}\mathbf{u}_i\\
&=-\left(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3+\mathbf{e}_4\right),
\end{aligned}
$$

$$
\begin{aligned}
&|\mathbf{e}_2|\\
&\leq\sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\right\|\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|\\
&\leq\sqrt{NT}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\right\|\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|\\
&+\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{F}_{x,-1}^0}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\right\|\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|\\
&=\sqrt{NT}O_p\left(\delta_{NT}^{-4}\right),
\end{aligned}
$$

by $\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|=O_p\left(\delta_{NT}^{-2}\right)$, (A.20) and (A.23).

$$
\begin{aligned}
|\mathbf{e}_3|&\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{F}_x^0}{\sqrt{T}}\right\|\|\mathbf{G}\|\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|\\
&=\sqrt{N}O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

by using again $\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)'\mathbf{u}_i}{T}\right\|=O_p\left(\delta_{NT}^{-2}\right)$.

$$
\begin{aligned}
|\mathbf{e}_4|&\leq\sqrt{N}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}}\mathbf{F}_x^0}{\sqrt{T}}\right\|\left\|\mathbf{G}\mathbf{G}'-\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\|\\
&=\sqrt{N}O_p\left(\delta_{NT}^{-2}\right),
\end{aligned}
$$

by (A.28). Consider now $\mathbf{e}_1$ which can be written as

$$
\begin{aligned}
\mathbf{e}_1 &= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)}{T} \mathbf{G}' \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \mathbf{G} \mathbf{G}' \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \left[\mathbf{G} \mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right] \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&= \mathbf{a}_1 + \mathbf{a}_2.
\end{aligned}
$$

Start with $\mathbf{a}_2$ which is given by

$$
\begin{aligned}
|\mathbf{a}_2| &\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \right\| \left\| \mathbf{G} \mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_x^{0\prime} \mathbf{u}_i}{T} \right\| \\
&\leq \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_{i,-1}' \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \right\| \left\| \mathbf{G} \mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}} \right\| \left\| \frac{\mathbf{u}_i}{\sqrt{T}} \right\| \\
&+ \sqrt{NT} \frac{1}{N} \sum_{i=1}^{N} \left\| \frac{\mathbf{V}_{i,-1}' \mathbf{F}_{x,-1}^0}{\sqrt{T}} \right\| \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \left(\hat{\mathbf{F}}_x - \mathbf{F}_x^0 \mathbf{G} \right)}{T} \right\| \\
&\times \left\| \mathbf{G} \mathbf{G}' - \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}} \right\| \left\| \frac{\mathbf{u}_i}{\sqrt{T}} \right\| \\
&= \sqrt{NT} O_p \left(\delta_{NT}^{-4} \right),
\end{aligned}
$$

by (A.20), (A.23) and (A.28). As for $\mathbf{a}_1$ we have

$$
\begin{aligned}
\mathbf{a}_1 &= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \left(\hat{\mathbf{F}}_x \mathbf{G}^{-1} - \mathbf{F}_x^0 \right)}{T} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{1}{NT} \sum_{\ell=1}^{k} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \mathbf{F}_x^0}{T} \boldsymbol{\gamma}_{\ell j}^0 \mathbf{v}_{\ell j}' \hat{\mathbf{F}}_x \hat{\mathbf{Q}} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{1}{NT} \sum_{\ell=1}^{k} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell j}}{T} \boldsymbol{\gamma}_{\ell j}^{0\prime} \mathbf{F}_x^{0\prime} \hat{\mathbf{F}}_x \hat{\mathbf{Q}} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}} \sum_{i=1}^{N} \frac{1}{NT} \sum_{\ell=1}^{k} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{M}_{F_{x,-1}^0} \mathbf{v}_{\ell j}}{T} \mathbf{v}_{\ell j}' \hat{\mathbf{F}}_x \hat{\mathbf{Q}} \left(\frac{\mathbf{F}_x^{0\prime} \mathbf{F}_x^0}{T} \right)^{-1} \mathbf{F}_x^{0\prime} \mathbf{u}_i \\
&= \mathbf{d}_1 + \mathbf{d}_2 + \mathbf{d}_3,
\end{aligned}
$$

$$
\begin{aligned}
&|\mathbf{d}_1| \\
&\le \frac{1}{N}\sum_{i=1}^{N}\left\| \frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{F}_x^0}{\sqrt{T}} \left(\frac{1}{\sqrt{N}}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\gamma_{\ell j}^0\frac{\mathbf{v}_{\ell j}'\hat{\mathbf{F}}_x}{T}\right)\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\right\| \\
&\le \frac{1}{N}\sum_{i=1}^{N}\left\| \frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{F}_x^0}{\sqrt{T}}\right\| \left\|\frac{1}{\sqrt{N}}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\gamma_{\ell j}^0\frac{\mathbf{v}_{\ell j}'\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\right\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{\sqrt{T}}\right\| \\
&+\frac{1}{\sqrt{T}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\left\| \frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{F}_x^0}{\sqrt{T}}\right\|\left\|\frac{1}{\sqrt{N}}\sum_{j=1}^{N}\gamma_{\ell j}^0\frac{\mathbf{v}_{\ell j}'\mathbf{F}_x^0}{\sqrt{T}}\right\|\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\delta_{NT}^{-2}\right)+O_p\left(N^{-1/2}\right)+O_p\left(T^{-1/2}\right)=O_p\left(\delta_{NT}^{-1}\right).
\end{aligned}
$$

by (A.25).

$$
\begin{aligned}
\mathbf{d}_2 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\gamma_{\ell j}^{0\prime}\frac{\mathbf{F}_x^{0\prime}\hat{\mathbf{F}}_x}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\gamma_{\ell j}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{V}_j}{T}\boldsymbol{\Gamma}_{xj}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}.
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{d}_3 &= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\mathbf{v}_{\ell j}'\hat{\mathbf{F}}_x\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\mathbf{v}_{\ell j}'\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&+ \frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\mathbf{v}_{\ell j}'\mathbf{F}_x^0\mathbf{G}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \mathbf{c}_1+\mathbf{c}_2,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{c}_1 &= \frac{1}{\sqrt{NT}}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\mathbf{v}_{\ell j}'\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&- \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&= \mathbf{b}_1+\mathbf{b}_2.
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{b}_1| &\le \sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\right\| \\
&\le \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'}{\sqrt{T}}\right\|\left\|\mathbf{M}_{F_{x,-1}^0}\right\|\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{\sqrt{T}}\right\|\left\|\hat{\mathbf{Q}}\right\| \\
&\times\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\delta_{NT}^{-1}\right),
\end{aligned}
$$

by (A.17).

$$
\begin{aligned}
&|\mathbf{b}_2| \\
&\le \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\right\| \\
&\le \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)}{T}\right\|\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{\sqrt{T}}\right\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \\
&+ \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{F}_{x,-1}^0}{\sqrt{T}}\right\|\left\|\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_{x,-1}^{0\prime}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)}{\sqrt{T}}\right\| \\
&\times\left\|\frac{\left(\hat{\mathbf{F}}_x-\mathbf{F}_x^0\mathbf{G}\right)}{\sqrt{T}}\right\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \\
&= O_p\left(\delta_{NT}^{-1}\right).
\end{aligned}
$$

by (A.12), (A.13) and (A.17).

$$
\begin{aligned}
\mathbf{c}_2 &= \frac{1}{\sqrt{NT}}\frac{1}{NT}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{v}_{\ell j}}{T}\mathbf{v}_{\ell j}'\mathbf{F}_x^0\mathbf{G}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\mathbf{F}_x^{0\prime}\mathbf{u}_i \\
&= \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{F}_x^0\mathbf{G}}{\sqrt{T}}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&- \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)\mathbf{F}_x^0\mathbf{G}}{T}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&= \mathbf{b}_1+\mathbf{b}_2.
\end{aligned}
$$

$$
\begin{aligned}
&|\mathbf{b}_1| \\
&\leq \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{F}_x^0\mathbf{G}}{\sqrt{T}}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\right\| \\
&\leq \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{F}_x^0\mathbf{G}}{\sqrt{T}}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\sum_{r=1}^{m_x}\frac{\mathbf{F}_x^{0\prime}\mathbf{f}_{x,r}^0}{T}\gamma_{ri}^0\right\| \\
&+ \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{F}_x^0\mathbf{G}}{\sqrt{T}}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\sum_{r=1}^{m_y}\frac{\mathbf{F}_x^{0\prime}\mathbf{f}_{y,r}^0}{T}\lambda_{ri}^0\right\| \\
&+ \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\bar{\boldsymbol{\Sigma}}_{kNT}\mathbf{F}_x^0\mathbf{G}}{\sqrt{T}}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\boldsymbol{\varepsilon}_i}{T}\right\| \\
&\leq \frac{1}{\sqrt{N}}\sqrt{\frac{N}{T}}\sum_{r=1}^{m_x}\left(\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\gamma_{ri}^0\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\right\|\left\|\frac{\mathbf{f}_{x,r}^0}{\sqrt{T}}\right\|\right)\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left\|\frac{\mathbf{F}_x^0}{\sqrt{T}}\right\|^2\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\| \\
&+ \frac{1}{\sqrt{N}}\sqrt{\frac{N}{T}}\sum_{r=1}^{m_y}\left(\left\|\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\lambda_{ri}^0\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\right\|\left\|\frac{\mathbf{f}_{y,r}^0}{\sqrt{T}}\right\|\right)\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left\|\frac{\mathbf{F}_x^0}{\sqrt{T}}\right\|^2\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\| \\
&+ \frac{1}{\sqrt{T}}\sqrt{\frac{N}{T}}\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}\boldsymbol{\varepsilon}_i}{\sqrt{T}}\right\|\right)\lambda_{\max}\left(\bar{\boldsymbol{\Sigma}}_{kNT}\right)\left\|\frac{\mathbf{F}_x^0}{\sqrt{T}}\right\|\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\|\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\| \\
&= O_p\left(N^{-1/2}\right) + O_p\left(T^{-1/2}\right).
\end{aligned}
$$

$$
\begin{aligned}
|\mathbf{b}_2| &\leq \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)\mathbf{F}_x^0}{T}\mathbf{G}\hat{\mathbf{Q}}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T}\right\| \\
&\leq \frac{1}{\sqrt{T}}\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\mathbf{V}_{i,-1}'}{\sqrt{T}}\right\|\left\|\mathbf{M}_{F_{x,-1}^0}\right\|\left\|\frac{\sqrt{N}\left(\boldsymbol{\Sigma}_{kNT}-\bar{\boldsymbol{\Sigma}}_{kNT}\right)\mathbf{F}_x^0}{\sqrt{T}}\right\|\|\mathbf{G}\|\left\|\hat{\mathbf{Q}}\right\| \\
&\times\left\|\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\right\|\left\|\frac{\mathbf{F}_x^{0\prime}}{\sqrt{T}}\right\|\left\|\frac{\mathbf{u}_i}{\sqrt{T}}\right\| \\
&= O_p\left(T^{-1/2}\right),
\end{aligned}
$$

by (A.10). By adding everything together we therefore have

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})\mathbf{u}_i \\
&= \sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^0}\mathbf{V}_j}{T}\boldsymbol{\Gamma}_{xj}^{0\prime}\left(\boldsymbol{\Upsilon}_{kN}^0\right)^{-1}\left(\frac{\mathbf{F}_x^{0\prime}\mathbf{F}_x^0}{T}\right)^{-1}\frac{\mathbf{F}_x^{0\prime}\mathbf{u}_i}{T} \\
&+ \sqrt{N}O_p\left(\delta_{NT}^{-2}\right) + O_p\left(\delta_{NT}^{-1}\right).
\end{aligned}
$$

Next consider the third term in (A.3). By following the same steps as in the discussion above and by replacing $\mathbf{M}_{F_{x,-1}^0}\mathbf{V}_{i,-1}$ with $\mathbf{V}_{i,-1}$, $(\mathbf{M}_{\hat{F}_x}-\mathbf{M}_{F_x^0})$ with $(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^0})$ and $\mathbf{u}_i$ with $\mathbf{M}_{\hat{F}_x}\mathbf{u}_i$, we get

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\left(\mathbf{M}_{\hat{F}_{x,-1}}-\mathbf{M}_{F_{x,-1}^{0}}\right)\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}}{T}\\
&+\sqrt{N}O_{p}\left(\delta_{NT}^{-2}\right)+O_{p}\left(\delta_{NT}^{-1}\right)\\
&=\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\left(\mathbf{M}_{\hat{F}_{x}}-\mathbf{M}_{F_{x}^{0}}\right)\mathbf{u}_{i}}{T}\\
&+\sqrt{N}O_{p}\left(\delta_{NT}^{-2}\right)+O_{p}\left(\delta_{NT}^{-1}\right)\\
&=\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}\\
&+\sqrt{N}O_{p}\left(\delta_{NT}^{-2}\right)+O_{p}\left(\delta_{NT}^{-1}\right),
\end{aligned}
$$

where the last equality is by Lemma 5. Collecting the results together, we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{\hat{F}_{x,-1}}\mathbf{M}_{\hat{F}_{x}}\mathbf{u}_{i}\\
&=\frac{1}{\sqrt{NT}}\sum_{i=1}^{N}\mathbf{V}_{i,-1}'\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j,-1}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{F}_{x,-1}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x,-1}^{0\prime}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}}{T}\\
&+\sqrt{\frac{T}{N}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\mathbf{V}_{i,-1}'\mathbf{V}_{j}}{T}\mathbf{\Gamma}_{xj}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{\mathbf{F}_{x}^{0\prime}\mathbf{F}_{x}^{0}}{T}\right)^{-1}\frac{\mathbf{F}_{x}^{0\prime}\mathbf{u}_{i}}{T}\\
&+\sqrt{N}O_{p}\left(\delta_{NT}^{-2}\right)+O_{p}\left(\delta_{NT}^{-1}\right),
\end{aligned}
$$

which is the required result in (A.41). The result in (A.40) can be shown by following the similar steps as discussed above. ■

Proof of Lemma 12. We first show (A.44), (A.45) and (A.46). Under Assumptions 1-3, 4(i)(ii), 5(i)(ii)(iv), as $(N,T)\xrightarrow{j}\infty$ such that $N/T\to c$ with $0<c<\infty$, we have

$$
\begin{aligned}
&\left\|\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\mathbf{\Gamma}_{xj}^{0}\mathbf{V}_{j,-1}'\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{u}_{i}\right\|\\
&=\left\|\frac{1}{\sqrt{NT}}\frac{1}{N}\sum_{\ell=1}^{k}\sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\boldsymbol{\gamma}_{\ell j}^{0}\mathbf{v}_{\ell j,-1}'\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}(\mathbf{F}_{x}^{0}\boldsymbol{\gamma}_{i}^{0}+\mathbf{F}_{y}^{0}\boldsymbol{\lambda}_{i}^{0}+\boldsymbol{\varepsilon}_{i})\right\|\\
&\leq\frac{1}{\sqrt{N}}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\boldsymbol{\gamma}_{\ell j}^{0}\mathbf{v}_{\ell j,-1}'\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\mathbf{F}_{y}^{0}\right)\boldsymbol{\lambda}_{i}^{0}\right\|\\
&+\frac{1}{\sqrt{N}}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\mathbf{\Gamma}_{xi}^{0\prime}\left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1}\left(\frac{1}{\sqrt{NT}}\sum_{\ell=1}^{k}\sum_{j=1}^{N}\boldsymbol{\gamma}_{\ell j}^{0}\mathbf{v}_{\ell j,-1}'\mathbf{M}_{F_{x,-1}^{0}}\mathbf{M}_{F_{x}^{0}}\boldsymbol{\varepsilon}_{i}\right)\right\|\\
&=O_{p}\left(N^{-1/2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \frac{1}{NT} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} \mathbf{u}_i \right\| \\
&= \left\| \frac{1}{NT} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} (\mathbf{F}_x^0 \boldsymbol{\gamma}_i^0 + \mathbf{F}_y^0 \boldsymbol{\lambda}_i^0 + \boldsymbol{\varepsilon}_i) \right\| \\
&\leq \frac{1}{\sqrt{N}} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} \mathbf{F}_y^0}{T} \boldsymbol{\lambda}_i^0 \right\| \\
&+ \frac{1}{\sqrt{N}} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} \boldsymbol{\varepsilon}_i}{T} \right\| \\
&= O_p\left(N^{-1/2}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T} \right\| \\
&= \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} (\mathbf{F}_x^0 \boldsymbol{\gamma}_i^0 + \mathbf{F}_y^0 \boldsymbol{\lambda}_i^0 + \boldsymbol{\varepsilon}_i)}{T} \right\| \\
&\leq \left\| \frac{1}{N} \sum_{\ell=1}^{k} \sum_{r=1}^{m_y} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{f}_{y,r}^0}{T} \lambda_{ri}^0 \right\| \\
&+ \left\| \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \boldsymbol{\varepsilon}_i}{T} \right\| \\
&\leq \sum_{r=1}^{m_y} \sum_{\ell=1}^{k} \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ri}^0 \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \left\| \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{f}_{y,r}^0}{T} \right\| \\
&+ \sqrt{\frac{N}{T}} \frac{1}{N} \sum_{\ell=1}^{k} \sum_{i=1}^{N} \left\| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}^{\prime} \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \left\| \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \right\| \left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \right\| \left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\| \\
&= O_p(N^{-1/2}) + O_p(T^{-1/2}),
\end{aligned}
$$

since $\left\| \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \right\|$, $\left\| \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \right\|$, $\left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{f}_{y,r}^0}{T} \right\|$ and $\left\| \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \boldsymbol{\varepsilon}_i}{\sqrt{T}} \right\|$ are $O_p(1)$ and also

$$
\begin{aligned}
&\left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ri}^{0} \frac{\tilde{\mathbf{V}}_{i,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&\leq \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_{ri}^{0} \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| + \left\| \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \lambda_{ri}^{0} \mathbf{\Gamma}_{xi}^{0\prime} (\mathbf{\Upsilon}_{kN}^{0})^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&\leq \frac{1}{\sqrt{N}} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{ri}^{0} \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell i,-1}}{T} \gamma_{\ell i}^{0\prime} \right\| + \frac{1}{\sqrt{T}} \left\| \frac{1}{N} \sum_{i=1}^{N} \sum_{j\neq i}^{N} \lambda_{ri}^{0} \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell j,-1}}{\sqrt{T}} \gamma_{\ell j}^{0\prime} \right\| \\
&+ \frac{1}{\sqrt{N}} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{ri}^{0} \mathbf{\Gamma}_{xi}^{0\prime} \right\| \left\| (\mathbf{\Upsilon}_{kN}^{0})^{-1} \right\| \left\| \frac{1}{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xj}^{0} \frac{\mathbf{V}_{j,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&+ \frac{1}{\sqrt{NT}} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \lambda_{ri}^{0} \mathbf{\Gamma}_{xi}^{0\prime} \right\| \left\| (\mathbf{\Upsilon}_{kN}^{0})^{-1} \right\| \left\| \frac{1}{N} \sum_{j=1}^{N} \sum_{n\neq j}^{N} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{v}_{\ell j,-1}}{\sqrt{T}} \gamma_{\ell j}^{0\prime} \right\| \\
&= O_p(N^{-1/2}) + O_p(T^{-1/2}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&\leq \left\| \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| + \left\| \frac{1}{\sqrt{N}} \frac{1}{N} \sum_{j=1}^{N} \sum_{n=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} (\mathbf{\Upsilon}_{kN}^{0})^{-1} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&\leq \frac{1}{\sqrt{N}} \left\| \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell i,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| + \frac{1}{\sqrt{T}} \left\| \frac{1}{\sqrt{N}} \sum_{j\neq i}^{N} \frac{\mathbf{V}_{i,-1}' \mathbf{v}_{\ell j,-1}}{\sqrt{T}} \gamma_{\ell j}^{0\prime} \right\| \\
&+ \frac{1}{\sqrt{N}} \left\| \mathbf{\Gamma}_{xi}^{0\prime} \right\| \left\| (\mathbf{\Upsilon}_{kN}^{0})^{-1} \right\| \left\| \frac{1}{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xj}^{0} \frac{\mathbf{V}_{j,-1}' \mathbf{v}_{\ell j,-1}}{T} \gamma_{\ell j}^{0\prime} \right\| \\
&+ \frac{1}{\sqrt{NT}} \left\| \mathbf{\Gamma}_{xi}^{0\prime} \right\| \left\| (\mathbf{\Upsilon}_{kN}^{0})^{-1} \right\| \left\| \frac{1}{N} \sum_{j=1}^{N} \sum_{n\neq j}^{N} \mathbf{\Gamma}_{xn}^{0} \frac{\mathbf{V}_{n,-1}' \mathbf{v}_{\ell j,-1}}{\sqrt{T}} \gamma_{\ell j}^{0\prime} \right\| \\
&= O_p(N^{-1/2}) + O_p(T^{-1/2}),
\end{aligned}
$$

for every $i = 1, 2, ..., N$. So, we have:

$$
\frac{1}{\sqrt{NT}} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \mathbf{\Gamma}_{xj}^{0} \mathbf{V}_{j,-1}' \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_{x}^{0}} \mathbf{u}_i = o_p(1),
$$

$$
\frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\tilde{\mathbf{V}}_{i,-1}' \mathbf{V}_{j,-1}}{T} \mathbf{\Gamma}_{xj}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{M}_{F_x^0} \mathbf{u}_i}{T} = o_p(1),
$$

$$
\frac{1}{NT} \sum_{i=1}^{N} \mathbf{\Gamma}_{xi}^{0\prime} \left(\mathbf{\Upsilon}_{kN}^{0}\right)^{-1} \left(\frac{\mathbf{F}_{x,-1}^{0\prime} \mathbf{F}_{x,-1}^{0}}{T} \right)^{-1} \mathbf{F}_{x,-1}^{0\prime} \bar{\mathbf{\Sigma}}_{kNT,-1} \mathbf{M}_{F_{x,-1}^{0}} \mathbf{M}_{F_x^0} \mathbf{u}_i = o_p(1),
$$

as required. The remaining results in (A.42) and (A.43) are shown in a similar way. ■

Proof of Lemma 13. This is derived in a similar manner based on the proofs of Lemmas A.11 and A.12, provided in Bai (2009, p.16-19 of the supplement). ■

Proof of Lemma 14. The proof is obtained in the same manner as that of Lemma 13. ■

Proof of Lemma 15. Using the identity $\hat{\mathbf{u}}_i = \mathbf{u}_i - \mathbf{W}_i\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)$ we have

$$\begin{aligned}
\frac{1}{NT}\sum_{i=1}^{N}\hat{\boldsymbol{\xi}}_{\hat{F}iT}\hat{\boldsymbol{\xi}}'_{\hat{F}iT} &= \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}'_{\hat{F}iT} \\
&\quad - \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)'\mathbf{W}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i \\
&\quad - \frac{1}{NT}\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{W}_i\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\boldsymbol{\xi}'_{\hat{F}iT} \\
&\quad + \frac{1}{NT}\sum_{i=1}^{N}\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{W}_i\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right)'\mathbf{W}'_i\mathbf{M}_{\hat{F}_x}\hat{\mathbf{Z}}_i \\
&= \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}'_{\hat{F}iT} - \mathbf{E}_1 - \mathbf{E}_2 + \mathbf{E}_3.
\end{aligned}$$

We have

$$\|\mathbf{E}_1\| \leq \sqrt{T}\left\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right\|\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{W}_i}{T}\right\|\frac{\left\|\boldsymbol{\xi}'_{\hat{F}iT}\right\|}{\sqrt{T}} = O_p\left(\frac{1}{\sqrt{N}}\right),$$

since $\left\|\frac{\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{W}_i}{T}\right\| \leq \left\|\frac{\hat{\mathbf{Z}}'_i\mathbf{W}_i}{T}\right\| + \left\|\frac{\hat{\mathbf{Z}}'_i\hat{\mathbf{F}}_x}{T}\right\|\left\|\frac{\hat{\mathbf{F}}'_x\mathbf{W}_i}{T}\right\| = O_p(1)$. Similarly, $\|\mathbf{E}_2\| = O_p\left(\frac{1}{\sqrt{N}}\right)$. Also

$$\|\mathbf{E}_3\| \leq T\left\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right\|^2\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{\hat{\mathbf{Z}}'_i\mathbf{M}_{\hat{F}_x}\mathbf{W}_i}{T}\right\|^2 = O_p\left(\frac{1}{N}\right).$$

Thus,

$$\frac{1}{NT}\sum_{i=1}^{N}\hat{\boldsymbol{\xi}}_{\hat{F}iT}\hat{\boldsymbol{\xi}}'_{\hat{F}iT} = \frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}'_{\hat{F}iT} + o_p(1).$$

as required. ■

Proof of Lemma 16. The proof is obtained by replacing $\hat{\boldsymbol{\xi}}_{\hat{F}iT}$, $\boldsymbol{\xi}_{\hat{F}iT}$, $\mathbf{Z}_i$, $\hat{\mathbf{Z}}_i$ by $\hat{\tilde{\boldsymbol{\xi}}}_{\hat{F}iT}$, $\tilde{\boldsymbol{\xi}}_{\hat{F}iT}$, $\hat{\tilde{\mathbf{Z}}}_i$ and $\tilde{\mathbf{Z}}_i$, respectively, and following the same steps as in the proof of Lemma 15. ■

Proof of Lemma 17. By Lemma 15 and Proposition 1 we have

$$\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}'_{\hat{F}iT} = \frac{1}{NT}\sum_{i=1}^{N}\mathbf{Z}'_i\mathbf{M}_{F_x}\mathbf{u}_i\mathbf{u}'_i\mathbf{M}_{F_x}\mathbf{Z}_i + o_p(1).$$

Noting that $E\left(\mathbf{Z}'_j\mathbf{M}_{F_x}\mathbf{u}_j\mathbf{u}'_i\mathbf{M}_{F_x}\mathbf{Z}_i\right) = \mathbf{0}$ for all $i \neq j$ and using Lemma 1, $\frac{1}{NT}\sum_{i=1}^{N}\mathbf{Z}'_i\mathbf{M}_{F_x}\mathbf{u}_i\mathbf{u}'_i\mathbf{M}_{F_x}\mathbf{Z}_i \xrightarrow{p} \lim_{N,T\to\infty}\frac{1}{NT}\sum_{i=1}^{N}E\left(\mathbf{Z}'_i\mathbf{M}_{F_x}\mathbf{u}_i\mathbf{u}'_i\mathbf{M}_{F_x}\mathbf{Z}_i\right)$, which yields $\frac{1}{NT}\sum_{i=1}^{N}\boldsymbol{\xi}_{\hat{F}iT}\boldsymbol{\xi}'_{\hat{F}iT} - \boldsymbol{\Omega} = o_p(1)$ when $(N,T) \to \infty$ jointly, as required. Also it is easily seen that the same result hold for the uncentered version, namely, $\frac{1}{NT}\sum_{i=1}^{N}\left(\boldsymbol{\xi}_{\hat{F}iT} + \frac{\mathbf{b}}{\sqrt{NT}}\right)\left(\boldsymbol{\xi}_{\hat{F}iT} + \frac{\mathbf{b}}{\sqrt{NT}}\right)' - \boldsymbol{\Omega} = o_p(1)$. ■

Proof of Lemma 18. Suppose that Assumption 6 holds true. It makes it clear that for the uncentered version the following result holds $\frac{1}{NT}\sum_{i=1}^{N}\left(\tilde{\boldsymbol{\xi}}_{\hat{F}iT} + \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}\right)\left(\tilde{\boldsymbol{\xi}}_{\hat{F}iT} + \frac{\tilde{\mathbf{b}}}{\sqrt{NT}}\right)' - \tilde{\boldsymbol{\Omega}} = o_p(1)$. ■

Proof of Lemma 19. First of all, by Lemma 6, $\hat{\mathbf{A}}_{NT} - \frac{1}{N}\sum_{i=1}^{N}\mathbf{A}_{i,T} = o_p(1)$ and $\hat{\mathbf{B}}_{NT} - \frac{1}{N}\sum_{i=1}^{N}\mathbf{B}_{i,T} = o_p(1)$, then applying Lemma 1 yields the required results. ■