Name Your Own Price at Priceline.com: Strategic Bidding and Lockout Periods

Chia-Hui Chen*

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Abstract

A buyer suggests prices to $N$ sellers in a time period and buys from the seller who accepts first. The number of bidding chances is determined by how frequently the buyer can make an offer. We show that with no limit on the frequency and no discounting, the price path is either kept low most of the time with big jumps at the end, or increasing gradually over time. Which type of path occurs in equilibrium depends on the buyer’s trade-off between committing to a price ceiling and finely screening the sellers’ costs. With discounting, restricting the frequency mitigates the delay caused by the hesitation to raise bids in the first type of path and thus might benefit the buyer. Our analysis provides insight into the existence of complex contractual procedures in bargaining, the lockout period design at Priceline.com, and the prevalence of last-minute deals on the Internet.

1 Introduction

Priceline.com, known for its Name Your Own Price (NYOP) system, is a website devoted to helping travelers obtain discount rates for travel-related items such as airline tickets and hotel stays. The NYOP mechanism works as follows. First, a customer enters a bid that specifies the general characteristics of the desired item (travel dates, location, hotel rating, etc.) and the price that she is willing to pay.

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Next, Priceline.com either communicates the customer’s bid to participating sellers or accesses their private database to determine whether Priceline.com can satisfy the customer’s specified terms and the bid price. If a seller accepts the bid, the offer cannot be cancelled. If no seller accepts the bid, the customer can rebid either by changing the desired specifications or by waiting for a minimum period of time, the lockout period, before submitting a new, higher price offer. The lockout period is 24 hours for a hotel, three days for rental cars, and seven days for an airline ticket. Priceline says in its seller’s guideline that the rule is designed to protect the sellers. Our analysis suggests that the lockout period may often benefit the buyer, because it allows the buyer to commit to fewer rounds of bidding.\(^1\)

To represent the Priceline auction, we use a dynamic model in which a single buyer suggests prices to \(N\) potential sellers for a finite number of rounds. The number of rounds \(T\) determines the length of the lockout period. By letting \(T\) go to infinity, we can also consider the case of no lockout period. For simplicity, we assume that the buyer’s valuation is known. The sellers’ costs are privately known and independently drawn from a common distribution. Employing the model, we characterize the equilibrium price path, the timing of the transaction, and how the lockout period affects the equilibrium outcome and the buyer’s payoff.

The model studied here can be applied to many other economic settings: for example, when a company chooses an outsourcing partner among several external providers, when a government agency procures goods and services among a pool of potential sellers, or when a producer determines which union of performers to hire. In each of these cases, the costs of the sellers are private information, and the buyer bids for the services or goods until a seller accepts. The bargaining process in those situations cannot, of course, last forever. The number of chances that the buyer has to revise the price depends on how long it takes for the buyer to make an offer and sellers to respond. If the buyer needs to go through a complex contractual procedure to make a proposal, the number of chances is reduced. The complexity of the procedure is analogous to the lockout period design in Priceline’s case. Our analysis thus gives insight into why a certain level of complexity is necessary to enhance the welfare involved.

Flipping the model over, we can also consider the case when a seller with an indi-

\(^1\)To make a bid at Priceline, a customer must enter his credit card number, billing address, phone number, and email address. This requirement helps Priceline identify each customer and makes it difficult for a customer to create fake identities.
visible unit makes price offers to several interested buyers, so the seller is confronted with a dilemma similar to the one in a durable goods monopoly. However, there are two differences between our setting and a durable goods monopoly. First, there is a deadline in our environment, so the seller can wait until the end to set a monopoly price and thus, is endowed with some commitment power. Second, the seller cannot produce additional units, so the scarcity of the good causes competition among buyers. Due to the two differences, our model has very different equilibrium paths than those of the Coase conjecture\(^2\) and yields a positive profit for the seller. Real-world applications include an airline or a cruise line selling a seat to travelers, a landlord renting his apartment to tenants, etc. In those cases, there exists a deadline after which the object becomes valueless. Formerly, the seller could only advertise the price of the object in newspapers or on flyers; nowadays, the seller can post the price on his own website or other bulletin websites such as Craigslist. Hence, the price can be updated much more frequently than it used to be. Our result provides clues about how the price paths would look different under the two circumstances and why we see last-minute deals prevalent with the advent of the Internet.\(^3\)

In this paper, we first show that without a lockout period and with no discounting, the equilibrium bidding paths fall into two categories: (i) fully screening and (ii) reaching a price ceiling by jumps. In the case of fully screening, as \(T\) goes to infinity, sellers’ types are almost fully separated through a gradually increasing bid sequence submitted by the buyer. The buyer does not stop raising the bid until a seller accepts or the price reaches the maximum cost level. Therefore, the equilibrium is fully efficient. Case (ii), where the equilibrium path reaches a price ceiling by jumps, is the one of main interest. As \(T\) goes to infinity, although sellers with cost arbitrarily close to the minimum cost level are more and more finely separated, other types of sellers get pooled into a bounded number of cost intervals. In this case, as the buyer keeps her bids close to the price accepted by the minimum-cost seller until the very end, the price pattern is convexly increasing, and most of the trades (if any) are realized at the end.

Along the bidding process, the buyer’s bidding strategy influences the rate at which she learns about the sellers’ valuations. Ideally, the buyer would like to commit to a strategy that optimally reveals this information. If she could do that,\(^3\)

\(^2\)The Coase conjecture asserts that with no constraint on the rate of sales, the monopolist’s price will fall immediately to marginal cost in an infinite-period setting.

\(^3\)This application is related to the revenue management literature, in which the standard assumption is that the buyers are myopic. See Horner and Samuelson (2010) for more references.
she would gradually raise the price to price discriminate among the sellers and stop at the optimal reserve price, much like a Dutch auction, but in reverse. But when commitment is impossible, as we assume, the buyer cannot help but respond to the information revealed by rejections. As a consequence, the buyer cannot implement the optimal bidding path and faces the trade-off between finely screening the sellers’ costs and successfully committing to a price ceiling. If the benefit of having a price ceiling dominates the benefit of screening finely, the buyer wants to bid so that initial bids reveal little information, and to bid seriously only at the end. Then since only a few chances are left, there exist jumps between bids, and the price stops at a level lower than the maximum possible cost. So, the equilibrium path falls into the category that reaches a price ceiling by jumps. The last minute rush will lead to pooling and inefficient outcomes, because many sellers will accept simultaneously and the winner will be determined by lottery.

We also show that without a lockout period, the expected payoff of a buyer is weakly higher than that in a first-price reverse auction (where sellers submit their bids to a buyer) without a reserve price, but lower than that in a first-price reverse auction with the optimal reserve price. Moreover, when the expected payoff is strictly higher than that in a first-price reverse auction without a reserve price, the equilibrium bidding path is convexly increasing.

The lockout period, by reducing the number of bidding rounds, affects the process of information revelation. It makes the buyer bid more aggressively early on, because she does not need to be as concerned about the detrimental effects of learning the sellers’ information while still having many bidding opportunities. This can be especially valuable if the buyer moderately discounts the future, wanting to learn early about bookings. Thus, the lockout period can be advantageous to the buyer because it permits the buyer to commit to fewer rounds of bidding. However, the welfare effects are ambiguous in general. The finding that the lockout period can be valuable is in line with McAdams and Schwarz (2007)’s view that an intermediary can create value by offering a credible commitment device.

Our analysis is relevant to several strands of literatures. One of these is the literature on bargaining. The convexly increasing path characterized in the equilibrium with “a price ceiling reached by jumps” is related to the deadline effect observed in many bargaining processes and experimental evidence (see, e.g., Roth, Murnighan, and Schoumaker (1988)). It is observed that a high percentage of agreements are reached just before the deadline, and the frequency of disagreements is
non-negligible. Among the explanations provided in the literature, the environments and rationales proposed by Hart (1989) and Spier (1992) on strikes and pretrial negotiation are closest to ours. Both consider bilateral sequential bargaining models with one-sided incomplete information and a deadline. In Hart (1989), as in a durable good monopoly setting, both parties discount the future and would like to reach an agreement immediately. Since different types of informed parties value time differently, time behaves as a screen. In that setting, the deadline effect is observed when the time between offers is sufficiently large. But when the time interval approaches zero, the screening instrument is weakened, all settlement occurs immediately (the “Coase conjecture”), and the deadline effect disappears. In Spier (1992), since the plaintiff (the uninformed party) would like to settle as soon as possible while the defendant (the informed party) prefers to pay as late as possible, the total pie does not shrink over time as in Hart (1989). Therefore, time does not screen among different types of informed parties, and the deadline effect persists as the time interval approaches zero. In our model, since the buyer has only one unit of demand, instead of time, what screens among the sellers is the concern that another seller might accept the buyer’s offer first and get the chance to fulfill the trade. The screening instrument is not weakened when the time interval goes to zero, so the deadline effect persists in the limit.

The second related literature is that on the Coase conjecture. The environment studied here is similar to a durable goods monopoly, but with the roles of buyer and seller reversed. In a durable goods monopoly, the seller makes bids. Here the buyer does it. To avoid confusion, call the side that determines the price “the principal” and the other side “the agents”. Much of the durable goods theory focuses on the robustness of Coase’s conjecture and explores the conditions under which the consequence that a monopolist makes no profit does not occur. Kahn (1986) and McAfee and Wiseman (2008) show that with a capacity cost (i.e. an increased cost of increased production speed), the principal has the ability to restrict future sales and thus acquires positive profit. However, the static monopoly profit cannot be reached because the principal cannot avoid trade with all agents eventually. In our setting, the unit demand of the principal is like a capacity constraint, which endows the principal with commitment power and induces competition among agents. Moreover, a deadline exists, so the principal can commit to stopping the trade even though her demand is not fulfilled. Stokey’s (1981) discrete-time model also considers the case with a deadline and shows that the Coase conjecture still holds when the length of
the period shrinks. Her conclusion is different from ours because (i) in our model, as discussed previously, the concern of losing a chance to trade, instead of time, serves as a screen, and (ii) in our model, an agent derives the same utility no matter when the good is received, whereas in Stokey’s model, an agent derives less utility if the trade occurs close to the deadline, and this reduces the inclination to trade at the end. The two features, competition among agents due to the principal’s capacity constraint and an existing deadline, put the principal in a favorable position and allow profit higher than the static monopoly profit.

Lastly, our paper is closely related to Horner and Samuelson (2010). The two papers were developed independently. Both emphasize and characterize the two types of equilibrium paths. Assuming that the agents’ types are uniformly distributed, Horner and Samuelson prove the uniqueness of the equilibrium and explore the relation between the number of buyers and the type of the equilibrium in detail. Our paper shows that the equilibrium paths must be one of the two types for a large class of distributions including the uniform distribution. We also study discounting and discuss the inefficiency caused by late transactions, which is an important feature of the type of equilibrium in which the sellers are not almost fully separated.

The remainder of this paper is organized as follows. Section 2 describes the model and presents an example that motivates our research. Section 3 derives the equilibrium path. Section 4 characterizes the equilibrium bidding behavior. Section 5 incorporates waiting cost into the model to characterize the optimal lockout period for the buyer. Section 6 discusses extensions and concludes.

2 The Model and an Example

There are \( N \geq 2 \) sellers and one buyer in the market. The buyer has one unit of demand for the good provided by the sellers. The buyer’s reservation value for the good is \( v \), which is a common knowledge. Seller \( i \) privately knows his cost \( \theta^i \) to provide the good. Each \( \theta^i \) is independently and identically distributed on \([c, \overline{c}]\), where \( c \geq 0 \) and \( \overline{c} \leq v \), according to a distribution function \( F \). \( F \) is smooth, i.e. of class \( C^\infty \), and has a density \( f \) with full support. We further assume that \( x + \frac{F(x)}{f(x)} \) strictly increases in \( x \). A buyer’s payoff is \( v - b \), where \( b \) is his payment to the seller, if he gets the object, and 0 otherwise. All the players are risk neutral.

There is one platform allowing the buyer to submit a bid price to the sellers. The buyer is allowed to propose the price for \( T \) rounds. In round \( t \), the buyer announces
the price, and the sellers decide whether to accept or not. If $n$ sellers accept the price, each of them gets the chance to provide the good with probability $\frac{1}{n}$, and the game stops. If no seller accepts and $t < T$, the process proceeds to the next round, and the buyer submits a new price. If $t = T$, then the market closes and no further transaction can happen.

2.1 Equilibrium concept

The equilibrium concept used in this paper is the perfect Bayesian equilibrium. Only symmetric pure strategy equilibria are considered. Let $p_t$ be the price that the buyer offers the sellers in round $t$, and denote by $h_t = (p_1, p_2, \cdots, p_t)$ the history of the prices submitted by the buyer in the first $t$ rounds.

The buyer’s strategy is a set of functions $\{b_t(h_{t-1})\}_{t=1}^{T}$, where $b_t(h_{t-1})$ is the price that the buyer would submit in round $t$ given the price history $h_{t-1}$ and the fact that no seller accepts in the first $t - 1$ rounds. We prove in Proposition 5 in Appendix A that in equilibrium, if a seller with cost $x$ accepts in round $t$, a seller with cost $x_0 < x$ also accepts in round $t$. Therefore, a seller’s strategy can be summarized by a set of cutoff values of cost, $\{x_t(h_t)\}_{t=1}^{T}$, where $x_t(h_t)$ means that in round $t$, given $h_t$, a seller accepts the buyer’s offer if and only if his cost is less than or equal to $x_t(h_t)$. When no seller accepts in the first $t - 1$ rounds, let $y_t(h_{t-1})$ be the greatest lower bound of a seller’s cost believed by the other players given history $h_{t-1}$. Denote by $u^{0}_t(b, x | h_{t-1}, y_t(h_{t-1}))$ the buyer’s expected utility, and $u^{i}_t(b, x^i, x^j | h_t, \theta^i, y_t(h_{t-1}))$ seller $i$’s expected utility, where $x^{-i}$ is the other sellers’ strategy, $x^i$ is seller $i$’s strategy, and $\theta^i$ is the realization of seller $i$’s cost.

**Definition 1** A symmetric equilibrium is a $(b, y, x)$ that satisfies

(a) $y_{t+1}(h_t) = \max \{x_t(h_t), x_{t-1}(h_{t-1}), \cdots, x_1(h_1)\}$, \forall t, h_t, and

(b) $u^{0}_t(b, x | h_{t-1}, y_t(h_{t-1})) \geq u^{0}_t(b', x | h_{t-1}, y_t(h_{t-1}))$ and

$u^{i}_t(b, x, x | h_t, \theta^i, y_t(h_{t-1})) \geq u^{i}_t(b, x, x' | h_t, \theta^i, y_t(h_{t-1}))$, \forall b', x', t, h_t, h_{t-1}.

Condition (a) implies that in round $t$, the other players believe that seller $i$ is of some type that would not have accepted any price occurring on the historical path. Condition (b) means that players cannot do better by deviating from the equilibrium strategy.

$x^{-i}$ is a tuple consisting of the other sellers’ strategies. But when the other sellers use the same strategies, $x^{-i}$ can be represented by a single function.
Table 1: Equilibrium outcomes for $T = 1, 2, 3, 4, 5$.

<table>
<thead>
<tr>
<th>Buyer’s Payoff</th>
<th>$E(\tau)$</th>
<th>$x_{T-4}$</th>
<th>$x_{T-3}$</th>
<th>$x_{T-2}$</th>
<th>$x_{T-1}$</th>
<th>$x_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 1$</td>
<td>0.38490</td>
<td>0</td>
<td>0</td>
<td>0.4225</td>
<td>(0.4225)</td>
<td>0.4225</td>
</tr>
<tr>
<td>$T = 2$</td>
<td>0.40024</td>
<td>0.2972</td>
<td>0.1709</td>
<td>0.5212</td>
<td>(0.5212)</td>
<td>0.5212</td>
</tr>
<tr>
<td>$T = 3$</td>
<td>0.40111</td>
<td>0.4563</td>
<td>0.0597</td>
<td>0.2165</td>
<td>0.5475</td>
<td>(0.5475)</td>
</tr>
<tr>
<td>$T = 4$</td>
<td>0.40115</td>
<td>0.5826</td>
<td>0.0154</td>
<td>0.2165</td>
<td>0.5475</td>
<td>(0.5475)</td>
</tr>
<tr>
<td>$T = 5$</td>
<td>0.40115</td>
<td>0.6626</td>
<td>0.0154</td>
<td>0.2165</td>
<td>0.5475</td>
<td>(0.5475)</td>
</tr>
</tbody>
</table>

2.2 An Example

To obtain some intuition for the results developed below, we consider the example where $N = 2$, $v = 1$, and $F$ is a uniform distribution on $[0, 1]$. In this setting, in a first-price reverse auction, the buyer’s expected payoff is $\frac{1}{3}$; and if the buyer is allowed to set a reserve price, then by setting the reserve price at $\frac{1}{2}$, the buyer gets $\frac{5}{12}$, the same as the expected payoff realized in Myerson’s optimal mechanism.

In Table 1, we show the equilibrium paths of the cutoff $x_t$ and the price $b_t$, and the expected buyer’s payoffs when the number of rounds $T = 1, 2, 3, 4,$ and $5$. We assume that the game begins at time $0$ and ends at time $1$. If the buyer’s bid in the $t$th round is accepted, the transaction occurs at $(1 - t)$. Column $E(\tau)$ lists the expected transaction time conditional on the transaction occurring. There are several points worth noticing:

1. The buyer’s payoff increases in $T$; but the increment becomes smaller and smaller. Therefore, the profit of having one more bidding chance shrinks as $T$ increases. (proved in Proposition 3)

2. The cost cutoff in round $T - t$, $x_{T-t}$, increases in $T$ and converges when $T$ goes to infinity. (proved in Proposition 2)

3. Given $T$, the cutoff path $x_t$ and the bidding path $b_t$ are increasing. But with larger $T$, the rate of increase is small in the first few rounds, and big jumps

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5The numerical results have four digits of precision, so the numbers in the table might not be accurate enough to show small differences.
occur in the last few rounds. This represents one type of the equilibrium path. If we consider another example with $v = 1.5$, then the sequence of $x_t$ when $T = 5$ becomes $\{0.1318, 0.2823, 0.4600, 0.6678, 0.9125\}$. This represents another type of the equilibrium path along which $x_t$ increases steadily over time. (characterized in Theorem 1)

4. The payoff is lower than the payoff in a reverse auction with the optimal reserve price for all values of $T$; and when $T$ is large enough (in this example, when $T \geq 1$), the payoff is higher than the payoff in a reverse auction with no reserve price. (proved in Theorem 2)

5. In equilibrium the buyer does not get the object only if both sellers’ costs are above $x_T$. Therefore, the probability of the buyer getting the object increases in $T$, but the increment shrinks as $T$ increases. From the table, we see that when $T$ increases from 3 to 4, and to 5, neither the buyer’s payoff nor the probability that the buyer gets the object increases much. However, the expected transaction time is much later. This fact suggests that if the buyer has waiting cost and prefers earlier transactions, having fewer rounds might be good for him. The analysis in Section 5 confirms the conjecture.

3 Derivation of the Equilibrium Path

In this section, we derive the equilibrium path by solving a series of programs backward. The complete construction of the equilibrium strategy and belief is shown in the online appendix.

First let

$$\bar{F}(x) = 1 - F(x)$$

be the counter-cumulative distribution function. In the last period when $t = T$, a seller accepts the last-round bid $b_T$ as long as his cost is below $b_T$, so the cost cutoff $x_T = b_T$. Knowing this, and given the belief that all the sellers have costs higher than $x_{T-1}$, the buyer chooses $b_T$ to maximize his payoff:

$$V_T(x_{T-1}) = \max_{b_T \in [x_{T-1}, \bar{c}], x_T} (v - b_T) P(x_{T-1}, x_T),$$

$$s.t. b_T = x_T.$$ 

$$P(x_{T-1}, x_T) = \bar{F}(x_{T-1})^N - \bar{F}(x_T)^N$$

is the probability that all the sellers’ costs are
above \( x_{T-1} \) and the demand is fulfilled, given that sellers with costs between \( x_{T-1} \)
and \( x_T \) are willing to provide the good. The maximizers, \( \tilde{b}_T (x_{T-1}) \) and \( \tilde{x}_T (x_{T-1}) \),
are the buyer’s price and the sellers’ cost cutoff in the continuation equilibrium given belief \( x_{T-1} \):

\[
(\tilde{b}_T (x_{T-1}), \tilde{x}_T (x_{T-1})) \in \arg \max_{b_T, x_T \in [x_{T-1}, \bar{x}]} (v - b_T) P (x_{T-1}, x_T) \tag{P1}
\]

Next we proceed backward to period \( t = T - 1 \). Given belief \( x_{t-1} \), suppose that
the buyer chooses \( b_t \in [x_{t-1}, \bar{x}] \). The cost cutoff \( x_t \) in period \( t \) is determined by
finding the type of seller who is indifferent between accepting in period \( t \) and period \( t + 1 \):

\[
x_t \in \chi (b_t, x_{t-1}) \equiv \{ x \in [x_{t-1}, \bar{x}] \ | \ (b_t - x) G (x_{t-1}, x) = C_{t+1} (x) \} . \tag{1}
\]

\( C_{t+1} (x_t) \equiv (\tilde{b}_{t+1} (x_t) - x_t) G (x_t, \tilde{x}_{t+1} (x_t)) \) and \( (b_t - x_t) G (x_{t-1}, x_t) \) measure
the payoffs received by a seller with cost \( x_t \) if he accepts in period \( t + 1 \) and in period \( t \) respectively;
\( G (x_{t-1}, x_t) \equiv \sum_{n=0}^{N-1} \tilde{F} (x_t)^{N-1-n} \tilde{F} (x_{t-1})^n \) measures the probability
that a seller gets to sell the good if he accepts in period \( t \). However, \( \chi (b_t, x_{t-1}) \) in (1) might be empty. If \( b_t \) is so small that \( (b_t - x) G (x_{t-1}, x) < C_{t+1} (x) \)
for all \( x \in [x_{t-1}, \bar{x}] \), then no seller with cost in \( [x_{t-1}, \bar{x}] \) accepts in period \( t \), so \( x_t = x_{t-1} \).
A more difficult situation arises when

\[
\{ x \in [x_{t-1}, \bar{x}] \ | \ (b_t - x) G (x_{t-1}, x) < C_{t+1} (x) \} \neq \emptyset, \tag{2}
\]

\[
\{ x \in [x_{t-1}, \bar{x}] \ | \ (b_t - x) G (x_{t-1}, x) > C_{t+1} (x) \} \neq \emptyset, \text{ and}
\]

\[
\{ x \in [x_{t-1}, \bar{x}] \ | \ (b_t - x) G (x_{t-1}, x) = C_{t+1} (x) \} = \emptyset,
\]

which occurs only if \( C_{t+1} (x) \) is not continuous. In this case, we are not able to
find a cost cutoff in period \( t \) given \( b_t \).\footnote{Given belief \( x_{t-1} \), the conditional probability that a seller gets to sell the good if he accepts in
period \( t \) is \( \frac{G (x_{t-1}, x)}{\tilde{F} (x_{t-1})} \).} One way to solve the problem is to let
\footnote{Conditions in (2) imply that there does not exist \( x_t \) such that
\( (b - x) G (x_{t-1}, x_t) > (\tilde{b}_{t+1} (x_t) - x) G (x_t, \tilde{x}_{t+1} (x_t)) \text{ for } x < x_t \text{ and}
(\tilde{b}_{t+1} (x_t) - x) G (x_t, \tilde{x}_{t+1} (x_t)) \text{ for } x > x_t \).}
the buyer play mixed strategies in round \( t + 1 \). However, it can be shown that choosing \( b_t \in \{ b \mid \chi(\{b, x_{t-1}\}) = \phi \} \) is never an optimal strategy for the buyer (see the online appendix for more details), so without loss of generality, we can focus on \( b_t \in \{ b \mid \chi(\{b, x_{t-1}\}) \neq \phi \} \). Expecting (1), the buyer chooses \( \bar{b}_t(x_{t-1}) \) to maximize his expected payoff \( V_t(x_{t-1}) \):

\[
V_t(x_{t-1}) = \max_{b_t, x_t \in [x_{t-1}, \bar{x}]} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t)
\]

\[
s.t. (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t),
\]

\[
(\bar{b}_t(x_{t-1}), \bar{x}_t(x_{t-1})) \in \arg \max_{b_t, x_t \in [x_{t-1}, \bar{x}]} (v - b_t) P(x_{t-1}, x_t) + V_{t+1}(x_t)\] (P2)

\[
s.t. (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t).
\]

Applying the same procedure backward, we get a series of \( \bar{b}_t(x_{t-1}) \) and \( \bar{x}_t(x_{t-1}) \), for \( t = 1, 2, \cdots T \). But still, we need to make sure that solutions to (P1) and (P2) exist. This is proved in the following proposition. Note that there might be multiple solutions to programs (P1) and (P2). In that case, only those that ensure the existence of equilibrium can be candidates for \( \bar{b}_t(x_{t-1}) \) and \( \bar{x}_t(x_{t-1}) \) (see the proof of Proposition 1 for more details).

**Proposition 1** There exists a set of solutions \( \{ \bar{b}_t(x_{t-1}), \bar{x}_t(x_{t-1}) \} \) that solves programs (P1) and (P2) for all \( t \).

**Proof.** The details of the proof are in Appendix A. Here is the sketch. First, by Berge’s maximum theorem, \( V_T(x_{T-1}) \) is continuous, and the solution set of \( x_T \) for program (P1) is upper hemi-continuous. Therefore, we are able to pick \( \bar{x}_T(x_{T-1}) \) from the solution set such that \( C_T(x_{T-1}) \) is lower semi-continuous. Next, substituting the constraint into the objective function in round \( T - 1 \) in program (P2), the objective function is graph-continuous defined in Leininger (1984), and by Leininger’s generalized maximum theorem, \( V_{T-1} \) is upper semi-continuous, and the solution set of \( x_{T-1} \) exists and is upper hemi-continuous. Applying the same procedure backward, we guarantee the existence of a solution to each round-\( t \) program.

The following remark summarizes the procedure adopted to derive the equilibrium path.

**Remark 1** The equilibrium path \( \{(b_1, \cdots, b_T), (x_1, \cdots, x_T)\} \) can be found by solv-
ing the recursive program

\[ V_t(x_{t-1}) = \max_{b_t, x_t \in [x_{t-1}^L, x_{t-1}^U]} (v - b_t) P(c_t, x_t) + V_{t+1}(x_t) \]  
\[ \text{s.t.} (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1}(x_t), \]

where \( x_0 = c, \) \( V_{T+1}(x_T) = 0, \) and \( C_{T+1}(x_T) = 0. \) The value of the program with \( t = 1 \) is the buyer’s payoff in equilibrium.

Program P3 shows that the equilibrium path \{\( (b_1, \ldots, b_T), (x_1, \ldots, x_T) \)\} maximizes the buyer’s payoff but is subject to two constraints. The first one is the sellers’ IC constraint, which exists in every mechanism and is shown in the constraint part of the program. The second constraint comes from the recursive form of the program. In each round, the buyer makes his bidding decision based on his current information and is not able to commit to a bidding path at the beginning. The second constraint keeps the buyer from achieving the outcome derived from the optimal mechanism stated by Myerson (1981).

4 Equilibrium Bidding Behavior

With \( T \) chances to submit prices, the buyer is able to segment the sellers in up to \( T \) groups according to their costs. However, the buyer cannot commit to a bidding path in advance, and in each round, will choose a price that maximizes his expected payoff based on his belief. Thus, the buyer would suffer from the inability to commit and get lower payoff than when commitment is possible. In this section, we focus on the case when there is no lockout period restriction so that the buyer can submit as many bids as desired. We first show that when committing to a bidding path is impossible, the optimal outcome for the buyer stated by Myerson (1981) might not be attainable. Next, we characterize the equilibrium bidding behavior and show that there are two possible types of equilibrium bidding paths. One incurs constant trades over time, and the other leads to late transactions.

4.1 Commitment and optimality

In this section, we show that Myerson’s optimal outcome, which can be achieved when the buyer can commit to a bidding path in advance, is not attainable without commitment if the optimal auction design involves setting a reserve price.
In our setting, a reverse Dutch auction with a reserve price $r$ such that $r + \frac{F_i(r)}{f_i(r)} = v$ implements Myerson’s optimal mechanism. In a reverse Dutch auction, the price is continuously increased and stops at the reserve price. Therefore, if under NYOP, the buyer can commit and the number of rounds is large enough, the buyer can roughly duplicate the price path in a reverse Dutch auction and receive a payoff approximately the same as in an optimal mechanism.

However, when commitment is not possible, even though the buyer is allowed to adjust the price as many times as desired, the maximum payoff resulting from the optimal mechanism is not approximately achievable. To see this, recall that on the equilibrium path, the last-round $b_T$ and $x_T$ can be found by solving

$$x_T = b_T = \arg \max_b (v - b) \left[ \bar{F} (x_{T-1})^N - \bar{F} (b)^N \right].$$

A necessary condition for $b_T$ is

$$\bar{F} (x_{T-1})^N = \bar{F} (b_T)^N + (v - b_T) N \bar{F} (b_T)^{N-1} f (b_T).$$  \hspace{1cm} (3)

Suppose the optimal auction involves setting a reserve price $r < \bar{r}$. If the optimal auction can be approximately implemented when $T$ goes to infinity, then it must be that $\lim_{T \to \infty} b_T = \lim_{T \to \infty} x_T = r$ and $\lim_{T \to \infty} x_{T-1} = r$. But by equation (3), if $\lim_{T \to \infty} b_T = r$, $\lim_{T \to \infty} x_{T-1} < r$, so the optimal auction cannot be approximately implemented.

### 4.2 Possible forms for the equilibrium paths when no lockout period restriction is imposed

In this section, we characterize the pattern of the equilibrium bidding path when $T \to \infty$ (i.e. when there is no lockout period restriction). The question is how the buyer designs a bidding path to discriminate between sellers. When commitment is possible, it is optimal for the buyer to induce sellers to reveal information about their costs gradually in every round. But when commitment is impossible, acquiring new information will change the buyer’s pricing strategy later on, and it is not clear whether doing so is beneficial for the buyer. In Theorem 1, we characterize the equilibrium paths. Although the equilibrium paths would be different in different environments, we show that the paths can be neatly classified into two types: either

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8If $v \geq \bar{r} + \frac{1}{f_i(\bar{r})}$, no reserve price is required to implement the optimal mechanism.
the sellers with different costs are almost fully separated so the sellers’ private information is revealed gradually over time, or they are pooled in intervals and most information about the sellers’ costs is revealed just before the deadline.

Before characterizing the equilibrium paths, we impose the following condition for subsequent discussion. The superscript \( T \) of \( x_t^{T-1} \) represents the total number of rounds.

**Condition 1** Assume that \( F \) is such that \( x_t^{T-1} \) defined in (P1) and (P2) is absolutely continuous on \([c, \bar{c}]\) for all \( t \) and \( T \).

We prove in Proposition 8 in the online appendix that if the distribution is of the form \( F(x) = \frac{x^{a}}{\bar{c}} \), where \( x \in [c, v] \), \( a \geq 1 \), Condition 1 holds; furthermore, \( x_t^{T-1} \) derived in (P1) and (P2) is uniquely defined for all \( t \) and \( T \). Since \( F \) is smooth, Condition 1 implies that \( x_t^{T-1} \) is also smooth for all \( t \) (proved in Proposition 7 in the online appendix). Thus, the objective functions and the constraints of the programs in Section 3 are differentiable, and the envelope theorem can be applied.

For convenience, we denote \( x_t \) and \( b_t \) on the equilibrium path when there are \( T \) rounds by \( x_T^T \) and \( b_T^T \). The following proposition shows a convergence property of \( x_T^T \) when \( T \) goes to infinity.

**Proposition 2** Assume Condition 1. Then \( \lim_{T \to \infty} x_T^T \) exists for all \( t \in \{0, 1, \ldots\} \).

**Proof.** Note that given any \( t \) and \( T \), \( x_T^T (\cdot) = x_{t+1}^{T+1} (\cdot) \) (defined in program P2 on page 11). When we increase the number of rounds from \( T \) to \( T + 1 \), \( x_1^{T+1} \geq x_0^{T+1} = x_T^T \). By Lemma 1 in Appendix A, \( x_1^{T+1} \geq x_0^{T+1} \) implies \( x_{T+1}^{T+1} \geq x_T^{T-1} \) for all \( t \). Hence, \( x_T^{T-1} \) increases in \( T \). Furthermore, \( x_T^{T-1} \) has an upper bound \( \bar{c} \), so we conclude that \( \lim_{T \to \infty} x_T^{T-1} \) exists. \( \blacksquare \)

Proposition 2 shows that \( x_T^{T-1} \) converges as \( T \to \infty \) with \( t \) held fixed. Intuitively, there are two different ways in which this could happen. One is that trade could be taking place gradually so that \( x_T^{T-1} \approx \bar{c} \) for \( T \) large enough. Another is that the pattern of sales could look like the one we saw in the example in Section 2.2 where most trade occurred at the end. The main result of this section shows formally that there are two different possibilities as we conjecture. To state the result, we need a preliminary definition:

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9Horner and Samuelson (2010) prove that if the distribution is of the form \( F(x) = \frac{x^{a}}{\bar{c}} \), where \( x \in [c, v] \), \( x_T^T (x_{t-1}) \) is uniquely defined, which implies that Condition 1 holds.
Let $X^T = \{x_t^T\}_{t=1}^T$, and let $|A|$ denote the number of elements in set $A$. The following defines a cluster point of the cutoff set $X^T$ when $T \to \infty$.

**Definition 2** $z \in [c, \overline{c}]$ is a cluster point if for any $\epsilon > 0$ and $M > 0$, there exists $T' = T'(\epsilon, M)$ such that for all $T > T'$, $|\{x \in X^T \mid |x - z| < \epsilon\}| > M$.

Definition 2 implies that if $z$ is a cluster point, the number of $x_t$’s falling around $z$ increases with $T$. Let $B$ be the set of cluster points, and $[c, \overline{c}] \setminus B$ be the complement of $B$.

**Theorem 1** Assume Condition 1.

1. The cluster point set $B$ is either the whole interval $[c, \overline{c}]$ or a single point $\{\underline{c}\}$, i.e. $B = [c, \overline{c}]$ or $\{\underline{c}\}$.

2. The cluster point set $B$ is a single point $\{\underline{c}\}$ if and only if the last period cutoff $x_T^T$ is bounded away from $\overline{c}$ when $T \to \infty$, i.e. $B = \{\underline{c}\}$ if and only if $\lim_{T \to \infty} x_T^T < \overline{c}$.

3. If $B = [c, \overline{c}]$, the buyer’s payoff is approximately the same as that in a first-price reverse auction without a reserve price.

**Proof.** The details of the proof are in Appendix B. Here is the sketch. Lemma 3 shows that if the number of rounds left in a continuation game starting with belief $x_{t-1}$ goes to infinity, then the difference between $x_t$ and $x_{t-1}$ goes to 0. So, if $a \in [c, \overline{c}]$, is a cluster point, any point $x < a$ must be a cluster point too. However, Lemma 7 shows that it cannot be the case that $a \in (c, \overline{c})$, $[c, a]$ belongs to the cluster point set $B$, and $(a, \overline{c}]$ belongs to the complement of $B$ because it violates the necessary condition under which the buyer chooses the optimal strategy for himself in every round. Therefore, the cluster point set is either $[c, \overline{c}]$ or $\{\underline{c}\}$. The third statement comes from the revenue equivalence principle.

The first statement of the theorem implies that there are only two possible equilibrium paths: one with the cluster point set $B$ equal to the whole interval $[c, \overline{c}]$ and one with the cluster point set equal to a single point $\{\underline{c}\}$. If the cluster point set is $[c, \overline{c}]$, then it is implied that the sellers are almost fully separated in equilibrium, and information about the sellers’ costs is revealed gradually over time. To separate the sellers, the buyer will increase the bids gradually and stop at $\overline{c}$, and
transactions occur constantly along the path. If the cluster point set is \( \{c\} \), then most cutoff points \( x_t \) cluster at \( c \), and only a few cutoff points spread around other places. In this case, sellers with costs within the same cutoff interval accept the same price, so they are not fully separated. In addition, most information about the sellers’ costs is revealed just before the deadline. Since the prices in the first many rounds are accepted by sellers with costs around \( c \), and the prices in the last few rounds are accepted by sellers with costs in higher intervals, by the revenue equivalence principle, we can derive the price path and show that the prices in the first many rounds are roughly the same, and there are big jumps in prices in the last few rounds. Therefore, if the cluster point set is \( \{c\} \), the equilibrium bidding path is convex. Moreover, since the bids in the first many rounds are only accepted by sellers with costs around \( c \), the transaction is more likely to occur in the last few rounds.

To see why these two types of paths arise in equilibrium, observe that under NYOP, the buyer is allowed to set up a price path so that \( \lim_{T \to \infty} b_T = \lim_{T \to \infty} x_T < \bar{c} \), which functions as a reserve price. But since there is no commitment, to sustain \( \lim_{T \to \infty} x_T < \bar{c} \), there must be jumps between cost cutoffs \( x_T, x_{T-1}, x_{T-2}, \ldots \), and thus, the buyer cannot finely screen the sellers’ costs. Therefore, the buyer faces the trade-off between finely screening the sellers’ costs and successfully committing to a reserve price. If the buyer chooses to give up the chance to have a reserve price, he will increase the price gradually to \( \bar{c} \) to reveal information about the sellers’ costs and almost fully separate the sellers. In this case, the cluster point set \( B \) is the whole interval \( [c, \bar{c}] \), and \( \lim_{T \to \infty} x_T = \bar{c} \). On the other hand, if the buyer chooses to have \( \lim_{T \to \infty} x_T < \bar{c} \), he has to restrict himself from getting too much information about the sellers’ costs at the beginning. Supposing he raises bids early so that sellers with higher cost also accept, then once the bid is rejected, he believes that the sellers’ costs are above a higher threshold and will raise bids further in the next rounds. In the end, \( \lim_{T \to \infty} b_T = \bar{c} \). Therefore, he has to keep the bids low most of the time so that his belief about the sellers’ costs does not change much until the last few rounds. In the end, since only a few chances remain, he cannot raise the bids to \( \bar{c} \), and there are big jumps between the cost cutoffs. In this case, the cluster point set \( B \) is a single point \( \{c\} \). And therefore, we have the second statement of the theorem saying that the event that \( \{c\} \) is the only cluster point occurs if and only if \( \lim_{T \to \infty} x_T \) is strictly lower than \( \bar{c} \). In other words, late transaction and late information revelation coincide with the possibility that the
buyer’s demand is not fulfilled.

The result could explain the puzzle proposed by Spann and Tellis (2006). They analyze bidding patterns in the data of a NYOP retailer in Germany that sells airline tickets for various airlines and allows multiple bidding. They argue that with positive bidding cost, the pattern should be concavely increasing because at the beginning, consumers try to increase the probability of successful bidding by bidding higher, but when the bids are closer to their reservation value, the increasing rate slows down; and with zero bidding cost, the pattern should reflect linearly increasing bids. However, the result shows that only 36% of the data fit the first pattern and 5% fit the second pattern. 23% of the data fit the pattern which is convexly increasing, so they conclude that consumer behavior on the internet is not so rational. Nevertheless, a convexly increasing pattern corresponds to the case $B = \{c\}$ in Theorem 1. Thus, a convex path can actually occur in a fully rational environment.\footnote{After the buyer’s waiting cost is incorporated in the next section, all three patterns can occur in our model with different parameters.} In addition to the convex bidding path, the case $B = \{c\}$ also implies that most transactions occur near the end. This is related to the deadline effect that has been observed in many negotiation processes such as bargaining during strikes and pretrial negotiation. Our model thus provides insight into this phenomenon.

### 4.3 Factors that affect the type of the equilibrium path

As shown in the previous section, there are two types of equilibrium paths. Which type of path would occur depends on the distribution of the sellers’ cost $\tilde{F}$, the buyer’s value $v$, and the number of sellers $N$, parameters that affect the buyer’s trade-offs between finely screening the sellers’ costs and having a reserve price. With higher $v$, the buyer values the good more highly and cannot stand the risk of not getting the good, so setting a reserve price is less profitable for the buyer. On the other hand, when $N$ is larger, the probability that there exists at least one seller with a low cost increases, and competition among the sellers forces them to accept lower prices too, so a buyer also benefits less from setting a reserve price. Therefore, we expect that a path that almost fully screens the sellers is more likely to occur when $v$ is large and when $N$ is large.

The intuition regarding how $N$ affects the equilibrium path is confirmed in Horner and Samuelson (2010). They prove that when $v = 1$ and a seller’s cost is uniformly distributed on $[0, 1]$, an equilibrium path with $\lim_{T \to \infty} x_T^T < \bar{c}$ occurs if
Figure 1: Path of $x_t$

The relationship between $v$ and the type of the equilibrium path is proved in Section 4.4. Here we use Figure 1 for illustration. Figure 1 shows the path of $x_t$ for different values of $v$ when $T = 20$, $N = 2$, and a seller’s cost is uniformly distributed on $[0, 1]$. With $v = 1$, $v = 1.2$, and $v = 1.4$, the optimal reserve prices are 0.5, 0.6, and 0.7 respectively. So when $v = 1$, the buyer is more inclined to have $x_T$ much lower than $\bar{c} = 1$; and in equilibrium, a seller with cost higher than 0.1 would not sell the good until the last two periods, which implies that transactions are much more likely to occur in the last two periods. On the other hand, when $v = 1.4$, the loss of having $x_T$ much lower than $\bar{c} = 1$ dominates the benefit, so in equilibrium, the buyer raises bids gradually to a price close to 1, and transactions occur constantly in every period.

While it is difficult to derive the equilibrium path for $T \to \infty$, we can compare the buyer’s payoff when $T$ is finite with the payoff in a reverse auction and get an idea about whether $\lim_{T \to \infty} x_T^T < \bar{c}$ or $\lim_{T \to \infty} x_T^T = \bar{c}$ occurs in equilibrium.

In Proposition 3, we first show that the buyer’s payoff increases with the number of rounds $T$. Then we show that if for some finite period $M$, the buyer’s expected
payoff is higher than in a reverse auction, then when $T$ goes to infinity, $\lim_{T \to \infty} x_T^T < \bar{c}$, and sellers with different costs are not almost fully separated. For example, when $N = 2$, $v = 1$, and $F(x)$ is uniformly distributed on $[0, 1]$, the expected payoff of the buyer is $\frac{1}{3}$ in a first-price reverse auction. But if the buyer is allowed to submit the price once, he can optimally choose $b = 0.4225$ and get expected payoff $0.3849$. Thus, we know that, when $T \to \infty$, $x_T^T$ is bounded away from $\bar{c}$.

**Proposition 3**  
1. The buyer’s payoff increases with $T$, and the payoff converges when $T \to \infty$.  
2. If $\lim_{T \to \infty} x_T^T = \bar{c}$, there does not exist a finite number $M$ such that the buyer’s expected payoff when there are $M$ rounds is higher than that in a first-price reverse auction without a reserve price.

**Proof.** We first show that the buyer’s payoff increases with $T$, and the payoff converges when $T \to \infty$. When the number of rounds increases from $M$ to $M + 1$, the buyer can submit price $c$ in the first round and then in the remaining rounds, do the same thing as when there are $M$ rounds. Following this strategy, the buyer’s payoff is the same as when $T = M$, and he might be able to do better by using other strategies. Therefore, the buyer’s payoff is weakly increasing with $T$. Moreover, the buyer’s payoff is bounded by the payoff in Myerson’s optimal mechanism, so the payoff converges when $T \to \infty$.

If the buyer’s payoff when $T = M$ is higher than that in a first-price reverse auction without a reserve price, by the first statement, the buyer’s payoff when $T \to \infty$ is weakly higher than when $T = M$. Hence, by the third statement of Theorem 1, $\lim_{T \to \infty} x_T^T = \bar{c}$ does not occur in equilibrium. □

### 4.4 Payoff comparison among different mechanisms

When there is no waiting cost, as shown in the first statement of Proposition 3, having more rounds does not hurt the buyer because the buyer can choose to waste the additional bidding chances in the beginning. Therefore, setting a lockout period does not benefit the buyer when the buyer has no time preference.

However, Proposition 3 no longer holds if the buyer prefers to close the transaction earlier. We can find some clues from the equilibrium path characterized in Section 4.2. Suppose that the buyer realizes his demand $M$ days in advance. If allowed to submit bids many times a day, under some circumstances, the buyer would not submit serious bids until the last day, and so successful transactions only
occur at the end. If only one bid is allowed a day, then transactions will occur much earlier, but the negative impact on the buyer’s payoff is infinitesimal. This intuition is formalized and analyzed in the next section when the buyer has waiting cost.

Based on the analysis above, we can also characterize the buyer’s payoff with different types of equilibrium paths and obtain an upper bound and a lower bound for the buyer’s expected payoff under NYOP. Note that there might exist multiple equilibria. We consider the equilibrium that yields the highest payoff to the buyer. By Theorem 1, we know that if \( \lim_{T \to \infty} x_T = \bar{c} \), the buyer’s payoff is approximately the same as that in a first-price reverse auction without a reserve price. The following theorem further proves that if \( \lim_{T \to \infty} x_T < \bar{c} \), the buyer’s payoff is higher than that in a first-price auction without a reserve price. The intuition is as follows: \( \lim_{T \to \infty} x_T < \bar{c} \) implies that the buyer keeps the price low in the first many rounds so that only sellers with costs around \( \bar{c} \) accept. The buyer could instead have a bidding path that increases more aggressively from the beginning while the sellers expect the buyer’s behavior correctly, so that in the end, \( \lim_{T \to \infty} x_T = \bar{c} \). Since he chooses not to do so, it must be that he can get a higher payoff by keeping the price low in the first many rounds.

**Theorem 2** Consider the equilibrium that yields the highest payoff to the buyer. When \( T \to \infty \), if on the equilibrium path, \( \lim_{T \to \infty} b_T = \lim_{T \to \infty} x_T < \bar{c} \), the buyer’s expected payoff is strictly greater than that in a reverse auction without a reserve price. Thus, when \( T \to \infty \), the buyer’s expected payoff is between the payoff in a reverse auction without a reserve price and the payoff in a reverse auction with the optimal reserve price.

**Proof.** Note that when \( T \to \infty \), a path that almost fully separates sellers and satisfies sellers’ IC constraint is a feasible solution candidate to program P3 (it is the stationary solution to program P3 when \( T = \infty \), see Appendix A, Proposition 6) and it brings the buyer almost the same expected payoff as in a reverse auction with no reserve price. Therefore, if the solution to program P3 is the path with \( \lim_{T \to \infty} b_T = \lim_{T \to \infty} x_T < \bar{c} \), it must yield a higher value to the program than in a reverse auction with no reserve price. This proves the first statement. The second statement follows from Theorem 1, the discussion in Section 4.1, and the first statement. \( \blacksquare \)

We can consider the mechanism used at Hotwire.com as a first-price reverse auction without a reserve price. Hotels submit their prices to Hotwire.com, and
Hotwire.com picks the lowest one and announces it on the website. Then, customers see the price and decide whether to buy or not. Therefore, we should expect that customers get higher expected savings under NYOP.

With the help of Theorem 2, we are able to prove that a path almost fully screening the sellers is more likely to occur when \( v \) is large, an intuition discussed in Section 4.3.

**Proposition 4** Suppose that \( \bar{F} \) is such that for all \( v \), there is a unique equilibrium. Then there exists a threshold \( \bar{v} \) such that an equilibrium path with \( \lim_{T \to \infty} x_T^T < \bar{v} \) (i.e. the cluster point set \( B = \{c\} \)) occurs if and only if \( v < \bar{v} \).

**Proof.** Along with the fact that \( \bar{F} \) is smooth, we impose the condition of uniqueness of the equilibrium so that the envelope theorem and Theorem 2 can be applied. In period \( T - 1 \), given \( v \) and belief \( x_{T-2} \), the buyer’s payoff is

\[
V_{T-1} (x_{T-2}, v) = (v - \bar{x}_{T-1} (x_{T-2}, v)) \left[ \bar{F} (x_{T-2})^N - \bar{F} (\bar{x}_{T-1} (x_{T-2}, v))^N \right] - C_T (\bar{x}_{T-1} (x_{T-2}, v)) \left[ \bar{F} (x_{T-2}) - \bar{F} (\bar{x}_{T-1} (x_{T-2}, v)) \right] + V_T (\bar{x}_{T-1} (x_{T-2}, v), v).
\]

By the envelope theorem, \( \frac{dV_{T-1}}{dv} = \left( \bar{F} (x_{T-2})^N - \bar{F} (x_{T-1})^N \right) + \frac{\partial V_T}{\partial v} \), where \( \frac{\partial V_T}{\partial v} = \bar{F} (x_{T-1})^N - \bar{F} (x_T)^N \), so \( \frac{dV_{T-1}}{dv} = \bar{F} (x_{T-2})^N - \bar{F} (x_T)^N \). For clarification, given \( v \), let \( x_T^T (v) \) be the equilibrium cutoff in period \( T \), and \( \pi^T (v) \) and \( \pi (v) \) be the buyer’s payoffs under NYOP with \( T \) rounds and in a reverse auction without a reserve price respectively. Applying similar procedures to derive \( \frac{dV_{T-2}}{dv} \), \( \frac{dV_{T-3}}{dv} \), \( \ldots \), we can conclude that \( \frac{d\pi^T (v)}{dv} = 1 - \bar{F} (x_T^T (v))^N \leq 1 \). Note that \( \frac{d\pi}{dv} = 1 \).

We show that it cannot be the case that for \( v' > v \), \( \lim_{T \to \infty} x_T^T (v') < \bar{v} \) and \( \lim_{T \to \infty} x_T^T (v) = \bar{v} \). If that is the case, by Theorem 2, \( \lim_{T \to \infty} \pi^T (v') > \pi (v') \) and \( \lim_{T \to \infty} \pi^T (v) = \pi (v) \).

\[
\lim_{T \to \infty} \pi^T (v) = \lim_{T \to \infty} \pi^T (v') - \int_v^{v'} \frac{d\lim_{T \to \infty} \pi^T (x)}{dx} \, dx \tag{4}
\]

and

\[
\pi (v) = \pi (v') - \int_v^{v'} \frac{d\pi (x)}{dx} \, dx. \tag{5}
\]

Since \( \lim_{T \to \infty} \pi^T (v') > \pi (v') \) and \( \frac{d\lim_{T \to \infty} \pi^T (x)}{dx} \leq \frac{d\pi (x)}{dx} \), by (4) and (5), \( \lim_{T \to \infty} \pi^T (v) > \pi (v) \), a contradiction. Therefore, there is a threshold \( \bar{v} \) such that \( \lim_{T \to \infty} x_T^T (v) < \bar{v} \) if and only if \( v < \bar{v} \).
5 Optimal Lockout Period When Waiting is Costly

At Priceline, when a bid is rejected, a customer has to wait for a period of time to submit another bid, but some other NYOP websites in Europe allow customers to rebid immediately once their bids are rejected. In this section, we examine the conditions under which having a lockout period restriction benefits customers.

5.1 Model with discounting and the equilibrium path

In reality, buyers would like to pin down their travel plans as early as possible, so late transactions actually incur some waiting costs. Therefore, we incorporate buyers’ waiting cost into the model in Section 2 and show that setting an appropriate lockout period rule may benefit the buyer. However, we assume that sellers have no preference for early or late transactions.

The model is modified as follows. The buyer realizes his demand for the good at time 0 and tries to fulfill the demand in time period \([0, M]\). After time \(M\), the buyer no longer needs the good. If the buyer gets the good at price \(B\) at time \(t\), his utility is \(\delta^{\frac{1}{T}} (v - B)\), where \(\delta \in (0, 1)\) is the discount factor for the whole time period \([0, M]\). The platform sets a lockout period rule which regulates how frequently the buyer can submit a bid. If the lockout period is \(s\), the buyer can submit bids for \(\left\lfloor \frac{M}{s} \right\rfloor\) times, that is, \(T = \left\lfloor \frac{M}{s} \right\rfloor\).

When waiting is costly, the buyer is more anxious to close the transaction early. However, the buyer’s trade-offs between finely screening the sellers’ costs and successfully committing to a reserve price still exist. If the buyer tries to commit to a reserve price, he has to bear the cost of delaying the transaction as well as not being able to screen the sellers. On the contrary, if the buyer gives up having a reserve price, he can raise the price aggressively to close the transaction; and moreover, if the buyer has unlimited chances to revise his bids, the sellers’ costs can be finely screened in a minute.

Apparently, a buyer with lower \(\delta\) (who discounts the future more heavily) will find committing to a reserve price more costly and hence, choose to raise the bids aggressively. This is illustrated in Figure 2. Figure 2 shows the paths of \(x_t\) for different values of \(\delta\) when \(v = 1\), \(T = 50\), \(N = 2\), and a seller’s cost is uniformly distributed on \([0, 1]\). We can see that with lower \(\delta\), the path becomes more concave, and a trade is more likely to occur in early periods.
5.2 Optimal lockout period

Proposition 3 shows that without discounting, having more rounds does not hurt the buyer because the buyer can always waste the additional bidding chances in the beginning. However, this strategy postpones the trade and cannot be implemented without cost when the buyer discounts the future. Therefore, with a discount factor $\delta < 1$, the buyer’s payoff might not monotonically increase with the number of rounds, and setting a lockout period might increase the buyer’s payoff.

As we see in Figure 2, the equilibrium path might be concave or convex when there is no lockout period. Given different types of equilibrium paths, we first discuss how imposing a lockout period rule affects the buyer’s payoff. Then we extend the discussion to different parameter settings and support the discussion with numerical examples. We focus on the environment in which Myerson’s optimal mechanism involves setting a reserve price. If setting a reserve price is unnecessary, having more rounds always benefits the buyer because it helps the buyer separate the sellers better and be able to close the transaction earlier.
Convex path of $x_t$ If without a lockout period, the path of $x_t$ is convex, then the last-round price is lower than $\bar{c}$, and most transactions occur late. If there is a lockout period, the buyer has fewer bidding chances and will bid seriously from the beginning, so transactions occur earlier. However, the buyer also loses chances to separate sellers with cost around $c$.

Concave path of $x_t$ If without a lockout period, the path of $x_t$ is concave, which implies that the buyer raises the bid aggressively. With a lockout period, the buyer cannot raise the bid all the way up to $\bar{c}$, so there is a reserve-price-like effect. But the lockout period limits the buyer’s bidding chances so that the buyer cannot separate the sellers well, and it also prevents the buyer from bidding aggressively and getting the object early.

Next, we apply the above discussion to see in what circumstances having a lockout period can benefit the buyer. We consider scenarios with different values of $\delta$ and $N$ in Observation 1 and Observation 2 respectively. Each observation is followed by numerical examples.

**Observation 1** Setting an appropriate lockout period increases the buyer’s payoff when $\delta$ is in the middle range.

We consider the example where $N = 2$, $v = 1.2$, and a seller’s cost is uniformly distributed on $[0, 1]$. The following table summarizes $\lim_{T \to \infty} x_T$, the limit of the last round cutoff when $T$ goes to infinity, $T^*$, the number of rounds that maximizes the buyer’s payoff, and $\pi(T^*)$, the corresponding buyer’s payoff, given different values of $\delta$.

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>$\lim_{T \to \infty} x_T$</th>
<th>$T^*$</th>
<th>$\pi(T^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.96</td>
<td>0.808</td>
<td>3</td>
<td>0.5473</td>
</tr>
<tr>
<td>0.90</td>
<td>1</td>
<td>2</td>
<td>0.5399</td>
</tr>
<tr>
<td>0.85</td>
<td>1</td>
<td>$\infty$</td>
<td>0.5333</td>
</tr>
</tbody>
</table>

The result shows that setting a lockout period so that the buyer has three and two bidding chances maximizes the buyer’s payoff when $\delta = 0.96$ and 0.9 respectively. With $\delta = 0.96$, when there is no lockout period, $\lim_{T \to \infty} x_T < 1$, so the equilibrium path of $x_t$ is mostly convex, and the transaction is very likely to occur late. This is
illustrated by the dotted line in Figure 3, which shows the cutoff path when there are 50 rounds. The three stars in Figure 3 represent the equilibrium path when there are only 3 rounds. Comparing the two paths, we can see that by setting a lockout period so that the buyer is allowed to bid thrice, the buyer benefits from having early transactions but suffers from not being able to separate sellers with costs around $c$. When $\delta = 1$, $\lim_{T \to \infty} x_T < 1$ as well, so the pros and cons of a lockout period rule are similar. However, waiting is more costly in the case of $\delta = 0.96$ than in the case of $\delta = 1$. Therefore, when $\delta = 0.96$, the benefit of setting an appropriate lockout period dominates the loss; but when $\delta = 1$, the loss dominates the benefit.

With $\delta = 0.9$, when there is no lockout period, $\lim_{T \to \infty} x_T = 1$, so the equilibrium path of $x_t$ is concave, and transactions occur early. This is illustrated by the dotted line in Figure 4, which shows the cutoff path when there are 50 rounds. The two stars in Figure 4 represent the equilibrium path when there are only 2 rounds. We can observe that, by setting a lockout period, the buyer benefits from having a last-round price lower than $\tau$, which functions like a reserve price, but suffers from not being able to close the transaction early and separate sellers finely. When $\delta \leq 0.85$, $\lim_{T \to \infty} x_T = 1$ as well, so the pros and cons of the lockout period rule are similar. But since waiting is more costly when $\delta$ is smaller, the loss caused by not being able to close the transaction early dominates the benefit when $\delta \leq 0.85$, while the benefit dominates the loss when $\delta = 0.9$. The example thus indicates that the buyer benefits from setting an appropriate lockout period when $\delta$ is in the middle range.

In addition to the situation mentioned above, setting a lockout period is especially valuable for the buyer when having a reserve price benefits the buyer a lot. Consider another example where $v = 1$ and other parameters, $N$ and $F$, are the same as in the previous example. The optimal reserve price is 0.5. Having a reserve price can improve the buyer’s payoff greatly. In this case, when $\delta$ is lower than 0.62, $\lim_{T \to \infty} x_T = 1$, so the buyer’s payoff when there is no lockout period is at most $\frac{1}{3}$ (the payoff in a reverse auction with no reserve price). On the other hand, the buyer’s payoff when only one bidding chance is allowed is 0.3849 for all $\delta$. Therefore, when $\delta < 0.62$, setting an appropriate lockout period always benefits the buyer.

**Observation 2** Setting a lockout period increases the buyer’s payoff when $N$ is small.

We consider the example where $v = 1$, $\delta = 0.95$, and a seller’s cost is uniformly
Figure 3: Path of $x_t$ when there are 50 rounds and when there are 3 rounds, $\delta = 0.96$.

Figure 4: Path of $x_t$ when there are 50 rounds and when there are 2 rounds, $\delta = 0.9$. 
distributed on \([0, 1]\). The following table summarizes \(T^\ast\), the number of rounds that maximizes the buyer’s payoff, and \(\pi(T^\ast)\), the corresponding buyer’s payoff, given different values of \(N\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>(T^\ast)</th>
<th>(\pi(T^\ast))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>0.3949</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>0.5065</td>
</tr>
<tr>
<td>4</td>
<td>(\infty)</td>
<td>0.6000</td>
</tr>
</tbody>
</table>

The result shows that when the number of sellers is 2 or 3, setting an appropriate lockout period so that the buyer has only a few bidding chances increases the buyer’s payoff; but if the number of sellers is larger than 3, having more rounds is better for the buyer. The reason is as follows. As we discussed in Section 4.3, when \(N\) is large, having a reserve price does not increase the buyer’s payoff much, so without a lockout period, the buyer does not hesitate to raise the bid from the beginning. Therefore, \(\lim_{T \to \infty} x_T = 1\), the equilibrium path of \(x_t\) is concave, and transactions occur early. Similar to what is shown in Figure 4, by setting a lockout period, the buyer benefits from having a last-round price lower than \(\bar{r}\), which functions like a reserve price, but suffers from not being able to close the transaction early and separate sellers finely. However, when \(N\) is large, having a reserve price is not important to the buyer, so the loss dominates the benefit, and the buyer is better off without a lockout period.

The above discussion shows that the optimal lockout period varies with the environment. An appropriate lockout period increases the buyer’s payoff when the number of sellers is small and when the buyer’s discount factor is in the middle range. Priceline’s lockout period rule seems to hurt customers by restricting their rebidding opportunities, but in fact, a customer with waiting costs might find it beneficial.

6 Conclusion and Discussion

This paper analyzes the Name Your Own Price (NYOP) mechanism adopted by Priceline. We characterize the buyer’s and the sellers’ equilibrium strategies and show that Priceline’s lockout period restriction, a design to protect sellers that seems to hurt customers, can actually benefit a customer with a moderate discount factor. The analysis also provides insight into the existence of complex contractual procedures in bargaining and the prevalence of last-minute deals on the Internet.
We show that when there is no lockout period and no waiting cost, the equilibrium paths can be categorized into two classes. In the first class, the cluster point set of the sellers’ cost cutoffs in all rounds is the whole cost interval \([c, \bar{c}]\), which implies that sellers with different costs are almost fully separated and information about the sellers’ cost is revealed gradually over time. In this case, the buyer raises bids constantly, the ending price is the highest possible cost \(\bar{c}\), and the buyer’s payoff is approximately the same as the payoff in a reverse auction without a reserve price. In the second class, the cluster point set is a single point \(\{c\}\), which implies that sellers with different costs are pooled in intervals except the one with the lowest possible cost, and information about the sellers’ costs is barely revealed in the first many rounds. In this case, the buyer does not raise the bid much until the very end, the ending price is lower than \(\bar{c}\), and the buyer’s payoff is greater than the payoff in a reverse auction without a reserve price. In the second type of equilibrium paths, most transactions occur just before the deadline. The delay of transactions incurs waiting cost if the buyer has time preference. Therefore, setting a lockout period might actually benefit a buyer by moving transactions forward.

Discussion on the modeling approach and Priceline’s mechanism As in many other applied papers, our model is a simplification and an abstraction of the real world, so it does not capture some features of reality. First of all, the buyer’s value is probably not known by the sellers. However, in reality, the information seems less imperfect than the information about the sellers’ costs (the lowest prices they are willing to accept), which depend on the amounts of the excess inventory left over from their traditional retail channels. Therefore, while assuming that the sellers’ costs are private information, we assume that the buyer’s valuation is publicly known. This assumption also allows us to get around the signaling issue and focus on how the bidding path is designed to elicit the sellers’ information.

Next, although Priceline classifies hotels into different star ratings to ease customers’ quality concern when buying through NYOP, there might still be diversity within the same rating class. Hence, a customer might worry about the adverse selection problem, that is, if he bids low, only those whose quality is at the low end will accept. The adverse selection problem might increase a customer’s starting bid as well as add twists to the subsequent bidding path. Accounting for the quality dimension is an interesting topic for future research, and as suggested in the procurement literature (see, e.g., Manelli and Vincent (1995), Morand and Thomas...
(2002), and Asker and Cantillon (2010)), a more sophisticated trading procedure might be necessary to achieve efficiency and to increase total surplus. This can explain why we recently saw offer-counter-offer negotiation mechanisms arising on eBay and iOffer.com.

**Extensions**  This paper indicates some interesting directions for future research. Based on our analysis, one might be curious about whether Priceline could do better by adopting other measures, such as restricting the number of bidding chances instead of the frequency of bidding. Moreover, one might extend the model to consider the cases when the buyer has private information, when there are multiple buyers bidding at the same time, and when the quality of the units provided by different sellers is not consistent, so that the adverse selection problem exists. The following are some of our conjectures about the equilibrium path in such extended circumstances.

When the buyer’s value is private information, signaling issues arise. Since types with lower value are more inclined to bid low and have \( x_T \) lower than \( \bar{c} \), if sellers know that the buyer is of a lower type, they will accept lower prices. Knowing this, high types might try to imitate low types, and low types might try to bid lower in order to be distinguished from high types. Therefore, a convexly increasing path is still likely to happen, and the lockout period rule might still benefit the buyer.

When there are multiple buyers bidding at the same time (but the number of sellers is still larger than the number of buyers), everyone will try to get a unit from the seller with the lowest cost first. In the beginning, competition among the buyers raises the price higher than it would have been if there were only one buyer, and the sellers also raise the price thresholds that they are willing to accept. Once a buyer’s offer is accepted by a seller, both leave the market, and the price drops to a new level. Note that the price drops suddenly and is lower than it would have been if no buyer had gotten a unit, which also means that the remaining sellers are willing to accept a lower price after a buyer and a seller leave the market. This is because the remaining sellers have rejected all prices offered before, so all players believe that the remaining units will be sold at higher costs; if at the same time, the sellers asked for a higher price after a buyer had left, then the remaining buyers would have to pay an even higher price. This would make the buyers try harder to get the first unit, and the prices in the beginning would be driven up further. Therefore, along the equilibrium path, when there are more buyers staying in the
market, the sellers ask for higher prices but the units sold are provided by sellers with lower costs; and when fewer buyers are left in the market, the sellers ask for lower prices and the units are sold by sellers with higher costs. This is an interesting phenomenon waiting for future theoretical and empirical investigation.

Exploring these extensions will bring us one step closer to the reality and help us better understand the NYOP mechanism as well as other bargaining and price determination processes in the real world.

\section*{Appendix}

\textbf{Proposition 5} In a symmetric pure strategy equilibrium, if a seller with cost \( x \) accepts in round \( t \), then a seller with cost \( x' < x \) also accepts in round \( t \).

\textbf{Proof.} Suppose that at the beginning of round \( t \), a seller believes that the other sellers’ costs are in set \( S_t \subset [c, \overline{c}] \). After seeing the current price \( b_t \), it is expected that a seller with cost in set \( A_t \subset S_t \) accepts in round \( t \), and if no seller accepts, the buyer will offer \( b_{t+1} \) in round \( t+1 \), and a seller with cost in set \( A_{t+1} \subset S_{t+1} \equiv S_t \setminus A_t \) will accept. If a seller with cost \( x \) accepts in round \( t \) instead of round \( t+1 \), it implies that the seller gets higher expected payoff by accepting in round \( t \) than by accepting in round \( t+1 \), i.e.,

\begin{equation}
(b_t - x) \sum_{n=0}^{N-1} \frac{1}{n + 1} \frac{(N-1)!}{n! (N-n-1)!} \Pr(x \in A_t \mid x \in S_t)^n (1 - \Pr(x \in A_t \mid x \in S_t))^{N-n-1} > \Pr(x \in S_{t+1} \mid x \in S_t)^N - 1 \times \Pr(x \in A_{t+1} \mid x \in S_{t+1})^n (1 - \Pr(x \in A_{t+1} \mid x \in S_{t+1}))^{N-n-1}
\end{equation}

\begin{align}
&= (b_t - x) \sum_{n=0}^{N-1} \frac{1}{N!} \frac{(N-1)!}{n! (N-n-1)!} (\Pr(x \in A_t \mid x \in S_t)^n (1 - \Pr(x \in A_t \mid x \in S_t))^{N-n-1}) \\
&\Rightarrow (b_t - x) \sum_{n=0}^{N-1} \frac{\Pr(x \in S_t) - \Pr(x \in A_t)^N}{N \Pr(x \in S_t)} \\
&> (b_{t+1} - x) \sum_{n=0}^{N-1} \frac{\Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1})^N}{N \Pr(x \in S_{t+1})}.
\end{align}

Since \( \Pr(x \in S_t) - \Pr(x \in A_t) > \Pr(x \in S_{t+1}) - \Pr(x \in A_{t+1}) \) and \( \Pr(x \in S_t) > \)}
Proof of Proposition 1. In the last period, recall that
\[
V_T(x_{T-1}) = \max_{x_T \in [x_{T-1}, \bar{x}]} (v - x_T) P(x_{T-1}, x_T),
\]
\[
\bar{x}_T(x_{T-1}) \in X_T(x_{T-1}) = \arg \max_{x_T \in [x_{T-1}, \bar{x}]} (v - x_T) P(x_{T-1}, x_T),
\]
and
\[
C_T(x_{T-1}) = (\bar{x}_T(x_{T-1}) - x_{T-1})G(x_{T-1}, \bar{x}_T(x_{T-1})).
\]
By Berge's maximum theorem, we know that $V_T(x_{T-1})$ is continuous and $X_T(x_{T-1})$ is upper hemi-continuous. In period $t$, $t < T$, let
\[
\phi_t(x_{t-1}, x_t) = (v - x_t) \left[ \bar{F}(x_{t-1})^N - \bar{F}(x_t)^N \right] - C_{t+1}(x_t) \left[ \bar{F}(x_{t-1}) - \bar{F}(x_t) \right] + V_{t+1}(x_t),
\]
\[
\alpha(x_{t-1}) = [x_{t-1}, \bar{x}].
\]
Then
\[
V_t(x_{t-1}) = \max_{x_t \in \alpha(x_{t-1})} \phi_t(x_{t-1}, x_t),
\]
\[
\bar{x}_t(x_{t-1}) \in X_t(x_{t-1}) = \arg \max_{x_t \in \alpha(x_{t-1})} \phi_t(x_{t-1}, x_t).
\]
We show that by picking a proper $\bar{x}_t (x_{t-1})$ from $X_t (x_{t-1}), t \leq T$, each round-$t$ program has a solution.

First observe that for upper hemi-continuous correspondence $X_T$, we are able to find (i) $n_T$ closed intervals $[a_k, a_{k+1}], k = 1, \cdots, n_T$, such that $\cup_k [a_k, a_{k+1}] = [\underline{c}, \overline{c}]$, and (ii) $n_T$ continuous functions $x_T, k : [a_k, a_{k+1}] \rightarrow [\underline{c}, \overline{c}], k = 1, \cdots, n_T$, such that $x_T, k (x) \in X_T (x), \forall x \in [a_k, a_{k+1}]$. Let

$$
\bar{x}_T (x_{T-1}) = \begin{cases} 
\bar{x}_T, k (x_{T-1}), \text{ if } x_{T-1} \in (a_k, a_{k+1}) \\
\arg \min_{x \in \{x_T, k (x_{T-1}), x_T, k+1 (x_{T-1})\}} (x - x_{T-1}) G (x_{T-1}, x), \text{ if } x_{T-1} = a_{k+1}, k < n_T 
\end{cases}
$$

$$
C_T (x_{T-1}) = (\bar{x}_T (x_{T-1}) - x_{T-1}) G (x_{T-1}, \bar{x}_T (x_{T-1})),
$$

$$
\bar{b}_T (x_{T-1}) = \bar{x}_T (x_{T-1}).
$$

$C_T$ is lower semi-continuous and $V_T$ is continuous, so $\phi_{T-1}$ is upper semi-continuous. Note that $\phi_{T-1}$ is graph-continuous with respect to $\alpha$, which is defined in Leininger (1984). So by Leininger’s generalized maximum theorem, $V_{T-1}$ is upper semi-continuous, and $X_{T-1}$ is upper hemi-continuous.

Similarly, since $X_{T-1}$ is upper hemi-continuous, we are able to find (i) $n_{T-1}$ closed intervals $[a'_k, a'_{k+1}], k = 1, \cdots, n_{T-1}$, such that $\cup_k [a'_k, a'_{k+1}] = [\underline{c}, \overline{c}]$, and (ii) $n_{T-1}$ continuous functions $x_{T-1, k} : [a'_k, a'_{k+1}] \rightarrow [\underline{c}, \overline{c}], k = 1, \cdots, n_{T-1}$, such that $x_{T-1, k} (x) \in X_{T-1} (x), \forall x \in [a'_k, a'_{k+1}]$. Let

$$
\bar{x}_{T-1} (x_{T-2}) = \begin{cases} 
\bar{x}_{T-1, k} (x_{T-2}), \text{ if } x_{T-2} \in (a'_k, a'_{k+1}) \\
\arg \min_{x \in \{x_{T-1, k} (x_{T-2}), x_{T-1, k+1} (x_{T-2})\}} (x - x_{T-2}) G (x_{T-2}, x) + C_T (x), \text{ if } x_{T-2} = a'_{k+1} 
\end{cases}
$$

$$
C_{T-1} (x_{T-2}) = (\bar{x}_{T-1} (x_{T-2}) - x_{T-2}) G (x_{T-2}, \bar{x}_{T-1} (x_{T-2})) + C_T (\bar{x}_{T-1} (x_{T-2})),
$$

$$
\bar{b}_{T-1} (x_{T-2}) = \bar{x}_{T-1} (x_{T-2}) + \frac{C_T (\bar{x}_{T-1} (x_{T-2}))}{G (\bar{x}_{T-2}, \bar{x}_{T-1} (x_{T-2})).}
$$

$C_{T-1}$ is lower semi-continuous and $V_{T-1}$ is upper semi-continuous, so $\phi_{T-2}$ is upper semi-continuous. Note that $\phi_{T-2}$ is graph-continuous with respect to $\alpha$. Applying the same procedure backward, we conclude that there exists a set of solutions $\{\bar{b}_t (x_{t-1}), \bar{x}_t (x_{t-1})\}_t$ that solves program P1 and P2 for all $t$. \qed

**Lemma 1** Assume Condition 1. $\bar{x}_t (x_{t-1})$ (defined in program P2) increases in $x_{t-1}$.  

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\textbf{Proof.} It is easy to check that $\pi_t^T (x_{t-1})$ defined in program P1 increases in $x_{t-1}$.

With $t < T$, $\pi_t^T (x_{t-1})$ is derived from program P2. Let

$$\varphi (x_t, x_{t-1}) \equiv (v - b_t (x_t; x_{t-1})) \left[ \tilde{F} (x_{t-1})^N - \tilde{F} (x_t)^N \right] + V_{t+1}^T (x_t),$$

where $b_t (x_t; x_{t-1}) = \frac{C_{t+1}^T (x_t)}{\tilde{F} (x_t)^{N-1} + \tilde{F} (x_t)^{N-2} F (x_{t-1}) + \cdots + F (x_{t-1})^{N-1}} + x_t.$

$$\frac{\partial \varphi (x_t, x_{t-1})}{\partial x_t} = - \left[ \tilde{F} (x_{t-1}) - \tilde{F} (x_t) \right] \left[ \left( \tilde{F} (x_{t-1})^{N-1} + \cdots + \tilde{F} (x_t)^{N-1} \right) + \frac{dC_{t+1}^T (x_t)}{dx_t} \right]$$

$$+ f (x_t) \left( \pi_{t+1} (x_t) - x_t \right) \left[ N \tilde{F} (x_t)^{N-1} - \left( \tilde{F} (\pi_{t+1} (x_t))^N - 1 \right) \right]$$

and

$$\frac{\partial^2 \varphi (x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} = f (x_t) \left[ N \tilde{F} (x_{t-1})^{N-1} + \frac{dC_{t+1}^T (x_t)}{dx_t} \right].$$

For any $x_t, x_{t-1},$ and $x'_{t-1} \in [x_{t-1}, x_t]$, if $\left( \tilde{F} (x_{t-1})^{N-1} + \cdots + \tilde{F} (x_t)^{N-1} \right) + \frac{dC_{t+1}^T (x_t)}{dx_t} < 0$ (which also implies $\left( \tilde{F} (x'_{t-1})^{N-1} + \cdots + \tilde{F} (x_t)^{N-1} \right) + \frac{dC_{t+1}^T (x_t)}{dx_t} < 0$), then $\frac{\partial \varphi (x_t, x_{t-1})}{\partial x_t} > 0$ and $\frac{\partial \varphi (x_t, x'_{t-1})}{\partial x_t} > 0$ if $\left( \tilde{F} (x_{t-1})^{N-1} + \cdots + \tilde{F} (x_t)^{N-1} \right) + \frac{dC_{t+1}^T (x_t)}{dx_t} > 0$, then

$$N \tilde{F} (x_{t-1})^{N-1} + \frac{dC_{t+1}^T (x_t)}{dx_t} > 0,$$ so $\frac{\partial^2 \varphi (x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} > 0$. Since $\frac{\partial^2 \varphi (x_t, x_{t-1})}{\partial x_t \partial x_{t-1}} > 0$, $\frac{\partial \varphi (x_t, x_{t-1})}{\partial x_t}$ is maximized when $x = \pi_t^T (x_{t-1})$ and when $x = \pi_t^T (x'_{t-1})$ respectively, the facts that $\frac{\partial \varphi (x_t, x_{t-1})}{\partial x_t} > 0$ implies $\frac{\partial \varphi (x_t, x_{t-1})}{\partial x_t} > 0$ and that $\pi_t^T (x_{t-1})$ is continuous lead to the conclusion that $\pi_t^T (x_{t-1})$ increases in $x_{t-1}$. \hspace{1cm} \blacksquare$

\textbf{Proposition 6} A path that fully separates sellers is a stationary solution to program P3 when $T = \infty$.

\textbf{Proof.} If the buyer fully separates sellers, we can rewrite the necessary condition
(11) for a stationary solution as

\[ 0 = \left[ \bar{F}(x - dx) - \bar{F}(x) \right] \left[ -\bar{F}(x - dx)^{N-1} - \cdots - \bar{F}(x)^{N-1} - C'(x) \right] \]
\[ - f(x) \, dx \left[ \bar{F}(x + dx)^{N-1} + \cdots + \bar{F}(x)^{N-1} - N\bar{F}(x)^{N-1} \right] \]
\[ \Rightarrow 0 = f(x) \, dx \left[ -N\bar{F}(x)^{N-1} - \frac{(N-1)N}{2} \bar{F}(x)^{N-2} f(x) \, dx - C'(x) \right] \]
\[ - f(x) \, dx \left[ N\bar{F}(x)^{N-1} - \frac{(N-1)N}{2} \bar{F}(x)^{N-2} f(x) \, dx - N\bar{F}(x)^{N-1} \right], \]

where \( dx \) is a positive number that can be arbitrarily small. Note that \( \frac{C(x)}{N} \) can be considered as the information rent given to a seller with cost \( x \). In our setting, in an incentive compatible mechanism that fully separates sellers with different costs, the information rent \( R(x) \) has the property that \( R'(x) = -\bar{F}(x)^{N-1} \), so \( \frac{C(x)}{N} = -\bar{F}(x)^{N-1} \). Therefore, the necessary condition holds. Given \( x_{t-1} \) and that \( dx = x_{t+1} - x_t \) is arbitrarily small, if \( x_t - x_{t-1} \notin O(dx) \), i.e. \( x_t - x_{t-1} \) does not go to 0 as \( dx \), then (11) is negative. So the necessary condition is also sufficient, and a path that fully separates sellers is a stationary solution. ■

### B Appendix

We use Lemmas 2, 3, and 4 to prove Lemma 6 and Lemma 7, and use Lemmas 6, 7, and 4 to prove Theorem 1. We occasionally add superscript \( T \) to \( V_t(x) \) and \( C_t(x) \) (defined in Section 3) for clarification. Note that for two sets \( (t, T) \) and \( (t', T') \), if \( T - t = T' - t' \), then \( V^T_t(x) = V^{T'}_{t'}(x) \) and \( C^T_t(x) = C^{T'}_{t'}(x) \). So, we let \( c_k(x) = C^{T-k+1}_T(x) \).

**Proof of Theorem 1.** By Lemma 7, if \( B \neq \{\varepsilon\} \), there does not exist \( a \in (\underline{\varepsilon}, \bar{\varepsilon}) \) such that \( (a, \bar{\varepsilon}) \subset [\underline{\varepsilon}, \bar{\varepsilon}] \setminus B \). Then by Lemma 6, \( B = [\underline{\varepsilon}, \bar{\varepsilon}] \). So the first statement is proved. The third statement follows from the revenue equivalence principle. For the second statement, if \( \lim_{T \to -\infty} x^T_T < \bar{\varepsilon} \), it must be that \( B = \{\varepsilon\} \). On the other hand, if \( B = \{\varepsilon\} \), there exists \( t < \infty \) such that \( \bar{\varepsilon} - \lim_{T \to -\infty} x^T_{T-t} > 0 \). By Lemma 4, \( \lim_{T \to -\infty} x^T_T < \bar{\varepsilon} \). ■

**Lemma 2** Assume Condition 1. If \( \epsilon \equiv x^T_{t+1} - x^T_t > 0 \), then \( \delta \equiv x^T_t - x^T_{t-1} > 0 \). The value of \( \delta \) depends on \( x^T_t \), \( \epsilon \), and \( T - t \).
Proof. Given any $t$, $T$ and given belief $x_{t-1}$, the continuation equilibrium $x^*_t$ and $b^*_t$ are derived from
\[
V^T_t (x_{t-1}) = \max_{x_t} (v - b_t) \left[ \bar{F}(x_{t-1})^N - \bar{F}(x_t)^N \right] + V^T_{t+1} (x_t), \tag{P4}
\]
where $b_t = \frac{C^T_{t+1} (x_t)}{F(x_t)^{N-1} + F(x_t)^{N-2}F(x_{t-1}) + \cdots + F(x_{t-1})^{N-1} + x_t}$.

The solution $x^*_t$ must satisfy the first-order condition
\[
0 = \left[ \bar{F}(x_{t-1}) - \bar{F}(x^*_t) \right] \left[ - \left( \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1} \right) - C^T_{t+1} (x^*_t) \right] - C^T_{t+1} (x^*_t) f (x^*_t) + N (v - x^*_t) \bar{F}(x^*_t)^{N-1} f (x^*_t) + V^T_{t+1} (x^*_t). \tag{8}
\]

Note that
\[
V^T_{t+1} (x_t) = \max_{\{b_{t+1}, x_{t+1}\}} (v - b_{t+1} (x_{t+1}; x_t)) \left[ \bar{F}(x_t)^N - \bar{F}(x_{t+1})^N \right] + V^T_{t+2} (x_{t+1}) \tag{9}
\]
where $b_{t+1} (x_{t+1}; x_t) = \frac{C^T_{t+2} (x_{t+1})}{F(x_{t+1})^{N-1} + F(x_{t+1})^{N-2}F(x_t) + \cdots + F(x_t)^{N-1} + x_t}$.

Let $x^*_t (x_t)$ be the solution to program (9). By the envelope theorem,
\[
V^T_{t+1} (x_t) = -N \bar{F}(x_t)^{N-1} f (x_t) (v - x^*_t) + f (x_t) C^T_{t+2} (x^*_t; t+1). \tag{10}
\]

Plugging into (8), we get
\[
0 = \left[ \bar{F}(x_{t-1}) - \bar{F}(x^*_t) \right] \left[ -C^T_{t+1} (x^*_t) - \left( \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1} \right) \right] - \left[ C^T_{t+1} (x^*_t) - C^T_{t+2} (x^*_t) \right] f (x^*_t) - N \bar{F}(x^*_t)^{N-1} f (x^*_t) [x^*_t - x^*_{t+1}] \\
= - \left[ \bar{F}(x_{t-1}) - \bar{F}(x^*_t) \right] \left[ \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1} \right] + c_{T-t} (x^*_t) \\
+ f (x^*_t) (x^*_{t+1} - x^*_t) \left[ N \bar{F}(x^*_t)^{N-1} - (\bar{F}(x^*_t)^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1}) \right]. \tag{11}
\]

If $x^*_t = x^*_{t+1} - \epsilon$,
\[
(x^*_{t+1} - x^*_t) \left[ N \bar{F}(x^*_t)^{N-1} - (\bar{F}(x^*_t)^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1}) \right] \text{ is strictly positive, and so by (11),} \\
\left[ \bar{F}(x_{t-1}) - \bar{F}(x^*_t) \right] \left[ \bar{F}(x_{t-1})^{N-1} + \cdots + \bar{F}(x^*_t)^{N-1} \right] + c_{T-t} (x^*_t) \text{ is strictly positive. Proposition 7 in the online appendix implies that } c_{T-t} (x) \text{ exists and is bounded. Therefore, by (11), if } x^*_{t+1} > 0, x^*_t - x_{t-1} > 0. \text{ Moreover, the}
The difference between \( x_{t-1} \) and \( x_t^* \) only depends on \( x_t^* \), \( \epsilon (\epsilon = x_{t+1}^* - x_t^*) \), and \( T - t \). 

**Lemma 3** Assume Condition 1. In a continuation game starting from round \( t \) \((t < T - 1)\) with the belief that the greatest lower bound of a seller’s cost is \( x_{t-1} \), when the number of rounds left in the continuation game goes to infinity, \( x_t \) approaches to \( x_{t-1} \) on the continuation equilibrium path but \( x_t \neq x_{t-1} \).

**Proof.** In the continuation game, the equilibrium path \( \{x_t\}_{t \leq \tau \leq T} \) and \( \{b_t\}_{t \leq \tau \leq T} \) are derived from program P4. As \( T \rightarrow \infty \), the value of the program converges, so the additional payoff a buyer can get by adding one more round goes to 0. In the following proof, we show that when one more round is added to the continuation game, the additional payoff the buyer can get does not go to 0 as \( T \rightarrow \infty \) if there exists \( \epsilon > 0 \) such that \( x_t > x_{t-1} + \epsilon \) for all \( T \). Since the buyer’s payoff must converge, when \( T \rightarrow \infty \), \( x_t \rightarrow x_{t-1} \).

Let \( \{x_t^*, b_t^*\}_{t \leq \tau \leq T} \) be the equilibrium path when there are \( T - t \) rounds left, which can be derived from program P4. If we add a constraint \( x_t = x_{t-1} \) to P4 and let \( \{x_t', b_t'\}_{t \leq \tau \leq T} \) be the solution to the program, then the buyer’s payoff and \( \{x_t', b_t'\}_{t+1 \leq \tau \leq T} \) would be the same as those in the continuation game with \( T - t - 1 \) rounds left. The value of the program is

\[
V_t (x_{t-1}) = (v - b_t') \left[ \bar{F}(x_{t-1})^N - \bar{F}(x_t')^N \right] + V_{t+1} (x_t'), \text{ where } x_t' = x_{t-1}.
\]

Without the constraint, \( x_t' \) can be increased by \( \epsilon \), and \( V_t (x_{t-1}) \) increases approximately by

\[
\left[ (v - b_t') N \bar{F}(x_t')^{N-1} f(x_t') + V_{t+1} (x_t') \right] \epsilon.
\]

\[
= \left[ (v - b_t') N \bar{F}(x_t')^{N-1} f(x_t') - N \bar{F}(x_t')^{N-1} (v - x_{t+1}) f(x_t') 
+ (b_t' - x_{t+1}) (\bar{F}(x_{t+1})^{N-1} + \bar{F}(x_{t+1})^{N-2} \bar{F}(x_t') + \cdots + \bar{F}(x_t')^{N-1}) f(x_t') \right] \epsilon.
\]

\[
= (x_{t+1} - x_t') \left[ N \bar{F}(x_t')^{N-1} - (\bar{F}(x_{t+1})^{N-1} + \bar{F}(x_{t+1})^{N-2} \bar{F}(x_t') + \cdots + \bar{F}(x_t')^{N-1}) \right] f(x_t') \epsilon.
\]

The second equation comes from \((b_t' - x_{t+1}) N \bar{F}(x_t')^{N-1} = (b_t' + x_t') (\bar{F}(x_t')^{N-1} + \cdots + \bar{F}(x_{t+1})^{N-1})\).

Therefore, if \( x_{t+1}' > x_t' + \epsilon \), increasing the number of rounds from \( T - t - 1 \) to \( T - t \) strictly increases the buyer’s payoff, and the additional payoff does not go to 0 as \( T \rightarrow \infty \).”

**Lemma 4** Given any \( T \) and \( t < \infty \), if \( x_{T-1}^T < \bar{c} \), then \( x_T^T < \bar{c} \) and \( x_T^T - x_{T-1}^T > 0 \).
Proof. Suppose that \( x_{T-1}^T < \tau \). When \( t = 1 \), given that \( x_{T-1}^T < \tau \), by (3), \( x_T^T < \tau \) and \( x_T^T - x_{T-1}^T > 0 \). When \( t = 2 \), given that \( x_{T-2}^T < \tau \), if \( x_{T-1}^T = \tau \), then \( x_T^T = \tau \). This implies that the buyer pays for the good at a price higher than or equal to \( \tau \), which is not optimal for the buyer and cannot happen in equilibrium. Therefore, \( x_{T-1}^T < \tau \), and we can apply the result we get in the case when \( t = 1 \).

Applying the same argument to the case where \( t = 3, 4, \ldots \), we can conclude that, for any \( t \), if \( x_{T-t}^T < \tau \), then \( x_T^T < \tau \) and \( x_T^T - x_{T-t}^T > 0 \). ■

Recall that \( B \) is the set of cluster points defined on page 15, and \([c, \tau] \setminus B\) is the complement of \( B \). Note that set \( B \) is closed.

Lemma 5 \textit{The set of cluster points \( B \) is closed.}

Proof. If \( B \) is not closed, there exists \( y \in [c, \tau] \setminus B \) such that for any \( \epsilon > 0 \), there is \( z \in B \) such that \( |z - y| < \frac{\epsilon}{2} \). Since \( z \in B \), given \( \epsilon \), for any \( M \), there exists \( T' \) such that for all \( T > T' \), \( \| \{ x \in X^T \mid |x - z| < \frac{\epsilon}{2} \} \| > M \). But then for all \( T > T' \), \( \| \{ x \in X^T \mid |x - y| < \epsilon \} \| > M \), so \( y \in B \), a contradiction. Therefore, \( B \) is closed. ■

Lemma 6 \textit{Assume Condition 1. If} \( a \in B \), \( [c, a] \subset B \).

Proof. By Lemma 3, \( c \in B \). If not the whole interval \([c, a]\) belongs to \( B \), by Lemma 5, there exist \( b, c \) such that \((b, c) \subset [c, \tau] \setminus B\) and \( b, c \in B \). For any \( z \in (b, c) \), there exist \( \epsilon \) and \( M \) such that there does not exist \( T' \) such that for all \( T > T' \), \( \| \{ x \in X^T \mid |x - z| < \epsilon \} \| > M \).

Given \( z \), \( \epsilon \), and \( M \), let \( E_1 \equiv \{ \epsilon_1 \mid \text{There does not exist } T' \text{ such that for all } T > T', \| \{ x \in X^T \mid x \in (z - \epsilon_1, z + \epsilon) \} \| > M \} \) and let \( \epsilon_1 \equiv \sup (E_1) \).\(^{11}\) Note that there exists an infinite sequence \( \{ T_1, T_2, \ldots \} \) such that \( \| \{ x \in X^T \mid x \in (z - \epsilon_1, z + \epsilon) \} \| \leq M \) if \( T = T_1, T_2, \ldots \). Since \( a \in B \) (the number of \( x_i \)'s falling around \( a \) increases as \( T \to \infty \)), there exists an infinite subsequence \( \{ S_1, S_2, \ldots \} \subset \{ T_1, T_2, \ldots \} \) such that for those \( T = S_1, S_2, \ldots \), the number of \( x_i \)'s falling in \( (z - \epsilon_1, \tau) \) increases with \( T \).\(^{12}\) By Lemma 3 and Condition 1, unboundedly many \( x_i \)'s \in \( (z - \epsilon_1, \tau) \) will approach \( z - \epsilon_1 \) when \( T = S_1, S_2, \ldots \) goes to infinity.\(^{13}\)

---

\(^{11}\) \(E_1\) is not empty because \( \epsilon \in E_1 \).

\(^{12}\) Let \( B_1 \) be the cluster point set in \([z - \epsilon_1, \tau]\). If there does not exist such a subsequence, this implies that the number of \( x_i \)'s in \([z - \epsilon_1, \tau]\) is bounded as \( T \to \infty \). Therefore, the number of \( x_i \)'s in \([z - \epsilon_1, \tau] \setminus B_1\) decreases as \( T \to \infty \), but this cannot be true because \( T \) is unbounded, and the number of \( x_i \)'s is non-negative.

\(^{13}\) Note that there exists a sequence \( \{ x_i \mid x_i \in X^{S_i}, x_i \leq z - \epsilon_1 \} \}_{i \in \mathbb{N}} \) such that \( \lim_{i \to \infty} x_i = z - \epsilon_1 \) since \( \epsilon_1 \equiv \sup (E_1) \).
Therefore, the number of \( x_i \)'s falling in \( (z - \varepsilon, z + \varepsilon) \) cannot be less than \( M \), a contradiction. \( \blacksquare \)

**Lemma 7** Assume Condition 1. If there exists \( a \in (\underline{c}, \overline{c}) \) such that \( (a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \), then \( [\underline{c}, \overline{c}] \setminus B = (a, \overline{c}) \).

**Proof.** By Lemma 6, we only need to show that it cannot be the case that \( (a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \) and \( (a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \), first, supposing \( (a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \) and \( (a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \), we show that when \( T \to \infty \), there exists \( x \in X_T \), \( x > a \), that is arbitrarily close to \( a \).

Since \((a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \), there exists \( t < \infty \) such that \( \overline{c} - \lim_{T \to \infty} x_{t}^{T} > 0 \). By Lemma 4, \( \lim_{T \to \infty} x_{t}^{T} < \overline{c} \) and \( \lim_{T \to \infty} x_{t}^{T} - \lim_{T \to \infty} x_{t-1}^{T} > 0 \). By Lemma 2, \( \lim_{T \to \infty} x_{t-1}^{T} - \lim_{T \to \infty} x_{t-s}^{T} > 0 \), for all \( s < \infty \). Since \((a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \), it must be the case that \( \lim_{T \to \infty} \lim_{T \to \infty} x_{t-1}^{T} = a \), and \( \lim_{T \to \infty} [\lim_{T \to \infty} x_{t}^{T} - a] = 0 \). Since \([\underline{c}, \overline{c}] \subset B \) and \( \lim_{T \to \infty} \lim_{T \to \infty} x_{t}^{T} = a \), we can rewrite the necessary condition (11) for the optimality problem as

\[
0 = \left[ \tilde{F}(x - dx) - \tilde{F}(x) \right] - f(x) dx^{+} \left[ \tilde{F}(x + dx^{+}) - \tilde{F}(x) \right] - f(x) dx^{-} \left[ \tilde{F}(x - dx^{-}) - \tilde{F}(x) \right]
\]

where \( C(x) = \lim_{t \to \infty} c_{t}(x) \), \( x \in [\underline{c}, \overline{c}] \), and \( dx^{-} \) and \( dx^{+} \) are two positive numbers which can be arbitrarily small. For \( x \in (\underline{c}, a) \), since \([\underline{c}, a] \subset B \), \( dx^{-} \in O(dx^{+}) \) and \( dx^{+} \in O(dx^{-}) \), so an approximation of equation (12) is

\[
\Rightarrow 0 = f(x) dx^{-} \left[ -N\tilde{F}(x)^{N-1} - \frac{(N - 1)N}{2} \tilde{F}(x)^{N-2} f(x) dx^{-} - C'(x) \right] - f(x) dx^{+} \left[ N\tilde{F}(x)^{N-1} - \frac{(N - 1)N}{2} \tilde{F}(x)^{N-2} f(x) dx^{+} - N\tilde{F}(x)^{N-1} \right],
\]

and it implies \( C'(x) = -N\tilde{F}(x)^{N-1} \) for \( x \in (\underline{c}, a) \).

Since \((a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \) and \([\underline{c}, a] \subset B \), if the starting belief of a continuation game decreases from \( a \) to \( a - \varepsilon \), where \( \varepsilon \) is a small positive number, \( \lim_{T \to \infty} x_{t}^{T} \) does not change for all \( t \); but if the starting belief increases from \( a \) to \( a + \varepsilon \), \( \lim_{T \to \infty} x_{t}^{T} \) increases as well for all \( t \). \( C \) is composed of \( \lim_{T \to \infty} x_{t}^{T} \), \( t = 1, 2, \ldots \). Therefore, the left derivative and the right derivative of \( C(a) \) are not equal, i.e. \( \lim_{x \to a} \frac{C(x) - C(a)}{x - a} \neq \lim_{x \to a} \frac{C(x) - C(a)}{x - a} = -N\tilde{F}(a)^{N-1} \), and \( C'(x) \) is not continuous at \( a \). However, by Lemma 8, \( C'(x) \) is continuous at \( a \), a contradiction. Therefore, a path that \([\underline{c}, a] \subset B \) and \((a, \overline{c}) \subset [\underline{c}, \overline{c}] \setminus B \) cannot occur in equilibrium. Note that there must be at least
one cluster point in \([c, \overline{c}]\). Since only \(c\) can be in \(B\), \([c, \overline{c}] \setminus B = (c, \overline{c}]\).

**Lemma 8** *(Sublemma of Lemma 7)* Assume Condition 1. \(C'(x)\) is continuous at \(a\), where \(C'(x)\) and \(a\) are defined in Lemma 7.

**Proof.** \(C'(x) = \lim_{t \to -\infty} c'_t(x)\). Note that \(c_t(x) = C^T_{t-t+1}(x) = C^1_1(x)\), so

\[
c_1(x) = C^1_1(x) = (\pi^1_1(x) - x)G(x, \pi^1_1(x)),
\]

and for \(t \geq 2\)

\[
c_t(x_t) = C^t_1(x_t) = (\pi^t_1(x_t) - x_t)G(x_t, \pi^t_1(x_t)) + C^t_2(\pi^t_1(x_t)) = (\pi^t_1(x_t) - x_t)G(x_t, \pi^t_1(x_t)) + c_{t-1}(\pi^t_1(x_t)).
\]

Then

\[
c'_t(x_t) = \left[ (\pi''_1(x_t) - 1) G(x_t, \pi''_1(x_t)) \right. \\
\left. + (\pi''_1(x_t) - x_t) \left( \frac{\partial G}{\partial x_t} + \frac{\partial G}{\partial \pi''_1(x_t)} \right) \right] + c'_{t-1}(\pi''_1(x_t)) \pi''_1(x_t).
\]  

(13)

First, let \(x_t = a\). Because \(\pi''_1(x_t) > a\), by the discussion in the last paragraph of the proof of Lemma 7, \(c'_{t-1}(\pi''_1(x_t))\) does not converge to \(-N \hat{F}(a)^N\), so by Lemma 9, \(\pi''_1(x_t)\) converges to 1. Moreover, \(\lim_{t \to -\infty} \pi''_1(x_t) = x_t = a\), and \(\{G(x_t, \pi''_1(x_t))\}_t\) and \(\{\frac{\partial G}{\partial x_t} + \frac{\partial G}{\partial \pi''_1(x_t)}\}_t\) are uniformly bounded, so (13) implies that \(\lim_{t \to -\infty} c'_t(a) = \lim_{x_t \to a} \lim_{t \to -\infty} c'_t(x)\). Next, because \([c, a] \subset B\) and \(\lim_{t \to -\infty} \lim_{T \to -\infty} x^T_{t-T} = a\), we can find a sequence \(\{x_t \in X^\prime\}_t\) such that \(x_t < a\), \(\lim_{t \to -\infty} x_t = a\), \(\pi''_1(x_t) \geq a\), and \(\lim_{t \to -\infty} \pi''_1(x_t) = a\). Similarly, \(\lim_{t \to -\infty} \pi''_1(x_t) = 1\), \(\lim_{t \to -\infty} \pi''_1(x_t) = a\), and \(\{G(x_t, \pi''_1(x_t))\}_t\) and \(\{\frac{\partial G}{\partial x_t} + \frac{\partial G}{\partial \pi''_1(x_t)}\}_t\) are uniformly bounded, so (13) implies that \(\lim_{x_t \to a} \lim_{t \to -\infty} c'_t(x) = \lim_{x_t \to a} \lim_{t \to -\infty} c'_t(x)\). Therefore, \(C'(x) = \lim_{t \to -\infty} c'_t(x)\) is continuous at \(a\). ■

**Lemma 9** *(Sublemma of Lemma 7)* Assume Condition 1. If \(c'_{t-1}(\pi''_1(x_t))\) does not converge to \(-N \hat{F}(a)^N\) when \(t \to \infty\) and \(\lim_{t \to -\infty} x_t = a\), then \(\pi''_1(x_t)\) converges to 1 when \(t \to \infty\).


**Proof.** By the first-order condition

\[ 0 = \phi_{t+1}\left(\bar{x}_t^1(x_t), x_t\right) \]

\[ = -\left[\bar{F}(x_t) - \bar{F}(\bar{x}_t^1(x_t))\right] \left[\left(\bar{F}(x_t)^{N-1} + \cdots + \bar{F}(\bar{x}_t^1(x_t))^{N-1}\right) + c_{t-1}(\bar{x}_t^1(x_t))\right] \]

\[ + f\left(\bar{x}_t^1(x_t)\right) \left[\left(\bar{x}_t^2(x_t) - \bar{x}_t^1(x_t)\right)\right] \]

\[ \times \left[N\bar{F}(\bar{x}_t^1(x_t))^N - (\bar{F}(\bar{x}_t^1(x_t)))^{N-1} + \cdots + \bar{F}(\bar{x}_t^1(x_t))^{N-1}\right] \]

and the Implicit Function Theorem, \( \frac{d\bar{x}_t^1(x_t)}{dx_t} = -\frac{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)/\partial x_t}{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)/\partial x_t} \). When \( t \to \infty \) and \( \lim_{t \to \infty} x_t = a \), by Lemma 3, \( \bar{x}_t^1(x_t) \) and \( \bar{x}_t^2(x_t) \) both converge to \( a \). Therefore, \( \frac{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)}{\partial x_t} \) and \( -\frac{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)}{\partial x_t} \) both converge to \( f(a)\left[N\bar{F}(a)^{N-1} + \lim_{t \to \infty} c_{t-1}(\bar{x}_t^1(x_t))\right] \), and \( \frac{d\bar{x}_t^1(x_t)}{dx_t} = -\frac{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)/\partial x_t}{\partial\phi_{t+1}(\bar{x}_t^1(x_t), x_t)/\partial x_t} \) converges to 1. \( \blacksquare \)

**References**


C Online Appendix

C.1 Construction of the Equilibrium

In Section 3, \( \bar{\pi}_t (x_{t-1}) \) characterizes the equilibrium cutoff path. But to characterize a seller’s strategy or the belief regarding the cost cutoff, we need to consider the situations when the buyer submits off-equilibrium bids, and when a seller deviates from the equilibrium strategy and does not accept a price that he is supposed to accept.

The following assumption is made to ensure the existence of pure strategy equilibrium. Without the assumption, we are still able to construct an equilibrium in which mixed strategies are applied off the equilibrium path. Therefore, Assumption 1 is not necessary for an equilibrium to exist.

If the assumption fails to hold, it is implied that \( C_{t+1} (x_t) \) is not continuous at some \( a \in [x_{t-1}, \bar{a}] \) and the problem described in (2) arises. In this case, the discontinuity of \( C_{t+1} \) at \( a \) suggests that in the continuation game starting in round \( t+1 \) with belief \( a \), there are two feasible continuation equilibrium paths: one results in \( C_{t+1} (a) = \lim_{x \downarrow a} C_{t+1} (x) \), and the other results in \( C_{t+1} (a) = \lim_{x \uparrow a} C_{t+1} (x) \).

In this case, the buyer can play a mixed strategy in round \( t+1 \), assigning positive weights on the two prices that lead to the two feasible continuation equilibrium paths. By choosing a proper weight \( q \), \( (b_t - a) G (x_{t-1}, a) = q \lim_{x \downarrow a} C_{t+1} (x) + (1-q) \lim_{x \uparrow a} C_{t+1} (x) \), so \( x_t = a \) is the cutoff in period \( t \). However, such a \( b_t \) is not an optimal strategy for the buyer. The buyer can choose \( b_t' \) such that \( (b_t' - a) G (x_{t-1}, a) = \min \{ \lim_{x \downarrow a} C_{t+1} (x) , \lim_{x \uparrow a} C_{t+1} (x) \} \) and get a higher expected payoff in period \( t \). Therefore, we can ignore this kind of situation when deriving the equilibrium path in Section 3.

**Assumption 1** Given \( x_{t-1} \), assume that there exists \( b \) such that if \( b_t \in [b, \bar{b}] \), there exists \( x_t \in [x_{t-1}, \bar{x}] \) such that \( (b_t - x_t) G (x_{t-1}, x_t) = C_{t+1} (x_t) \), and if \( b_t < b \), \( (b_t - x_t) G (x_{t-1}, x_t) < C_{t+1} (x_t) \) for all \( x_t \in [x_{t-1}, \bar{x}] \).

---

\(^{14}\)When the distribution \( F (x) \) is of the form \( F (x) = \left( \frac{x - \bar{x}}{\bar{v} - \bar{x}} \right)^a \) where \( x \in [\bar{x}, \bar{v}] \), \( \bar{v} \leq v \), and \( a \geq 1 \), \( C_{t+1} (x_t) \) is continuous.
Let

\[
\hat{x}_T (b_T, x_{T-1}) = \begin{cases} 
\overline{c} & \text{if } b_T > \overline{c} \\
 b_T & \text{if } x_{T-1} \leq b_T \leq \overline{c} \\
x_{T-1} & \text{if } b_T < x_{T-1}
\end{cases}
\]

and \(\hat{x}_T (b_T, x_{T-1}) = \begin{cases} 
\overline{c} & \text{if } b_T > \overline{c} \\
b_T & \text{if } b_T \leq \overline{c}
\end{cases}\) .

For \(t < T\), let

\[
\hat{x}_t (b_t, x_{t-1}) = \begin{cases} 
\overline{c} & \text{if } b_t \geq \overline{c} \\
x_{t-1} & \text{if } b_t < \overline{b} \text{ (defined in Assumption 1)} \\
\hat{x}_t (x_{t-1}) & \text{if } b_t = \overline{b}_t (x_{t-1})
\end{cases}
\]

otherwise,

\[
\hat{x}_t (b_t, x_{t-1}) \in \{ x_t \mid (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1} (x_t) \}.
\]

and

\[
\hat{x}_t (b_t, x_{t-1}) = \begin{cases} 
\overline{c} & \text{if } b_t \geq \overline{c} \\
 b_t - \frac{C_{t+1} (x_{t-1})}{G(x_{t-1}, x_{t-1})} & \text{if } b_t < \overline{b} \\
\hat{x}_t (x_{t-1}) & \text{if } b_t = \overline{b}_t (x_{t-1})
\end{cases}
\]

otherwise,

\[
\hat{x}_t (b_t, x_{t-1}) \in \{ x_t \mid (b_t - x_t)G(x_{t-1}, x_t) = C_{t+1} (x_t) \}.
\]

\(\hat{x}_t (b_t, x_{t-1})\) is for determining a player’s belief about the greatest lower bound of a seller’s cost, so \(\hat{x}_t (b_t, x_{t-1}) \geq x_{t-1}\), and \(\hat{x}_t (b_t, x_{t-1})\) is for determining a seller’s strategy. The difference between \(\overline{x}_t (x_{t-1})\) and \(\hat{x}_t (b_t, x_{t-1})\) (or \(\hat{x}_t (b_t, x_{t-1})\)) is that \(\overline{x}_t (x_{t-1})\) is determined at the same time when the buyer determines \(b_t\), and \(\hat{x}_t (b_t, x_{t-1})\) (or \(\hat{x}_t (b_t, x_{t-1})\)) is determined after the buyer submits off-equilibrium bids. If an off-equilibrium bid \(b_t\) is too high, all the sellers accept, so \(\hat{x}_t = \overline{x}_t = \overline{c}\). If \(b_t\) is too low, sellers with values higher than \(x_{t-1}\) do not accept, so the belief about the greatest lower bound of the sellers’ costs after all the sellers reject \(b_t\) is still \(x_{t-1}\), i.e. \(\hat{x}_t = x_{t-1}\). However, a seller with cost \(x < x_{t-1}\) might have deviated from the equilibrium strategy and did not accept a price that he should have accepted. In that case, a seller with cost lower than \(b_t - \frac{C_{t+1} (x_{t-1})}{G(x_{t-1}, x_{t-1})}\) still finds it preferable to accept in period \(t\): \((b_t - x)G(x_{t-1}, x_t) > C_{t+1} (x_{t-1})\), so \(\hat{x}_t = b_t - \frac{C_{t+1} (x_{t-1})}{G(x_{t-1}, x_{t-1})}\).
With \( \bar{b}_t (x_{t-1}) \), \( \bar{x}_t (b_t, x_{t-1}) \), and \( \bar{x}_t (b_t, x_{t-1}) \), Theorem 3 formally describes the equilibrium.

**Theorem 3** Assume Assumption 1. Let \( \bar{b}_t \) be as defined in \((P1)\), \((P2)\), and \( \bar{x}_t \), \( \bar{x}_t \) be as defined in \((P5)\), \((P6)\), and \((P7)\). The following \((b, y, x)\) is an equilibrium of the game.

\[
\begin{align*}
    b_t(h_{t-1}) & = \bar{b}_t(\bar{x}_{t-1}(p_{t-1}, \bar{x}_{t-2}(p_{t-2}, \ldots \bar{x}_1(p_1, \zeta) \ldots))), \\
    x_t(h_t) & = \bar{x}_t(p_t, \bar{x}_{t-1}(p_{t-1}, \ldots \bar{x}_1(p_1, \zeta) \ldots)), \\
    y_{t+1}(h_t) & = \bar{x}_t(p_t, \bar{x}_{t-1}(p_{t-1}, \ldots \bar{x}_1(p_1, \zeta) \ldots)).
\end{align*}
\]

**Proof.** First we show that \( u_i^t(b, x, x' | h_t, \theta^i, y_t(h_{t-1})) \geq u_i^t(b, x, x' | h_t, \theta^i, y_t(h_{t-1})) \).

For \( t = T \), it is obvious that following the equilibrium strategy \( x_T(h_T) \) is optimal for a seller. For \( t < T \), let \( x_{t-1} = x_{t-1}(h_{t-1}) \). In the continuation game, after \( b_t \) is submitted, the price path \((b_{t+1}, b_{t+2}, \ldots, b_T)\) and the belief path \((y_{t+1}, y_{t+2}, \ldots, y_T) = (x_t, x_{t+1}, \ldots, x_{T-1})\) can be found by solving programs \((P1)\) and \((P2)\).Seller \( i \)'s deviation does not affect \((b_{t+1}, \ldots, b_T)\) and \((y_{t+1}, \ldots, y_T)\). Suppose seller \( i \)'s cost \( \theta^i \) is in \((x_{s-1}, x_s], s \geq t \), so he should sell in round \( s \). If he accepts in round \( s' \neq s \), \( u_i^t(b, x, x' | h_t, \theta^i, y_t(h_{t-1})) = (b_{s'} - \theta^i) \frac{G(x_{s'-1}, x_{s'})}{NF(x_{s'-1})^{N-1}} \). If he follows \( x \) (accepts in round \( s \)), \( u_i^t(b, x, x | h_t, \theta^i, y_t(h_{t-1})) = (b_s - \theta^i) \frac{G(x_{s-1}, x_s)}{NF(x_{s-1})^{N-1}} \). If \( s' > s \), we know that

\[
\begin{align*}
    (b_s - x_s)G(x_{s-1}, x_s) & = (b_{s+1} - x_s)G(x_s, x_{s+1}) \\
    \vdots \\
    (b_{s'-1} - x_{s'-1})G(x_{s'-2}, x_{s'-1}) & = (b_{s'} - x_{s'-1})G(x_{s'-1}, x_{s'})
\end{align*}
\]

Since \( G(x_{s'-1}, x_{s'}) < NF(x_{s-1})^{N-1} < G(x_{s'-2}, x_{s'-1}) \), for any \( x < x_{s'-1}, (b_{s'-1} - x)G(x_{s'-2}, x_{s'-1}) > (b_{s'} - x)G(x_{s'-1}, x_{s'}) \). Applying the same argument, since \( \theta^i < x_s \leq \cdots \leq x_{s-1}, (b_s - \theta^i)G(x_{s-1}, x_s) > (b_{s+1} - \theta^i)G(x_s, x_{s+1}) \geq \cdots \geq (b_{s'} - \theta^i)G(x_{s'-1}, x_{s'}) \). On the other hand, if \( t \leq s' < s \), since \( \theta^i > x_{s-1} \geq \cdots \geq x_{s'}, (b_s - \theta^i)G(x_{s-1}, x_s) > (b_{s-1} - \theta^i)G(x_{s-2}, x_{s-1}) \geq \cdots \geq (b_{s'} - \theta^i)G(x_{s'-1}, x_{s'}) \). Therefore, \( (b_s - \theta^i) \frac{G(x_{s-1}, x_s)}{NF(x_{s-1})^{N-1}} > (b_{s'} - \theta^i) \frac{G(x_{s'-1}, x_{s'})}{NF(x_{s'-1})^{N-1}} \).

Next, we show that \( u_i^0(b, x | h_{t-1}, y_t(h_{t-1})) \geq u_i^0(b', x | h_{t-1}, y_t(h_{t-1})) \). For any \( t \) and any \( h_{t-1} \), given the sellers’ strategy \( x \), the buyer’s optimal strategy is to choose
\( p_t \) to maximize his conditional utility

\[
\max_{p_t} \frac{(v - p_t) P(x_{t-1}(h_{t-1}), x_t((h_{t-1}, p_t))) + V_{t+1}(x_t((h_{t-1}, p_t)))}{\bar{F}(x_{t-1}(h_{t-1}))^N}.
\]  \( \text{(P8)} \)

Under our construction of \( x_t(h_t) \), the solution to (P8) is the same as \( b_t = b_t(h_t) \) derived from (P2). Hence the strategy \( b_t \) constructed from (P2) is optimal.

\[ \mathbf{C.2 \ Other \ proofs} \]

\textbf{Proposition 7} Assume Condition 1, \( \pi_t^T(x_{t-1}) \) is smooth.

\textbf{Proof.} Suppose \( t = T \). The objective function of (P1)

\[
\varphi_T(x_T, x_{T-1}) = (v - x_T) \left[ \bar{F}(x_{T-1})^N - \bar{F}(x_T)^N \right]
\]

is smooth. Since \( \frac{\partial \varphi_T(x_T, x_{T-1})}{\partial x_T} \big|_{x_T=x_{T-1}} > 0 \) and \( \frac{\partial \varphi_T(x_T, x_{T-1})}{\partial x_T} \big|_{x_T=x_T} < 0 \), the solution \( x_T = \pi_T(x_{T-1}) \) is interior. Because \( \bar{F} \) is smooth, \( \frac{\partial \varphi_T(x_T, x_{T-1})}{\partial x_T} \) is continuous, so with interior solution, the following first-order condition must hold:

\[
0 = \phi_T(\pi_T^T(x_{T-1}), x_{T-1}) = -\bar{F}(x_{T-1})^N + \bar{F}(\pi_T^T(x_{T-1}))^N + (v - \pi_T^T(x_{T-1})) \left[ N \bar{F}(\pi_T^T(x_{T-1}))^{N-1} \right] f(\pi_T^T(x_{T-1})).
\]

With Condition 1, because the partial derivatives of \( \phi_T(x_T, x_{T-1}) \) are continuous and \( \frac{\partial \phi_T}{\partial x_T}(\pi_T^T(x_{T-1}), x_{T-1}) \neq 0 \) for \( x_{T-1} \in (\mathbb{C}, \mathcal{E}) \), the Implicit Function Theorem can apply. Therefore, \( \pi_T^T(x_{T-1}) \) is differentiable: \( \frac{d\pi_T^T(x_{T-1})}{dx_{T-1}} = -\frac{\partial \phi_T(\pi_T^T(x_{T-1}), x_{T-1})}{\partial x_T} / \frac{\partial \phi_T(\pi_T^T(x_{T-1}), x_{T-1})}{\partial x_T} \), and the fact that \( \phi_T(x_T, x_{T-1}) \) is smooth implies that \( \pi_T^T(x_{T-1}) \) is smooth.

Suppose \( t = T-1 \). Note that \( C_T(x_{T-1}) = (\pi_T(x_{T-1}) - x_{T-1}) G(x_{T-1}, \pi_T(x_{T-1})) \) is smooth, and so is the objective function of (P2),

\[
\varphi_t(x_t, x_{t-1}) = (v - x_t) \left[ \bar{F}(x_{t-1})^N - \bar{F}(x_t)^N \right] - C_{t+1}(x_t) \left[ \bar{F}(x_{t-1}) - \bar{F}(x_t) \right] + V_{t+1}(x_t).
\]

Since \( \frac{\partial \varphi_t(x_t, x_{t-1})}{\partial x_t} \big|_{x_t=x_{t-1}} > 0 \) and \( \frac{\partial \varphi_t(x_t, x_{t-1})}{\partial x_t} \big|_{x_t=x_t} < 0 \), the solution \( x_t = \pi_t^T(x_{t-1}) \) is interior. \( \frac{\partial \varphi_t(x_t, x_{t-1})}{\partial x_t} \) is continuous, so with interior solution, the following first-order
condition must hold:

\[
0 = \phi_t \left( \pi^T_t (x_{t-1}), x_{t-1} \right) \\
= - [\hat{F}(x_{t-1}) - \hat{F}(\pi^T_t (x_{t-1}))] \left[ (\hat{F}(x_{t-1})^{N-1} + \cdots + \hat{F}(\pi^T_t (x_{t-1}))^{N-1} + C'_t (\pi^T_t (x_{t-1})) \right] \\
+ f \left( \pi^T_t (x_{t-1}) \right) \pi^T_{t+1} (\pi^T_t (x_{t-1})) - \pi^T_t (x_{t-1})) \\
\times \left[ NF (\pi^T_t (x_{t-1}))^{N-1} - (\hat{F}(\pi^T_{t+1} (\pi^T_t (x_{t-1})))^{N-1} + \cdots + \hat{F}(\pi^T_t (x_{t-1}))^{N-1}) \right].
\]

Applying the same argument as in the case when \( t = T \), the Implicit Function Theorem implies that \( \pi^T_t (x_{t-1}) \) is smooth since \( \phi_t (x_t, x_{t-1}) \) is smooth. Following the same procedure backward, we prove that \( \pi^T_t (x_{t-1}) \) is smooth for all \( t \leq T \).

---

**Proposition 8** If the distribution is of the form \( \hat{F}(x) = \left( \frac{\bar{c} - x}{\bar{c} - \underline{c}} \right)^a \) where \( x \in [\underline{c}, \bar{c}] \), \( \bar{c} \leq v \), and \( a \geq 1 \), then Condition 1 holds, and \( \pi_t (x_{t-1}) \) is uniquely defined.

**Proof.** Define

\[
\pi_T (x_{T-1}, v) \in X_T (x_{T-1}, v) = \max_{x_T \in [x_{T-1}, \bar{c}]} (v - x_T) [\hat{F}(x_{T-1})^N - \hat{F}(x_T)^N],
\]

\[
\pi_t (x_{t-1}, v) \in X_t (x_{t-1}, v) = \max_{x_t \in [x_{t-1}, \bar{c}]} (v - x_t) [\hat{F}(x_{t-1})^N - \hat{F}(x_t)^N] \\
- C_{t+1} (x_t) [\hat{F}(x_{t-1}) - \hat{F}(x_t)] + V_{t+1} (x_t, v), \text{ where } t < T,
\]

\[
C_T^T (x_{T-1}, v) \equiv (\pi_T^T (x_{T-1}, v) - x_{T-1}) G (x_{T-1}, \pi_T^T (x_{T-1}, v)),
\]

\[
C_t^T (x_{t-1}, v) \equiv (\pi_t^T (x_{t-1}, v) - x_{t-1}) G (x_{t-1}, \pi_t^T (x_{t-1}, v)) + C_{t+1} (\pi_t^T (x_{t-1}, v), v), \text{ where } t < T,
\]

and

\[
\rho_t^T (x_{t-1}, v) \equiv f (x_{t-1}) (\pi_t^T (x_{t-1}, v) - x_{t-1}) \\
\times \left[ NF (x_{t-1})^{N-1} - (\hat{F}(\pi_t^T (x_{t-1}, v))^N + \cdots + \hat{F}(x_{t-1})^{N-1}) \right].
\]

Suppose that given \( v \) and \( x_{t-1} \), the equilibrium cost cutoffs in the next \( T-t+1 \) rounds are \( (x_T^T, x_{T+1}^T, \cdots , x_T^T) \). Due to the scaling property of the class of distributions, for any \( v' \) and \( x'_{t-1} \) such that \( \frac{v - x_{t-1}}{v' - x'_{t-1}} = \frac{v - x_{t-1}}{\bar{c} - x_{t-1}} \), there is a continuation equilibrium such that the equilibrium cost cutoffs \( (x'^T_t, x'^T_{t+1}, \cdots , x'^T_T) \) have the property that \( \frac{v - x'^T_t}{v' - x'^T_t} = \frac{v - x_{t-1}}{\bar{c} - x_{t-1}} \) for all \( \tau \geq t \).
First consider the case when \( v = \bar{v} \). Given any \( x_{t-1} \) and \( x_{t-1}' \), since \( \frac{\bar{v} - x_{t-1}'}{\bar{v} - x_{t-1}} \) for all \( t > T \), \( \frac{\partial C_t}{\partial x_{t-1}} \) and \( \rho^T_t (x_{t-1}, v) \) can be represented as

\[
\frac{\partial C_t^T (x_{t-1}, v)}{\partial x_{t-1}} = -h_t^T \bar{F} (x_{t-1})^{N-1} \quad \text{and} \quad \rho^T_t (x_{t-1}, v) = k_t^T \bar{F} (x_{t-1})^N,
\]

where \( h_t^T \) and \( k_t^T \) are two positive constants. In period \( T \), the objective function in (15) is concave in \( x_T \), so the solution is unique. For \( t < T \), when deriving \( \pi_t (x_{t-1}, v) \), the first-order condition is

\[
0 = \phi^T_t (x_t^*, x_{t-1}, v)
\]

\[
= - [\bar{F} (x_{t-1}) - \bar{F} (x_t^*)] \left[ (\bar{F} (x_{t-1})^{N-1} + \cdots + \bar{F} (x_t^*)^{N-1}) + \frac{\partial C_t^T (x_t^*, v)}{\partial x_t^*} \right]
\]

\[
+ \rho^T_{t+1} (x_t^*, v),
\]

where \( x_t^* = \pi_t (x_{t-1}, v) \). Note that

\[
[\bar{F} (x_{t-1}) - \bar{F} (x_t^*)] \left[ (\bar{F} (x_{t-1})^{N-1} + \cdots + \bar{F} (x_t^*)^{N-1}) + \frac{\partial C_t^T (x_t^*, v)}{\partial x_t^*} \right]
\]

\[
= \bar{F} (x_{t-1})^N - \bar{F} (x_t^*)^N + (\bar{F} (x_{t-1}) - \bar{F} (x_t^*)) \frac{\partial C_t^T (x_t^*, v)}{\partial x_t^*}.
\]

\[
\frac{\partial C^T_{t+1} (x_t^*, v)}{\partial x_t^*} = -h_{t+1}^T \bar{F} (x_t^*)^{N-1}, \quad \text{so for} \ x_t \in [x_{t-1}, \bar{v}], \quad \bar{F} (x_{t-1})^N - \bar{F} (x_t^*)^N + (\bar{F} (x_{t-1}) - \bar{F} (x_t^*)) \frac{\partial C^T_{t+1} (x_t^*, v)}{\partial x_t^*}
\]

either strictly increases in \( x_t^* \), or first decreases and then increases in \( x_t^* \). In addition, \( \phi^T_{t+1} (x_t^*, v) \) strictly decreases in \( x_t^* \), and

when \( x_t = x_{t-1} \),

\[
[\bar{F} (x_{t-1}) - \bar{F} (x_t^*)] \left[ (\bar{F} (x_{t-1})^{N-1} + \cdots + \bar{F} (x_t^*)^{N-1}) + \frac{\partial C^T_{t+1} (x_t^*, v)}{\partial x_t^*} \right] = 0 < \rho^T_{t+1} (x_t, v),
\]

when \( x_t = \bar{v} \),

\[
[\bar{F} (x_{t-1}) - \bar{F} (x_t^*)] \left[ (\bar{F} (x_{t-1})^{N-1} + \cdots + \bar{F} (x_t^*)^{N-1}) + \frac{\partial C^T_{t+1} (x_t^*, v)}{\partial x_t^*} \right] > \rho^T_{t+1} (x_t, v) = 0,
\]

so it is implied that there is only one \( x_t^* = \pi_t (x_{t-1}, v) \) satisfying the first-order condition, that is, the solution is unique. By Berge’s maximum theorem, the fact that the solution is unique for \( t = 1, 2, \ldots, T \) implies that given \( v = \bar{v}, \pi^T_t (x_{t-1}, v) \)

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is continuous in \( x_{t-1} \) for \( t = 1, 2, \cdots, T \) because the solution set \( X_t(x_{t-1}, v) \) is upper semi-continuous.

Then consider the case when \( v > \bar{v} \). In period \( T \), given any \( x_{T-1} \) and \( v \), with 
\[
\tilde{F}(x) = \left( \frac{v-x}{\bar{v}-x} \right)^a,
\]
the objective function in (15) is concave in \( x_T \), so the solution is unique, and by the maximum theorem, \( \pi_T^T(x_{T-1}, v) \) is continuous. Applying the Implicit Function Theorem as in the proof of Proposition 7, \( \pi_T^T(x_{T-1}, v) \) is smooth, so \( \frac{\partial \pi_T^T(x_{T-1}, v)}{\partial x_{T-1}} \) and \( \rho_T^T(x_{T-1}, v) \) are continuous.

Next we prove that for \( v > \bar{v} \) and \( t = T - 1 \), \( \pi_T^T(x_{T-1}, v) \) is continuous. By the maximum theorem, a necessary condition for \( \pi_T^T(x_{T-1}, v) \) not to be continuous is that for some \( x_{T-1} \) and \( v \), there is more than one \( x_t^* \) such that \( \phi_t^T(x_t^*, x_{t-1}, v) = 0 \). When \( v = \bar{v} \), \( \phi_t^T(x_t, v) = h_{t+1}^T(\pi(x_t))^N \) and \( \frac{\partial h_{t+1}^T(x_t, v)}{\partial x_t} = -h_{t+1}^T(\pi(x_t))^{N-1} \), and for any \( x_{T-1} \in [c, \bar{v}] \), there is only one \( x_t^* \in [x_{T-1}, \bar{v}] \) such that \( \phi_t^T(x_t^*, x_{T-1}, v) = 0 \). For \( t = T - 1 \), \( \phi_{T-1}^T(x_t, v) \) and \( \frac{\partial h_{T-1}^T(x_t, v)}{\partial x_t} \) are continuous, so there exists \( \epsilon \) such that for \( v \in (\bar{v}, \bar{v} + \epsilon) \), the shapes of \( \phi_{T-1}^T(x_t, v) \) and \( \frac{\partial h_{T-1}^T(x_t, v)}{\partial x_t} \) do not change much compared with the case when \( v = \bar{v} \), and thus for all \( x_{T-1} \in [c, \bar{v}] \), there is still only one \( x_t^* \) such that \( \phi_{T-1}^T(x_t^*, x_{T-1}, v) = 0 \), and \( \pi_{T-1}^T(x_{T-1}, v) \) is continuous. However, this implies that for all \( v \geq \bar{v} \) and \( x_{T-1} \in [c, \bar{v}] \), \( \pi_{T-1}^T(x_{T-1}, v) \) is continuous, due to the scaling property discussed in the second paragraph of the proof.\textsuperscript{15} Applying the Implicit Function Theorem as in the proof of Proposition 7, \( \pi_{T-1}^T(x_{T-1}, v) \) is smooth, so \( \frac{\partial \pi_{T-1}^T(x_{T-1}, v)}{\partial x_{T-1}} \) and \( \rho_{T-1}^T(x_{T-1}, v) \) are continuous. Applying the same argument backward, we can prove that \( \pi_t^T(x_{t-1}, v) \) is uniquely defined for all \( t = 1, 2, \cdots, T \), which implies that given \( v \), \( \pi_t^T(x_{t-1}) \) is continuous for all \( t = 1, 2, \cdots, T \).

\textsuperscript{15}For any \( v \geq \bar{v}, x_{t-1}, \) and sequences \( \{x_{t-1,k} \in [c, \bar{v}]\}_{k \in \mathbb{N}} \) and \( \{v_k \in [\bar{v}, \infty)\}_{k \in \mathbb{N}} \) converging to \( x_{t-1} \) and \( v \) respectively, there exist \( v' \in [\bar{v}, \bar{v} + \epsilon] \), \( x_{t-1}' \), and sequences \( \{x_{t-1,k}' \in [c, \bar{v}]\}_{k \in \mathbb{N}} \) and \( \{v_k' \in [\bar{v}, \infty)\}_{k \in \mathbb{N}} \) converging to \( x_{t-1}' \) and \( v' \) respectively such that \( \frac{v_{k}-x_{t-1,k}}{v_{k}-x_{t-1}} = \frac{v_{k}'-x_{t-1,k}'}{v_{k}'-x_{t-1}} = x_{t-1,k}' = \frac{x_{t-1,k}'}{x_{t-1,k}} \) for all \( k \). Then \( \pi_t^T(x_{t-1,k}, v_k) \) converges to \( \pi_t^T(x_{t-1}, v) \) since \( \frac{x_{t-1,k}'}{x_{t-1,k}} = \frac{\pi_t^T(x_{t-1,k}, v_k)}{\pi_t^T(x_{t-1,k}', v_k')} = \frac{\pi_t^T(x_{t-1,k}, v_k)}{\pi_t^T(x_{t-1,k}', v_k')} = \frac{\pi_t^T(x_{t-1}, v)}{\pi_t^T(x_{t-1}, v')} \), and \( \pi_t^T(x_{t-1,k}', v_k') \) converges to \( \pi_t^T(x_{t-1}', v') \).