Rationalizable Implementation of Correspondences[*]

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Abstract

A new condition, which we call uniform monotonicity, is shown to be necessary and almost sufficient for rationalizable implementation of correspondences. Uniform monotonicity is much weaker than Maskin monotonicity and reduces to it in the case of functions. Maskin monotonicity, the key condition for Nash implementation, had also been shown to be necessary for rationalizable implementation of social choice functions. Our conclusion is that the conditions for rationalizable implementation are not only starkly different from, but also much weaker than those for Nash implementation, when we consider social choice correspondences. Thus, dropping rational expectations significantly expands the class of rules that can be decentralized by communication-based economic institutions.

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1 Introduction

The design of institutions to be used by rational agents has been an important research agenda in economic theory. As captured by the notion of Nash equilib-

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rium, rationality is encapsulated in two aspects: these are (i) the best responses of agents to their beliefs, and (ii) that those beliefs are correct, the so-called rational expectations assumption. One can drop the latter and retain the former, moving then into the realm of rationalizability. One would conjecture that the design of institutions under rationalizable behavior, i.e., without insisting on rational expectations, should leave room for significantly different results than the theory based on equilibrium.[1] Settling this important question is our task in this paper. We show that dropping rational expectations significantly expands the class of rules that can be decentralized by communication-based institutions designed by the Central Authority for participating agents in the system.

The theory of Nash implementation has uncovered the conditions under which one can design a mechanism (or game form) such that the set of its Nash equilibrium outcomes coincides with a given *social choice correspondence* (henceforth, SCC). Indeed, Maskin (1999) proposes a well-known monotonicity condition, which we refer to as *Maskin monotonicity*. Maskin's (1999) main result shows that Maskin monotonicity is necessary and almost sufficient for Nash implementation.

Nash implementation is concerned with complete information environments, in which all agents know the underlying state and this fact is commonly certain among them. As a foundation of Nash equilibrium, Aumann and Brandenburger (1995) delineate a set of epistemic conditions under which the agents' strategic interaction always leads to a Nash equilibrium. Furthermore, Polak (1999) shows that when the agents' payoffs are commonly certain, as complete information environments prescribe, the Aumann-Brandenburger epistemic conditions imply common certainty of rationality.

Bernheim (1984) and Pearce (1984) independently propose rationalizability, a weaker solution concept than Nash equilibrium, by asking what are the strategic implications that come solely from common certainty of rationality. Brandenburger and Dekel (1987) allow for the agents' beliefs to be correlated and propose an even weaker version of rationalizability. Lipman (1994) extends the concept of rationalizability to games with infinite action sets. In this case, the set of all rationalizable strategies is fully characterized in terms of the strategies that survive the (possibly transfinite) iterative deletion of never best responses, taking limits as needed. Throughout the current paper, our discussion is entirely based upon Lipman's (1994) extension of correlated rationalizability of Brandenburger and Dekel (1987).

In a paper that was our starting point and motivation, Bergemann, Morris, and Tercieux (2011) –BMT in the sequel– recently consider the implementation of *social*

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[1]On the one hand, from the existence point of view, since rationalizability is a weaker solution concept, one would conjecture a more permissive theory. On the other hand, uniqueness would be harder to establish. Hence, the answer, a priori, is far from clear.

choice functions (henceforth, SCFs) under complete information in rationalizable strategies. By an SCF we mean a single-valued SCC. They show that Maskin monotonicity is necessary and almost sufficient for rationalizable implementation. This essentially would imply that rationalizable implementation is similar to Nash implementation. However, their result has one important caveat: BMT focus only on SCFs in their analysis (we note that rationalizability and single-valuedness amount to uniqueness of Nash equilibrium). In any attempt to extend their result, one should ponder the following observations: (1) Maskin's characterization on Nash implementation holds true regardless of whether we consider SCFs or SCCs; (2) Maskin monotonicity can be quite restrictive in the case of SCFs (see, e.g., Mueller and Satterthwaite (1977) and Saijo (1987)); and (3) Many interesting SCCs are Maskin monotonic, including the Pareto, Core, envy-free, constrained Walrasian or Lindhal correspondences, while any SCF selected from a Maskin monotonic SCC no longer inherits the property.[2]

Therefore, what we set out to resolve here is the question of how close rationalizable implementation really is to Nash implementation, without imposing the straightjacket of single-valuedness. In dealing with correspondences, we identify a new condition, which we call *uniform monotonicity*, basically closing the gap between necessity and sufficiency.[3] We show that uniform monotonicity is necessary (Theorem 1) and almost sufficient (Theorem 2) for rationalizable implementation of SCCs.[4] Our uniform monotonicity requires the lower contour sets to be nested across states "uniformly" over all outcomes in the range of the SCC. This setwise definition of monotonicity exhibits a clear contrast with Maskin monotonicity, which is a "pointwise" condition, in the sense that it requires the nestedness of the lower contour sets across states at any fixed outcome in the range of the SCC (see Subsection 4.1). Uniform monotonicity is logically weaker than Maskin monotonicity, and it is likely to be much weaker if the SCC contains many values in its range. However, both become equivalent in the case of SCFs. We also construct an example in which an SCC is rationalizably implementable by a finite mechanism, while it violates Maskin monotonicity at almost any outcome in the range of the SCC. In this sense, the SCC in the example is "very far from" being Nash implementable. Of course, as expected from our necessity result, we confirm that uniform monotonicity is satisfied for this SCC (Lemma 1).

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[2]The results in BMT (2011) contrasts with the much more permissive findings in Abreu and Matsushima (1994) for implementation in iterative elimination of weakly dominated strategies, or those in Abreu and Matsushima (1992), even though the latter are obtained for virtual or approximate implementation.

[3]A weaker version of this condition, based on the strict lower contour sets, first surfaced in Cabrales and Serrano (2011) under the name *weak quasimonotonicity;* see also its corrigendum, posted at http://www.econ.brown.edu/faculty/serrano/pdfs/2011GEB73-corrigendum.pdf.

[4]Theorem 2 assumes at least three agents and a "strong no worst alternative" condition.

Thus, rationalizable implementation is generally quite different from Nash implementation, and their alleged resemblance in BMT arose as an artifact of the assumption that only SCFs were being considered. This allows us to conclude that the design of economic institutions that rely on agents as best-responders, but which drop the rational expectations assumption, is possible for a significantly wider class of socially desirable rules. In drawing that landscape of possibilities, we have relied on a novel canonical mechanism that heavily exhibits the violation of rational expectations. In particular, we do not require the existence of Nash equilibrium in the mechanism, unlike BMT.

The rest of the paper is organized as follows. In Section 2, we introduce the general notation for the paper. Section 3 introduces rationalizability as our solution concept and defines the concept of rationalizable implementation. In Section 4, we propose and discuss uniform monotonicity, and show it to be necessary for rationalizable implementation. Section 5 illustrates by an example the conditions for rationalizable implementation and Nash implementation. In Section 6, we propose sufficient conditions for full implementation in rationalizable strategies, and provide a sketch of the proof to highlight the intuition behind our mechanism. Section 7 concludes. In the Appendix, we provide the proof of a claim (omitted from the main body of the paper), discuss the ordinal approach to rationalizable implementation as well as the role of finite mechanisms, and extend our results to the case of weak implementation.

2   Preliminaries

Let $N = \{1, \ldots, n\}$ denote the finite set of agents and Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is common knowledge among the agents. Let A denote the set of social alternatives, which are assumed to be independent of the information state. We shall assume that A is countable, and denote by $\Delta(A)$ the set of probability distributions over A.[5] Agent i's state dependent von Neumann-Morgenstern utility function is denoted $u_i : \Delta(A) \times \Theta \to \mathbb{R}$. We can now define an *environment* as $\mathcal{E} = (A, \Theta, (u_i)_{i \in N})$, which is implicitly understood to be common knowledge among the agents.

A (stochastic) *social choice correspondence* $F : \Theta \rightrightarrows \Delta(A)$ is a mapping from Θ to a nonempty compact subset of $\Delta(A)$.[6] The mapping F is called a *social*

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[5]It is easy to see that one can extend our arguments to a separable metric space of alternatives, focusing on its countable dense subset.

[6]The compact-valuedness of the SCC is used in our sufficiency results. We note, for instance, that it is consistent with the environment in Mezzetti and Renou (2012), who consider Nash implementation in terms of the support of the equilibrium, with finite A and deterministic SCCs. Although in their footnote 4 (p. 2360), they argue that their results extend to the case in which

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choice function if it is a single-valued social choice correspondence. In this case, we denote it by $f : \Theta \to \Delta(A)$. We henceforth use the acronyms SCC and SCF for both objects, respectively.

A *mechanism* (or *game form*) $\Gamma = ((M_i)_{i \in N}, g)$ describes a nonempty countable message space M_i for each agent $i \in N$ and an outcome function $g : M \to \Delta(A)$ where $M = M_1 \times \cdots \times M_n$.

3 Implementation in Rationalizable Strategies

We adopt *correlated rationalizability*, allowing the agents' beliefs to be correlated, as a solution concept and investigate the implications of implementation in *rationalizable* strategies. We fix a mechanism $\Gamma = (M, g)$ and define a message correspondence profile $S = (S_1, \ldots, S_n)$, where each $S_i \in 2^{M_i}$, and we write $\mathcal{S}$ for the collection of message correspondence profiles. The collection $\mathcal{S}$ is a lattice with the natural ordering of set inclusion: $S \le S'$ if $S_i \subseteq S_i'$ for all $i \in N$. The largest element is $\bar{S} = (M_1, \ldots, M_n)$. The smallest element is $\underline{S} = (\emptyset, \ldots, \emptyset)$.

We define an operator $b^\theta : \mathcal{S} \to \mathcal{S}$ to iteratively eliminate never best responses with $b^\theta = (b_1^\theta, \ldots, b_n^\theta)$ and b_i^θ is now defined as:

$$b_i^\theta(S) \equiv \left\{ m_i \in M_i \,\middle|\, \begin{array}{l} \exists \lambda_i \in \Delta(M_{-i}) \text{ such that} \\ (1)\lambda_i(m_{-i}) > 0 \Rightarrow m_j \in S_j \;\forall j \neq i; \\ (2)m_i \in \arg\max_{m_i'} \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i', m_{-i}); \theta) \end{array} \right\}$$

Observe that b^θ is increasing by definition: i.e., $S \le S' \Rightarrow b^\theta(S) \le b^\theta(S')$. By Tarski's fixed point theorem, there is a largest fixed point of b^θ, which we label $S^{\Gamma(\theta)}$. Thus, (i) $b^\theta(S^{\Gamma(\theta)}) = S^{\Gamma(\theta)}$ and (ii) $b^\theta(S) = S \Rightarrow S \le S^{\Gamma(\theta)}$. We can also construct the fixed point $S^{\Gamma(\theta)}$ by starting with $\bar{S}$ – the largest element of the lattice – and iteratively applying the operator b^θ. If the message sets are finite, we have

$$S_i^{\Gamma(\theta)} \equiv \bigcap_{k \ge 1} b_i^\theta \left(\left[b^\theta \right]^k (\bar{S}) \right)$$

In this case, the solution coincides with iterated deletion of strictly dominated strategies. But because the mechanism Γ may be infinite, transfinite induction may be necessary to reach the fixed point. It is useful to define

$$S_{i,k}^{\Gamma(\theta)} \equiv b_i^\theta \left(\left[b^\theta \right]^{k-1} (\bar{S}) \right),$$

A is a separable metric space and the SCC maps Θ into a countable dense subset of A, this is possible because they only insist on its implementation in terms of the "support" of the Nash equilibrium outcomes.

using transfinite induction if necessary. Thus, $S_i^{\Gamma(\theta)}$ is the set of messages surviving (transfinite) iterated deletion of never best responses of agent i. We refer the reader to Lipman (1994) for the formal treatment.

This is the central definition of implementability that we use in this paper:

Definition 1 (Full Rationalizable Implementation) *An SCC F is **fully implementable in rationalizable strategies** if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$,*

$$\bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} = F(\theta).$$

4   Uniform Monotonicity

In this section, we introduce a central condition to our results, which we term *uniform monotonicity*. We motivate it by comparing it to Maskin monotonicity, and we later show that uniform monotonicity is necessary for rationalizable implementation.

For the domain of complete information environments, Maskin (1999) proposes a monotonicity condition for *Nash implementation* where the set of Nash equilibrium outcomes is required to coincide with the SCC. This condition is often called *Maskin monotonicity*.

Definition 2 *An SCC F satisfies **Maskin monotonicity** if, for any states $\theta, \theta' \in \Theta$ and any $a \in F(\theta)$, if*

$$u_i(a, \theta) \geq u_i(z, \theta) \Rightarrow u_i(a, \theta') \geq u_i(z, \theta') \ \forall i \in N, \ \forall z \in \Delta(A),$$

then $a \in F(\theta')$.

Let D denote a countable subset of $\Delta(A)$ with a generic element d being a lottery over A. We denote the convex hull of D by

$$\text{co}(D) = \left\{ \{\alpha_d\}_{d \in D} \ \middle| \ \alpha_d \geq 0 \ \forall d \in D \ \text{and} \ \sum_{d \in D} \alpha_d = 1 \right\}.$$

Definition 3 *An SCC F satisfies **weak uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, if*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \ \ \forall a \in co(F(\theta)), \ \forall i \in N, \ \forall z \in \Delta(A),$$

then, $F(\theta) \subseteq F(\theta')$.

Remark: When we consider SCFs, $\mathrm{co}(F(\theta))$ becomes a singleton set. Therefore, in this case, the condition just defined reduces to Maskin monotonicity.

We slightly strengthen weak uniform monotonicity into the following:

Definition 4 *An SCC F satisfies **uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, if*

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \quad \forall a \in F(\theta),\ \forall i \in N,\ \forall z \in \Delta(A),$$

then $F(\theta) \subseteq F(\theta')$.

Remark: Note how, under expected utility, both conditions amount to the same thing, as requiring the nestedness of the lower contour sets over all $a \in F(\theta)$ or their convex hull is equivalent. However, it will be convenient to use the weak version for the proof of the necessity result, and the strong version for the proof of sufficiency.

4.1   Intuition and Examples

The comparison between Maskin monotonicity and uniform monotonicity is instructive. Maskin monotonicity always implies uniform monotonicity. The former checks for the "pointwise" inclusion, at an alternative $a \in F(\theta)$, of the lower contour sets of agents' preferences in state θ into those in θ', in order to determine whether that same alternative a should still remain in $F(\theta')$. The latter takes the entire set of alternatives $F(\theta)$ and checks "uniformly" whether, for each agent and $a \in F(\theta)$, his lower contour set at a in θ is contained in the lower contour set of a at θ', in order to determine that all outcomes in $F(\theta)$ should still be in $F(\theta')$. In other words, for an outcome $a \in F(\theta)$ to fall out of the SCC at θ' a preference reversal involving outcome a and another outcome $b \in \Delta(A)$ is required if the SCC is Maskin monotonic. If the SCC is uniformly monotonic, for $a \in F(\theta)$ and $a \notin F(\theta')$ to happen, all that is required is a preference reversal involving some pair $x \in F(\theta)$ and $y \in \Delta(A)$ and, importantly, x need not be the same as a. In this sense, uniform monotonicity is likely to be extremely weak in many settings because such "uniform inclusions" of lower contour sets will just be impossible, and the condition will be vacuously satisfied: for example, in a standard convex exchange economy (before extending it to expected utility preferences), if an SCC contains outcomes in which each agent is assigned bundles on different indifference curves (say a_i and b_i), it will generally be very difficult that the indifference curve through a_i at θ be nested into the one through the same bundle at θ', and at the same time, that the same nestedness happens for the indifference curves through bundle b_i. The reader is referred to Figure 1 for an illustration of this difficulty.
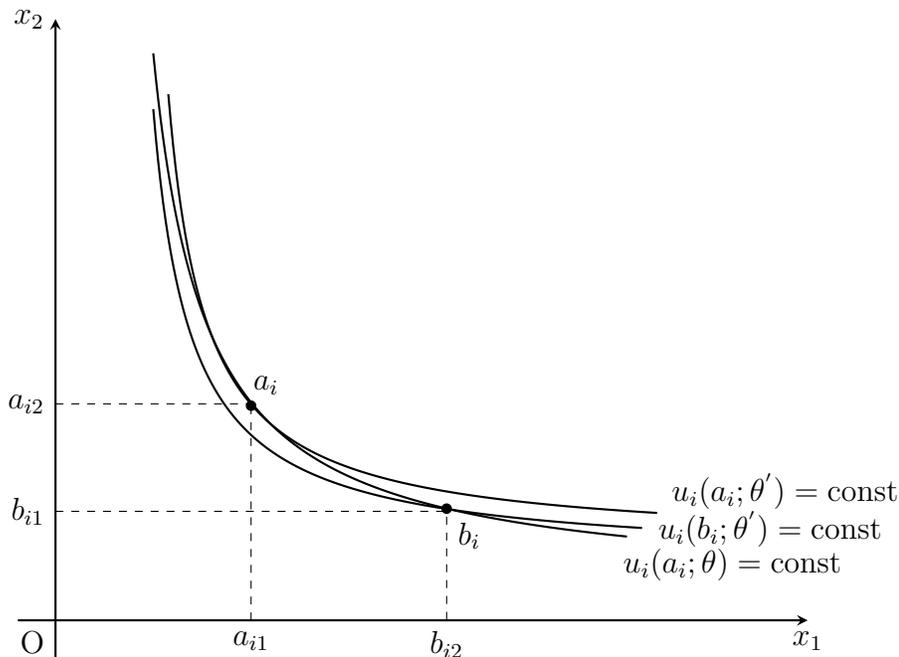
Figure 1: The Permissiveness of Uniform Monotonicity

In the figure, one can see that the nestedness of the lower contour sets at a_i from θ to θ' is satisfied, whereas the nestedness of the lower contour sets at b_i from θ to θ' is violated.

The same logic applies if one uses the probability simplex of lotteries over alternatives. With expected utility, the indifference map under any state consists of parallel straight lines. Maskin monotonicity is a trivial condition at points in the interior of the simplex, as the lower contour sets at any point are never nested (this was the key insight behind the very permissive results of virtual implementation (Abreu and Sen (1991), Matsushima (1988)), for instance). Thus, to make the argument of the relative permissiveness of uniform monotonicity, one should consider SCC's whose outcomes are at the boundaries of the simplex. Again, it will not be generally easy to have that *all* the lower contour sets at multiple boundary points in the simplex at state θ be nested into the corresponding lower contour sets at θ'.

This is not to say that uniform monotonicity is universally satisfied by all SCCs. Indeed, some SCCs may violate it. For instance, consider the *egalitarian-equivalent* allocation correspondence (henceforth, the EEA rule) in an exchange economy with continuous, convex, and strictly monotone preferences (define feasible allocations

with equality between total consumption and aggregate endowment).[7] Pazner and Schmeidler (1978) originally propose such an allocation rule and characterize it as the subset of feasible allocations for each of which there is a "reference" bundle on the ray that goes from the origin to the aggregate endowment vector such that each agent is indifferent between his assigned bundle and the reference bundle. Given the assumptions we imposed on the economy, the EEA rule is always nonempty, as the equal-division rule is egalitarian-equivalent. First we confirm that the EEA rule violates Maskin monotonicity. Let a_θ be an allocation specified by the EEA rule in state θ. Even if the nestedness of lower contour sets at a_θ across states is satisfied, as long as an agent's indifference curves at a_θ are not identical between two states, the original allocation a_θ no longer remains egalitarian-equivalent in the new state. Second, we argue that the EEA rule even violates uniform monotonicity (recall that uniform monotonicity is logically weaker than Maskin monotonicity). For the sake of expositional simplicity, consider the case where there are two agents and two commodities, each with the same aggregate amount. Assume further that agents have different Cobb-Douglas utility functions so that the contract curve (i.e., the set of Pareto efficient allocations) always lies either above or below the diagonal of the Edgeworth box. Then, we know that the equal-division rule is "not" Pareto efficient but as Pazner and Schmeidler (1978) show, there is a unique egalitarian equivalent allocation that is Pareto efficient. This implies that the EEA rule is genuinely a multi-valued correspondence consisting of these two allocations. Suppose the nestedness of the lower contour sets across states "over both outcomes in the EEA rule" needed for uniform monotonicity is satisfied. Note that the equal-division allocation continues to be egalitarian equivalent in the new state trivially. Let $\bar{z}_\theta$ be the reference bundle that corresponds to the unique Pareto efficient and egalitarian-equivalent allocation in state θ. In order for uniform monotonicity to hold, one must have that, given the reference bundle $\bar{z}_{\theta'}$, all agents' indifference curves through the assigned bundle in the new state θ' must continue to intersect at $\bar{z}_{\theta'}$. However, this cannot be guaranteed by the monotonic transformation of preferences we have for the hypothesis of uniform monotonicity. Therefore, the EEA rule violates uniform monotonicity.[8]

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[7]An allocation $(x_i)_{i \in N}$ is said to be *egalitarian-equivalent* if there is a bundle z such that z is indifferent to x_i for every $i \in N$.

[8]Dutta and Vohra (1993) show in their Theorem 2 that the EEA rule satisfies a condition of weak positive association, denoted by WPA_h, which is weaker than Maskin monotonicity. Clarifying the connection between WPA_h and uniform monotonicity might be an interesting open question, left for future research.

4.2   Necessity for Rationalizable Implementation

We proceed to state and prove our first result, which identifies a necessary condition for rationalizable implementation:

Theorem 1 *If an SCC F is fully implementable in rationalizable strategies, it satisfies weak uniform monotonicity.*

Proof: Suppose F is fully implementable in rationalizable strategies by a mechanism $\Gamma = (M, g)$. Fix two states $\theta, \theta' \in \Theta$ satisfying the following property:

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \ \forall a \in \mathrm{co}(F(\theta)), \ \forall i \in N, \ \forall z \in \Delta(A) \ (*)$$

Then, due to the hypothesis that F is implementable by Γ, we fix $m^* \in S^{\Gamma(\theta)}$, and we have that $g(m^*) \in F(\theta)$.

Fix $i \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}$, there exists $\lambda_i^{m_i^*, \theta} \in \Delta(M_{-i})$ satisfying the following two properties: (i) $\lambda_i^{m_i^*, \theta}(m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{\Gamma(\theta)}$ and $g(m_i^*, m_{-i}) \in F(\theta)$; and (ii) $\sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta)$ for each $m_i' \in M_i$.

We focus on the best response property of m_i^* summarized by inequality (ii). Fix $m_i' \in M_i$. Due to the construction of $\lambda_i^{m_i^*, \theta}$, we have that

$$\sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta)$$

$$u_i(a; \theta) \geq u_i(z^a; \theta),$$

where the two lotteries a and z^a are defined as

$$a = \sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) g(m_i^*, m_{-i}) \ \text{ and } \ z^a = \sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) g(m_i', m_{-i}).$$

Since $g(m_i^*, m_{-i}) \in F(\theta)$ for each m_{-i} with $\lambda_i^{m_i^*, \theta}(m_{-i}) > 0$, we have $a \in \mathrm{co}(F(\theta))$. Using Property $(*)$, we also obtain

$$u_i(a; \theta') \geq u_i(z^a; \theta').$$

Due to the choice of a and z^a and the hypothesis that $u_i(\cdot)$ is a von-Neumann-Morgenstern expected utility, we obtain the following:

$$\sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta') \geq \sum_{m_{-i}} \lambda_i^{m_i^*, \theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta').$$

Since this argument does not depend upon the choice of m_i', this shows that m_i^* is a best response to $\lambda_i^{m_i^*, \theta}$ in state θ' as well. Therefore, $m_i^* \in S_i^{\Gamma(\theta')}$. Since the

choice of agent i is arbitrary, we can conclude that $m^* \in S^{\Gamma(\theta')}$. Furthermore, since the choice of $m^* \in S^{\Gamma(\theta)}$ is also arbitrary, we have $S^{\Gamma(\theta)} \subseteq S^{\Gamma(\theta')}$. Finally, by full implementability, this implies that

$$F(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} \subseteq \bigcup_{m \in S^{\Gamma(\theta')}} \{g(m)\} = F(\theta').$$

The proof is thus complete. ∎

5 An Example

In this section, we show by example that rationalizable implementation can be very different from Nash implementation. We consider the following example. There are two agents $N = \{1, 2\}$; two states $\Theta = \{\alpha, \beta\}$; and a finite number K of pure outcomes $A = \{a_1, a_2 \ldots, a_K\}$ where $K \geq 4$.[9] Assume that it is commonly certain that both agents know the state, i.e., it is a complete information environment. Agent 1's utility function is given as follows: for each $k = 1, \ldots, K$,

$$u_1(a_k, \alpha) = u_1(a_k, \beta) = \begin{cases} 1 + K\varepsilon & \text{if } k = K, \\ 1 + (K - k)\varepsilon & \text{if } k \neq K, \end{cases}$$

where $\varepsilon \in (0, 1)$. Hence, agent 1 has state-uniform preferences over A and a_K is the best outcome in both states; a_1 is the second best outcome in both states; ...; and a_{K-1} is the worst outcome in both states for agent 1.

Agent 2's utility function in state α is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \alpha) = \begin{cases} 1 + (K + 1)\varepsilon & \text{if } k = K, \\ 1 + K\varepsilon & \text{if } k = 2, \\ 1 + k\varepsilon & \text{otherwise.} \end{cases}$$

In state β, agent 2's utility function is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \beta) = \begin{cases} 1 + (K + 1)\varepsilon & \text{if } k = K, \\ 1 & \text{if } k = 2, \\ 1 + k\varepsilon & \text{otherwise.} \end{cases}$$

Note that a_K is the best outcome for agent 2 in both states; a_2 is his second best outcome in state α but it is his worst outcome in state β; and a_{K-1} is his third best outcome in state α and it is his second best outcome in state β.

We consider the following SCC F: $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$ and $F(\beta) = \{a_K\}$.

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[9]This example builds upon the one discussed in the Concluding Remarks section of BMT (2011).

Claim 1 *For every outcome $a_k \in A$ with $a_k \neq a_2$,*

$$u_i(a_k, \alpha) \geq u_i(y, \alpha) \Rightarrow u_i(a_k, \beta) \geq u_i(y, \beta) \ \ \forall i = \{1, 2\}, \ \ \forall y \in \Delta(A).$$

Proof: Since agent 1 has state-uniform preferences, this claim is trivially true for agent 1. Thus, in what follows, we focus on agent 2. Take any lottery in the lower contour set of $a_k \in A \setminus \{a_2\}$ in state α. If that lottery did not contain a_2 in its support, it is still in the lower contour set of a_k in state β as no utilities have changed, and if it did contain a_2 in its support, since the utility of a_2 has decreased, it will also be in the lower contour set at β. This completes the proof. ∎

Fix $a_k \in A \setminus \{a_2, a_K\}$ arbitrarily. If F were to satisfy Maskin monotonicity, we would have $a_k \in F(\beta)$, which is not the case. Therefore, we confirm the violation of Maskin monotonicity by the SCC F at every $a_k \in A \setminus \{a_2, a_K\}$. As is clear from the construction, we can choose K arbitrarily large. Therefore, the violation of Maskin monotonicity is severe, measured by the number of alternatives that should remain in the social choice in state β given the relevant nestedness of agents' preferences across the two states. In this sense, this correspondence is "very far" from being Maskin monotonic.

Nevertheless, we claim that the SCC F is implementable in rationalizable strategies using a finite mechanism. Consider the following mechanism $\Gamma = (M, g)$ where $M_i = \{m_i^1, m_i^2, \dots, m_i^K\}$ for each $i = 1, 2$ and the deterministic outcome function $g(\cdot)$ is given in the table below:

$g(m)$		Agent 2					
		m_2^1	m_2^2	m_2^3	m_2^4	$\cdots$	m_2^{K-1}	m_2^K
	m_1^1	a_1	a_1	a_{K-2}	a_{K-3}	$\cdots$	a_2	a_{K-1}
	m_1^2	a_2	a_1	a_1	a_{K-2}	$\cdots$	a_3	a_{K-1}
	m_1^3	a_3	a_2	a_1	a_1	$\cdots$	a_4	a_{K-1}
Agent 1	m_1^4	a_4	a_3	a_2	a_1	$\cdots$	a_5	a_{K-1}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	m_1^{K-1}	a_1	a_{K-2}	a_{K-3}	a_{K-4}	$\cdots$	a_1	a_{K-1}
	m_1^K	a_{K-1}	a_{K-1}	a_{K-1}	a_{K-1}	$\cdots$	a_{K-1}	a_K

Claim 2 *The SCC F is fully implementable in rationalizable strategies by the mechanism Γ.*

Proof: In state α, all messages can be best responses. Therefore, no message can be discarded via the iterative elimination of never best responses. That is, the set of rationalizable message profiles $S^\Gamma(\alpha) = M$. This implies that the set of rationalizable outcomes in state α is $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$.

In state β, message m_2^K strictly dominates all other messages, $m_2^1, \ldots, m_2^{K-1}$ for agent 2. On the other hand, all messages for agent 1 can be a best response. In the second round of elimination of never best responses, m_1^K strictly dominates all other messages $m_1^1, \ldots, m_1^{K-1}$ for agent 1. Thus, we have $S^{\Gamma(\beta)} = \{(m_1^K, m_2^K)\}$. This implies that we have $F(\beta) = \{a_K\}$ as the unique rationalizable outcome in state β. This completes the proof. ∎

BMT (2011) show in their Proposition 1 that strict Maskin monotonicity is necessary for implementation in rationalizable strategies under complete information. It follows from the previous example that this crucially relies on the assumption that only SCFs were considered in BMT's main result. More specifically, we show that, while the failure of Maskin monotonicity is severe, implementation in rationalizable strategies is still possible by a finite mechanism. For completeness, we provide the following lemma.

Lemma 1 *The SCC F satisfies uniform monotonicity.*

Proof: Since agent 1 has state-uniform preferences, we only focus on agent 2 in the following argument. First, we set $\theta = \alpha$ and $\theta' = \beta$ in the definition of uniform monotonicity. We know that $F(\alpha) = \{a_1, \ldots, a_K\}$ and by Claim 1, for any $a \in F(\alpha) \backslash \{a_2, a_K\}$ and $i \in \{1, 2\}$, we have the corresponding monotonic transformation from α to β. For $a_2 \in F(\alpha)$ and $a_3 \in A$, however, we have

$$u_2(a_2; \alpha) > u_2(a_3; \alpha) \text{ and } u_2(a_2; \beta) < u_2(a_3; \beta).$$

Therefore, the condition needed for the monotonic transformation from α to β under uniform monotonicity is not satisfied. Hence, in this case, uniform monotonicity imposes no conditions on SCCs.

Second, we set $\theta = \beta$ and $\theta' = \alpha$ in the definition of uniform monotonicity. Since $F(\beta) = \{a_K\}$ and a_K is the best outcome for agent 2 in both states, we have that for any $y \in \Delta(A)$,

$$u_2(a_K; \beta) \geq u_2(y; \beta) \Rightarrow u_2(a_K; \alpha) \geq u_2(y; \alpha).$$

In this case, uniform monotonicity implies that $a_K \in F(\alpha)$, which is indeed the case. Thus, F satisfies uniform monotonicity. ∎

6 Sufficient Conditions for Full Implementation in Rationalizable Strategies

We turn in this section to our general sufficiency result. Before that, we introduce an additional condition.

Definition 5 *An SCC F satisfies the **strong no-worst-alternative** condition (henceforth, SNWA) if, for each $\theta \in \Theta$ and $i \in N$, there exists $z_i^\theta \in \Delta(A)$ such that, for each $a \in F(\theta)$,*

$$u_i(a; \theta) > u_i(z_i^\theta; \theta).$$

Remark: This condition is introduced by Cabrales and Serrano (2011). In words, SNWA says that the SCC never assign the worst outcome to any agent at any state. BMT (2011) use its SCF-version and call it the no-worst-alternative condition (NWA).

Lemma 2 *If an SCC F satisfies SNWA, then for each $i \in N$, there exists a function $z_i : \Theta \times \Theta \to \Delta(A)$ such that for all $\theta, \theta' \in \Theta$:*

$$u_i(a; \theta') > u_i(z_i(\theta, \theta'); \theta') \ \ \forall a \in F(\theta')$$

and whenever $\theta \neq \theta'$,

$$u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta).$$

   Proof: This is a straightforward extension of Lemma 2 in BMT (2011) to SCCs. We omit the proof. ∎

   For the sufficiency result we establish below, we propose the following mechanism $\Gamma = (M, g)$: each agent i sends a message $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6)$, where

- $m_i^1 \in \Theta$, i.e., a state;

- $m_i^2 = \{m_i^2[\theta]\}_{\theta \in \Theta}$ where $m_i^2[\theta] \in F(\theta)$, i.e., a state-dependent menu of socially desirable alternatives, understood as a recommendation to the designer;

- $m_i^3 = \{(m_i^3[\theta, 1], m_i^3[\theta, 2])\}_{\theta \in \Theta}$ where $m_i^3[\theta, 1] \in \Delta(A)$ and $m_i^3[\theta, 2] \in F(\theta)$, i.e., a state-dependent pair of alternatives, one of them in the SCC, understood as potential arguments for a challenge to the designer;

- $m_i^4 \in \Delta(A)$, i.e., a state-independent alternative, also understood as a challenge to the designer;

- $m_i^5 \in N$, i.e., a number chosen from $\{1, \ldots, n\}$, understood as a vote for some person to be the king;

- and $m_i^6 \in \mathbb{N}$, i.e., a positive integer.

The outcome function $g : M \to \Delta(A)$ is defined as follows: for each $m \in M$:

Rule 1. Consensus implements the recommendation made by the elected king: If there exists $\theta' \in \Theta$ such that $m_i^1 = \theta'$ and $m_i^6 = 1$ for all $i \in N$, then $g(m) = m_t^2[\theta']$ where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$.

Rule 2. An odd man out: If there exist $\theta' \in \Theta$ and $i \in N$ such that [a] $m_j^1 = \theta'$ and $m_j^6 = 1$ for all $j \neq i$, and [b] either $m_i^6 > 1$ or $m_i^1 \neq \theta'$, then the following subrules apply:

Rule 2-1. A nongreedy odd man out is heard in his challenge, although some bad outcomes are also implemented in the appeal process: If $u_i(m_t^2[\theta']; \theta') \geq u_i(m_i^3[\theta', 1]; \theta')$ and $m_t^2[\theta'] = m_i^3[\theta', 2]$ where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$, then

$$g(m) = \begin{cases} m_i^3[\theta', 1] & \text{with probability } m_i^6/(m_i^6 + 1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6 + 1) \end{cases}$$

Rule 2-2. A greedy odd man out is not heard in his challenge, and some bad outcomes are also implemented in the appeal process: Otherwise,

$$g(m) = \begin{cases} m_t^2[\theta'] & \text{with probability } m_i^6/(m_i^6 + 1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6 + 1) \end{cases}$$

where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$.

Rule 3. Stronger disagreements lead to the integer game, implementing potential disarray in the appeal/challenge process: In all other cases,

$$g(m) = \begin{cases} m_1^4 & \text{with probability } \frac{m_1^6}{n(m_1^6 + 1)} \\ m_2^4 & \text{with probability } \frac{m_2^6}{n(m_2^6 + 1)} \\ \vdots & \vdots \\ m_n^4 & \text{with probability } \frac{m_n^6}{n(m_n^6 + 1)} \\ \underline{z} & \text{with the remaining probability,} \end{cases}$$

where

$$\underline{z} = \frac{1}{n} \sum_{i \in N} \underline{z}_i \quad \text{and} \quad \underline{z}_i = \frac{1}{|\Theta|} \sum_{\theta \in \Theta} z_i^\theta.$$

Next, we state the general sufficiency result for full implementation in rationalizable strategies.

Theorem 2 *Suppose that there are at least three agents ($n \geq 3$). If an SCC F satisfies uniform monotonicity and SNWA, it is **fully** implementable in rationalizable strategies.*

Proof: We use the mechanism $\Gamma = (M, g)$ constructed above. The proof consists of Steps 1 through 4. Before going into the details of the proof, we briefly sketch its basic logic. We discuss the properties of our mechanism after providing the formal proof.

In Step 1, we show that any rationalizable message m_i involves $m_i^6 = 1$, that is, there must be at least consensus in the integer chosen. If this were not the case, either Rule 2 (odd man out) or Rule 3 (stronger disagreements) is triggered with probability one. By choosing the third and fourth components of the message appropriately, it is strictly better for agent i to announce an integer even higher than m_i^6, which contradicts the hypothesis that m_i is rationalizable.

In Step 2, we prove that any outcome in the range of the SCC can be supported by a rationalizable message profile. Let θ be the true state and fix $a \in F(\theta)$ arbitrarily. We construct the following message profile m: $m_i^1 = \theta$; $m_i^5 = 1$; and $m_i^6 = 1$ for every $i \in N$ and $m_1^2[\theta] = a$. Note first that no agent has an incentive to become the odd man out and induce Rule 2 by unilaterally deviating from m. Thus, the specification of the third and fourth components of the messages do not matter. Then, m induces Rule 1 with probability one and $g(m) = a$ where agent 1 is the king, the winner of the modulo game. The novelty of the argument is that we can make m_1 rationalizable because agent 1 believes that agent 2 is a "generous king" so as to choose agent 1's best outcome from $F(\theta)$. Similarly, we can also make m_2 rationalizable because agent 2 believes that agent 3 is a "generous king" so as to choose agent 2's best outcome from $F(\theta)$. We extend this argument to all agents so that we can make m rationalizable.

In Step 3, we show that every agent believes that all rationalizable message profiles induce Rule 1 (consensus also in the announced state) with probability one. Suppose, by way of contradiction, that agent i believes with positive probability that Rule 2 (odd man out) or Rule 3 (stronger disagreements) is triggered. By choosing the third and fourth components of the message appropriately and an integer in its sixth component sufficiently high, agent i is able to find an even better response against his belief. This is a contradiction. Step 3 implies that one can partition the set of rationalizable message profiles into separate components, $\theta, \theta', \theta'', \ldots$. For instance, in the θ' component, this is the announced state by each agent in the first item of their messages, which also determines the event to which each of them assigns probability 1. That is, in that component, each agent

i believes that all the others are using strategies of the form $(\theta', \cdot, \cdot, \cdot, \cdot, 1)$ with probability 1.

Finally, in Step 4, we prove that if m is a message profile such that $m_i = (\theta', \cdot, \cdot, \cdot, \cdot, 1) \in S_i^{\Gamma(\theta)}$ for each $i \in N$, then $g(m) \in F(\theta)$. If $\theta' = \theta$, this is trivially true. So, we assume $\theta' \neq \theta$. First, using the features of the canonical mechanism, a technical claim –Claim 3– shows that if one has a rationalizable message profile, one can modify it slightly in order to support any outcome in the range of the social choice correspondence. After that claim, the proof is by contradiction. If in state θ there were a rationalizable message profile whose outcome is not in $F(\theta)$, it must be the case that all agents are coordinating in a deception in which they are reporting state θ', and given previous steps in the proof, the outcome must be actually in $F(\theta')$. Uniform monotonicity allows us then to use the preference reversal for at least an agent and at least an alternative $a^* \in F(\theta')$. By the technical claim, this should also be supported by rationalizable messages, but we show it cannot.

Now, we proceed to the formal proof. Throughout, we denote the true state by θ

Step 1: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow m_i^6 = 1$.

Proof of Step 1: Let $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6) \in S_i^{\Gamma(\theta)}$. Suppose by way of contradiction that $m_i^6 > 1$. Then, for any profile of messages m_{-i} that agent i's opponents may play, (m_i, m_{-i}) will trigger either Rule 2 or Rule 3. We can partition the message profiles of all agents but i as follows:

$$M_{-i}^2 \equiv \left\{ m_{-i} \in M_{-i} \middle| \exists \theta' \in \Theta \text{ s.t. } m_j^1 = \theta', m_j^2[\theta'] \in F(\theta'), \text{ and } m_j^6 = 1 \ \forall j \neq i \right\}$$

denotes the set of messages of all agents but i in which Rule 2 is triggered, and

$$M_{-i}^3 \equiv M_{-i} \backslash M_{-i}^2$$

denotes the set of messages of all agents but i in which Rule 3 is triggered.

Suppose first that agent i has a belief $\lambda_i \in \Delta(M_{-i})$ under which Rule 3 is triggered with positive probability, so that $\sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) > 0$. If $u_i(m_i^4; \theta) > u_i(z_i^\theta; \theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}_i^6$ is chosen to be larger than m_i^6. In doing so, agent i decreases the probability that $\underline{z}$ is chosen in Rule 3. So, conditional on Rule 3, we have

$$\sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

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If $u_i(m_i^4; \theta) \leq u_i(z_i^\theta; \theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}_i^4 \in F(\theta)$ and $\hat{m}_i^6$ is chosen to be larger than m_i^6. Similarly, conditional on Rule 3, we obtain the same inequality.

Now suppose that agent i believes that Rule 2 will be triggered with positive probability, so that $\sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i}) > 0$. We again consider a deviation from m_i to $\hat{m}_i$ and observe that the choice of $\hat{m}_i^4$ does not affect the outcome of the mechanism conditional on Rule 2.

First, assume that $m_j^1 = \theta' \neq \theta$ for each $j \neq i$. Suppose $u_i(m_i^3[\theta', 1]; \theta) \geq u_i(z_i(\theta, \theta'); \theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^6$ larger than m_i^6 and keeping m_i unchanged otherwise. Since $u_i(m_i^3[\theta', 1]; \theta) \geq u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta)$, we have that conditional on Rule 2,

$$\sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(m_i, m_{-i}); \theta).$$

Otherwise, suppose that $u_i(m_i^3[\theta', 1]; \theta) < u_i(z_i(\theta, \theta'); \theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^3[\theta', 1] = z_i(\theta, \theta')$, $\hat{m}_i^6 > m_i^6 > 1$, and keeping m_i unchanged otherwise. Since $u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(m_i, m_{-i}); \theta).$$

Second, assume that $m_j^1 = \theta$ for each $j \neq i$. We choose $t^* \neq i$ and $m_{-i}^* \in \text{supp}(\lambda_i(\cdot))$ such that for each $j \neq i$ and $m_{-i} \in \text{supp}(\lambda_i(\cdot))$,

$$u_i(m_{t^*}^{*2}[\theta]; \theta) \geq u_i(m_j^2[\theta]; \theta).$$

Then, in this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^3[\theta, 1] = m_{t^*}^{*2}[\theta]$ and $\hat{m}_i^6 > m_i^6 > 1$, keeping m_i unchanged otherwise. Since $u_i(m_{t^*}^{*2}[\theta]; \theta) > u_i(z_i(\theta, \theta); \theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M_{-i}^2} \lambda_i(m_{-i})u_i(g(m_i, m_{-i}); \theta).$$

It follows that, in all cases, these choices of $\hat{m}_i$ strictly improve the expected payoff of agent i if either Rule 2 or Rule 3 is triggered. This implies that m_i is never a best response to any belief λ_i, which contradicts our hypothesis that $m_i \in S_i^{\Gamma(\theta)}$. ∎

Step 2: For any $\theta \in \Theta$ and $a \in F(\theta)$, there exists $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a$.

Proof of Step 2: Fix $\theta \in \Theta$ as the true state, and fix $a \in F(\theta)$. Define $m_1^* = (\theta, m_1^{*2}, m_1^{*3}, m_1^{*4}, 1, 1)$, where $m_1^{*2}[\theta] = a$. For each $j \in \{2, \ldots, n\}$, define $m_j^* = (\theta, m_j^{*2}, m_j^{*3}, m_j^{*4}, 1, 1)$, where $m_j^{*2}[\theta] = a_{j-1}(\theta)$, which denotes one of the maximizers of $u_{j-1}(\cdot; \theta)$ within all the outcomes in $F(\theta)$. Then, the constructed message profile m^* induces Rule 1 and agent 1 becomes the winner of the modulo game. We thus have $g(m^*) = a$ by construction. What remains to show is that $m^* \in S^{\Gamma(\theta)}$.

By construction of the mechanism, Rule 3 cannot be triggered by any unilateral deviation from Rule 1. So, the specification of m_i^{*4} does not affect our argument. Moreover, also by construction of the mechanism, no agent has an incentive to induce Rule 2 with a unilateral deviation from a truthful profile under Rule 1. So, effectively, the specification of m_i^{*3} does not affect our argument either.

We first show that m_1^* can be made a best response to some belief. Define $\lambda_1^* \in \Delta(M_{-1})$ as follows: for any $m_{-1} \in M_{-1}$, if $\lambda_1^*(m_{-1}) > 0$,

$$
\begin{aligned}
m_j^1 &= \theta; \\
m_j^2[\theta] &= a_{j-1}(\theta); \\
m_j^5 &= \begin{cases} 2 & \text{if } j = 2, \\ 1 & \text{otherwise}; \end{cases} \\
m_j^6 &= 1.
\end{aligned}
$$

for all $j \in \{2, \ldots, n\}$. Given this belief λ_1^* and m_1^*, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is generated. Therefore, m_1^* is a best response to λ_1^* so that it survives the first round of deletion of never best responses.

We next show that the support of λ_1^* is rationalizable. Assume $j \neq 1$. Define

$$
\bar{m}_j = \begin{cases} (\theta, \bar{m}_2^2, \bar{m}_2^3, \bar{m}_2^4, 2, 1) & \text{if } j = 2, \\ (\theta, \bar{m}_j^2, \bar{m}_j^3, \bar{m}_j^4, 1, 1) & \text{otherwise.} \end{cases}
$$

where $\bar{m}_j^2[\theta] = a_{j-1}(\theta)$. Define $\bar{\lambda}_2 \in \Delta(M_{-2})$ as follows: for any $m_{-2} \in M_{-2}$, if $\bar{\lambda}_2(m_{-2}) > 0$,

$$
\begin{aligned}
m_k^1 &= \theta; \\
m_k^2[\theta] &= a_{k-1}(\theta); \\
m_k^5 &= \begin{cases} 2 & \text{if } k = 1, \\ 1 & \text{otherwise}; \end{cases} \\
m_k^6 &= 1.
\end{aligned}
$$

for all $k \neq 2$. Then, given this belief $\bar{\lambda}_2$ and $\bar{m}_2$, agent 3 becomes the winner of the modulo game so that the outcome $a_2(\theta)$, which is the best one for agent 2, is

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realized. Therefore, $\bar{m}_2$ is a best response to $\bar{\lambda}_2$ so that it survives the first round of deletion of never best responses. Assume $j \in N \backslash \{1, 2\}$. Define $\bar{\lambda}_j \in \Delta(M_{-j})$ as follows: for any $m_{-j} \in M_{-j}$, if $\bar{\lambda}_j(m_{-j}) > 0$,

$$
\begin{aligned}
m_k^1 &= \theta; \\
m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1, \\ a_{k-1}(\theta) & \text{otherwise}; \end{cases} \\
m_k^5 &= \begin{cases} j+1 & \text{if } k = 1 \\ 1 & \text{otherwise}; \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq j$. Assume $j < n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent $j + 1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Assume, on the other hand, that $j = n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best one for agent n, is realized. Therefore, $\bar{m}_j$ is a best response to $\bar{\lambda}_j$ so that it survives the first round of deletion of never best responses. We can repeat this argument iteratively so that m_1^* survives the iterative deletion of never best responses. Hence, $m_1^* \in S_i^{\Gamma(\theta)}$.

Third, we shall show that, for each $j \neq 1$, m_j^* can be made a best response to some belief. For each $j \in \{2, \ldots, n\}$, define $\lambda_j^* \in \Delta(M_{-j})$ with support as follows:

$$
\begin{aligned}
m_k^1 &= \theta; \\
m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1, \\ a_{k-1}(\theta) & \text{otherwise}; \end{cases} \\
m_k^5 &= \begin{cases} j+1 & \text{if } k = 1, \\ 1 & \text{otherwise}; \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq j$. Given this belief λ_j^* and m_j^*, agent $j + 1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Therefore, for each $j \neq 1$, m_j^* is a best response to λ_j^* so that it survives the first round of deletion of never best responses.

Fourth, we will show that the support of λ_j^* is rationalizable. Consider $\bar{m}_1 = (\theta, \bar{m}_1^2, \bar{m}_1^3, \bar{m}_1^4, j+1, 1)$, where $\bar{m}_1^2[\theta] = a_n(\theta)$. Define $\bar{\lambda}_1 \in \Delta(M_{-1})$ as follows: for

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any $m_{-1} \in M_{-1}$, if $\bar{\lambda}_1(m_{-1}) > 0$,

$$
\begin{aligned}
m_k^1 &= \theta; \\
m_k^2[\theta] &= a_{k-1}(\theta); \\
m_k^5 &= \begin{cases} n+2-j & \text{if } k=2, \\ 1 & \text{otherwise;} \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq 1$. Given this belief $\bar{\lambda}_1$ and $\bar{m}_1$, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is realized. Therefore, $\bar{m}_1$ is a best response to $\bar{\lambda}_1$ so that it survives the first round of deletion of never best responses.

Consider agent $k \in N\backslash\{1,j\}$. We first assume $k < n$. Define $\bar{m}_k = (\theta, \bar{m}_k^2, \bar{m}_k^3, \bar{m}_k^4, 1, 1)$, where $\bar{m}_k^2[\theta] = a_{k-1}(\theta)$. Define $\bar{\lambda}_k \in \Delta(M_{-k})$ as follows: for any $m_{-k} \in M_{-k}$, if $\bar{\lambda}_k(m_{-k}) > 0$,

$$
\begin{aligned}
m_i^1 &= \theta; \\
m_i^2[\theta] &= a_{i-1}(\theta); \\
m_i^6 &= 1,
\end{aligned}
$$

for all $i \neq k$ and $\sum_{i \neq k} m_i^5 = n + k - 1$. Given this belief $\bar{\lambda}_k$ and $\bar{m}_k$, agent $k+1$ becomes the winner of the modulo game so that the outcome $a_k(\theta)$, which is the best one for agent k, is realized. Therefore, $\bar{m}_k$ is a best response to $\bar{\lambda}_k$ so that it survives the first round of deletion of never best responses.

Assume $n \neq j$. We define $\bar{m}_n = (\theta, \bar{m}_n^2, \bar{m}_n^3, \bar{m}_n^4, 1, 1)$ and $\bar{\lambda}_n \in \Delta(M_{-n})$ as follows: for any $m_{-n} \in M_{-n}$, if $\bar{\lambda}_n(m_{-n}) > 0$,

$$
\begin{aligned}
m_i^1 &= \theta; \\
m_i^2 &= \begin{cases} a_n(\theta) & \text{if } i=1 \\ a_{i-1}(\theta) & \text{otherwise;} \end{cases} \\
m_i^5 &= 1; \\
m_i^6 &= 1,
\end{aligned}
$$

for all $i \neq n$. Given this belief $\bar{\lambda}_n$ and $\bar{m}_n$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best for agent n, is realized. Therefore, $\bar{m}_n$ is a best response to $\bar{\lambda}_n$ so that it survives the first round of deletion of never best responses.

We conclude that the support of λ_j^* is rationalizable. So, we can repeat this argument iteratively so that for each $j \neq 1$, m_j^* survives the iterative deletion of never best responses. Therefore, $m_j^* \in S_j^{\Gamma(\theta)}$ for each $j \neq 1$. Since $m_1^* \in S_1^{\Gamma(\theta)}$, we obtain $m^* \in S^{\Gamma(\theta)}$. This completes the proof of Step 2. ∎

Step 3: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow \lambda_i(m_{-i}) = 0$ for any profile (m_i, m_{-i}) under Rules 2 or 3, where $\lambda_i \in \Delta(M_{-i})$ represents the belief held by i to which m_i is a best response.

Proof of Step 3: Suppose $m_i \in S_i^{\Gamma(\theta)}$. By Step 1, m_i has the form of $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1)$ for some $\theta' \in \Theta$, where the state θ' announced by different agents might be different. Given the message m_i, we define the set of messages of the remaining agents which trigger Rule 1, 2, or 3. Let M_{-i}^1 be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 1 and $M_{-i}^{2,i}$ be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 2 with agent i as the deviating player (odd man out).

We consider a given belief λ_i of agent i. If $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 0$, then Rule 2 or 3 will be triggered with probability one. Although Rule 2 can now be triggered with a "deviating agent (odd man out)" being different from i, it is easily checked that a similar argument to that in Step 1 applies so that the message m_i cannot be a best reply to λ_i. So, suppose that

$$0 < \sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) < 1.$$

For each $\tilde{\theta} \in \Theta$, define

$$\hat{m}_i^3(\tilde{\theta}) = \begin{cases} (m_{j*}^2[\theta'], m_i^3[\theta', 2]) & \text{if } \tilde{\theta} = \theta' \\ m_i^3[\tilde{\theta}] & \text{otherwise,} \end{cases}$$

where $j^* = \arg\max_{j \in N} u_i(m_j^2[\theta']; \theta)$. Define $\hat{m}_i^4 = \arg\max_{y \in \Delta(A)} u_i(y; \theta)$. We set $\hat{m}_i^6$ to be an integer sufficiently large. Define $\hat{m}_i = (\theta', m_i^2, \hat{m}_i^3, \hat{m}_i^4, m_i^5, \hat{m}_i^6)$ as i's alternative message in which we keep $m_i^1 = \theta', m_i^2$ and m_i^5 unchanged. Then, as $\hat{m}_i^6$ tends to infinity, agent i's expected utility from choosing $\hat{m}_i$ is approximately at least as high as

$$\sum_{m_{-i} \in M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta) + \sum_{m_{-i} \notin M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(\hat{m}_i^4; \theta),$$

which is strictly larger than i's expected payoff from choosing m_i. Hence, by choosing $\hat{m}_i^6$ large enough, $\hat{m}_i$ is a better response to λ_i (in words, the loss in Rule 2 can always be offset by a bigger gain in Rule 3). This is a contradiction.

So, if $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1) \in S_i^{\Gamma(\theta)}$, it follows that agent i must be convinced that each $j \neq i$ is choosing a message of the form $(\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1)$ and hence $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 1$. ∎

We introduce an additional piece of notation. For any $\theta, \theta' \in \Theta$ and $i \in N$, define

$$S_i^{\Gamma(\theta)}[\theta'] = \left\{ m_i \in S_i^{\Gamma(\theta)} \mid m_i^1 = \theta' \text{ and } m_i^6 = 1 \right\}.$$

Let $S^{\Gamma(\theta)}[\theta'] = \times_{i\in N} S_i^{\Gamma(\theta)}[\theta']$, and of course, $S^{\Gamma(\theta)} = \bigcup_{\theta'\in\Theta} S^{\Gamma(\theta)}[\theta']$.

Step 4: $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$.

Proof of Step 4: By Step 3, we know that if $m_i \in S_i^{\Gamma(\theta)}$, there exists $\theta' \in \Theta$ such that agent i both is using and is convinced that every agent j is using only messages of the form $m_j = (\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1)$. We begin with the following auxiliary claim, whose proof is relegated to the appendix:

Claim 3 *If there exists $\bar{m} \in S^{\Gamma(\theta)}[\theta']$, for any $a^* \in F(\theta')$, there also exists $m^* \in S^{\Gamma(\theta)}[\theta']$ such that $a^* = g(m^*)$.*

We thus proceed with the proof. If $\theta' = \theta$, by Rule 1 and the construction of the mechanism, $g(m) \in F(\theta)$. So, in what follows, we assume that $\theta' \neq \theta$. Suppose by way of contradiction that there exists $\bar{m} \in S^{\Gamma(\theta)}[\theta']$ such that $g(\bar{m}) \notin F(\theta)$. Since $g(\bar{m}) \in F(\theta')$, by uniform monotonicity, we know that there exist $i \in N$, $a^* \in F(\theta')$, and $z^* \in \Delta(A)$ such that $u_i(a^*; \theta') \geq u_i(z^*; \theta')$; and $u_i(a^*; \theta) < u_i(z^*; \theta)$.

By Claim 3, there exists $m^* \in S^{\Gamma(\theta)}[\theta']$ such that $g(m^*) = a^*$ and $m_j^* = (\theta', m_j^{*2}, m_j^{*3}, m_j^{*4}, m_j^{*5}, 1) \in S_j^{\Gamma(\theta)}$ for any $j \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}[\theta']$, there exists $\lambda_i \in \Delta(M_{-i})$ such that (i) $\lambda_i(m_{-i}) > 0 \Rightarrow m_j = (\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1)$ for any $j \neq i$ and (ii) $\sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(\tilde{m}_i, m_{-i}); \theta)$ for all $\tilde{m}_i \in M_i$. Define

$$\hat{m}_{-i}(m_i^*) \in \arg \max_{(m_i^*, m_{-i})\in S^{\Gamma(\theta)}[\theta']} u_i(g(m_i^*, m_{-i}); \theta).$$

Note that we have $g(m_i^*, \hat{m}_{-i}(m_i^*)) \in \arg\max_{a\in F(\theta')} u_i(a; \theta)$, i.e., $g(m_i^*, \hat{m}_{-i}(m_i^*))$ induces one of i's best outcomes under Rule 1 in which all agents unanimously announce θ' in state θ. Without loss of generality, assume that the winner of the modulo game that gives this great outcome to agent i is actually not agent i himself.[10]

Then, we define $\tilde{\lambda}_i \in \Delta(M_{-i})$ as follows: $\tilde{\lambda}_i(m_{-i}) = 0$ if and only if $m_{-i} \neq \hat{m}_{-i}(m_i^*)$. By construction of $\tilde{\lambda}_i$, we have that m_i^* must be a best response to the redefined belief $\tilde{\lambda}_i$. Since m_i^* is a best response to $\tilde{\lambda}_i$ and m_i^* triggers Rule 1 with probability one under $\tilde{\lambda}_i$, agent i should not in particular have an incentive to induce Rule 2. This implies that we must have $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(g(m_i', \hat{m}_{-i}(m_i^*)); \theta)$ for any m_i'.

We can organize the argument in two cases:

---

[10]This is indeed confirmed in the proof of Claim 3, where we explicitly construct m_i^* from $\bar{m}_i$.

Case 1. Assume $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$. Then, we define $\hat{m}_i^3$ as follows: for any $\tilde{\theta} \in \Theta$,

$$\hat{m}_i^3[\tilde{\theta}, 1] = \begin{cases} z^* & \text{if } \tilde{\theta} = \theta', \\ m_i^{*3}[\tilde{\theta}, 1] & \text{otherwise}, \end{cases}$$

and

$$\hat{m}_i^3[\tilde{\theta}, 2] = \begin{cases} a^* & \text{if } \tilde{\theta} = \theta', \\ m_i^{*3}[\tilde{\theta}, 2] & \text{otherwise}. \end{cases}$$

Define $\hat{m}_i = (\theta', m_i^{*2}, \hat{m}_i^3, m_i^{*4}, m_i^{*5}, \hat{m}_i^6)$ where we only change the third and sixth components of m_i^*, i.e., $\hat{m}_i^3$ and $\hat{m}_i^6$. With this choice of strategy, agent i changes the outcome with respect to using m_i^* only when the outcome under m_i^* was a^*. By choosing $\hat{m}_i^6$ sufficiently large, we conclude that $\hat{m}_i$ is an even better response than m_i^* to $\tilde{\lambda}_i$. This contradicts the hypothesis that m_i^* is a best response to $\tilde{\lambda}_i$. Therefore, we can conclude that $m_i^* \notin S_i^{\Gamma(\theta)}[\theta']$. By Claim 3, this further implies that $\bar{m}_i \notin S_i^{\Gamma(\theta)}[\theta']$. This contradicts our hypothesis that $\bar{m} \in S^{\Gamma(\theta)}[\theta']$.

Case 2. Assume, on the other hand, that $g(m_i^*, \hat{m}_{-i}(m_i^*)) \neq a^*$. We shall show that this case is impossible. In this case, relying on the strategy $\hat{m}_i$ as defined in Case 1, note that $g(\hat{m}_i, \hat{m}_{-i}(m_i^*)) = g(m_i^*, \hat{m}_{-i}(m_i^*))$ with probability $\hat{m}_i^6/(\hat{m}_i^6+1)$, as the only change happened upon a^* being the outcome.

For each $\varepsilon > 0$ and m_{-i}, we define

$$\lambda_i^\varepsilon(m_{-i}) = \begin{cases} 1-\varepsilon & \text{if } m_{-i} = \hat{m}_{-i}(m_i^*) \\ \varepsilon & \text{for one } \tilde{m}_{-i} : \tilde{m}_{-i}^1 = \theta', \tilde{m}_{-i}^2 = a^*, \tilde{m}_{-i}^5 = \hat{m}_{-i}^5(m_i^*), \tilde{m}_{-i}^6 = 1, \end{cases}$$

where we denote $\tilde{m}_j^1 = \theta'$ for all $j \neq i$ by $\tilde{m}_{-i}^1 = \theta'$ and the same notation applies to $\tilde{m}_{-i}^2, \tilde{m}_{-i}^5$, and $\tilde{m}_{-i}^6$. Since agent i, by our hypothesis, is not the winner of the modulo game under $(m_i^*, \hat{m}_{-i}(m_i^*))$, by construction of $\tilde{m}_{-i}$, we have $g(m_i^*, \tilde{m}_{-i}) = a^*$. Moreover, each $\tilde{m}_j$ is rationalizable because agent j can believe that agent i chooses j's best outcome in the place of $m_i^2[\theta']$ and i becomes the winner of the modulo game.

Consider again the strategy $\hat{m}_i$ as in Case 1. As $\hat{m}_i^6$ tends to infinity, we obtain

$$\sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta)$$
$$\approx (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon u_i(z^*; \theta)$$
$$> (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon u_i(a^*; \theta) \quad (\because u_i(z^*; \theta) > u_i(a^*; \theta))$$
$$= \sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta),$$

where the last equality follows from the fact that agent i is not the winner of the modulo game when the others are using the specified strategy with probability ε.

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Therefore, $\hat{m}_i$ is an even better response than m_i^* to λ_i^ε. We conclude that m_i^* is *not* a best response to λ_i^ε for any $\varepsilon > 0$.

Finally, to show impossibility, we claim that m_i^* is a best response to λ_i^ε as long as we choose $\varepsilon > 0$ sufficiently small. Consider an alternative message $m_i^{'}$ that induces Rule 2. Fix any such alternative message $m_i^{'}$. No matter how large $m_i^{'6}$ can be, one can choose $\varepsilon > 0$ small enough so that

$$\frac{1}{m_i^{'6} + 1} \left[u_i(g(m_i^*, \tilde{m}_{-i}); \theta) - u_i(z_i(\theta^{'}, \theta^{'}); \theta) \right]$$
$$> \varepsilon \left[u_i(g(m_i^{'}, \tilde{m}_{-i}); \theta) - u_i(g(m_i^*, \tilde{m}_{-i}); \theta) \right].$$

Indeed, this is so because $g(m_i^*, \tilde{m}_{-i}) = a^* \in F(\theta^{'})$, and hence, $u_i(g(m_i^*, \tilde{m}_{-i}); \theta) > u_i(z_i(\theta^{'}, \theta^{'}); \theta)$ by SNWA. Moreover, given agent i's belief λ_i^ε, m_i^* results in the best outcome with probability $1 - \varepsilon$ so that $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(g(m_i^{'}, \hat{m}_{-i}(m_i^*)); \theta)$. Therefore, once we choose $\varepsilon > 0$ small enough, m_i^* is made an even better response to λ_i^ε than $m_i^{'}$. Since this argument applies to any such alternative message $m_i^{'}$, agent i has no incentive to trigger Rule 2 himself. This establishes that m_i^* is a best response to λ_i^ε. ■

Steps 1 through 4 together imply that for each $\theta \in \Theta$,

$$\bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} = F(\theta).$$

This completes the proof of the theorem. ■

We make some comments on the properties of the mechanism we constructed. One novelty of the mechanism lies in the way we use the modulo game under the consensus Rule 1. Modulo games have been often used in the literature, but they are viewed as a substitute for the integer games where the person who announces the highest integer becomes a dictator. In fact, our Rule 3 on strong disagreements is a stochastic version of an integer game, used to knock out bad message profiles so that the agents end up making a unanimous announcement in equilibrium, as only Rule 1 (consensus) prevails. However, in our mechanism, the modulo game in Rule 1 is used to select an outcome rather than knock out bad message profiles. Since the rational expectations hypothesis is not needed here, each agent believes that the modulo game under Rule 1 always works in his favor but the resulting outcome is not necessarily what he expects to happen.

Our paper follows the classic implementation literature in allowing for arbitrary mechanisms. This is often justified in order to obtain a tight characterization, i.e., to have a small gap between necessary and sufficient conditions. Jackson (1992),

however, rightly argues that some of the power of implementation results derive from the fact that we have not imposed any restrictions on the mechanisms. In order to restrict attention to reasonable mechanisms, Jackson, Palfrey, and Srivastava (henceforth, JPS, 1994) propose the best response property, which requires that there be a best response for every possible belief that an agent might hold about other agents' strategies. Indeed, our mechanism does not satisfy this property as there exist no best responses when Rule 3 (strong disagreements) is triggered. As we discuss in Section 5, finite mechanisms sometimes suffice for rationalizable implementation. Finite mechanisms clearly satisfy the JPS best response property. We will have more to say about rationalizable implementation by finite mechanisms for the case of SCFs and postpone this discussion until we state our result for that case.

When focusing only on SCFs, we obtain the following result as a corollary of Theorem 2.

Corollary 1 *Suppose that there are at least three agents ($n \geq 3$). If an SCF f satisfies Maskin monotonicity and NWA, it is fully implementable in rationalizable strategies.*

Remark: BMT (2011) show in their Proposition 1 that strict Maskin monotonicity is necessary for rationalizable implementation of SCFs and further show in their footnote 5 that under NWA, strict Maskin monotonicity is equivalent to the standard Maskin monotonicity. Therefore, our result is consistent with BMT in the case of SCFs.

Proof: This follows because uniform monotonicity and SNWA reduce to Maskin monotonicity and NWA (the SCF-version used by BMT (2011)), respectively, as long as the social choice rule is single-valued. ∎

This is a logical strengthening of Proposition 2 of BMT (2011) because we completely dispense with the responsiveness of SCFs, which is assumed there. We say that an SCF f is *responsive* if, for any $\theta, \theta' \in \Theta$, whenever $\theta \neq \theta'$, $f(\theta) \neq f(\theta')$. We recall that BMT (2011) introduce their *best-response property*, and restrict attention to the mechanisms satisfying it when considering nonresponsive SCFs. It is easy to see that our canonical mechanism used in Theorem 2 satisfies the BMT best-response property; see Definition 6 of BMT (p.1267). [11]

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[11]In their proposition 3, BMT show that strict Maskin monotonicity* (BMT (p.1265)) is a necessary condition for rationalizable implementation of SCFs by a mechanism satisfying their best response property. Maskin monotonicity*, together with the modified version of NWA, implies strict Maskin monotonicity*, which is itself logically stronger than strict Maskin monotonicity. See p.1269 of BMT for the details. When considering environments with monetary transfers and

7   Concluding Remarks

By relying on a setwise condition requiring the nestedness of lower contour sets, a condition that we term uniform monotonicity, we have shown that rationalizable implementation of correspondences leads to a significantly more permissive theory than its counterpart using Nash equilibrium. This has been established for environments with at least three agents. The two-agent general sufficiency argument is likely handled by adding the usual requirement of nonempty intersections of lower contour sets; we chose instead to focus on a simple finite mechanism for a useful example. The extension to incomplete information environments should be our natural next step. Our conclusion is that, in the comparison with Nash equilibrium, dropping the rational expectations assumption while still retaining best-replies to beliefs, expands significantly the range of socially desirable rules that can be potentially decentralized. For a specific rule in a concrete environment, one should aim to construct a less abstract mechanism than our proposed canonical one, but we view our contribution as a way to draw the landscape of rules that could be in principle implemented.

Appendix

In this Appendix, we first provide the proof of Claim 3, which is part of Step 4 in the proof of Theorem 2. Second, we discuss ordinality issues. Third, we comment on the role of finite mechanisms, and fourth, we outline how one can extend our results (Theorems 1 and 2) to the case of weak implementation.

A.1.   Proof of Claim 3

We set $n + 1 \equiv 1$ and $0 \equiv n$. We construct a message profile m^*, which induces Rule 1 with probability one, and in which all agents unanimously announce θ' in the first component of their message, and agent $i + 1$ becomes the winner of the modulo game, as follows. First, for agent $i + 1$, define $m_{i+1}^* =$

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quasilinear preferences, Chen, Kunimoto, Sun, and Xiong (2017) show that an SCF is rationalizably implementable by a finite mechanism if and only if it satisfies Maskin monotonicity*. That paper also obtains the following result: an SCF is Nash implementable by a finite mechanism if and only if it satisfies Maskin monotonicity. Hence, in some classes of economic environments, when we only deal with SCFs and finite mechanisms, Nash implementation yields results that are more permissive than rationalizable implementation. For the case of SCCs, the result for Nash implementation is extended. In particular, if an SCC is deterministic and the set of pure outcomes is finite, the SCC is Nash implementable by a finite mechanism if and only if it satisfies Maskin monotonicity. What is unknown is whether the rationalizable implementation result can be extended to SCCs in a similar fashion.

$(\theta', m_{i+1}^{*2}, \bar{m}_{i+1}^3, \bar{m}_{i+1}^4, i+1, 1)$, where $m_{i+1}^{*2}[\theta'] = a^*$. Second, for each $j \in N\backslash\{i+1\}$, define $m_j^* = (\theta', m_j^{*2}, \bar{m}_j^3, \bar{m}_j^4, 1, 1)$ such that

$$m_j^{*2}[\theta'] = \begin{cases} a_{j-1}(\theta, \theta') & \text{if } j \neq 1 \\ a_n(\theta, \theta') & \text{if } j = 1, \end{cases}$$

where $a_{j-1}(\theta, \theta') \in \arg\max_{a \in F(\theta')} u_{j-1}(a; \theta)$, which denotes one of the maximizers of $u_{j-1}(\cdot; \theta)$ within all the outcomes in $F(\theta')$. What remains to show is that $m^* \in S^{\Gamma(\theta)}$. We proceed to do so.

First, we show that m_{i+1}^* is a best response to some belief. Define $\lambda_{i+1}^* \in \Delta(M_{-(i+1)})$ with support as follows:

$$
\begin{aligned}
m_j^1 &= \theta'; \\
m_j^2[\theta'] &= a_{j-1}(\theta, \theta'); \\
m_j^5 &= \begin{cases} 2 & \text{if } j = i+2, \\ 1 & \text{otherwise}; \end{cases} \\
m_j^6 &= 1,
\end{aligned}
$$

for all $j \in N\backslash\{i+1\}$. Given λ_{i+1}^* and m_{i+1}^*, agent $(i+2)$ becomes the winner of the modulo game. Thus, it generates the best possible outcome for agent $(i+1)$ conditional on Rule 1. Since there exists $\bar{m}_{i+1} \in S_{i+1}^{\Gamma(\theta)}[\theta']$ and m_{i+1}^* differs from $\bar{m}_{i+1}$ only in the second and fifth components of the message, m_{i+1}^* is a best response to λ_{i+1}^*.

Second, we show that $\lambda_{i+1}^*(m_{-(i+1)}) > 0 \Rightarrow m_{-(i+1)} \in S_{-(i+1)}^{\Gamma(\theta)}[\theta']$, which means that each m_j in the support of λ_{i+1}^* is rationalizable. Define $\lambda_{i+2}^* \in \Delta(M_{-(i+2)})$ with support as follows:

$$
\begin{aligned}
m_k^1 &= \theta'; \\
m_k^2[\theta'] &= a_{k-1}(\theta, \theta'); \\
m_k^5 &= \begin{cases} i+3 & \text{if } k = i+1, \\ 1 & \text{otherwise}; \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq i+2$. Then, given λ_{i+2}^* and m_{i+2}^*, agent $(i+3)$ becomes the winner of the modulo game so that the best outcome for agent $(i+2)$ conditional on Rule 1 is realized. Assume $j \in N\backslash\{i+1, i+2\}$. Define $\lambda_j^* \in \Delta(M_{-j})$ with support as

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follows:

$$
\begin{aligned}
m_k^1 &= \theta'; \\
m_k^2[\theta'] &= \begin{cases} a_n(\theta,\theta') & \text{if } k = 1, \\ a_{k-1}(\theta,\theta') & \text{otherwise;} \end{cases} \\
m_k^5 &= \begin{cases} j+1 & \text{if } k = i+1, \\ 1 & \text{otherwise;} \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq j$. Assume $j < n$. Then, given λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the best outcome for agent j is realized conditional on Rule 1. Assume, on the other hand, that $j = n$. Then, given λ_n^* and m_n^*, agent 1 becomes the winner of the modulo game so that the best outcome for agent n is realized conditional on Rule 1. We know that (i) by our hypothesis and Step 3, there exist $\bar{m}_j \in S_j^{\Gamma(\theta)}[\theta']$ together with the belief $\bar{\lambda}_j$ to which $\bar{m}_j$ is a best response and which induces Rule 1 with probability one; (ii) m_j^* differs from $\bar{m}_j$ only in the second and fifth components of the message; and (iii) m_j^* generates the best outcome for himself conditional on Rule 1 given the belief λ_j^*. Therefore, we have that the support of λ_{i+1}^* is rationalizable. That is, for each $j \neq i+1$, $\lambda_{i+1}^*(m_{-(i+1)}) > 0 \Rightarrow m_{-(i+1)} \in S_{-(i+1)}^{\Gamma(\theta)}[\theta']$. Thus, $m_{i+1}^* \in S_{i+1}^{\Gamma(\theta)}$.

Third, we show that, for each $j \neq i+1$, m_j^* can be made a best response to some belief. Fix $j \neq i+1$. Define $\lambda_j^* \in \Delta(M_{-j})$ with support as follows:

$$
\begin{aligned}
m_k^1 &= \theta'; \\
m_k^2[\theta'] &= \begin{cases} a_n(\theta,\theta') & \text{if } k = 1, \\ a_{k-1}(\theta,\theta') & \text{otherwise;} \end{cases} \\
m_k^5 &= \begin{cases} j+1 & \text{if } k = i+1, \\ 1 & \text{otherwise;} \end{cases} \\
m_k^6 &= 1,
\end{aligned}
$$

for all $k \neq j$. Given λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the best outcome for agent j conditional on Rule 1 is realized. Since we know that there exists $\bar{m}_j \in S_j^{\Gamma(\theta)}[\theta']$ together with $\bar{\lambda}_j$ to which $\bar{m}_i$ is a best response and which induces Rule 1 with probability one and m_j^* differs from $\bar{m}_j$ only in the second and fifth components of the message, m_j^* is a best response to λ_j^*.

Fourth, we claim that for each $j \neq i+1$, the support of λ_j^* is rationalizable, i.e., $\lambda_j^*(m_{-j}) > 0 \Rightarrow m_{-j} \in S_{-j}^{\Gamma(\theta)}[\theta']$. Consider $\hat{m}_{i+1} = (\theta', \hat{m}_{i+1}^2, \bar{m}_{i+1}^3, \bar{m}_{i+1}^4, j+1, 1)$, as a generic element in the support of λ_j^*, where $\hat{m}_{i+1}^2[\theta'] = a_n(\theta,\theta')$ if $i+1 = 1$ (i.e.,

$i = n$) or $a_i(\theta, \theta')$ otherwise. Define $\hat{\lambda}_{i+1} \in \Delta(M_{-(i+1)})$ with support as follows:

$$m_k^1 = \theta';$$

$$m_k^2[\theta'] = \begin{cases} a_n(\theta, \theta') & \text{if } k = 1, \\ a_{k-1}(\theta, \theta') & \text{otherwise;} \end{cases}$$

$$m_k^5 = \begin{cases} n + 2 - j & \text{if } k = i + 2, \\ 1 & \text{otherwise;} \end{cases}$$

$$m_k^6 = 1,$$

for all $k \neq i + 1$. Given $\hat{m}_{i+1}$ and $\hat{\lambda}_{i+1}$, agent $(i + 2)$ becomes the winner of the modulo game so that the best outcome for agent $(i + 1)$ conditional on Rule 1 is realized. Fix $k \in N\backslash\{i + 1, j\}$. Consider $\hat{m}_k = (\theta', \hat{m}_k^2, \bar{m}_k^3, \bar{m}_k^4, 1, 1)$, as a generic element in the support of λ_j^* where $\hat{m}_k^2[\theta'] = a_n(\theta, \theta')$ if $k = 1$ or $a_{k-1}(\theta, \theta')$ otherwise. Define $\hat{\lambda}_k \in \Delta(M_{-k})$ with support as follows:

$$m_h^1 = \theta';$$

$$m_h^2[\theta'] = \begin{cases} a_n(\theta, \theta') & \text{if if } h = 1, \\ a_{k-1}(\theta, \theta') & \text{otherwise;} \end{cases}$$

$$m_h^5 = \begin{cases} k + 1 & \text{if } h = i + 1, \\ 1 & \text{otherwise;} \end{cases}$$

$$m_h^6 = 1,$$

for all $h \neq k$. Given $\hat{m}_k$ and $\hat{\lambda}_k$, agent $(k + 1)$ becomes the winner of the modulo game so that the best outcome for agent k conditional on Rule 1 is realized. We know that: (i) by our hypothesis and Step 3, there exists $\bar{m}_k \in S_k^{\Gamma(\theta)}[\theta']$ together with the belief $\bar{\lambda}_k$ to which $\bar{m}_k$ is a best response and which induces Rule 1 with probability one; (ii) $\hat{m}_k$ differs from $\bar{m}_k$ only in the second and fifth components of the message; and (iii) $\hat{m}_k$ generates the best outcome for himself conditional on Rule 1 given the belief $\hat{\lambda}_k$. Therefore, for each $j \neq i + 1$, we have that the support of λ_j^* is rationalizable. That is, for each $j \neq i + 1$, $\lambda_j^*(m_{-j}) > 0 \Rightarrow m_{-j} \in S_{-j}^{\Gamma(\theta)}[\theta']$. This implies that $m_j^* \in S_j^{\Gamma(\theta)}[\theta']$ for each $j \neq i + 1$.

In sum, we conclude that we have $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a^*$, as desired. ∎

A.2. Ordinality

The theory of implementation often associates each state with a profile of *ordinal* preferences and does not introduce any cardinal representation. In this subsection, we identify a state θ with a profile of ordinal preferences $(\succeq_i^\theta)_{i \in N}$ over A. This

is the ordinal approach considered in Mezzetti and Renou (2012) in the context of Nash implementation. BMT (2011) also discuss the ordinal approach to rationalizable implementation. Specifically, we refer the reader to Section 6 of Mezzetti and Renou (2012) and Section 5 of BMT (2011). We say that $u = (u_1, \ldots, u_n)$ is a cardinal representation of $(\succeq_i^\theta)_{i \in N, \theta \in \Theta}$ if, for each $a, a' \in A, i \in N$, and $\theta \in \Theta$, we have $u_i(a; \theta) \geq u_i(a'; \theta) \Leftrightarrow a \succeq_i^\theta a'$. A deterministic SCC F is *ordinally* fully implementable in rationalizable strategies if it is fully implementable in rationalizable strategies "independently of the cardinal representation."

We come to the next couple of definitions:

Definition 6 *A deterministic SCC F satisfies* **ordinal** *(weak) uniform monotonicity if it satisfies (weak) uniform monotonicity for any cardinal representation.*

With its weak version, we can show the following necessity result, whose proof is omitted:

Proposition 1 *If a deterministic SCC F is ordinally fully implementable in rationalizable strategies, it satisfies ordinal weak uniform monotonicity.*

Here is our next definition:

Definition 7 *A deterministic SCC F satisfies* **ordinal** *SNWA if, for each $\theta \in \Theta$ and $i \in N$, there exists a pure alternative $z_i^\theta \in A$ such that $a \succ_i^\theta z_i^\theta$ for each $a \in F(\theta)$.*

The proof of this sufficiency result is also omitted:

Proposition 2 *Suppose that there are at least three agents ($n \geq 3$). If a deterministic SCC F satisfies ordinal uniform monotonicity and ordinal SNWA, it is ordinally fully implementable in rationalizable strategies.*

A.3. Finite Mechanisms

Recall that our sufficiency result for rationalizable implementation (Theorem 2) employs an *infinite* implementing mechanism. In this section, we elaborate on the role of finite implementing mechanisms. We show by example that there is a finite mechanism that achieves Nash implementation but fails rationalizable implementation. Of course, this does not necessarily imply that one cannot achieve rationalizable implementation by a finite mechanism. Rather, the point is that a Nash-implementing finite mechanism need not achieve rationalizable implementation.

There are two agents $N = \{1, 2\}$; two states $\Theta = \{\alpha, \beta\}$; and a finite number of pure outcomes $A = \{a_1, a_2, a_3\}$. Agent 1's strict preference relation over A is given as follows:

$$a_1 \succ_1^\alpha a_2 \succ_1^\alpha a_3 \text{ and } a_2 \succ_1^\beta a_1 \succ_1^\beta a_3.$$

Agent 2's strict preference relation over A is given as follows:

$$a_1 \succ_2^\alpha a_3 \succ_2^\alpha a_2 \text{ and } a_1 \succ_2^\beta a_3 \succ_2^\beta a_2.$$

Note that Agent 2 has state-independent preferences. We consider the following SCC: $F(\alpha) = \{a_1, a_3\}$ and $F(\beta) = \{a_3\}$. We first claim that the SCC F satisfies Maskin monotonicity. When we move from β to α, we know that $F(\beta) \subseteq F(\alpha)$. So, we consider the case of moving from α to β. Fix $a_1 \in F(\alpha)$. Since there is a preference reversal around a_1 such that $a_1 \succ_1^\alpha a_2$ and $a_2 \succ_1^\beta a_1$, going from α to β is not a monotonic transformation of preferences around a_1. Thus, no conditions are imposed on F so that it satisfies Maskin monotonicity. This further implies that F satisfies uniform monotonicity as well.

We construct a finite mechanism that implements the SCC F in Nash equilibrium. Consider the following mechanism $\Gamma = (M, g)$ where $M_1 = \{m_1^1, m_1^2, m_1^3\}$; $M_2 = \{m_2^1, m_2^2\}$; and the deterministic outcome function $g(\cdot)$ is given in the table below:

$g(m)$		Agent 2	
		m_2^1	m_2^2
	m_1^1	a_1	a_3
Agent 1	m_1^2	a_2	a_3
	m_1^3	a_3	a_3

It can be easily shown that $(m_1^1, m_2^1), (m_1^2, m_2^2)$, and (m_1^3, m_2^2) are pure strategy Nash equilibria in the game $\Gamma(\alpha)$ and (m_1^2, m_2^2) and (m_1^3, m_2^2) are pure strategy Nash equilibria in the game $\Gamma(\beta)$. As long as we are concerned with pure strategy Nash equilibria, we can see that the mechanism Γ Nash implements the SCC F.

Suppose that there is a nontrivial mixed strategy Nash equilibrium in the game $\Gamma(\alpha)$. Assume that agent 2 chooses a pure strategy in the supposed mixed equilibrium. Then, agent 1 has an incentive to randomize over the messages only if agent 2 chooses m_2^2 where a_3 is the resulting outcome no matter what agent 1 does. Therefore, any such mixed strategy equilibrium outcome, if it exists, is consistent with the SCC. Now, let us consider a mixed strategy equilibrium where agent 2 uses a genuinely mixed strategy: choose m_2^1 with probability q and m_2^2 with probability $1 - q$ where $q \in (0, 1)$. In the game $\Gamma(\alpha)$, for any $q \in (0, 1)$, m_1^2 cannot be a best response to agent 2's strategy $(q, 1 - q)$. Hence, in such a mixed

strategy equilibrium, m_1^2 is never played with positive probability. Then, any such mixed strategy equilibrium outcome, if it exists, only generates a_1 or a_3, which are consistent with the SCC F.

Suppose that there is a nontrivial mixed strategy equilibrium in the game $\Gamma(\beta)$. Assume that agent 2 chooses a pure strategy in the supposed mixed equilibrium. Then, agent 1 has an incentive to randomize over the messages only if agent 2 chooses m_2^2 where a_3 is the resulting outcome no matter what agent 1 does. Therefore, any such mixed strategy equilibrium outcome, if it exists, is consistent with the SCC. Now, let us consider a mixed strategy equilibrium where agent 2 uses a genuinely mixed strategy: choose m_2^1 with probability q and m_2^2 with probability $1-q$ where $q \in (0,1)$. In the game $\Gamma(\beta)$, for any $q \in (0,1)$, only m_1^2 is a strict best response to agent 2's strategy $(q, 1-q)$. In other words, agent 2 should choose m_1^2 with probability one. However, if agent 1 chooses m_1^2 for sure, agent 2 is better off by switching to choosing a pure strategy m_2^2. Therefore, there is no such mixed strategy equilibrium. This implies that the SCC F is Nash implementable by the finite mechanism Γ even in the sense of Mezzetti and Renou (2012), who require every outcome in the support of Nash equilibria to be consistent with the SCC.

Next, we show that the mechanism Γ does not implement the SCC F in rationalizable strategies. Consider the game $\Gamma(\beta)$. First, m_1^2 is a best response to the belief that agent 2 chooses m_2^1. Second, m_2^1 is a best response to the belief that agent 1 chooses m_1^1. Third, m_1^1 is a best response to the belief that agent 2 chooses m_2^2. Therefore, both m_1^2 and m_2^1 survive the first round of deletion of never best responses. We can repeat this argument so that $m_1^2 \in S_1^{\Gamma(\beta)}$ and $m_2^1 \in S_2^{\Gamma(\beta)}$. Since both m_1^2 and m_2^1 are rationalizable messages, we obtain that $g(m_1^2, m_2^1) = a_2$, which is inconsistent with $F(\beta) = \{a_3\}$. Therefore, the SCC F is *not* rationalizably implementable by the mechanism Γ.

We can easily modify this example by adding to A one more pure outcome a_4, which is a strictly worse outcome for anyone than $\{a_1, a_2, a_3\}$ in any state. With this modification, the SCC F now satisfies SNWA. It is possible that the SCC in this example is implementable in rationalizable strategies. In particular, note that the only assumption missing from our Theorem 2 is the presence of two agents. Adding a third agent, say with state-independent preferences identical to agent 2, but with only one message in the mechanism, would still allow us to retain all the features of the example. In that modification, Theorem 2 would implement F using the infinite mechanism in its proof. We do not know whether one could prove Theorem 2 on the basis of a finite canonical mechanism. On the other hand, the example in Section 5 is also based on a finite mechanism, and it re-enforces the paper's main message of rationalizable implementation being significantly more permissive than Nash implementation, even for a two-agent environment.

In closing, we recall that BMT (2011) introduce the *best-response property*, and restrict attention to the mechanisms satisfying it when considering nonresponsive SCFs. It is easy to see that our canonical mechanism used in Theorem 2 satisfies the best-response property. Of course, as long as we have a finite mechanism that achieves rationalizable implementation, that finite mechanism will satisfy it as well. This is confirmed in the example we discussed in Section 5.

A.4. Weak Implementation in Rationalizable Strategies

Next, we provide the definition of *weak rationalizable implementation*.

Definition 8 (Weak Rationalizable Implementation) *An SCC F is **weakly implementable in rationalizable strategies** if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$, the following two conditions hold: (1) $S^{\Gamma(\theta)} \neq \emptyset$; and (2) for each $m \in S^{\Gamma(\theta)}$, then $g(m) \in F(\theta)$.*

We begin by proposing a condition for weak implementation in rationalizable strategies:

Definition 9 *An SCC F satisfies **weak K-uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, there exists a nonempty set $K(\theta) \subseteq F(\theta)$ such that if*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \quad \forall a \in co(K(\theta)), \forall i \in N, \forall z \in \Delta(A),$$

then, $K(\theta) \subseteq F(\theta')$.

Remark: When we consider SCFs, $co(K(\theta))$ becomes a singleton set. Therefore, in this case, the condition just defined also reduces to Maskin monotonicity.

We slightly strengthen weak K-uniform monotonicity into the following:

Definition 10 *An SCC F satisfies K-**uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, there exists a nonempty set $K(\theta) \subseteq F(\theta)$ such that, if*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \quad \forall a \in K(\theta), \forall i \in N, \forall z \in \Delta(A),$$

then, $K(\theta) \subseteq F(\theta')$.

The same comment we made after the definition of uniform monotonicity applies here. Therefore, under expected utility, weak K-uniform monotonicity is equivalent to K-uniform monotonicity. Note also that K-uniform monotonicity is logically weaker than uniform monotonicity, itself weaker than Maskin monotonicity. All three reduce to the same condition when one considers single-valued rules.

The necessity result for weak implementation follows:

Theorem 3 *If an SCC F is weakly implementable in rationalizable strategies, it satisfies weak K-uniform monotonicity.*

Proof: Suppose F is weakly implementable in rationalizable strategies by a mechanism $\Gamma = (M, g)$. Fix two states $\theta, \theta' \in \Theta$. Define

$$K(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\}.$$

Assume the following property:

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \ \ \forall a \in \mathrm{co}(K(\theta)), \ \forall i \in N, \ \forall z \in \Delta(A) \ \ (*)$$

Then, due to the hypothesis that F is weakly implementable by Γ, we fix $m^* \in S^{\Gamma(\theta)}$, and we have that $g(m^*) \in K(\theta)$.

Fix $i \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}$, there exists $\lambda_i^{m_i^*,\theta} \in \Delta(M_{-i})$ satisfying the following two properties: (i) $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{\Gamma(\theta)}$ and $g(m_i^*, m_{-i}) \in F(\theta)$; and (ii) $\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta)$ for each $m_i' \in M_i$.

We focus on the best response property of m_i^* summarized by inequality (ii). Fix $m_i' \in M_i$. Due to the construction of $\lambda_i^{m_i^*,\theta}$, we have that

$$\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \ \geq \ \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta)$$

$$u_i(a; \theta) \ \geq \ u_i(z^a; \theta),$$

where the two lotteries a and z^a are defined as

$$a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i^*, m_{-i}) \ \text{ and } \ z^a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i', m_{-i}).$$

Since $g(m_i^*, m_{-i}) \in K(\theta)$ for each m_{-i} with $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0$, we have $a \in \mathrm{co}(K(\theta))$. Using Property $(*)$, we also obtain

$$u_i(a; \theta') \geq u_i(z^a; \theta').$$

Due to the choice of a and z^a and the hypothesis that $u_i(\cdot)$ is a von-Neumann-Morgenstern expected utility, we obtain the following:

$$\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta') \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta').$$

Since this argument does not depend upon the choice of m_i', this shows that m_i^* is a best response to $\lambda_i^{m_i^*,\theta}$ in state θ' as well. Therefore, $m_i^* \in S_i^{\Gamma(\theta')}$. Since the

choice of agent i is arbitrary, we can conclude that $m^* \in S^{\Gamma(\theta')}$. Furthermore, since the choice of $m^* \in S^{\Gamma(\theta)}$ is also arbitrary, we have $S^{\Gamma(\theta)} \subseteq S^{\Gamma(\theta')}$. Finally, by weak implementability, this implies that

$$K(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} \subseteq \bigcup_{m \in S^{\Gamma(\theta')}} \{g(m)\} \subseteq F(\theta').$$

The proof is thus complete. ∎

Next, we state the general sufficiency result for weak implementation in rationalizable strategies.

Theorem 4 *Suppose that there are at least three agents ($n \geq 3$) and the set of pure outcomes A is finite. If a deterministic SCC F satisfies K-uniform monotonicity and SNWA, it is **weakly** implementable in rationalizable strategies.*

Remark: In the sufficiency result below, we need the compactness of $K(\theta)$. To guarantee this, we assume A is finite and only consider deterministic SCCs. We view this as a technical requirement for the result.

Proof: By K-uniform monotonicity, for each $\theta \in \Theta$, we have a nonempty set $K(\theta) \subseteq F(\theta)$. Since the SCC F is deterministic and the set of pure outcomes A is finite, we have that $K(\theta)$ is a finite set for each $\theta \in \Theta$. We construct a mechanism $\Gamma = (M, g)$ such that each agent i sends a message $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6)$ where $m_i^1 \in \Theta, m_i^2 = \{m_i^2[\theta]\}_{\theta \in \Theta}$ where $m_i^2[\theta] \in K(\theta)$, $m_i^3 = \{(m_i^3[\theta, 1], m_i^3[\theta, 2])\}_{\theta \in \Theta}$ where $m_i^3[\theta, 1] \in \Delta(A)$ and $m_i^3[\theta, 2] \in K(\theta)$, $m_i^4 \in \Delta(A), m_i^5 \in N$, and $m_i^6 \in \mathbb{N}$. This mechanism is essentially the same as the one proposed in the proof of Theorem 2. The only change we have made from the message space proposed in the proof of Theorem 2 is that we use $K(\theta)$ in the spaces of $m_i^2[\theta]$ and $m_i^3[\theta, 2]$. The outcome function $g : M \to \Delta(A)$ is defined exactly as for the case of full implementation in the proof of Theorem 2. The proof consists of Steps I through IV, parallel to the steps for the full implementation proof.

Step I: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow m_i^6 = 1$.

Step II: For any $\theta \in \Theta$ and $a \in K(\theta)$, there exists $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a$.

Step III: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow \lambda_i(m_{-i}) = 0$ for any profile (m_i, m_{-i}) under Rules 2 or 3, where $\lambda_i \in \Delta(M_{-i})$ represents the belief held by agent i to which m_i is a best response.

Step IV: $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$.

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Steps I through IV together imply that for each $\theta \in \Theta$ and $m \in M$: (i) $S^{\Gamma(\theta)} \neq \emptyset$ and (ii) $m \in S^{\Gamma(\theta)} \Rightarrow g(m) \in F(\theta)$. Thus, this completes the proof of Theorem 4. ∎

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