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STRATEGY-PROOF, INDIVIDUALLY RATIONAL AND SYMMETRIC SOCIAL CHOICE FUNCTION FOR DISCRETE PUBLIC GOOD ECONOMIES

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Strategy–Proof, Individually Rational and Symmetric Social Choice Function for Discrete Public Good Economies

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* I began this work when I was in the Institute of Social and Economic Research at Osaka University. I appreciate the good research environment of the institute.

Section 1. Introduction.

We consider the problem of choosing an allocation in an economy with one private good and one (pure) public good. The public good is produced using the private good as an input which are collected from agents. The set Y of possible production levels of the public good is discrete, that is, $Y = \{0, 1, \cdots, \bar{y}\}, \bar{y} < +\infty$. The feasible set is a subset of $\mathbb{R}^n_+ \times Y$, which depends on the production technology. Each agent has a classical preference[1]. When agents as a society have to select from the feasible set, the procedure must take into account agents' preferences. Procedures are formally represented as functions from the set of possible preference profiles into the feasible set, and they are called social choice functions, or schemes. However, preferences are usually privately known. Therefore, selfish agents will possibly try to strategically misrepresent their preferences so as to manipulate the final outcome in their favor. As a result of such strategic behavior, the actual outcome may be far from satisfactory from a social point of view. Thus it is important for a social choice function to be immune from such strategic behavior. If it is immune from unilateral strategic behavior, then it is said to be strategy–proof. The conditions of "individual rationality" and "symmetry" are also important as distributional requirements. Individual rationality says that the social choice function never assigns an outcome which makes some agent worse off than he was in his initial situation. Symmetry says that if two agents have the same preference, they are treated equally. These conditions exclude unfair social choice functions such as dictatorship. Our purpose is to identify the strategy–proof, individually rational and symmetric social choice function.

A scheme of convex cost sharing is such that agents are required to share the cost of the public good according to predetermined convex cost share functions of production of the public good. Given the cost share functions, agents are faced with non–linear budget

[1] That is, each agent cares only about his consumption of the private good and the level of the public good, and his preference is continuous, strictly monotone and strictly convex.

constraints, from which their demands for the public good are derived. A scheme of convex cost sharing is said to be conservative or determined by the minimum demand principle if the minimum amount demanded for the public good is produced. Recently, the schemes of convex and conservative cost sharing have been characterized. Moulin (1994) used the stronger condition of strategy–proofness called coalitional strategy–proofness; it says that a social choice function is immune from coalitional strategic behavior. He established that a social choice function is coalitionally strategy–proof, individually rational and symmetric if and only if it is a scheme of equal and convex cost sharing determined by the minimum demand principle. Serizawa (1994) used a condition of non–bossiness[2], which was introduced by Satterthwaite and Sonnenschein (1981); it says that by changing his announced preferences, no agent can change an agent's consumption of the private good without changing his own consumption. He established that a social choice function is strategy–proof, individually rational and non–bossy if and only if it is a scheme of convex cost sharing determined by the minimum demand principle.

Coalitional strategy–proofness used in Moulin (1994)'s characterization is very strong. The question remains open whether his characterization still holds or not if we weaken coalitional strategy–proofness into strategy–proofness. On the other hand, non–bossiness used in Serizawa (1994)'s characterization seems to be a complex condition. Thus the questions also remains open whether we can substitute it by a simple, weak and economically meaningful condition. In this paper, we solve these two questions. We use a weaker condition of symmetry, which we call unanimous symmetry; it says that if all agents have the same preference, they are treated equally. We establish that

a social choice function is strategy–proof, individually rational and unanimously symmetric if and only if it is a scheme of equal and convex cost sharing determined by the minimum demand principle.

[2] Satterthwaite and Sonnenschein (1981) required this condition for the simplicity of schemes. For further justification, refer to their paper.

In Section 2, we set up the model and define important notions. In Section 3, we study how agents share the cost of the public good when a social choice function satisfies strategy–proofness, individual rationality and unanimously symmetry. In Section 4, we define the minimum demand principle formally and prove our main result.

Section 2. Model.

The set of agents is $N = \{1, 2, \dots, n\}$. A coalition is a subset of N. Given a coalition $N' \subsetneq N$ and an agent $i \in N$, we denote the coalition $N \backslash N'$ by $-N'$ and the coalition $N \backslash \{i\}$ by $-i$. There are one private good and one (pure) public good. The public good is produced using the private good as an input which are collected from agents. The set Y of possible production levels of the public good is discrete, that is, $Y = \{0, 1, \cdots, \bar{y}\}$, $\bar{y} < +\infty$. For each agent $i \in N$, we denote agent i's cost share of the public good in terms of the private good by x_i, and the amount of the public good by y. The cost function C of the public good is a increasing function from Y to $\mathbb{R}_+$ such that $C(0) = 0$. We do not assume free–disposability. Thus the feasible set is the set $Z = \{z = (x_1, \dots, x_n, y) \in \mathbb{R}^n_+ \times Y \mid C(y) = \sum_i x_i\}$.

Given $z, z', \dots \in Z$, denote the i–th coordinates of $z, z', \dots$ by $x_i, x_i', \dots$ respectively and the last coordinates of $z, z', \dots$ by $y, y', \dots$ respectively. Let $e = (0, \dots, 0)$. Given $z, z' \in Z$, let $z^i = (-x_i, y)$, $[z, z']_i = \{z'' \in Z \mid \exists \lambda \in [0, 1]$ such that $z^{i''} = \lambda \cdot z^i + (1-\lambda) \cdot z^{i'}\}$, and $[z, z'] = \bigcap_{i \in N} [z, z']_i$. Given z, z', and z'', we write $z \geqq_i z'$ if $z^i \geqq z^{i'}$, and $z'' \geqq_i [z, z']_i$ if there is $\bar{z} \in [z, z']_i$ such that $z'' \geqq_i \bar{z}$. Let U^i be the set of preferences on $\mathbb{R}^2_+$ which are represented by continuous, strictly quasi–concave, decreasing in the first coordinate and increasing in the second coordinate, where the first and second coordinates of $\mathbb{R}^2_+$ denote agent i's cost share and the amount of the public good respectively. For the simplicity of notation, we treat $u^i \in U^i$ as a function on $\mathbb{R}^{n+1}_+$ although u^i actually depends only on the i–th and last coordinates. Given $z \in Z$ and $u^i \in U^i$, let $UC(u^i, z) = \{z' \in Z \mid u^i(z') \geqq u^i(z)\}$. Let $U = U^1 \times \dots \times U^n$. Given $N' \subsetneq N$ and $i \in N$, let $U^{N'} = \times_{j \in N'} U^j$. We denote generic elements of U, $U^{N'}$ and U^{-i} by u, $u^{N'}$ and u^{-i} respectively. If $u = (u^1, \dots, u^n) \in U$, $N' \subsetneq N$ and $i \in N$ are given previously, $u^{N'}$ denotes $(u^j)_{j \in C}$ and u^{-i} denotes $(u^j)_{j \in N \setminus \{i\}}$. Given $Z' \in Z$ and $u \in U$, let $B^i(u, Z') = \{z \in Z' \mid \forall z' \in Z, u^i(z) \geqq u^i(z')\}$ and $B^i_y(u, Z')$ be the projection of $B^i(u, Z')$ on Y. When Z' is fixed, we treat $B^i(u, Z')$ and $B^i_y(u, Z')$ as correspondences from U.

Definition A social choice function or scheme is a function f from U to Z.

Note that since we define the feasible set without free–disposability, social choice functions are always budget–balancing. Thus from now we always assume budget–balance even though we do not mention it explicitly. Given a social choice function $f: U \longrightarrow Z$ and $u \in U$, we write $f(u) = (f_1(u), \dots, f_n(u), f_y(u))$, $f^i(u) = (f_i(u), f_y(u))$ and $f^{-i}(u) = ((f_j(u))_{j \neq i}, f_y(u))$. Let Z_f be the image of f and Y_f be the projection of Z_f on Y.

Preferences are usually privately known. Therefore selfish agents may try to misrepresent their preferences so as to obtain an outcome they prefer. As a result of such strategic behaviors, the actual outcome may be far from satisfactory from a social point of view. The condition of "strategy–proofness" requires that a social choice function be immune from such strategic behaviors; no agent can increase his utility by (unilateral) manipulation. "Coalitional strategy–proofness" is a stronger condition; by coalitional manipulation no coalition can increase the utility of any member in the coalition without decreasing of the utility of some other member in it.

Definition: A social choice function f is (individually) strategy–proof if

$$\forall u \in U, \forall i \in N, \forall \hat{u}^i \in U^i, \ u^i(f(u)) \geqq u^i(f(\hat{u}^i, u^{-i})),$$

and it is weakly coalitionally strategy–proof if

$\forall u \in U, \forall N' \subseteq N, \forall \hat{u}^{N'} \in U^{N'}, \exists i \in N'$ s.t. $u^i(f(u)) \geqq u^i(f(\hat{u}^{N'}, u^{-N'}))$,

and it is coalitionally strategy–proof if

$\forall u \in U, \forall N' \subseteq N, \forall \hat{u}^{N'} \in U^{N'}$,

$$[\exists i \in N' \text{ s.t. } u^i(f(\hat{u}^{N'}, u^{-N'})) > u^i(f(u))] \Rightarrow [\exists j \in N' \backslash \{i\} \text{ s.t. } u^j(f(\hat{u}^{N'}, u^{-N'})) < u^j(f(u))].$$

Next we consider distributional requirements of social choice functions. A social choice function is "individually rational" if it never assigns an allocation which makes some agent worse off than he would be by consuming no public good and paying nothing. This condition requires that the fruit of cooperation be nonnegatively distributed among all agents. Thus it excludes the situation where some agent is exploited when he joins the society. A social choice function is "symmetric" if it treats two agents equally when they have the same preference; it is "unanimously symmetric" if it treats all agents equally when they all have the same preference. These two conditions require a social choice function not to treat agents unfairly.

Definition: A social choice function f is individually rational if

$\forall u \in U, \forall i \in N, \ u^i(f(u)) \geqq u^i(e)$.

Definition[3]: A social choice function f is symmetric if for any $u \in U$, for any $i \in N$ and for any $j \in N$,

$[u^i = u^j] \Rightarrow f_i(u) = f_j(u)$.

Definition: A social choice function f is unanimously symmetric if for any $u \in U$,

$[\forall i \in N, \forall j \in N, u^i = u^j] \ \Rightarrow \ [\forall i \in N, \forall j \in N, f_i(u) = f_j(u)]$.

[3] Symmetry is called "anonymity" in Moulin (1994).

Remark: Unanimous symmetry is weaker than symmetry.

Generally we say that some agent is "bossy" if he intervenes in what he is not concerned with. Since agent i's preference depends only on z^i, agent i is not concerned with z^{-i}. Thus here, non–bossiness implies that by changing his announced preferences, no agent can change the other agents' consumptions z^{-i} without changing his consumption z^i.

Definition (Satterthwaite and Sonnenschein, 1981): A social choice function f is non–bossy if

$$\forall u \in U, \forall i \in N, \forall \hat{u}^i \in U^i, [\, f^{-i}(u) \neq f^{-i}(\hat{u}^i, u^{-i}) \Rightarrow f^i(u) \neq f^i(\hat{u}^i, u^{-i}) \,].$$

Fact 1: Coalitional strategy–proofness implies non–bossiness.

Proof. Let f be a coalitionally strategy–proof social choice function. Let $u \in U$, $i \in N$ and $\hat{u}^i \in U^i$ be such that $f^i(u) = f^i(\hat{u}^i, u^{-i})$. We want to show that $f^{-i}(u) = f^{-i}(\hat{u}^i, u^{-i})$. By contradiction, suppose $f^{-i}(u) \neq f^{-i}(\hat{u}^i, u^{-i})$. Since $f^i(u) = f^i(\hat{u}^i, u^{-i})$ implies $f_y(u) = f_y(\hat{u}^i, u^{-i})$, there is $j \in N$ such that $f_j(u) \neq f_j(\hat{u}^i, u^{-i})$. Then since $f_y(u) = f_y(\hat{u}^i, u^{-i})$, $u^j(f(u)) > u^j(f(\hat{u}^i, u^{-i}))$ or $u^j(f(u)) < u^j(f(\hat{u}^i, u^{-i}))$. Since $u^i(f(u)) = u^i(f(\hat{u}^i, u^{-i}))$, this contradicts coalitional strategy–proofness for $\{i, j\}$.

Q.E.D.

Section 3. Schemes of Cost Sharing.

In this section, we study how agents share the cost of public good when a social choice function satisfies strategy–proofness, individual rationality and unanimously symmetry. A cost share function is a function $t_i: Y \rightarrow \mathbb{R}_+$. A social choice function f is a scheme of cost sharing if there are cost share functions t_i, $i \in N$, such that for any $z \in Z_f$, for any $i \in N$, $x_i = t_i(y)$ and $\sum_i t_i(y) = C(y)$.

Definition: A social choice function f is a scheme of convex cost sharing if f is a scheme of cost sharing and its cost share functions t_i, $i \in N$, are convex on Y_f, that is,

$\forall i \in N, \forall y \in Y_f, \forall y' \in Y_f, \forall y'' \in Y_f,$

$y < y'' < y' \Longrightarrow t_i(y'') \leqq \lambda \cdot t_i(y) + (1-\lambda) \cdot t_i(y'),$

where $\lambda = (y' - y'')/(y' - y)$.

Definition: A social choice function f is a scheme of equal cost sharing if for any $z \in Z_f$ and for any $i \in N$, $x_i = C(y)/n$.

Proposition 1: If a social choice function is strategy–proof, individually rational and unanimously symmetric, then it is a scheme of equal cost sharing.

Proof Let f be a strategy–proof, individually rational and unanimously symmetric social choice function, $u \in U$ and $z = f(u)$. We want to show that for any $i \in N$, $x_i = C(y)/n$. By contradiction, suppose not. Then without loss of generality, we may let $x_1 < C(y)/n$ by $\sum_{i \in N} x_i = C(y)$.

In this paragraph, we define utility functions on $\mathbb{R}_+ \times Y$ to specify them precisely. Since C is increasing and $x_1 < C(y)/n$, there is $\hat{u}^1 \in U^1$ such that

(1) for any $(x_1', y') \in [0, C(\bar{y})] \times Y$, if $\hat{u}^1(x_1', y') \geqq \hat{u}^1(C(\bar{y}), y)$, $y' \geqq y$;

(2) for any $(x_1', y') \in \mathbb{R}_+ \times Y$, if $y' \geqq y$ and $\hat{u}^1(x_1', y') \geqq \hat{u}^1(x_1, y)$, $x_1' < C(y')/n$.

Let $\hat{u}^2 = \hat{u}^1$, that is,

(1′) for any $(x_2', y') \in [0, C(\bar{y})] \times Y$, if $\hat{u}^2(x_2', y') \geqq \hat{u}^2(C(\bar{y}), y)$, $y' \geqq y$;

(2′) for any $(x_2', y') \in \mathbb{R}_+ \times Y$, if $y' \geqq y$ and $\hat{u}^2(x_2', y') \geqq \hat{u}^2(x_1, y)$, $x_2' < C(y')/n$.

Similarly specify $\hat{u}^j$, $j = 3, \cdots, n$. Then by unanimous symmetry, we have that

(3) $f_1(\hat{u}) = C(f_y(\hat{u}))/n$.

In this paragraph, we show that $f_y(\hat{u}) \geqq y$. If $f_y(\hat{u}^1, u^{-1}) < y$, then by (1),

$$\hat{u}^1(f(u)) \geqq \hat{u}^1(C(\bar{y}), y) > \hat{u}^1(f(\hat{u}^1, u^{-1})).$$

This contradicts strategy–proofness. Thus $f_y(\hat{u}^1, u^{-1}) \geqq y$. If $f_y(\hat{u}^1, \hat{u}^2, u^{-\{1,2\}}) < y$, then by (1′),

$$\hat{u}^2(f(\hat{u}^1, u^{-1})) \geqq \hat{u}^2(C(\bar{y}), y) > \hat{u}^2(f(\hat{u}^1, \hat{u}^2, u^{-\{1,2\}})).$$

This contradicts strategy–proofness. Thus $f_y(\hat{u}^1, \hat{u}^2, u^{-\{1,2\}}) \geqq y$. Repeating this argument for $j = 3, \cdots, n$, we have that (4) $f_y(\hat{u}) \geqq y$.

It follows from (2), (3) and (4) that $\hat{u}^1(f(\hat{u})) < \hat{u}^1(z)$. Thus there is $\tilde{u}^1 \in U^1$ such that

(5) for any $z' \in Z$, if $\tilde{u}^1(z') \geqq \tilde{u}^1(z)$, $y' \geqq y$;

(6) for any $z' \in Z$, if $\tilde{u}^1(z') \geqq \tilde{u}^1(e)$ and $y' \geqq y$, $\hat{u}^1(z') > \hat{u}^1(f(\hat{u}))$.

By (5) and strategy–proofness for $\tilde{u}^1$; $f_y(\tilde{u}^1, u^{-1}) \geqq y$. By (1′) and strategy–proofness for $\hat{u}^2$, $f_y(\tilde{u}^1, \hat{u}^2, u^{-\{1,2\}}) \geqq y$. Repeating this argument for $j = 3, \cdots, n$, we have: $f_y(\tilde{u}^1, \hat{u}^{-1}) \geqq y$. Thus it follows from (6) and individual rationality for $\tilde{u}^1$ that $\hat{u}^1(f(\tilde{u}^1, \hat{u}^{-1})) > \hat{u}^1(f(\hat{u}))$. This contradicts strategy–proofness. Q.E.D.

Figure enters here.

We say that $\hat{u}^i \in U^i$ is <u>a Maskin Monotonic Transformation of u^i at z</u> if (i) $UC(\hat{u}^i, z) \subseteq UC(u^i, z)$ holds, and (ii) $z' \in UC(\hat{u}^i, z)$ and $z^{i\prime} \neq z^i$ together imply that $u^i(z') > u^i(z)$. Let $M(u^i, z)$ be <u>the set of Maskin Monotonic Transformations of u^i at z</u>.

<u>Corollary 1</u>: If a social choice function is strategy–proof, individually rational and unanimously symmetric, then it is weakly coalitionally strategy–proof.

<u>Proof.</u> Let f be a strategy–proof, individually rational and unanimously symmetric social

choice function. Then by Proposition 1, it is a scheme of equal coat sharing. By contradiction, suppose that there are $N' \subsetneq N$, $u \in U$ and $\tilde{u}^{N'} \in U^{N'}$ such that for each $i \in N'$, $u^i(f(\tilde{u}^{N'}, u^{-N'})) > u^i(f(u))$. Then for each $i \in N'$, there is $\hat{u}^i \in M(u^i, f(u)) \cap M(\tilde{u}^i, f(\tilde{u}^{N'}, u^{-N'}))$. Without loss of generality, let $N' = \{1, \cdots, n'\}$.

In this paragraph, we show that $f(\hat{u}^1, u^{-1}) = f(u)$ and $f(\hat{u}^1, \tilde{u}^{N' \setminus \{1\}}; u^{-N'})) = f(\tilde{u}^{N'}, u^{-N'}))$. By contradiction, $f(\hat{u}^1, u^{-1}) \neq f(u)$. Then since f is a scheme of equal cost sharing, $f_y(\hat{u}^1, u^{-1}) \neq f_y(u)$. Strategy–proofness for agent 1 implies $\hat{u}^1(f(\hat{u}^1, u^{-1})) \geqq \hat{u}^1(f(u))$. Then the condition (ii) of Maskin Monotonic Transformation implies $u^1(f(\hat{u}^1, u^{-1})) > u^1(f(u))$. This contradict strategy–proofness. Thus $f(\hat{u}^1, u^{-1}) = f(u)$. Similarly we can show that $f(\hat{u}^1, \tilde{u}^{N' \setminus \{1\}}; u^{-N'})) = f(\tilde{u}^{N'}, u^{-N'}))$.

Repeating the argument of the above paragraph for $i = 2, \cdots, n'$, we have that $f(\hat{u}^{N'}, u^{-N'}) = f(u)$ and $f(\hat{u}^{N'}, u^{-N'}) = f(\tilde{u}^{N'}, u^{-N'})$. Since $f(\tilde{u}^{N'}, u^{-N'}) \neq f(u)$, this is a contradiction. Q.E.D.

Proposition 2: If a social choice function is strategy–proof, individually rational and unanimously symmetric, then it is a scheme of equal and convex cost sharing.

Proof Let f be a strategy–proof, individually rational and unanimously symmetric social choice function. By Proposition 1, f is a scheme of equal cost sharing. Thus for each agent $i \in N$, there is the cost share function $t_i : Y \to \mathbb{R}_+$ such that for any $z \in Z_f$, $x_i = t_i(y) = C(y)/n$. We will prove that for $i \in N$, t_i is convex on Y_f. By contradiction, suppose that t_1 is not convex on Y_f. Then there are $z \in Z$, $\dot{z} \in Z_f$ and $\dot{z}' \in Z_f$ such that $\dot{y}' < y < \dot{y}$ and

(1) for any $z' \in Z_f$ with $\dot{y}' < y' < \dot{y}$, $[\dot{z}, \dot{z}']_1 >_1 z'$[4].

Let $u^1 \in U^1$ be such that (2) for any $z' \in Z_f$, if $u^1(z') \geqq u^1(\dot{z})$, then $z' \geqq_1 \dot{z}$, and

4 We write $[\hat{z}, \hat{z}'] >_i z$ when there is $z' \in [\hat{z}, \hat{z}']$ such that $z^{i\prime} > z^i$.

(3) for any $z' \in Z_f$, if $u^1(z') \geqq u^1(z)$, then $y' \geqq y$, and (4) for any $z' \in Z_f$, if $u^1(z') \geqq u^1(\dot{z}')$, then $y' \geqq \dot{y}'$. For each $j \in N \setminus \{1\}$, let $u^j \in U^j$ be such that (5) for any $z' \in Z_f$, if $u^j(z') \geqq u^j(z)$, then $z' \geqq_j z$, (6) for any $z' \in Z_f$, if $u^j(z') > u^j(\dot{z})$, then $z' \geqq_j [z, \dot{z}]$, (7) for any $z' \in Z_f$, if $u^j(z') \geqq u^j(\dot{z}')$, then $y' \geqq \dot{y}'$.

We claim that (8) $f(u) = z$. By (3) and (5), we have that for any $j \in N$ and for any $z' \in Z_f$ with $y' < y$, $u^j(z) > u^j(z')$. Thus by weakly coalitional strategy–proofness for N (Corollary 1), we have: (9) $f_y(u) \geqq y$. By contradiction, suppose that $f_y(u) > y$. Then $f_2(u) > x_2$. By (6), we have: (10) $u^2(z) > u^2(f(u))$. Let $\tilde{u}^2 \in U^2$ be such that (11) for any $z' \in Z_f$, if $\tilde{u}^2(z') \geqq \tilde{u}^2(e)$, then $z' \geqq_2 [e, z]$ and (12) for any $z' \in Z_f$, if $\tilde{u}^2(z') \geqq \tilde{u}^2(z)$, then $z' \geqq_2 z$. It follows from individual rationality for agent 2 and (10) that $f_2(\tilde{u}^2, u^{-2}) \geqq x_2$. And similarly to (9), by (3), (5) and (12), we have: $f_y(\tilde{u}^2, u^{-2}) \geqq y$. Thus $f(\tilde{u}^2, u^{-2}) = z$, so that (10) implies that $u^2(f(\tilde{u}^2, u^{-2})) > u^2(f(u))$. This contradicts strategy–proofness for agent 2, so that $f_y(u) = y$. Therefore (8) holds.

Let $\dot{z}'' \in [\dot{z}, \dot{z}']$ be such that $\dot{y}'' = y$. Let $\tilde{u}^1 \in U^1$ be such that (13) for any $z' \in Z_f$, if $\tilde{u}^1(z') \geqq \tilde{u}^1(\dot{z})$, then $z' \geqq_1 \dot{z}$, and (14) for any $z' \in Z_f$, if $\tilde{u}^1(z') \geqq \tilde{u}^1(\dot{z}')$, then $z' \geqq_1 [\dot{z}, \dot{z}']$.

By contradiction, suppose that $\tilde{u}^1(\dot{z}') > \tilde{u}^1(f(\tilde{u}^1, u^{-1}))$. Let $\hat{u}^1 \in U^1$ be such that (15) for any $z' \in Z_f$, if $\hat{u}^1(z') \geqq \hat{u}^1(\dot{z}')$, then $z' \geqq_1 \dot{z}'$, and (16) for any $z' \in Z$, if $\hat{u}^1(z') \geqq \hat{u}^1(e)$, then $z' \geqq_1 [e, \dot{z}']$. Since for any $z' \in Z_f$ with $y' < \dot{y}$, $\hat{u}^1(\dot{z}') > \hat{u}^1(z')$ and $u^j(\dot{z}') > u^j(z')$, $j \in N \setminus \{1\}$, weakly coalitional strategy–proofness for N implies $f_y(\hat{u}^1, u^{-1}) \geqq \dot{y}'$. It follows from individual rationality for agent 1 and (16) that $f_1(\hat{u}^1, u^{-1}) = \dot{x}_1'$. Thus $f(\hat{u}^1, u^{-1}) = \dot{z}'$, so that $\tilde{u}^1(f(\hat{u}^1, u^{-1})) > \tilde{u}^1(f(\tilde{u}^1, u^{-1}))$. This contradicts strategy–proofness for agent 1. Therefore (17) $\tilde{u}^1(f(\tilde{u}^1, u^{-1})) \geqq \tilde{u}^1(\dot{z}')$.

Then (1), (14) and (17) imply that $f(\tilde{u}^1, u^{-1}) = \dot{z}$ or $f(\tilde{u}^1, u^{-1}) = \dot{z}'$. By contradiction, suppose that $f(\tilde{u}^1, u^{-1}) = \dot{z}'$. Then for any $j \in N \setminus \{1\}$, $u^j(\dot{z}) > u^j(f(\tilde{u}^1, u^{-1}))$ and $\tilde{u}^1(\dot{z}) > \tilde{u}^1(f(\tilde{u}^1, u^{-1}))$. Since $\dot{z} \in Z_f$, this contradicts weakly coalitional strategy–proofness for N. Thus $f(\tilde{u}^1, u^{-1}) = \dot{z}$. Hence (2) and (8) imply that $u^1(f(\tilde{u}^1, u^{-1})) > u^1(f(u))$, contradicting strategy–proofness for agent 1. Q.E.D.

Proposition 2 says that when the cost function C is not convex, the production range Y_f of a strategy–proof, individually rational and unanimously symmetric social choice function cannot be equal to Y.

Corollary 2: Let the cost function C be strictly concave. If a social choice function f is strategy–proof, individually rational and unanimously symmetric, then there is $y \in Y$ such that $Y_f \subsetneq \{e, y\}$ (the production range contains at most two levels).

Section 4. Main Result.

In this section, we define "the minimum demand principle" formally and establish our main result.

Definition: Given $Z' \subsetneq Z$, a tie–breaking rule $b^i(\cdot, Z')$ for agent i is a selection from $B^i(\cdot, Z')$.

Given subsets A and A′ in Y, we write $A \leqq A'$ if $\min A \leqq \min A'$ and $\max A \leqq \max A'$. Similarly to Barbera and Jackson (1993), we can establish the following fact.

Fact 2: Let f be a scheme of convex cost sharing and $i \in N$. A tie–breaking rule $b^i(\cdot, Z_f)$ is strategy–proof if and only if the condition (*) below holds:

(*) for any $u \in U$, $j \in N\backslash\{i\}$ and $\hat{u}^j \in U^j$,

$$[b^i_y(u, Z_f) < b^i_y(\hat{u}^j, u^{-j}, Z_f)] \implies [B^j_y(u, Z_f) \leqq B^i_y(u, Z_f) \leqq B^j_y(\hat{u}^j, u^{-j}, Z_f)].$$

The detail of the proof is given in Appendix.

When a social choice function f is a scheme of cost sharing, each agent i has his budget curve $\{z \in Z \mid x_i = t_i(y)\}$, from which his demand for the public good is derived. We consider schemes of cost sharing which produce the public good as much as the minimum demand.

Definition[5]: A scheme f of convex cost sharing is determined by the minimum demand principle if there are tie–breaking rules $b^i(\cdot, Z_f)$, $i \in N$, satisfying the above condition (*) such that for any $u \in U$, $f_y(u) = \min_{i \in N} \{b^i_y(u, Z_f)\}$.

Theorem 1 (Moulin, 1994) A social choice function is coalitionally strategy–proof, individually rational and symmetric if and only if it is a scheme of equal and convex cost sharing determined by the minimum demand principle.

Theorem 2 (Serizawa, 1994)[6] Let the cost function C be convex. A social choice function is strategy–proof, individually rational and non–bossy if and only if it is a scheme of convex cost sharing determined by the minimum demand principle.

Theorem 3 A social choice function f is strategy–proof, individually rational and unanimously symmetric if and only if it is a scheme of equal and convex cost sharing

[5]Moulin (1994) defined the minimum demand principle slight differently and called it "conservativeness". He defined as follows.
Definition: Let a social choice function f be a scheme of convex cost sharing. f is determined by the minimum demand principle if for any $u \in U$, $f_y(u) =$

$\min\{\max B^i_y(u) \mid i \in N\}$.

[6] His result is more general and can be applied to the case where C is not convex. Here the convexity assumption of C is made for the simplicity of exposition of his result.

determined by the minimum demand principle.

Proof of Theorem 3 The "if" part of Theorem is straightforward. Thus we have to consider the "only if" part of the proof. By Proposition 2, f is a scheme of equal and convex cost sharing. Thus we have only to show the minimum demand principle. For the simplicity of notation, we omit Z_f of $B^i(\cdot, Z_f)$ and $b^i(\cdot, Z_f)$, $i \in N$. Given $u \in U$, let $z^m_f(u) \in Z_f$ be such that $z^m_f(u) = (t_1(y^m_f(u)), \cdots, t_n(y^m_f(u)), y^m_f(u)) \in \bigcup_{i \in N} B^i(u)$ and for $z \in \bigcup_{i \in N} B^i(u)$, $y^m_f(u) \leqq y$.

Note that (1) for any $j \in N$, for any $u \in U$ and for any $z \in Z_f$ with $y < y^m_f(u)$, $u^j(z^m_f(u)) > u^j(z)$. We claim that (2) for any $u \in U$ and $i \in N$, if $z^m_f(u) \in B^i(u)$, then $f(u) \in B^i(u)$. By contradiction, suppose that there are $u \in U$ and $i \in N$ such that $z^m_f(u) \in B^i(u)$ and $u^i(f(u)) < u^i(z^m_f(u))$. (1) and weakly coalitional strategy–proofness (Corollary 1) imply that $f_y(u) \geqq y^m_f(u)$. Note that there is $\hat{u}^i \in U^i$ such that $B^i((\hat{u}^i, u^{-i}), Z_f) = \{z^m_f(u)\}$ and for any $z \in Z_f$, if $\hat{u}^i(z) \geqq \hat{u}^i(e)$, then $z \geqq_i [e, z^m_f(u)]$. Then by (1), weakly coalitional strategy–proofness implies that $f_y(\hat{u}^i, u^{-i}) \geqq y^m_f(u)$. Thus since individual rationality for agent i implies that $f(\hat{u}^i, u^{-i}) \geqq_i [e, z^m_f(u)]$, it holds that $f(\hat{u}^i, u^{-i}) = z^m_f(u)$. Therefore $u^i(f(\hat{u}^i, u^{-i})) > u^i(f(u))$, contradicting strategy–proofness for agent i.

In this paragraph, we construct selections b^i, $i \in N$, from B^i, $i \in M$, such that for any $u \in U$, $f_y(u) = \min_{i \in N}\{b^i_y(u)\}$. It follows from (1) and weakly coalitional strategy–proofness that for any $u \in U$, $f_y(u) \geqq y^m_f(u)$. First consider the case where $u \in U$ is such that $f_y(u) = y^m_f(u)$. For each $i \in N$, let $b^i(u)$ be such that $b^i_y(u) = \min B^i_y(u)$. Then it is trivial that $f_y(u) = \min_{i \in N}\{b^i_y(u)\}$. Next consider the case where $u \in U$ is such that $f_y(u) > y^m_f(u)$. Let $N' = \{i \in N \mid z^m_f(u) \in B^i(u)\}$. By (2), for any $i \in N'$, $f(u) \in B^i(u)$. Thus let $b^i(u) = f(u)$ for each $i \in N'$, and $b^i(u)$ be such that $b^i_y(u) = \min B^i_y(u)$ for each $i \in N \backslash N'$. Since $(\min B^i_y(u), \max B^i_y(u)) \cap Y_f = \emptyset$ for any $i \in N'$, it follows that $y^m_f(u) = \min B^i_y(u)$ and $f_y(u) = \max B^i_y(u)$ for $i \in N'$, and $f_y(u) \leqq b^i_y(u)$ for $i \in N \backslash N'$. Thus $f_y(u)$

$= \min_{i \in N}\{b^i_y(u)\}$.

As the last step, we establish that each b^i satisfies (*). Let $i \in N$, $u \in U$, $j \in N\backslash\{i\}$, $\hat{u}^j \in U^j$ be such that $b^i_y(u) < b^i_y(\hat{u}^j, u^{-j})$. We want to show that $B^j_y(u) \leqq B^i_y(u) \leqq B^j_y(\hat{u}^j, u^{-j})$. Note that $f_y(u) \leqq b^i_y(u) < b^i_y(\hat{u}^j, u^{-j})$ and $f_y(\hat{u}^j, u^{-j}) \leqq b^i_y(\hat{u}^j, u^{-j})$. By contradiction, suppose that $f_y(\hat{u}^j, u^{-j}) < b^i_y(\hat{u}^j, u^{-j})$. Since $(\min B^i_y(u), \max B^i_y(u)) \cap Y_f = \emptyset$, it must be the case that $\min B^i_y(u) = b^i_y(u)$ and $\max B^i_y(u) = b^i_y(\hat{u}^j, u^{-j})$. Thus $f_y(\hat{u}^j, u^{-j}) \leqq b^i_y(u) < b^i_y(\hat{u}^j, u^{-j})$. This contradicts the construction of b^i. Therefore $f_y(u) \leqq b^i_y(u) < b^i_y(\hat{u}^j, u^{-j}) = f_y(\hat{u}^j, u^{-j})$. By contradiction, suppose that $\max B^j_y(u) > \max B^i_y(u)$. Then since $\max B^j_y(u) > \max B^i_y(u) \geqq b^i_y(\hat{u}^j, u^{-j}) = f_y(\hat{u}^j, u^{-j})$ and $f_y(\hat{u}^j, u^{-j}) > b^i_y(u) \geqq f_y(u)$, it follows from convexity of the cost share functions that $u^j(f(\hat{u}^j, u^{-j})) > u^j(f(u))$, contradicting strategy–proofness. Therefore, $\max B^j_y(u) \leqq \max B^i_y(u)$. Similarly $\min B^j_y(u) \leqq \min B^i_y(u)$. Hence $B^j_y(u) \leqq B^i_y(u)$. By the same way, we can show that $B^i_y(u) \leqq B^j_y(\hat{u}^j, u^{-j})$. Q.E.D.

<u>Remark</u> By Corollary 1, the scheme of equal and convex cost sharing determined by the minimum demand principle is weakly coalitionally strategy–proof, but not coalitionally strategy–proof. If we use Moulin (1994)'s definition of the minimum demand principle[7], the scheme is coalitionally strategy–proof.

Since the scheme of equal and convex cost sharing determined by the minimum demand principle is not Pareto–efficient, we have Corollary 3.

<u>Corollary 3</u>[8]: There is no social choice function which is strategy–proof, individually rational, unanimously symmetric and Pareto–efficient.

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[7] See Footnote 5.

[8] This is a generalization of Ledyard and Robert (1974).

Appendix.

Proof of Fact 2 For simplicity of notation, hereafter we omit Z_f of $b^i(\cdot, Z_f)$, $B^i(\cdot, Z_f)$ and $B^j(\cdot, Z_f)$, and we treat $B^i(\cdot, Z_f)$ and $B^j(\cdot, Z_f)$ as correspondences from U^i and U^j respectively. Since f is convex cost sharing, for each $i \in N$, there is the cost share functions t_j, $j \in N$, which are convex on Y_f.

Part I: "only if" Let $u \in U$, $j \in N\backslash\{i\}$, $\hat{u}^j \in U^j$ and $b^i_y(u) < b^i_y(\hat{u}^j, u^{-j})$. Since u^i is strictly quasi–concave and t_i is convex on Y_f, $B^i_y(u^i) = \{b^i_y(u), b^i_y(\hat{u}^j, u^{-j})\} = [b^i_y(u), b^i_y(\hat{u}^j, u^{-j})] \cap Y_f$. By contradiction, suppose that $\min B^j_y(u^j) > b^i_y(u)$. Then since u^j is strictly quasi–concave and t_j is convex on Y_f and $b^i_y(\hat{u}^j, u^{-j}) > b^i_y(u)$, it follows that $u^j(b^i(\hat{u}^j, u^{-j})) > u^j(b^i(u))$. This contradicts strategy–proofness. Thus $\min B^j_y(u^j) \leqq b^i_y(u)$. This and convexity of t_j together imply that $\max B^j_y(u^j) \leqq \max B^i_y(u^i)$, and so $B^j_y(u^j) \leqq B^i_y(u^i)$. Similarly we can show $B^i_y(u^i) \leqq B^j_y(\hat{u}^j)$.

Part II: "if" Let $u \in U$, $j \in N$ and $\hat{u}^j \in U^j$. We want to show $u^j(b^i(u)) \geqq u^j(b^i(\hat{u}^j, u^{-j}))$. Thus we may consider only the case where $b^i(u) \notin B^j(u^j)$.

By contradiction, suppose that $\min B^j_y(u^j) \leqq b^i_y(u) \leqq \max B^j_y(u^j)$. Then $b^i(u) \notin B^j(u^j)$ implies that $\min B^j_y(u^j) < b^i_y(u) < \max B^j_y(u^j)$. Since $(\min B^j_y(u^j), \max B^j_y(u^j)) \cap Y_f = \emptyset$ and $b^i_y(u) \in Y_f$, this is a contradict. Therefore $\max B^j_y(u^j) < b^i_y(u)$ or $b^i_y(u) < \min B^j_y(u^j)$. Consider the case where $\max B^j_y(u^j) < b^i_y(u)$. Then $\max B^j_y(u^j) < \max B^i_y(u^i)$ and it follows from (*) that $b^i_y(\hat{u}^j, u^{-j}) \geqq b^i_y(u)$. But then since t_j is convex on Y_f, $u^j(b^i(u)) \geqq u^j(b^i(\hat{u}^j, u^{-j}))$. Similarly we can show that $u^j(b^i(u)) \geqq u^j(b^i(\hat{u}^j, u^{-j}))$ in the case where $\min B^j_y(u^j) > b^i_y(u)$. Q.E.D.

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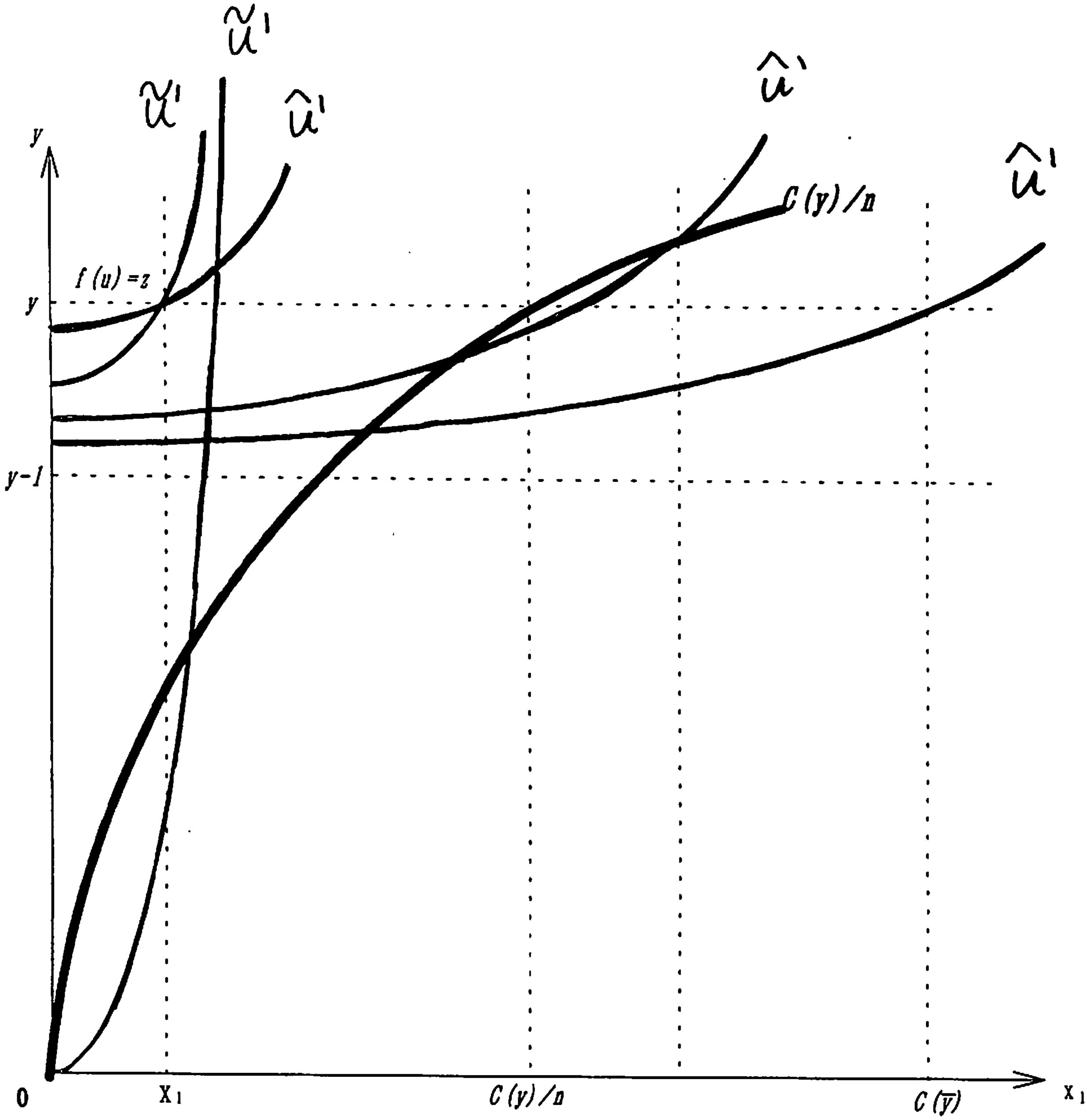

Figure: Illustration of the proof of Proposition 1.

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