10 Exponential and Logarithmic Functions

10.1 The Nature of Exponential Functions

In power expressions such as x^3 or x^5, the exponents are constants. A function whose independent variable appears in the role of an exponent such as 3^x is called an exponential function.

- Simple exponential function

$$y = f(t) = b^t \tag{1}$$

y: the dependent variables, t: the independent variable, b: a fixed *base* of the exponent.

- Generalized Exponential Function

$$y = f(t) = ab^{ct} \tag{2}$$

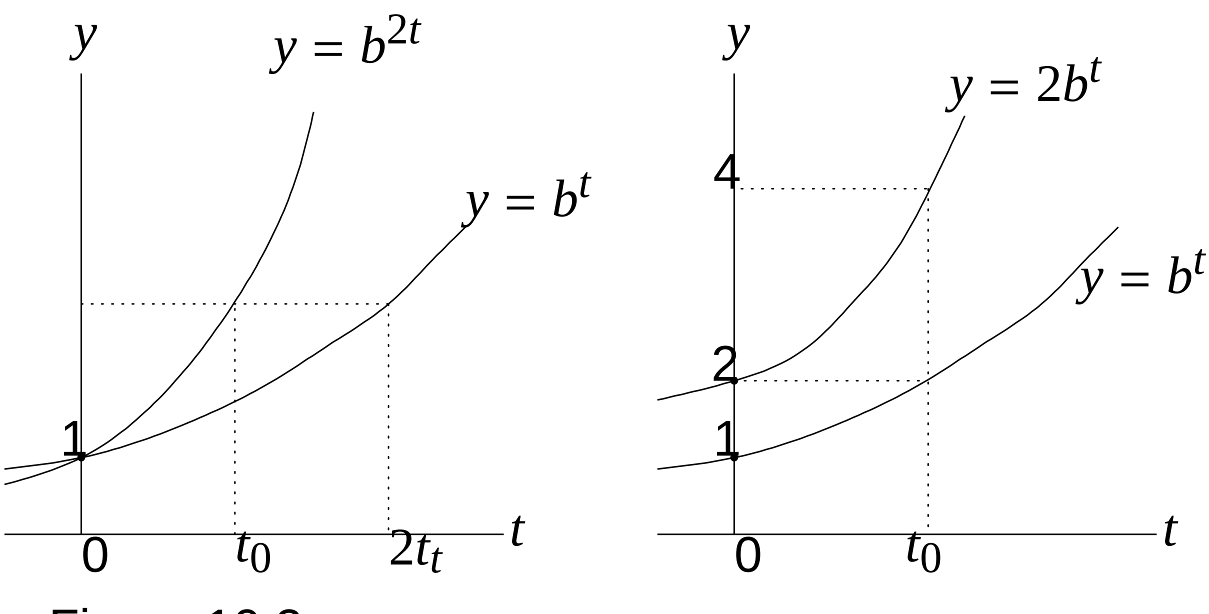

Figure 10.2

● A Preferred Base

As far as mathematical manipulations are concerned, a certain irrational number denoted by symbol e is more convenient than others.

$$e = 2.71828...$$

● Natural Exponential Function

$$y = e^t, \qquad y = e^{3t}, \qquad y = Ae^{rt}$$
$$y = \exp(t), \quad y = \exp(3t), \quad y = A\exp(rt)$$

● The Derivative of Natural Exponential Function

$$\frac{d}{dt}e^t = e^t, \quad \frac{d}{dt}Ae^{rt} = rAe^{rt}$$

10.2 Natural Exponential Functions and the Problem of Growth

- The Number e

Consider the following function:

$$f(m) = \left(1 + \frac{1}{m}\right)^m \tag{3}$$

The function $f(m)$ is increasing in m.

$$f(1) = \left(1 + \frac{1}{1}\right)^1 = 2$$
$$f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25$$
$$f(3) = \left(1 + \frac{1}{3}\right)^3 = 2.37037...$$
$$f(4) = \left(1 + \frac{1}{4}\right)^4 = 2.44141...$$
$$\vdots$$

The function of $f(m)$ is bounded from above.

$$
\begin{aligned}
f(m) &= 1 +_m C_1 \frac{1}{m} +_m C_2 \frac{1}{m^2} + \quad \cdots \quad +_m C_m \frac{1}{m^m} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \quad \cdots \quad + \frac{1}{m!}\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{m-1}{m}\right) \\
&\leq 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \quad \cdots \quad + \frac{1}{m!} \\
&\leq 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \quad \cdots \quad + \frac{1}{2^{m-1}} = 1 + \frac{1 - \frac{1}{2^m}}{1 - \frac{1}{2}} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&= 1 + 2 = 3 \qquad (4)
\end{aligned}
$$

Because $f(m)$ is bounded from above and is monotonically increasing in m, $f(m)$ converges to a certain number if m is increased indefinitely.

$$\text{Definition of } e : \qquad e \equiv \lim_{m\to\infty} f(m) = 2.71828... \qquad (5)$$

• The approximation value of e

Consider the Maclaurin series of $\phi(x) = e^x$

$$\phi(x) = \phi(0) + \frac{\phi'(0)}{1!}x + \frac{\phi''(0)}{2!}x^2 + \cdots + \frac{\phi^{(n)}(0)}{n!}x^n + R_n$$
$$= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n \quad (6)$$
$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{e^p}{(n+1)!}x^{n+1} \quad (7)$$

Since $R_n \to 0$ as $n \to \infty$, we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \quad (8)$$

Substituting unity into x, we find that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$
$$= 2 + 0.5 + 0.1666667 + \cdots$$
$$= 2.7182819 \quad (9)$$

● An Economic Interpretation of e

The number e can be interpreted as the result of a special process of interest compounding.

Suppose that, starting out with a principal of \$1, we find a banker to offer us the interest rate of 100% per annum. If interest is to be compounded once a year, the value of our asset at the end of the year will be \$2.

$$\begin{aligned} V(1) &= \text{initial principal} \times (1 + \text{interest rate}) \\ &= 1 \times \left(1 + \frac{100\%}{1}\right)^1 = 2 \end{aligned} \tag{10}$$

Suppose that interest is compounded semiannually. Then, we have

$$V(2) = \left(1 + \frac{100\%}{2}\right) \times \left(1 + \frac{100\%}{2}\right) = \left(1 + \frac{1}{2}\right)^2 \tag{11}$$

If the frequency of compounding in 1 year is m, our year end asset value is

$$V(m) = \left(1 + \frac{1}{m}\right)^m \tag{12}$$

In the limiting case where $m \to \infty$, the value of the asset at the end of 1 year will be

$$\lim_{m \to \infty} V(m) = e. \tag{13}$$

The number of e can be interpreted as the year-end value to which a principal of \$1 will grow if interest at the rate of 100% per annum is compounded continuously.

10.3 Logarithms

- The Meaning of Logarithm

$$y = b^t \quad \Longleftrightarrow \quad t = \log_b y \tag{14}$$

The log of y to the base b is the power to which the base b must be raised to attain the value y.

⟨ Examples ⟩

$$\begin{aligned}
\log_4 16 &= \log_4 4^2 = 2 \\
\log_{10} 1000 &= \log_{10} 10^3 = 3 \\
\log_{10} 0.01 &= \log_{10} 10^{-2} = -2 \\
\ln e^2 &= \log_e e^2 = 2 \\
\ln 1 &= \log_e e^0 = 0 \\
\ln \frac{1}{e} &= \log_e e^{-1} = -1
\end{aligned}$$

- Rules of Logarithms

Rule I : $\ln(uv) = \ln u + \ln v \quad (u, v > 0)$

Rule II : $\ln(u/v) = \ln u - \ln v \quad (u, v > 0)$

Rule III : $\ln u^a = a \ln u \quad (u > 0)$

Rule IV : $\log_b u = (\log_b e)(\log_e u) \quad (u > 0)$

Rule V : $\log_b e = \frac{1}{\log_e b}$

Proof of Rule I

$$uv = e^{\ln u} e^{\ln v} = e^{\ln u + \ln v} \quad \text{and} \quad uv = e^{\ln uv}$$

$$\Rightarrow \ln uv = \ln u + \ln v$$

10.4 Logarithm Functions

● Log Functions and Exponential Functions

Log functions are inverse functions of certain exponential functions.

$$t = \log_b y \quad \text{and} \quad t = \ln y \tag{15}$$

For example, the above two log functions are indeed the respective inverse functions of the exponential functions:

$$y = b^t \qquad \text{and} \quad y = e^t \tag{16}$$

The inverse of $y = Ae^{rt}$ can be obtained by taking the natural log of both sides of this exponential function and then solving for t:

$$\ln y = \ln(Ae^{rt}) = \ln A + rt \ln e = \ln A + rt \tag{17}$$

hence

$$t = \frac{\ln y - \ln A}{r} \tag{18}$$

As inverse functions of monotonically increasing functions, logarithmic functions must also be monotonically increasing.

$$\begin{aligned} \ln y_1 = \ln y_2 &\iff y_1 = y_2 \\ \ln y_1 > \ln y_2 &\iff y_1 > y_2 \end{aligned} \tag{19}$$

• The Graphical Form

Given the graph of the exponential function $y = e^t$, we can obtain the graph of the corresponding log function by replotting the original graph with the two axes transposed. The two curves are seen to be mirror images of each other with reference to the 45° line drawn through the origin.

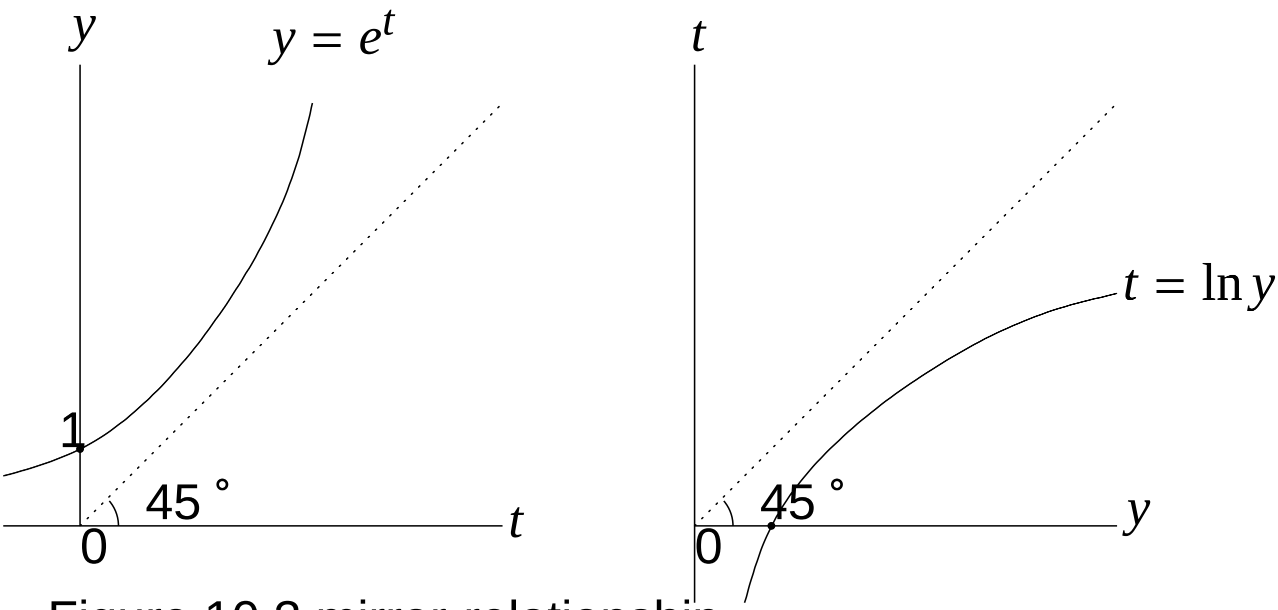

Figure 10.3 mirror-relationship

For any base,

$$\left.\begin{array}{l} 0 < y < 1 \\ y = 1 \\ y > 1 \end{array}\right\} \iff \left\{\begin{array}{l} \log y < 0 \\ \log y = 0 \\ \log y > 0 \end{array}\right. \quad (20)$$

• Base Conversion

Let us consider the conversion of Ab^{ct} into Ae^{rt}.

$$e^r = b^c \quad \Rightarrow \quad \ln e^r = \ln b^c \quad \Rightarrow \quad r = c \ln b \qquad (21)$$

Thus, Ab^{ct} can be rewritten in the natural-base form,

$$y = Ae^{(c \ln b)t} \qquad (22)$$

10.5 Derivatives of Exponential and Logarithmic Functions

- Log-Function Rule

$$\frac{d}{dt}\ln t = \frac{1}{t} \tag{23}$$

- Exponential-Function Rule

$$\frac{d}{dt}e^t = e^t \tag{24}$$

- The Rules Generalized

$$\begin{aligned} \frac{d}{dt}e^{f(t)} &= f'(t)e^{f(t)} \\ \frac{d}{dt}\ln f(t) &= \frac{f'(t)}{f(t)} \end{aligned} \tag{25}$$