11. The Case of More than One Choice Variable

In this chapter, we develop a way of finding the extreme values of an objective function that involves two or more choice variables. For functions of several variables, extreme values are again of two kinds: (1) absolute or global and (2) relative or local. Our focus will be heavily on relative extrema, and we often drop the adjective “relative”.

11.1 The Differential Version of Optimization

The discussion of optimization conditions for problems with a single choice variable was couched entirely in terms of *derivatives*. To prepare for the discussion of problems with two or more choice variables, it would be helpful to know how those conditions can equivalently be expressed in terms of *differentials*.

• First-Order Condition

Consider the function $z = f(x)$, as depicted in Figure 11.1. At the maximum point A as well as the minimum point B, the value of z must be stationary.

In other words,

it is a necessary condition for an extremum of z that $dz = 0$ instantaneously as x varies ($dx \neq 0$).

While the condition $dz = 0$ is necessary, it is clearly not sufficient for either a maximum or a minimum.

Recall that the differential of $z = f(x)$ is

$$dz = f'(x)dx. \tag{1}$$

Clearly, the first-order condition $dz = 0$ is equivalent to the derivative version of the first-order condition

$$\frac{dz}{dx} = 0, \qquad \text{or} \qquad f'(x) = 0. \tag{2}$$

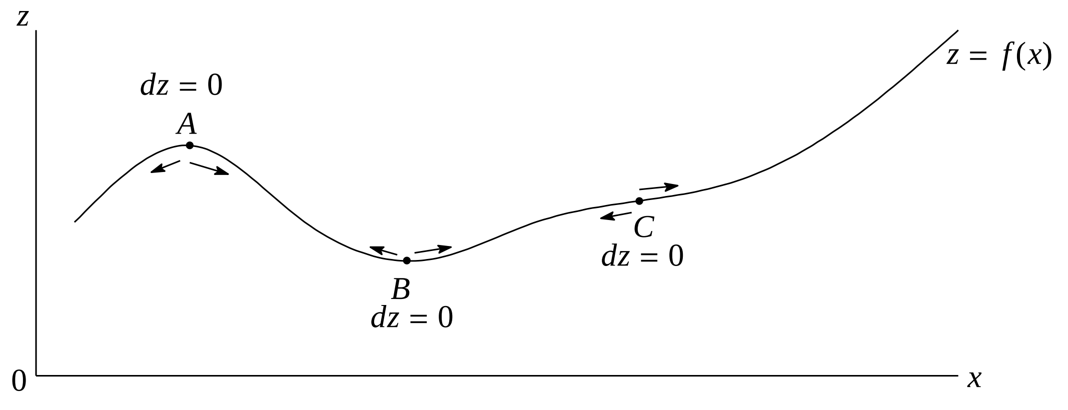

Figure 11.1

• Second-Order Condition

A maximum point A has the graphical property that as we slide along the curve infinitesimally toward the left ($dx < 0$) and the right ($dx > 0$) of A, we are decreasing in both directions. A sufficient condition for this is that $d^2z < 0$ on both sides of A in the immediate neighborhood of A.

Since $dz = 0$ at point A, $d^2z < 0$ at points on the two sides of A means dz is invariably decreasing as we move away from A in either direction.

In other words,

$$d(dz) < 0 \quad \text{or} \quad d^2z < 0 \quad \text{for arbitrary nonzero values of } dx.$$

This condition constitutes the differential version of the second-order sufficient condition for a maximum. Note that the negativity of d^2z is *sufficient* but *not necessary*, for a maximum of z.

The second-order conditions

For maximum of z: $f''(x) < 0$
For minimum of z: $f''(x) > 0$

can be translated, respectively, into

$$\left.\begin{array}{ll} \text{For mamimum of } z: & d^2z < 0 \\ \text{For minimum of } z: & d^2z > 0 \end{array}\right\} \text{for arbitrary nonzero values of } dx.$$

11.2 Extreme Values of A Function of Two Variables

With *two* choice variables, the graph of the function $z = f(x, y)$ becomes a surface in a 3-space, and while the extreme values are still to be associated with peaks and bottoms, these "hills" and "valleys" themselves now take on a three-dimensional character. They will be shaped like domes and bowls, respectively. The two diagrams in Figure 11.2 serve to illustrate. Point A constitutes a maximum. Similarly, point B constitutes a minimum.

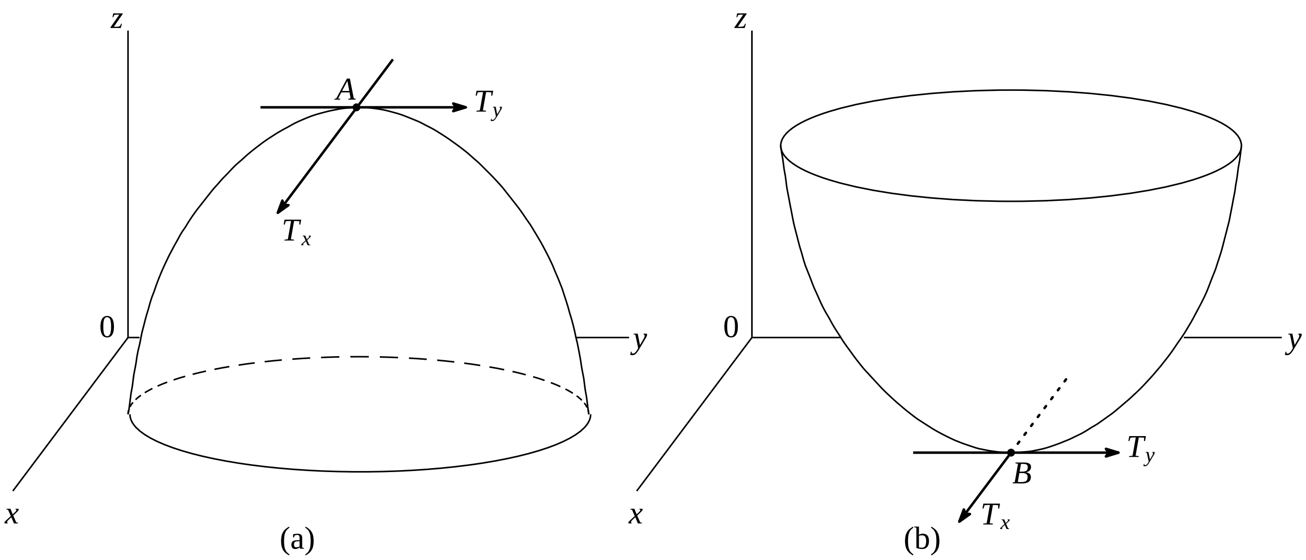

Figure 11.2

• First-Order Condition

For the function

$$z = f(x, y) \tag{3}$$

the first-order necessary condition for an extremum again involves $dz = 0$. But since there are two independent variables here, dz is now a total differential. In the present two-variable case, the total differential is

$$dz = f_x dx + f_y dy. \tag{4}$$

Thus, the first-order condition should be modified to the form

$$dz = f_x dx + f_y dy = 0 \quad \text{for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \tag{5}$$

In order to satisfy the above condition, it is necessary and sufficient that the two partial derivatives f_x and f_y be simultaneously equal to zero.

$$f_x = f_y = 0. \tag{6}$$

There is a simple graphical interpretation of this condition. With reference to point A in Figure 11.2 (a), to have $f_x = 0$ at that point means that the tangent line T_x, drawn through A and parallel to the xz plane (holding y constant), must have a zero slope. Similarly, to have $f_y = 0$ at point A means that the tangent line T_y, drawn through A and parallel to the yz plane, must also have a zero slope.

Note that the first-order condition is necessary but not sufficient as in the earlier discussion. This can easily be seen from the diagram in Figure 11.3. At point C, both T_x and T_y have zero slopes, but this point does not qualify as an extremum: whereas it is a *minimum* when viewed against the background of the yz plane, it turns out to be a *maximum* when looked at against the xz plane.

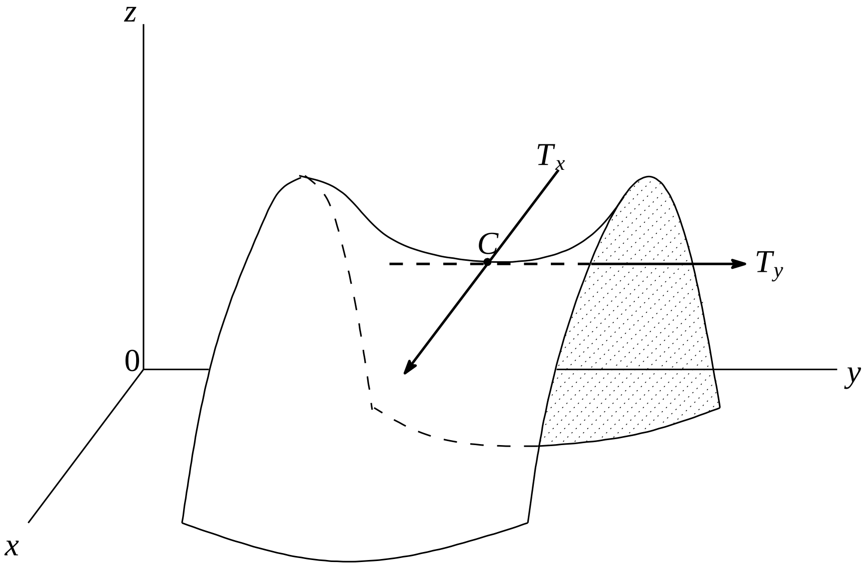

Figure 11.3

• Second-Order Partial Derivatives

The function $z = f(x, y)$ can give rise to two first-order derivatives,

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}. \tag{7}$$

Since f_x is itself a function of x, we can measure the rate of change of f_x with respect to x. A particular second-order partial derivative is denoted by

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right). \tag{8}$$

Similarly,

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right). \tag{9}$$

Recall that f_x is also a function of y and that f_y is also a function of x. Hence, there can be written two more second partial derivatives:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right). \tag{10}$$

These are called *cross partial derivatives*. Even though f_{xy} and f_{yx} have been separately defined, they will have identical values, $f_{xy} = f_{yx}$, as long as the two cross partial derivatives are both continuous.

Example 1
Find the four second-order partial derivatives of

$$z = x^3 + 5xy - y^2. \tag{11}$$

The first-order derivatives are

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y. \tag{12}$$

Upon further differentiation, we get

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2. \tag{13}$$

As expected, f_{yx} and f_{xy} are identical.

• Second-Order Total Differential

To obtain the second-order total differential d^2z, we merely apply the definition of a differential to dz (4) itself.

$$\begin{aligned} d^2z \equiv d(dz) &= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\ &= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\ &= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\ &= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2 \\ &= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \end{aligned} \tag{14}$$

Example 2
Given $z = x^3 + 5xy - y^2$, find dz and d^2z. The first-order derivatives are

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y. \tag{15}$$

Substituting these into (4), we find

$$dz = (3x^2 + 5y)dx + (5x - 2y)dy. \tag{16}$$

The second-order derivatives are

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2. \tag{17}$$

Substituting these into (14), we find

$$\begin{aligned} d^2z &= 6x dx^2 + 2 \cdot 5 dx dy + (-2) dy^2 \\ &= 6x dx^2 + 10 dx dy - 2 dy^2 \end{aligned} \tag{18}$$

• Second-Order Condition

In the two-variable case, $d^2z < 0$ at a stationary point would identify the point as the peak of a dome in a 3-space. Thus, once the first-order necessary condition is satisfied, the second-order sufficient condition for a maximum of $z = f(x, y)$ is

$$d^2z < 0 \text{ for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \tag{19}$$

The rationale behind (19) can be explained by means of Figure 11.4, which depicts the bird's-eye view of a surface. Let point A on the surface satisfy the first-order condition (5). Then point A is a prospective candidate for a maximum. If an infinitesimal movement away from A in any direction along the surface invariably results in a decrease in z—that is, if $dz < 0$ for arbitrary values of dx and dy, not both zero—A is a peak of a dome. Given that $dz = 0$ at other points in the neighborhood of A amounts to the condition that dz is decreasing, that is, $d^2z \equiv d(dz) < 0$, for arbitrary values of dx and dy, not both zero.

A positive d^2z at a stationary point, on the other hand, is associated with the bottom of a bowl. The second-order sufficient condition for a minimum of $z = f(x, y)$ is

$$d^2z > 0 \text{ for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \tag{20}$$

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. In the two-variable case, for any values of dx and dy, not both zero,

$$d^2z \begin{cases} < 0 & \iff f_{xx} < 0; \; f_{yy} < 0; \; \text{and} \; f_{xx} f_{yy} > f_{xy}{}^2; \\ > 0 & \iff f_{xx} > 0; \; f_{yy} > 0; \; \text{and} \; f_{xx} f_{yy} > f_{xy}{}^2; \end{cases} \tag{21}$$

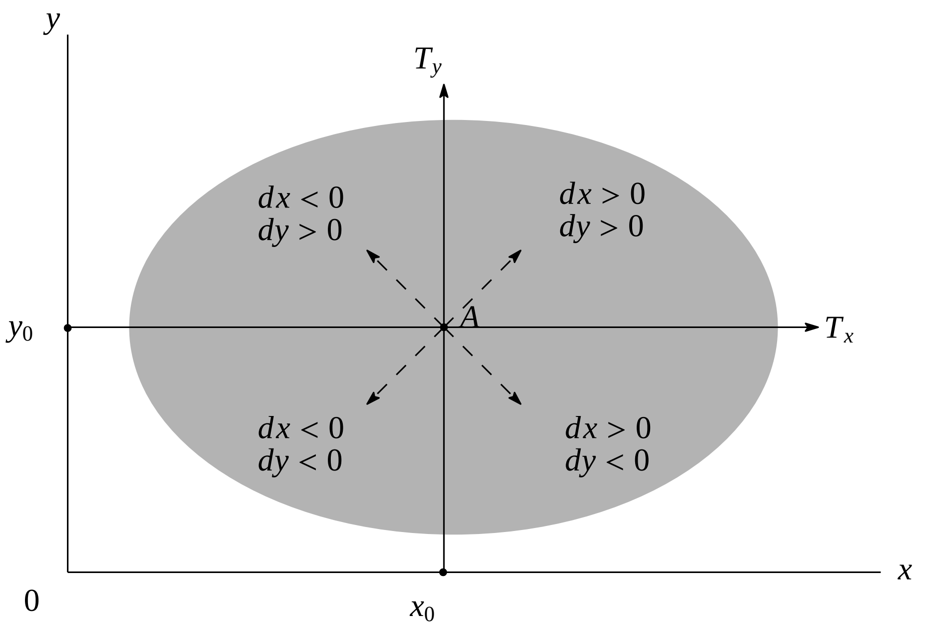

Figure 11.4

Table 11.1 Conditions for relative extremum: $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient condition	$f_{xx}, f_{yy} < 0$ and $f_{xx} f_{yy} > f_{xy}{}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx} f_{yy} > f_{xy}{}^2$

Example 3
Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. The first and second partial derivatives are

$$\begin{aligned} f_x &= 24x^2 + 2y - 6x, \\ f_y &= 2x + 2y, \\ f_{xx} &= 48x - 6, \\ f_{yy} &= 2, \\ f_{xy} &= 2. \end{aligned}$$

The first-order conditions are

$$\begin{aligned} f_x &= 24x^2 + 2y - 6x = 0, \\ f_y &= 2x + 2y = 0. \end{aligned}$$

The solutions for the above simultaneous equations are

$$x_1^* = 0, \qquad y_1^* = 0, \tag{22}$$

and

$$x_2^* = \frac{1}{3}, \qquad y_2^* = -\frac{1}{3}. \tag{23}$$

When $x_1^* = y_1^* = 0$, we have that

$$f_{xx} = -6, \quad f_{yy} = 2.$$

So $f_{xx} f_{yy}$ is negative and necessarily less than $f_{xy}{}^2 \geq 0$. This fails the second-order condition.

When $x_2^* = 1/3$ and $y_2^* = -1/3$, we have that

$$f_{xx} = 10, \quad f_{yy} = f_{xy} = 2.$$

Thus, all three parts of second-order condition for a minimum are satisfied. By setting $x_2^* = 1/3$ and $y_2^* = -1/3$ in the given function, we can obtain as a minimum of z the value $z^* = 23/27$.

11.3 Quadratic Forms——An Excursion

The expression for d^2z on the last line of (14) exemplifies what are known as *quadratic forms*, for which there exist established criteria for determining whether their signs are always positive, negative, nonpositive, or nonnegative, for arbitrary values of dx and dy, not both zero.

We define a *form* as a polynomial expression in which each component term has a uniform degree.

Example

Linear form	$4x - 9y + z$
Quadratic form	$4x^2 - xy + 3y^2$ $x^2 + 2xy - yw + 7w^2$

• Second-Order Total Differential as a Quadratic Form

If we consider the differentials dx and dy in (14) as variables and the partial derivatives as coefficient, i.e. if we let

$$\begin{array}{lll} u \equiv dx & v \equiv dy & \\ a \equiv f_{xx} & b \equiv f_{yy} & h \equiv f_{xy}[= f_{yx}] \end{array} \tag{24}$$

then the second-order total differential

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

can easily be identified as a quadratic form q in the two variables u and v:

$$q = au^2 + 2huv + bv^2 \tag{25}$$

Note that, in this quadratic form, $dx \equiv u$ and $dy \equiv v$ are cast in the role of variables, whereas the second partial derivatives are treated as constants.

• Positive and Negative Definiteness

A quadratic form q is said to be

$$\left.\begin{array}{l} \textit{Positive definite} \\ \textit{Positive semidefinite} \\ \textit{Negative definite} \\ \textit{Negative semidefinite} \end{array}\right\} \text{ if } q \text{ is invariably } \left\{\begin{array}{ll} \text{positive} & (> 0) \\ \text{nonnegative} & (\geq 0) \\ \text{negative} & (< 0) \\ \text{nonpositive} & (\leq 0) \end{array}\right. \tag{26}$$

regardless of the values of variables in the quadratic form, not all zero. Clearly, the cases of positive and negative definiteness of $q = d^2z$ are related to the second-order sufficient conditions for a minimum and a maximum, respectively.

• Determinantal Test for Sign Definiteness

For the two-variable case, determinantal conditions for the sign definiteness of q are relatively easy to derive.

We can rewrite (25) as follows:

$$\begin{aligned} q &= au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2 \\ &= a\left(u^2 + 2\frac{h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2 \\ &= a\left(u + \frac{h}{a}v\right)^2 + \frac{ab - h^2}{a}v^2 \end{aligned}$$

Since the variables u and v appear only in squares, the sign of q is independent of the values of these variables. We can predicate the sign of q entirely on the values of coefficients a, b, and h as follows:

$$q \text{ is} \left\{ \begin{array}{c} \text{positive definite} \\ \text{negative definite} \end{array} \right\} \text{ iff } \left\{ \begin{array}{c} a > 0 \\ a < 0 \end{array} \right\} \text{ and } ab - h^2 > 0. \tag{27}$$

The condition just derived may be stated more succinctly by the use of determinants. If we use the matrix representation, the quadratic form (25) can be rearranged into the following format:

$$q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{28}$$

Then, the condition (27) can be alternatively expressed as:

$$q \text{ is} \left\{ \begin{array}{c} \text{positive definite} \\ \text{negative definite} \end{array} \right\} \text{ iff } \left\{ \begin{array}{c} |a| > 0 \\ |a| < 0 \end{array} \right\} \text{ and } \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0 \tag{29}$$

The determinant $|a|$ is equal to a. The determinant $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is equal to $ab - h^2$.

When (29) is translated, via (24), into terms of the second-order total differential d^2z, we have

$$q \text{ is} \left\{ \begin{array}{c} \text{positive definite} \\ \text{negative definite} \end{array} \right\} \text{ iff } \left\{ \begin{array}{c} |f_{xx}| > 0 \\ |f_{xx}| < 0 \end{array} \right\} \text{ and } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}{}^2 > 0. \tag{30}$$

Since the latter inequality implies that f_{xx} and f_{yy} are required to take the same sign, we see that this is precisely the second-order sufficient condition presented in Table 11.1. The determinant with the second-order partial derivatives as its elements $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ is called a *Hessian determinant* (or simply a *Hessian*).

Example 1
Is $q = 5u^2 + 3uv + 2v^2 (= 5u^2 + 2 \times 1.5uv + 2v^2)$ either positive or negative definite?

$$5 > 0 \quad \text{and} \quad \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 5 \times 2 - 1.5^2 = 10 - 2.25 = 7.75 > 0. \tag{31}$$

Therefore q is positive definite.

11.4 Objective Functions with More than Two Variables

Let us specifically consider a function of three choice variables,

$$z = f(x_1, x_2, x_3). \tag{32}$$

• First-Order Condition for Extremum

As our earlier discussion suggests, to have a maximum or minimum of z, it is necessary that $dz = 0$ for arbitrary values of dx_1, dx_2 and dx_3, not all zero. The value of dz is now

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3. \tag{33}$$

The only way to guarantee a zero dz for arbitrary values of dx_1, dx_2 and dx_3, not all zero, is to have

$$f_1 = f_2 = f_3 = 0. \tag{34}$$

• Second-Order Condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, this will be suffice to establish that value of z as a minimum. Similarly, the negative definiteness of d^2z is a sufficient condition for the stationary value to be a maximum.

The expression for d^2z can be obtained by differentiating dz in (33).

$$\begin{aligned} d(dz) = d^2z &= \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3 \\ &= \frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1 \\ &\quad + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2 \\ &\quad + \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3 \\ &= \quad f_{11}dx_1^2 \quad + f_{12}dx_1dx_2 + f_{13}dx_1dx_3 \\ &\quad + f_{21}dx_2dx_1 + f_{22}dx_2^2 \quad + f_{23}dx_2dx_3 \\ &\quad + f_{31}dx_3dx_1 + f_{32}dx_3dx_2 + f_{33}dx_3^2 \end{aligned} \tag{35}$$

In determining the positive or negative definiteness of dz, we must again regard dx_i as variables and the derivatives F_{ij} as coefficients. The coefficients in (35) give rise to the symmetric Hessian determinant

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \tag{36}$$

whose leading principal minors may be denoted by

$$|H_1| = \begin{vmatrix} f_{11} \end{vmatrix}, \qquad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, \qquad |H_3| = |H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}. \tag{37}$$

Thus, on the basis of the determinantal criteria for positive and negative definiteness, we may state the second-order sufficient condition for an extremum of z as follows:

$$z^* \text{ is } \left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\} \tag{38}$$

$$\text{if} \left\{ \begin{matrix} |H_1| < 0; & |H_2| > 0; & |H_3| < 0 & (d^2z \text{ negative definite}) \\ |H_1| > 0; & |H_2| > 0; & |H_3| > 0 & (d^2z \text{ positive definite}) \end{matrix} \right. \tag{39}$$

In using this condition, we must evaluate all the leading principal minors at the stationary point where $f_1 = f_2 = f_3 = 0$.

Example 1

Find the extreme value(s) of

$$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2. \tag{40}$$

The first-order conditions for extremum are

$$f_1 = 4x_1 + x_2 + x_3 = 0$$
$$f_2 = x_1 + 8x_2 = 0$$
$$f_3 = x_1 + 2x_3 = 0.$$

This homogeneous linear-equation system has the single solution $x_1^* = x_2^* = x_3^* = 0$. This means that there is only on stationary value, $z^* = 2$.

The Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} \tag{41}$$

whose leading principal minors are all positive:

$$|H_1| = 4, \qquad |H_2| = 31, \qquad |H_3| = 54. \tag{42}$$

Thus we can conclude, by (39), that $z^* = 2$ is a minimum.