11. The Case of More than One Choice Variable

We develop a way of finding the extreme values of an objective function that involves two or more choice variables.

11.1 The Differential Version of Optimization

• First-Order Condition

It is a necessary condition for an extremum of z that $dz = 0$ instantaneously as x varies ($dx \neq 0$).

While the condition $dz = 0$ is necessary, it is clearly not sufficient for either a maximum or a minimum.

Recall that the differential of $z = f(x)$ is $dz = f'(x)dx$. Clearly, the first-order condition $dz = 0$ is equivalent to

$$\frac{dz}{dx} = 0, \qquad \text{or} \qquad f'(x) = 0. \tag{2}$$

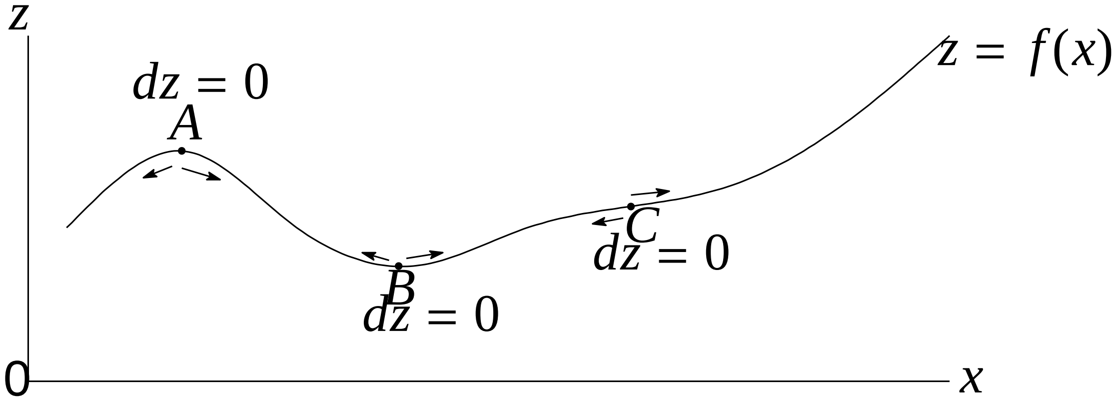

Figure 11.1

• Second-Order Condition

$d(dz) < 0$ or $d^2z < 0$ for arbitrary nonzero values of dx.

This condition constitutes the differential version of the second-order sufficient condition for a maximum. Note that the negativity of d^2z is *sufficient* but *not necessary*, for a maximum of z.

The second-order conditions

For maximum of z: $f''(x) < 0$

For minimum of z: $f''(x) > 0$

can be translated, respectively, into

$$\left.\begin{array}{ll} \text{For mamimum of } z: & d^2z < 0 \\ \text{For minimum of } z: & d^2z > 0 \end{array}\right\} \text{ for arbitrary nonzero}$$

values of dx.

11.2 Extreme Values of A Function of Two Variables

With *two* choice variables, the graph of the function $z = f(x, y)$ becomes a surface in a 3-space, and while the extreme values are still to be associated with peaks and bottoms, these “hills” and “valleys” themselves now take on a three-dimensional character. They will be shaped like domes and bowls, respectively. The two diagrams in Figure 11.2 serve to illustrate. Point A constitutes a maximum. Similarly, point B constitutes a minimum.

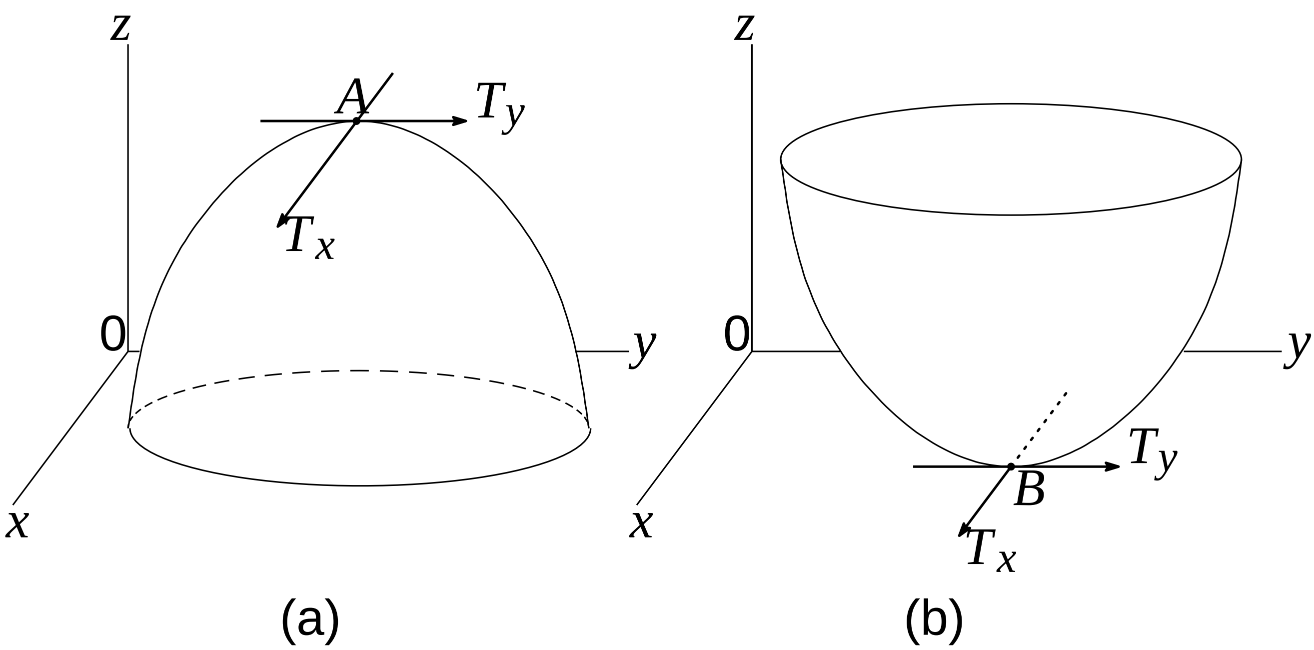

Figure 11.2

• First-Order Condition

For the function

$$z = f(x, y) \tag{3}$$

the first-order necessary condition for an extremum again involves $dz = 0$. The first-order condition should be modified to the form

$$dz = f_x dx + f_y dy = 0$$
$$\text{for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \tag{5}$$

From the above condition, we have

$$f_x = f_y = 0. \tag{6}$$

Note that the first-order condition is necessary but not sufficient as in the earlier discussion.

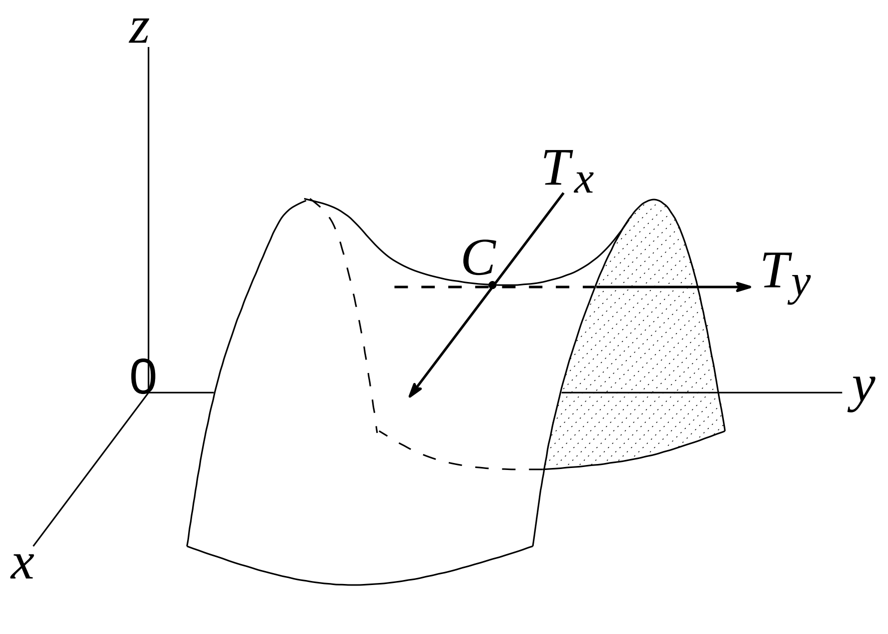

Figure 11.3

• Second-Order Partial Derivatives

The function $z = f(x, y)$ can give rise to two first-order derivatives,

$$f_x \equiv \frac{\partial z}{\partial x} \quad \text{and} \quad f_y \equiv \frac{\partial z}{\partial y}. \tag{7}$$

A particular second-order partial derivative is denoted by

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right). \tag{8}$$

Similarly,

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right). \tag{9}$$

There can be written two more second partial derivatives:

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right). \tag{10}$$

These are called *cross partial derivatives*. Even though f_{xy} and f_{yx} have been separately defined, they will have identical values, $f_{xy} = f_{yx}$, as long as the two cross partial derivatives are both continuous.

Example 1

Find the four second-order partial derivatives of

$$z = x^3 + 5xy - y^2. \qquad (11)$$

The first-order derivatives are

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y. \qquad (12)$$

Upon further differentiation, we get

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2. \qquad (13)$$

As expected, f_{yx} and f_{xy} are identical.

• Second-Order Total Differential

$$
\begin{aligned}
d^2z \equiv d(dz) &= \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy \\
&= \frac{\partial}{\partial x}(f_x dx + f_y dy)dx + \frac{\partial}{\partial y}(f_x dx + f_y dy)dy \\
&= (f_{xx}dx + f_{xy}dy)dx + (f_{yx}dx + f_{yy}dy)dy \\
&= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2 \\
&= f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2 \qquad (14)
\end{aligned}
$$

Example 2

Given $z = x^3 + 5xy - y^2$, find dz and d^2z. The first-order derivatives are

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y. \qquad (15)$$

Substituting these into $dz = f_x dx + f_y dy$, we find

$$dz = (3x^2 + 5y)dx + (5x - 2y)dy. \qquad (16)$$

The second-order derivatives are

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2. \qquad (17)$$

Substituting these into (14), we find

$$
\begin{aligned}
d^2z &= 6xdx^2 + 2 \cdot 5dxdy + (-2)dy^2 \\
&= 6xdx^2 + 10dxdy - 2dy^2 \qquad (18)
\end{aligned}
$$

• Second-Order Condition

Once the first-order necessary condition is satisfied, the second-order sufficient condition for a maximum of $z = f(x,y)$ is

$$d^2z < 0$$
$$\text{for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \quad (19)$$

The second-order sufficient condition for a minimum of $z = f(x,y)$ is

$$d^2z > 0$$
$$\text{for arbitrary values of } dx \text{ and } dy, \text{ not both zero.} \quad (20)$$

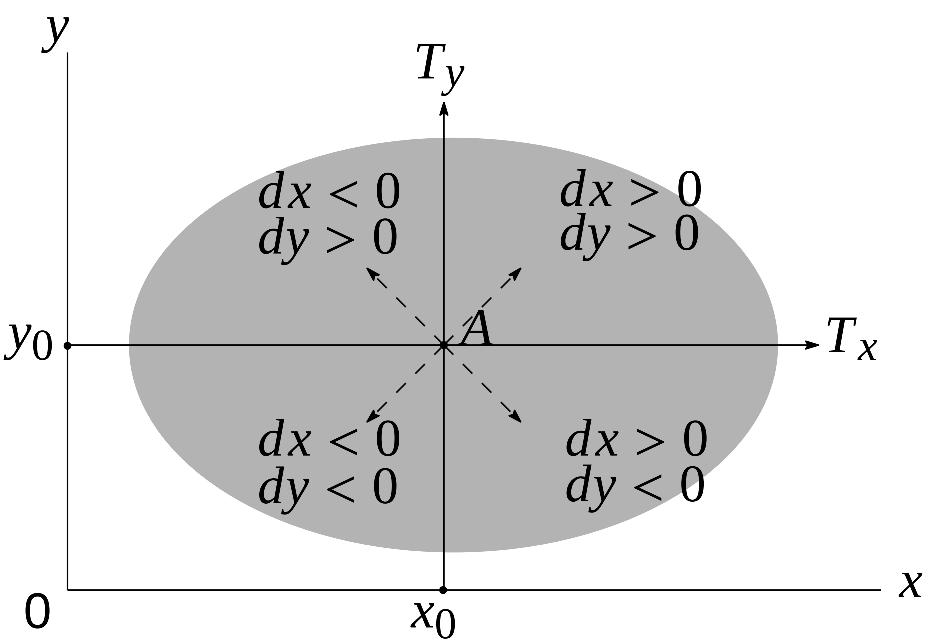

Figure 11.4

For operational convenience, second-order differential conditions can be translated into equivalent conditions on second-order derivatives. In the two-variable case, for any values of dx and dy, not both zero,

$$d^2z \begin{cases} < 0 & \Longleftrightarrow \quad f_{xx} < 0; \quad f_{yy} < 0; \quad \text{and} \quad f_{xx}f_{yy} > f_{xy}^2; \\ > 0 & \Longleftrightarrow \quad f_{xx} > 0; \quad f_{yy} > 0; \quad \text{and} \quad f_{xx}f_{yy} > f_{xy}^2; \end{cases} \tag{21}$$

Table 11.1 Conditions for relative extremum: $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient condition	$f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$	$f_{xx}, f_{yy} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$

Example 3

Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. The first and second partial derivatives are

$$f_x = 24x^2 + 2y - 6x, \quad f_y = 2x + 2y, \quad f_{xx} = 48x - 6,$$
$$f_{yy} = 2, \quad \text{and} \quad f_{xy} = 2.$$

The first-order conditions are

$$f_x = 24x^2 + 2y - 6x = 0,$$
$$f_y = 2x + 2y = 0.$$

The solutions for the above simultaneous equations are

$$x_1^* = 0, \qquad y_1^* = 0, \tag{22}$$

and

$$x_2^* = \frac{1}{3}, \qquad y_2^* = -\frac{1}{3}. \tag{23}$$

When $x_1^* = y_1^* = 0$, we have that

$$f_{xx} = -6, \qquad f_{yy} = 2.$$

So $f_{xx} f_{yy}$ is negative and necessarily less than ${f_{xy}}^2 \geq 0$. This fails the second-order condition.

When $x_2^* = 1/3$ and $y_2^* = -1/3$, we have that

$$f_{xx} = 10, \qquad f_{yy} = f_{xy} = 2.$$

Thus, all three parts of second-order condition for a minimum are satisfied. By setting $x_2^* = 1/3$ and $y_2^* = -1/3$ in the given function, we can obtain as a minimum of z the value $z^* = 23/27$.

11.3 Quadratic Forms——An Excursion

The expression for d^2z on the last line of (14) exemplifies what are known as *quadratic forms*, for which there exist established criteria for determining whether their signs are always positive, negative, nonpositive, or nonnegative, for arbitrary values of dx and dy, not both zero.

We define a *form* as a polynomial expression in which each component term has a uniform degree.

Example

Linear form	$4x - 9y + z$
Quadratic form	$4x^2 - xy + 3y^2$ $x^2 + 2xy - yw + 7w^2$

• Second-Order Total Differential as a Quadratic Form

If we consider the differentials dx and dy in (14) as variables and the partial derivatives as coefficient, i.e. if we let

$$\begin{aligned} &u \equiv dx \qquad v \equiv dy \\ &a \equiv f_{xx} \qquad b \equiv f_{yy} \qquad h \equiv f_{xy}[= f_{yx}] \end{aligned} \tag{24}$$

then the second-order total differential

$$d^2 z = f_{xx} dx^2 + 2 f_{xy} dx dy + f_{yy} dy^2$$

can easily be identified as a quadratic form q in the two variables u and v:

$$q = au^2 + 2huv + bv^2 \tag{25}$$

Note that, in this quadratic form, $dx \equiv u$ and $dy \equiv v$ are cast in the role of variables, whereas the second partial derivatives are treated as constants.

• Positive and Negative Definiteness

A quadratic form q is said to be

$$\left.\begin{array}{l} \textit{Positive definite} \\ \textit{Positive semidefinite} \\ \textit{Negative definite} \\ \textit{Negative semidefinite} \end{array}\right\} \text{if } q \text{ is invariably} \left\{\begin{array}{ll} \text{positive} & (>0) \\ \text{nonnegative} & (\geq 0) \\ \text{negative} & (<0) \\ \text{nonpositive} & (\leq 0) \end{array}\right. \quad (26)$$

regardless of the values of variables in the quadratic form, not all zero. Clearly, the cases of positive and negative definiteness of $q = d^2z$ are related to the second-order sufficient conditions for a minimum and a maximum, respectively.

• Determinantal Test for Sign Definiteness

We can rewrite (25) as follows:

$$
\begin{aligned}
q &= au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2 \\
&= a\left(u^2 + 2\frac{h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2 \\
&= a\left(u + \frac{h}{a}v\right)^2 + \frac{ab - h^2}{a}v^2
\end{aligned}
$$

We can predicate the sign of q entirely on the values of coefficients a, b, and h as follows:

$$
q \text{ is} \left\{ \begin{array}{c} \text{positive definite} \\ \text{negative definite} \end{array} \right\} \text{ iff } \left\{ \begin{array}{c} a > 0 \\ a < 0 \end{array} \right\} \text{ and } ab - h^2 > 0. \tag{27}
$$

If we use the matrix representation, the quadratic form (25) can be rearranged into the following format:

$$q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{28}$$

The condition (27) can be alternatively expressed as:

$$q \text{ is} \begin{Bmatrix} \text{positive definite} \\ \text{negative definite} \end{Bmatrix} \text{ iff } \begin{Bmatrix} |a| > 0 \\ |a| < 0 \end{Bmatrix} \text{ and } \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0 \tag{29}$$

The determinant $|a|$ is equal to a. The determinant $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$ is equal to $ab - h^2$.

When (29) is translated, via (24), into terms of the second-order total differential d^2z, we have

$$q \text{ is} \left\{ \begin{array}{l} \text{positive definite} \\ \text{negative definite} \end{array} \right\}$$

$$\text{iff}$$

$$\left\{ \begin{array}{l} |f_{xx}| > 0 \\ |f_{xx}| < 0 \end{array} \right\} \text{ and } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}{}^2 > 0. \quad (30)$$

Since the latter inequality implies that f_{xx} and f_{yy} are required to take the same sign, we see that this is precisely the second-order sufficient condition presented in Table 11.1. The determinant with the second-order partial derivatives as its elements $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$ is called a *Hessian determinant* (or simply a *Hessian*).

Example 1

Is $q = 5u^2 + 3uv + 2v^2 (= 5u^2 + 2 \times 1.5uv + 2v^2)$ either positive or negative definite?

$$5 > 0 \text{ and } \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 5 \times 2 - 1.5^2 = 10 - 2.25 = 7.75 > 0. \qquad (31)$$

Therefore q is positive definite.

11.4 Objective Functions with More than Two Variables

Let us specifically consider a function of three choice variables,

$$z = f(x_1, x_2, x_3). \tag{32}$$

• First-Order Condition for Extremum

As our earlier discussion suggests, to have a maximum or minimum of z, it is necessary that $dz = 0$ for arbitrary values of dx_1, dx_2 and dx_3, not all zero. The value of dz is now

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3. \tag{33}$$

The only way to guarantee a zero dz for arbitrary values of dx_1, dx_2 and dx_3, not all zero, is to have

$$f_1 = f_2 = f_3 = 0. \tag{34}$$

● Second-Order Condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function.

The expression for d^2z can be obtained by differentiating dz in (33).

$$
\begin{aligned}
d(dz) = d^2z &= \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3 \\
&= \frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1 \\
&\quad + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2 \\
&\quad + \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3 \\
&= f_{11}dx_1{}^2 + f_{12}dx_1dx_2 + f_{13}dx_1dx_3 \\
&\quad + f_{21}dx_2dx_1 + f_{22}dx_2{}^2 + f_{23}dx_2dx_3 \\
&\quad + f_{31}dx_3dx_1 + f_{32}dx_3dx_2 + f_{33}dx_3{}^2
\end{aligned} \tag{35}
$$

The coefficients in (35) give rise to the symmetric Hessian determinant

$$
|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \tag{36}
$$

whose leading principal minors may be denoted by

$$|H_1| = \begin{vmatrix} f_{11} \end{vmatrix}, \quad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix},$$

$$|H_3| = |H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}. \tag{37}$$

Thus, on the basis of the determinantal criteria for positive and negative definiteness, we may state the second-order sufficient condition for an extremum of z as follows:

$$z^* \text{ is } \left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\} \tag{38}$$

$$\text{if} \begin{cases} |H_1| < 0; \quad |H_2| > 0; \quad |H_3| < 0 \quad (d^2z \text{ negative definite}) \\ |H_1| > 0; \quad |H_2| > 0; \quad |H_3| > 0 \quad (d^2z \text{ positive definite}) \end{cases} \tag{39}$$

In using this condition, we must evaluate all the leading principal minors at the stationary point where $f_1 = f_2 = f_3 = 0$.

Example 1

Find the extreme value(s) of

$$z = 2{x_1}^2 + x_1 x_2 + 4{x_2}^2 + x_1 x_3 + {x_3}^2 + 2. \qquad (40)$$

The first-order conditions for extremum are

$$f_1 = 4x_1 + x_2 + x_3 = 0$$
$$f_2 = x_1 + 8x_2 = 0$$
$$f_3 = x_1 + 2x_3 = 0.$$

This homogeneous linear-equation system has the single solution $x_1^* = x_2^* = x_3^* = 0$. This means that there is only on stationary value, $z^* = 2$.

The Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} \qquad (41)$$

whose leading principal minors are all positive:

$$|H_1| = 4, \qquad |H_2| = 31, \qquad |H_3| = 54. \qquad (42)$$

Thus we can conclude, by (39), that $z^* = 2$ is a minimum.