8. Comparative-Static Analysis of General-Function Models

8.1 Differentials

• Differentials and Derivatives

By definition, the derivative $dy/dx = f'(x)$ is the limit of a difference quotient:

$$\frac{dy}{dx} = f'(x) = \lim_{\triangle x \to 0} \frac{\triangle y}{\triangle x} \tag{1}$$

As in Figure 8.1, we have

$$\frac{dy}{dx} = \text{ slope of tangent } AD = f'(x). \tag{2}$$

and, after multiplying through by dx, we get the change of y with respect to an infinitesimal change in x.

$$dy \equiv \left(\frac{dy}{dx}\right) dx \quad \text{or} \quad dy \equiv f'(x)dx. \tag{3}$$

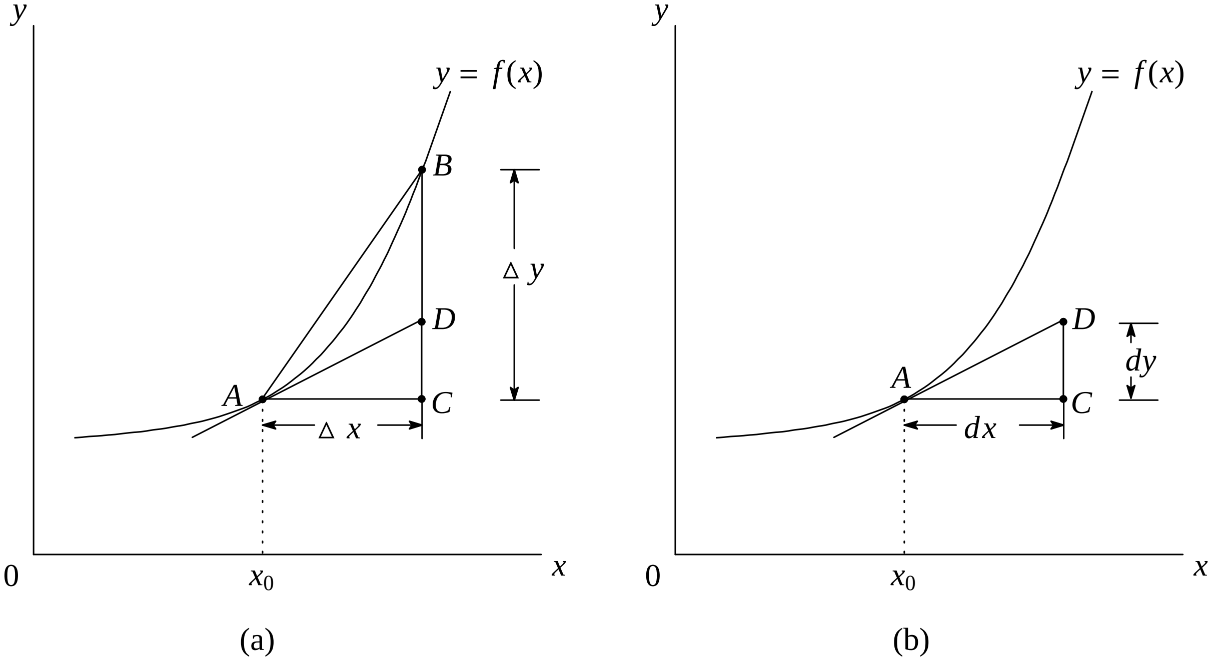

Figure 8.1

Example 1

Given $y = 3x^2 + 7x - 5$, find dy.

$$dy = \left(\frac{dy}{dx}\right) dx = (6x + 7)dx \tag{4}$$

• Differentials and Point Elasticity

For a demand function $Q = f(P)$, the *point elasticity* of demand is defined as:

$$\epsilon_d \equiv \frac{dQ/dP}{Q/P} \tag{5}$$

In general, for any function $y = f(x)$, the point elasticity of y with respect to x is defined as

$$\epsilon_{yx} \equiv \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}} \tag{6}$$

Example 2

Find ϵ_d if the demand function is $Q = 100 - 2P$.

$$\frac{Q}{P} = \frac{100 - 2P}{P} \qquad \text{and} \qquad \frac{dQ}{dP} = -2 \tag{7}$$

Thus, we obtain

$$\epsilon_d = \frac{-2}{(100 - 2P)/(P)} = \frac{-P}{50 - P} \tag{8}$$

Partial Differentiation

• Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, ..., x_n) \tag{9}$$

where the variables $x_i (i = 1, 2, ..., n)$ are all independent of one another. If the variable x_1 undergoes a change $\triangle x_1$ while $x_1, x_2, ..., x_n$ all remain fixed, there will be a corresponding change in y, namely $\triangle y$.

$$\frac{\triangle y}{\triangle x_1} = \frac{f(x_1 + \triangle x_1, x_2, ..., x_n) - f(x_1, x_2, ..., x_n)}{\triangle x_1} \tag{10}$$

If we take the limit of $\triangle y / \triangle x_1$ as $\triangle x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1. We define the *partial derivative* of y with respect to x_i as

$$f_{x_i} \equiv \frac{\partial y}{\partial x_i} \equiv \lim_{\triangle x_i \to 0} \frac{\triangle y}{\triangle x_i}. \tag{11}$$

• Geometric Interpretation of Partial Derivatives

Let us consider a production function $Q = Q(K, L)$. We can define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, as shown in Figure 8.2. Let us hold capital fixed at the level K_0 and consider only variations on the input L. The slope of the curve K_0CDA represents the rate of change of Q with respect to changes in L while K is held constant at K_0. It is clear that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C.

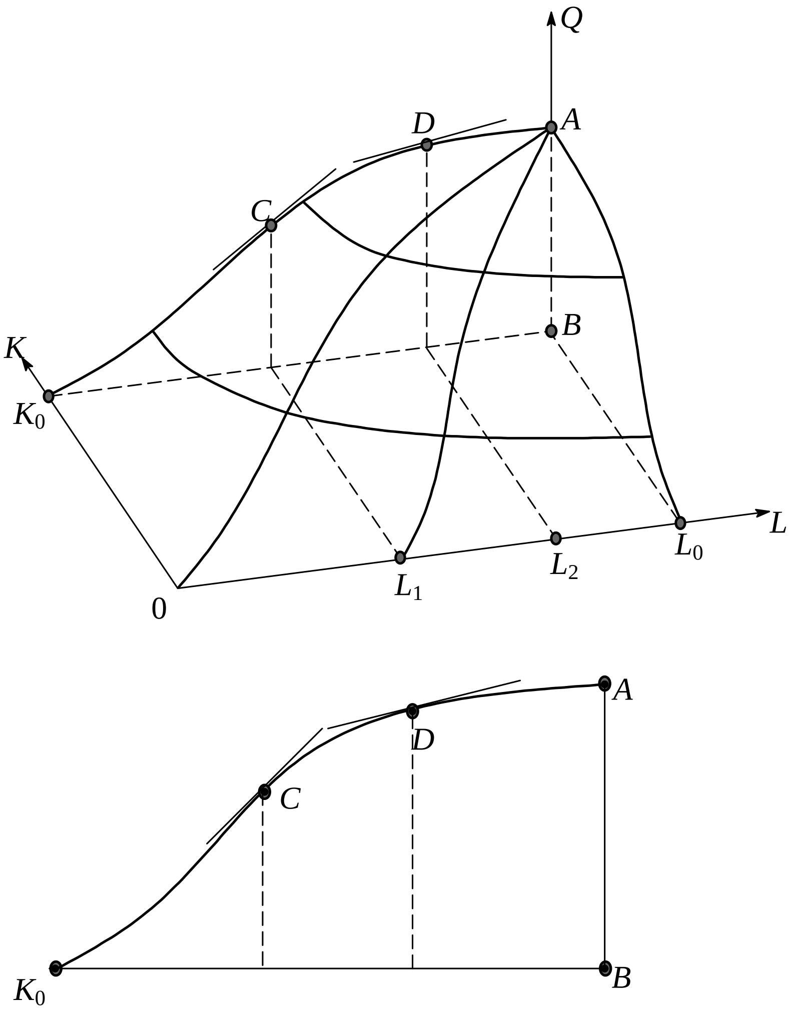

Figure 8.2

8.2 Total Differentials

Consider a saving function

$$S = S(Y, i) \tag{12}$$

where S is savings, Y is national income, and i is interest rate. The total change in S with respect to infinitesimal changes in Y and i is equal to

$$dS = \frac{\partial S}{\partial Y}dY + \frac{\partial S}{\partial i}di \quad \text{or} \quad dS = S_Y dY + S_i di \tag{13}$$

The expression dS is called the *total differential* of the saving function. Geometrically, total differential corresponds to the tangential plane as shown in Figure 8.3,

For the more general case of a function of n independent variables such as

$$U = U(x_1, x_2, \ldots x_n), \tag{14}$$

the total differential of this function can be written as

$$dU = \frac{\partial U}{\partial x_1}dx_1 + \frac{\partial U}{\partial x_2}dx_2 + \cdots + \frac{\partial U}{\partial x_n}dx_n \tag{15}$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \cdots + U_n dx_n = \sum_{i=1}^{n} U_i dx_i. \tag{16}$$

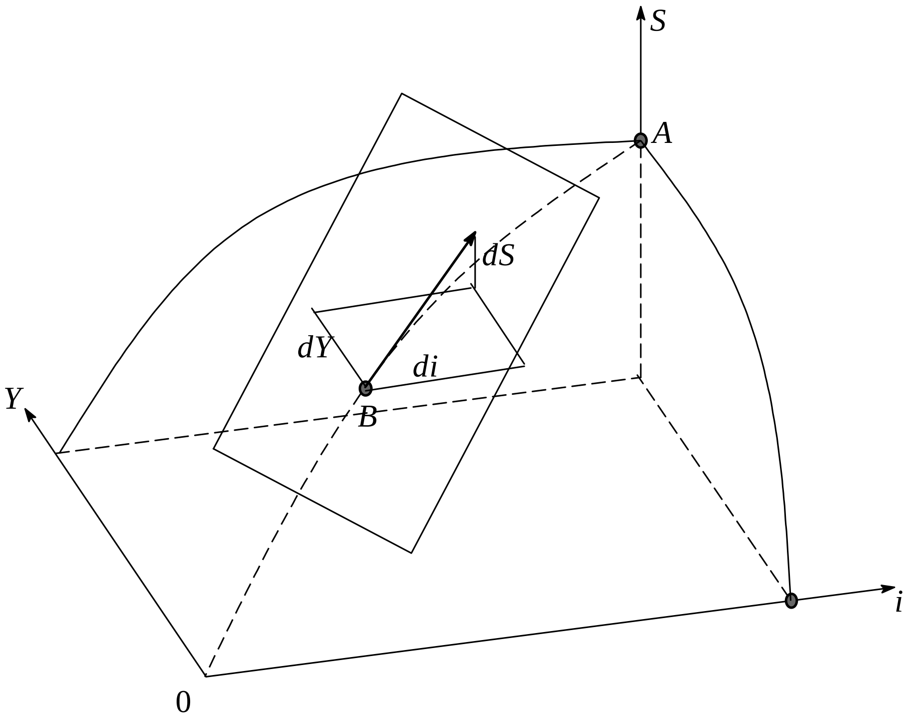

Figure 8.3

Example 1
(a)$U(x_1, x_2) = ax_1 + bx_2$
(b)$U(x_1, x_2) = x_1^2 + x_2^3 + x_1 x_2$
The total differentials are as follows:
(a)

$$\frac{\partial U}{\partial x_1} = U_1 = a, \quad \frac{\partial U}{\partial x_2} = U_2 = b \tag{17}$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2. \tag{18}$$

(b)

$$\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2, \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1 \tag{19}$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2. \tag{20}$$

8.3 Rules of Differentials

Let c be a constant and u and v be two functions.

Rule I	$dc = 0$	(constant-function rule)
Rule II	$d(cu^n) = cnu^{n-1}du$	(power-function rule)
Rule III	$d(u \pm v) = du \pm dv$	(sum-difference rule)
Rule IV	$d(uv) = v\,du + u\,dv$	(product rule)
Rule V	$d\left(\frac{u}{v}\right) = \frac{1}{v^2}(v\,du - u\,dv)$	(quotient rule)

8.4 Total Derivatives

• Finding the Total Derivative

Let us consider any function

$$y = f(x, w) \quad \text{where} \quad x = g(w) \tag{21}$$

The three variables y, x and w related to one another as in Figure. 8.4.

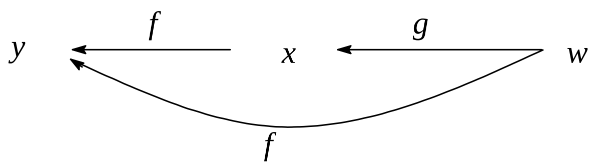

Figure 8.4

The variable w can affect y through two channels: (1) *directly*, via the function f, and (2) *indirectly*, via the function g and then f. To obtain this total derivative, we first differentiate y totally to get the total differential $dy = f_x sdx + f_w dw$. When both sides of this equation are divided by the differential dw, the result is

$$\begin{aligned} \frac{dy}{dw} &= \underbrace{f_x \frac{dx}{dw}}_{indirect\ \ effect} + \underbrace{f_w \frac{dw}{dw}}_{direct\ \ effect} \\ &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \end{aligned} \tag{22}$$

Example 1 Find the total derivative dy/dw, given the function

$$y = f(x, w) = 3x - w^2 \quad \text{where} \quad x = g(w) = 2w^2 + w + 4 \tag{23}$$

By virtue of (22), the total derivative should be

$$\begin{aligned} \frac{dy}{dw} &= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \\ &= 3(4w + 1) + (-2w) = 10w + 3 \end{aligned} \tag{24}$$

• A Variation on the Theme

Let us consider any function

$$y = f(x_1, x_2, w) \quad \text{where} \quad \begin{cases} x_1 &= g(w) \\ x_2 &= h(w) \end{cases} \tag{25}$$

The variable w can affect y through three channels: (1)indirectly, via the function g and then f, (2)again, indirectly, via the function h and then f, (3) directly via f. The total derivative of y with respect to w is given by

$$\begin{aligned} \frac{dy}{dw} &= f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw} \\ &= \frac{\partial y}{\partial x_1} \frac{dx_1}{dw} + \frac{\partial y}{\partial x_2} \frac{dx_2}{dw} + \frac{\partial y}{\partial w} \end{aligned} \tag{26}$$

Example 2 Let the production function be

$$Q = Q(K, L, t) \tag{27}$$

where, aside from the two inputs K and L, there is a third argument t, denoting time. Since capital and labor, too, can change over time, we may write

$$K = K(t) \quad \text{and} \quad L = L(t). \tag{28}$$

Then, the rate of change of output with respect to time can be expressed as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t} \qquad \text{or} \qquad \frac{dQ}{dt} = Q_K K'(t) + Q_L L'(t) + Q_t. \tag{29}$$

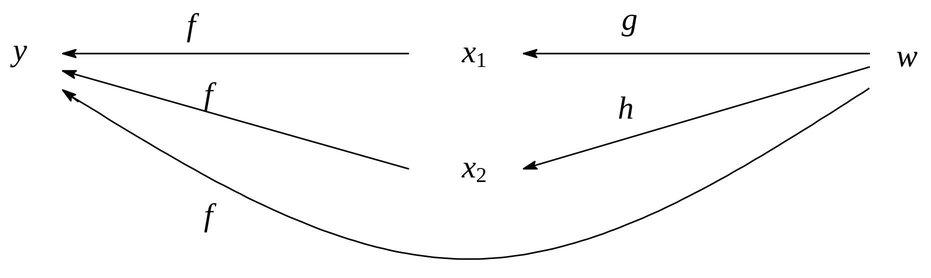

Figure 8.5

8.5 Derivatives of Implicit Functions

The concept of total differentials can also enable us to find the derivatives of so-called "implicit-functions".

• Implicit Functions

A function given in the form of $y = f(x)$, say,

$$y = f(x) = 3x^4 \tag{30}$$

is called an *explicit function*, because the variable y is explicitly expressed as a function of x. If this function is written alternatively in the equivalent form

$$y - 3x^4 = 0, \tag{31}$$

we no longer have an explicit function. Rather, the function (30) is then only *implicitly* defined by the equation (31). We call the function $y = f(x)$ implied by (31) an *implicit function*.

Because the left side of (31) is a function of the two variables y and x, it can be denoted in general by

$$F(y, x) = 0. \tag{32}$$

Of course, there may be more than two arguments in the F function. For instance, we may encounter an equation $F(y, x_1, x_2, ..., x_m) = 0$. Such an equation *may* also define an implicit function $y = f(x_1, x_2, ..., x_m)$.

In general, an explicit function, say, $y = f(x)$, can always be transformed into an equation $F(y, x) = 0$ by simply transposing the $f(x)$ expression to the left side of the equal sign. However, the reverse transformation is not always possible. For instance, the equation

$$F(y, x) = x^2 + y^2 - 9 = 0 \tag{33}$$

implies not a function, but a relation, because (33) plots as a circle as shown in Figure 8.6. Hence, no unique value of y corresponds to each value of x.

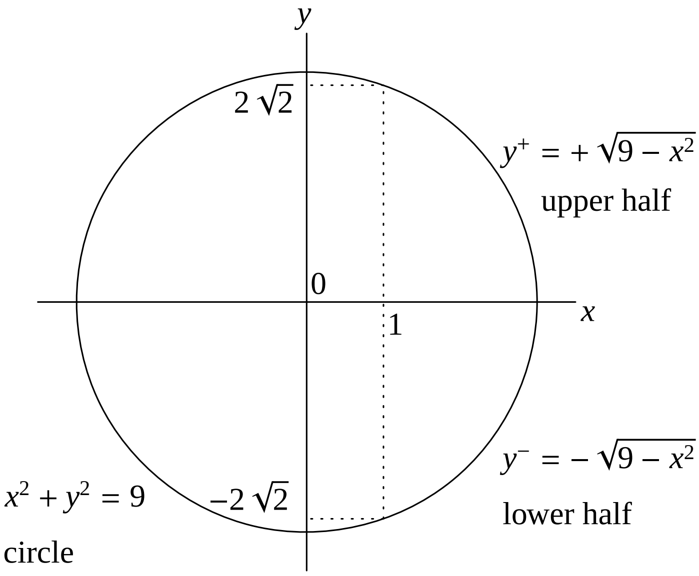

Figure 8.6

However, if we restrict y to nonnegative values, then we will have the upper half of the circle that does constitute a function

$$y = +\sqrt{9 - x^2}. \tag{34}$$

Similarly, the lower half of the circle constitutes another function

$$y = -\sqrt{9 - x^2}. \tag{35}$$

In view of this uncertainty, it becomes of interest to ask whether there are known general conditions under which we can be sure that a given equation in the form of

$$F(y, x_1, ..., x_m) = 0 \tag{36}$$

does indeed define an implicit function

$$y = f(x_1, ..., x_m). \tag{37}$$

Implicit Function Theorem

Given (36),

(A) if the function F has continuous partial derivatives $F_y, F_1, ..., F_m$,

and

(B) if F_y is non zero at a point $(y_0, x_{10}, ..., x_{m0})$ satisfying the equation (36),

there exists an m-dimensional neighborhood of $(y_0, x_{10}, ..., x_{m0})$, N, in which y is an implicitly defined function of the variables $x_1, ..., x_m$ in terms of (37). The function (37) has the following properties:

(i) $y_0 = f(x_{10}, ..., x_{m0})$

(ii) for every m-tuple $(x_1, ..., x_m)$ in the neighborhood N, the equation (36) is satisfied,

$$F(\underbrace{f(x_1, ..., x_m)}_{y}, x_1, ..., x_m) = 0. \tag{38}$$

The above equation has the status of an *identity* in that neighborhood.

and

(iii) the implicit function f is continuous and has continuous partial derivatives $f_1, ..., f_m$.

Let us apply this theorem to the equation of the circle (33). First, we verify that the conditions (A) and (B) are satisfied.

(A) $F_y = 2y$ and $F_x = 2x$ are continuous.

(B) Since F_y is nonzero except when $y = 0$, the condition (B) is satisfied except at $(-3, 0)$ and $(3, 0)$.

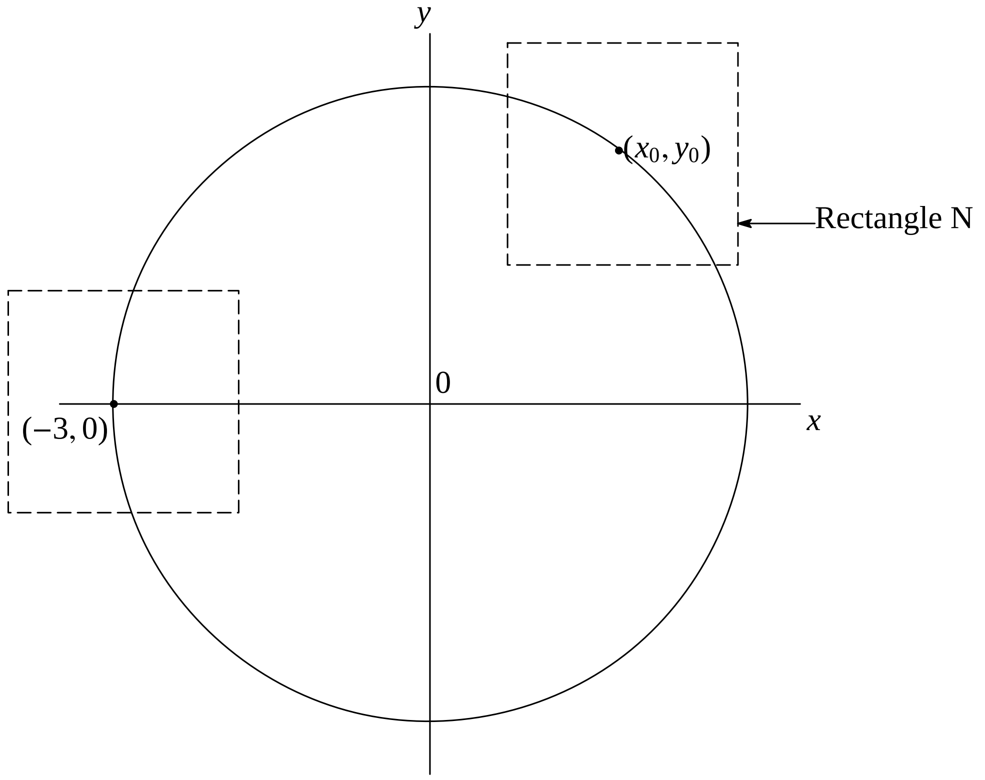

Figure 8.6b

Thus, around any point on the circle except $(-3, 0)$ and $(3, 0)$, we can construct a neighborhood in which the equation (33) defines an implicit function $y = f(x)$. See Figure 8.6b. In the rectangle N, an unique value of y corresponds to each value of x. Hence, the implicit function is defined. However, two values of y correspond to each value of x around (-3,0) and (3,0).

Several things should be noted about the implicit-function theorem.

1. The conditions cited in the theorem are sufficient (but not necessary) conditions.
   If we happen to find $F_y = 0$ at a point satisfying (36), this does not mean that an implicit function does not exist around that point.

2. Even if the existence of an implicit function f is assured, the theorem gives no clue as to the specific form the function f. Nor, does it tell us the exact size of the neighborhood.

• Derivatives of Implicit Functions

If the equation $F(y, x_1, ..., x_m) = 0$ can be solved for y, we can explicitly write out the function $y = f(x_1, ..., x_m)$, and find its derivatives. But what if the given equation $F(y, x_1, ..., x_m) = 0$ cannot be solved for y explicitly? In this case, if under the terms of the implicit function theorem an implicit function is known to exist, we can still obtain the desired derivatives without solving for y first. To do this, we make use of the following basic facts:

1. if two expressions are *identically* equal, their respective total differentials must be equal;

2. if we divide dy by dx_1 and let all the other differentials $(dx_2, ..., dx_m)$ be zero, the quotient can be interpreted as the partial derivative $\partial y/\partial x_1$.

Recall that $F(y, x_1, ...x_m) = 0$ has the status of an *identity* in the neighborhood where the implicit function is defined. Thus, the total differentials of both sides must be equal.

$$dF = d0 \tag{39}$$

or

$$F_y dy + F_{x_1} dx_1 + \cdots + F_{x_m} dx_m = 0. \tag{40}$$

Suppose that only y and x_i are allowed to vary. Then, we have that

$$F_y dy + F_{x_i} dx_i = 0. \tag{41}$$

Dividing both sides by dx_i and solving for dy/dx_i, we get

$$\left.\frac{dy}{dx_i}\right|_{other\ variables\ constant} \equiv \frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y} \tag{42}$$

Consider an equation $F(y, x_1, x_2) = 0$. Geometrically, the partial derivative $\partial y/\partial x_1$ corresponds to the slope of the tangent line of the contour line as shown in Figure 8.7.

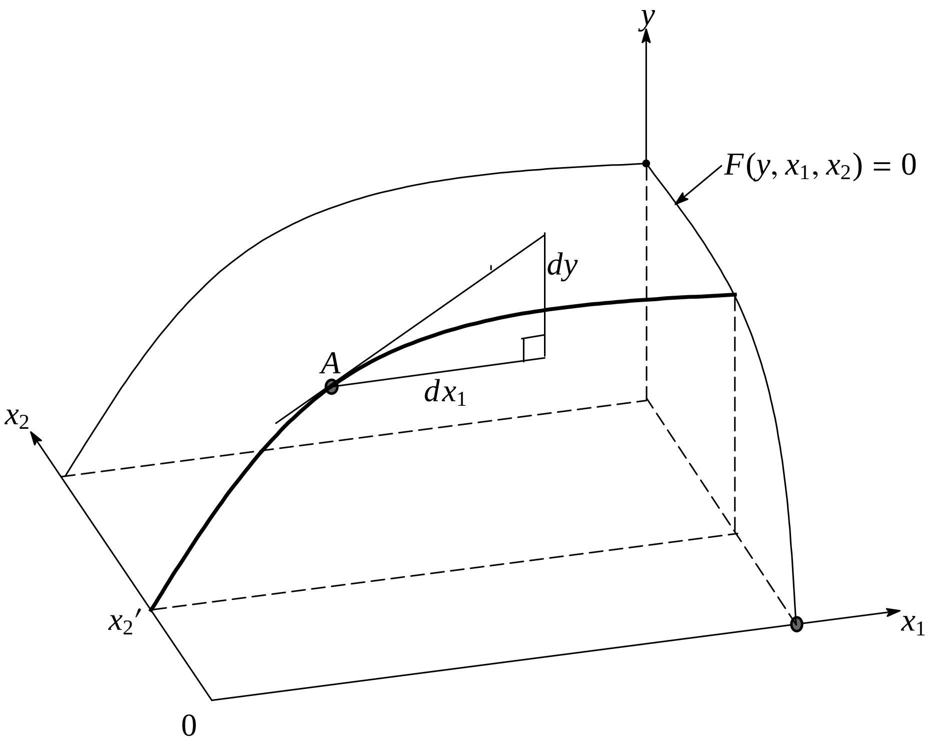

Figure 8.7

Example 1 Find $\partial y/\partial x$ for any implicit function that may be defined by the equation

$$F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0. \tag{43}$$

F_y, F_x, and F_w are all obviously continuous.

$$F_y = 3y^2x^2 + xw, \tag{44}$$

$$F_x = 2y^3x + yw, \tag{45}$$

$$F_w = 3w^2 + yx. \tag{46}$$

At a point such as $(1,1,1)$, F_y is nonzero. The existence of an implicit function $y = f(x, w)$ is assured around that point at least. Since the total differentials of both sides are equal,

$$F_y dy + F_x dx + F_w dw = 0. \tag{47}$$

Setting $dw = 0$ and rearranging the above equation, we obtain

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}. \tag{48}$$

At the point (1,1,1), this derivative has the value $-\frac{3}{4}$.