

8. Comparative-Static Analysis of General-Function Models

8.1 Differentials

- Differentials and Derivatives

The derivative $dy/dx = f'(x)$ is the limit of a difference quotient:

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad (1)$$

As in Figure 8.1, we have

$$\frac{dy}{dx} = \text{slope of tangent } AD = f'(x). \quad (2)$$

After multiplying through by dx , we get the change of y with respect to an infinitesimal change in x .

$$dy \equiv \left(\frac{dy}{dx}\right) dx \quad \text{or} \quad dy \equiv f'(x) dx. \quad (3)$$

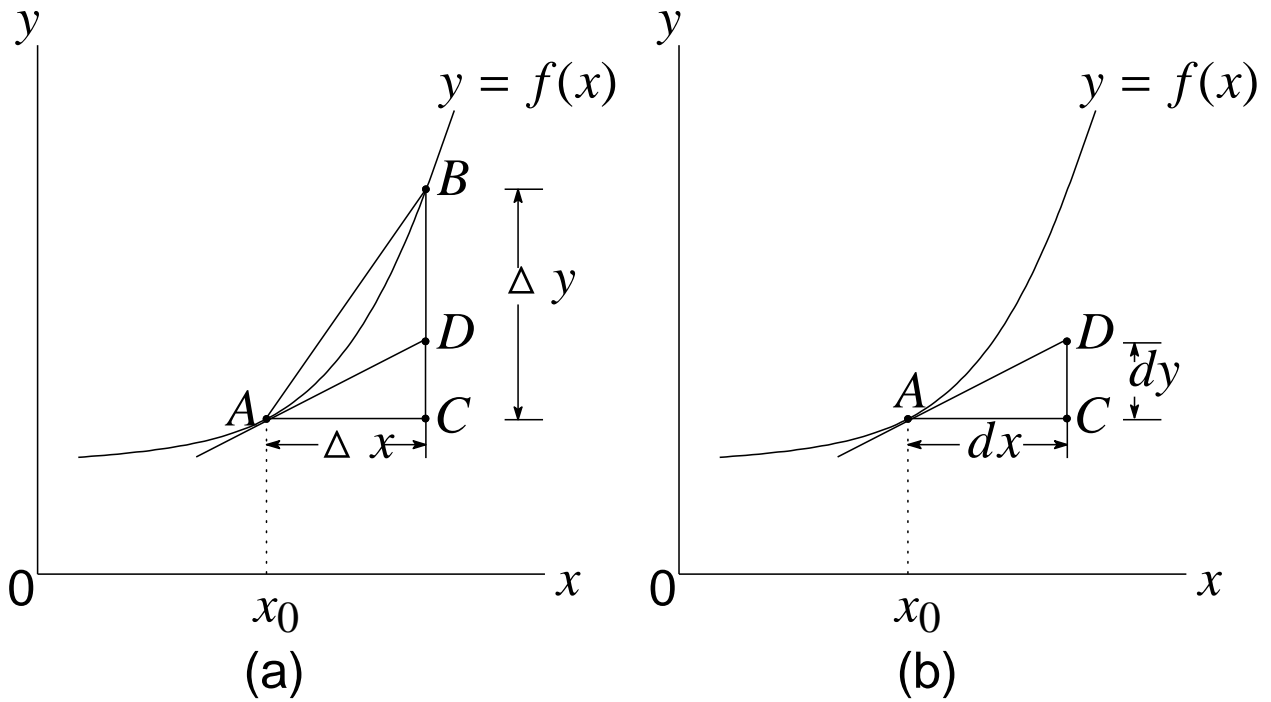


Figure 8.1

Example 1

Given $y = 3x^2 + 7x - 5$, find dy .

$$dy = \left(\frac{dy}{dx} \right) dx = (6x + 7)dx \quad (4)$$

- Differentials and Point Elasticity

For a demand function $Q = f(P)$, the *point elasticity* of demand is defined as:

$$\epsilon_d \equiv \frac{dQ/dP}{Q/P} \quad (5)$$

In general, for any function $y = f(x)$, the point elasticity of y with respect to x is defined as

$$\epsilon_{yx} \equiv \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}} \quad (6)$$

Example 2

Find ϵ_d if the demand function is $Q = 100 - 2P$.

$$\frac{Q}{P} = \frac{100 - 2P}{P} \quad \text{and} \quad \frac{dQ}{dP} = -2 \quad (7)$$

Thus, we obtain

$$\epsilon_d = \frac{-2}{(100 - 2P)/(P)} = \frac{-P}{50 - P} \quad (8)$$

Partial Differentiation

- Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \dots, x_n) \quad (9)$$

If the variable x_1 undergoes a change Δx_1 while x_2, \dots, x_n all remain fixed, there will be a corresponding change in y , namely Δy .

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1} \quad (10)$$

Partial derivative of y with respect to x_i :

$$f_{x_i} \equiv \frac{\partial y}{\partial x_i} \equiv \lim_{\Delta x_i \rightarrow 0} \frac{\Delta y}{\Delta x_i}. \quad (11)$$

- Geometric Interpretation of Partial Derivatives

Consider a production function $Q = Q(K, L)$. Let us hold capital fixed at the level K_0 and consider only variations on the input L . The slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L .

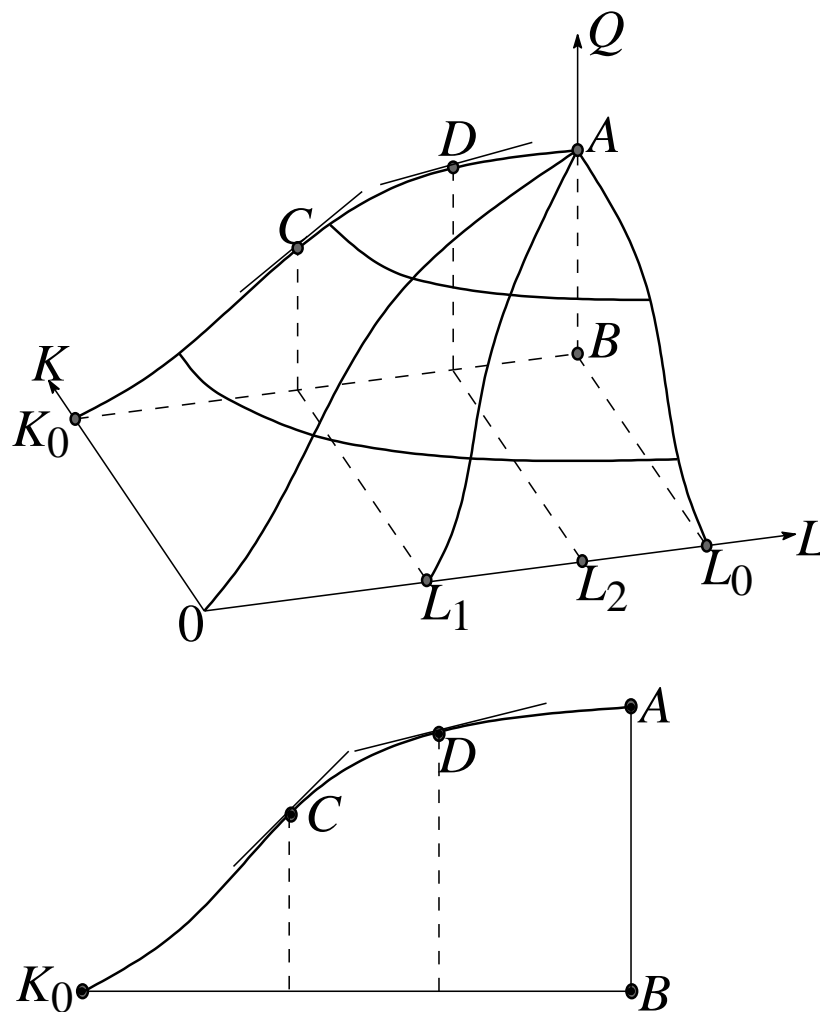


Figure 8.2

8.2 Total Differentials

Consider a saving function

$$S = S(Y, i) \quad (12)$$

S : savings, Y : national income, and i : interest rate.

⟨ *Total differential of the saving function* ⟩

The total change in S with respect to infinitesimal changes in Y and i :

$$dS = \frac{\partial S}{\partial Y}dY + \frac{\partial S}{\partial i}di \quad \text{or} \quad dS = S_Y dY + S_i di \quad (13)$$

Geometrically, total differential corresponds to the tangential plane as shown in Figure 8.3,

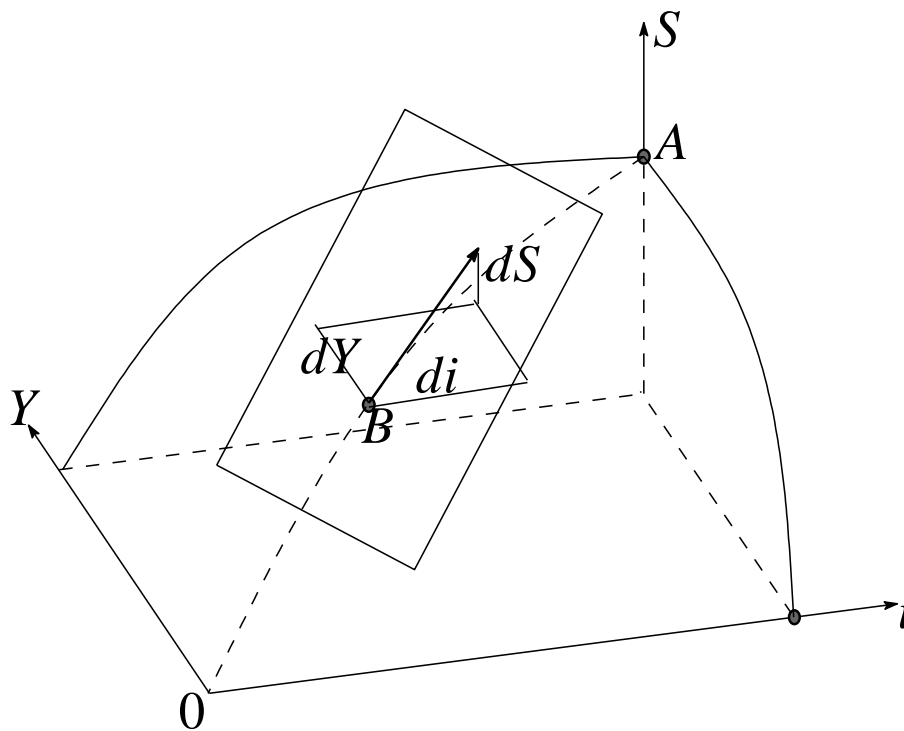


Figure 8.3

For the more general case of a function of n independent variables such as

$$U = U(x_1, x_2, \dots, x_n), \quad (14)$$

the total differential of U can be written as

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \dots + \frac{\partial U}{\partial x_n} dx_n \quad (15)$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \dots + U_n dx_n = \sum_{i=1}^n U_i dx_i. \quad (16)$$

Example 1

(a) $U(x_1, x_2) = ax_1 + bx_2$

(b) $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$

The total differentials are as follows:

(a)

$$\frac{\partial U}{\partial x_1} = U_1 = a, \quad \frac{\partial U}{\partial x_2} = U_2 = b \quad (17)$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = a dx_1 + b dx_2. \quad (18)$$

(b)

$$\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2, \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1 \quad (19)$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2) dx_1 + (3x_2^2 + x_1) dx_2. \quad (20)$$

8.3 Rules of Differentials

Let c be a constant and u and v be two functions.

Rule I $dc = 0$ (constant-function rule)

Rule II $d(cu^n) = cnu^{n-1}du$ (power-function rule)

Rule III $d(u \pm v) = du \pm dv$ (sum-difference rule)

Rule IV $d(uv) = v du + u dv$ (product rule)

Rule V $d\left(\frac{u}{v}\right) = \frac{1}{v^2}(v du - u dv)$ (quotient rule)

8.4 Total Derivatives

- Finding the Total Derivative

Let us consider any function

$$y = f(x, w) \quad \text{where} \quad x = g(w) \quad (21)$$

The three variables y , x and w related to one another as in Figure. 8.4.

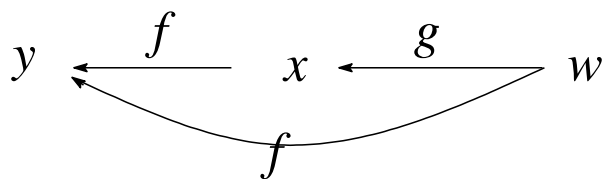


Figure 8.4

The variable w can affect y through two channels:

- (1) *directly*, via the function f .
- (2) *indirectly*, via the function g and then f .

To obtain this total derivative, we first differentiate y totally to get the total differential $dy = f_x dx + f_w dw$. Then we divide both sides of this equation by the differential dw .

$$\begin{aligned} \frac{dy}{dw} &= \underbrace{f_x \frac{dx}{dw}}_{\text{indirect effect}} + \underbrace{f_w \frac{dw}{dw}}_{\text{direct effect}} \\ &= \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w} \end{aligned} \quad (22)$$

Example 1

Find the total derivative dy/dw , given the function

$$y = f(x, w) = 3x - w^2 \quad \text{where} \quad x = g(w) = 2w^2 + w + 4 \quad (23)$$

By virtue of (22), the total derivative should be

$$\begin{aligned} \frac{dy}{dw} &= \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w} \\ &= 3(4w + 1) + (-2w) = 10w + 3 \end{aligned} \quad (24)$$

- A Variation on the Theme

Let us consider any function

$$y = f(x_1, x_2, w) \quad \text{where} \quad \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases} \quad (25)$$

The variable w can affect y through three channels:

- (1) indirectly, via the function g and then f .
- (2) indirectly, via the function h and then f .
- (3) directly via f .

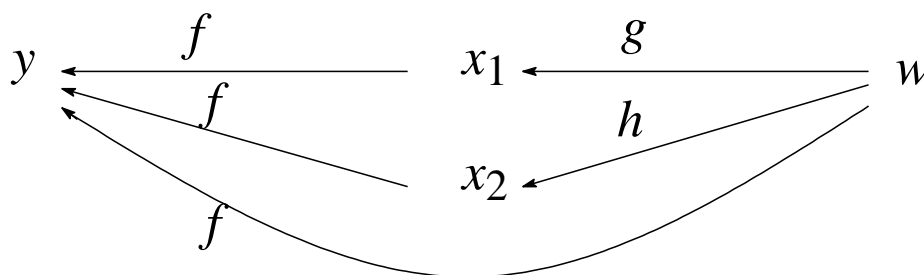


Figure 8.5

The total derivative of y with respect to w is given by

$$\begin{aligned} \frac{dy}{dw} &= f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw} \\ &= \frac{\partial y}{\partial x_1} \frac{dx_1}{dw} + \frac{\partial y}{\partial x_2} \frac{dx_2}{dw} + \frac{\partial y}{\partial w} \end{aligned} \quad (26)$$

Example 2

Let the production function be

$$Q = Q(K, L, t) \quad (27)$$

where, aside from the two inputs K and L , there is a third argument t , denoting time. Since capital and labor, too, can change over time, we may write

$$K = K(t) \quad \text{and} \quad L = L(t). \quad (28)$$

Then, the rate of change of output with respect to time can be expressed as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K} \frac{dK}{dt} + \frac{\partial Q}{\partial L} \frac{dL}{dt} + \frac{\partial Q}{\partial t} \quad (29.1)$$

or

$$\frac{dQ}{dt} = Q_K K'(t) + Q_L L'(t) + Q_t. \quad (29.2)$$

8.5 Derivatives of Implicit Functions

The concept of total differentials can also enable us to find the derivatives of so-called “implicit-functions”.

- Implicit Functions

A function given in the form of $y = f(x)$, say,

$$y = f(x) = 3x^4 \quad (30)$$

is called an *explicit function* because the variable y is explicitly expressed as a function of x . If this function is written alternatively in the equivalent form

$$y - 3x^4 = 0, \quad (31)$$

we no longer have an explicit function. The function (30) is then only *implicitly* defined by the equation (31). The function $y = f(x)$ implied by (31) is called an *implicit function*.

(31) can be denoted in general by

$$F(y, x) = 0. \quad (32)$$

In case where there are more than two arguments in the F function:

$$F(y, x_1, x_2, \dots, x_m) = 0$$

Such an equation *may* also define an implicit function $y = f(x_1, x_2, \dots, x_m)$.

- An explicit function, say, $y = f(x)$, can always be transformed into an equation $F(y, x) = 0$.
- The reverse transformation is not always possible.

For instance, the equation

$$F(y, x) = x^2 + y^2 - 9 = 0 \quad (33)$$

implies not a function, but a relation, because (33) plots as a circle as shown in Figure 8.6. Hence, no unique value of y corresponds to each value of x .

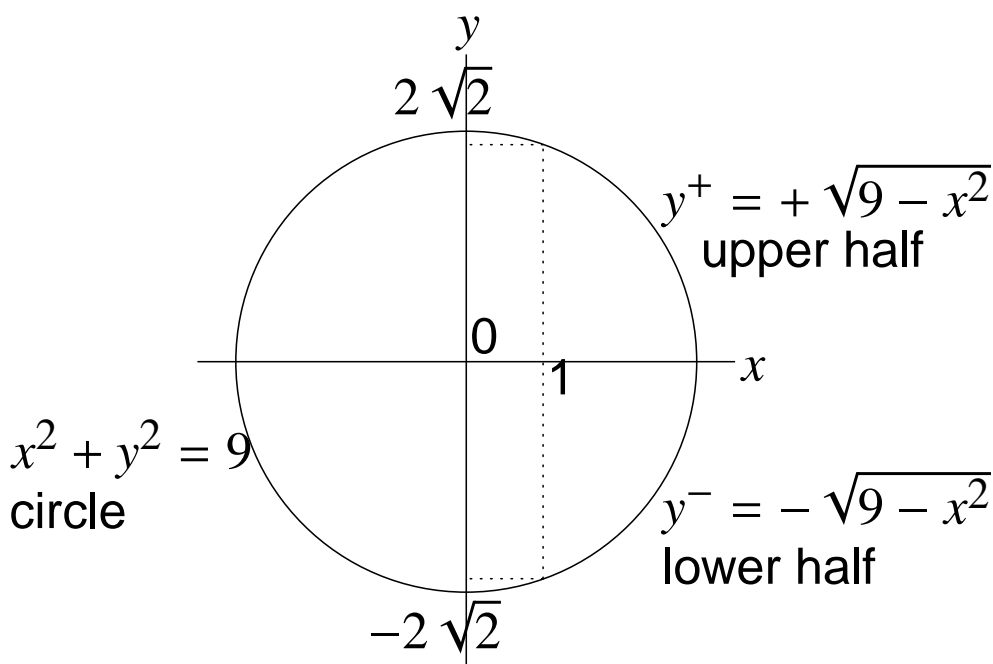


Figure 8.6

If we restrict y to nonnegative values, then we will have the upper half of the circle that does constitute a function

$$y = +\sqrt{9 - x^2}. \quad (34)$$

Similarly, the lower half of the circle constitutes another function

$$y = -\sqrt{9 - x^2}. \quad (35)$$

In view of this uncertainty, it becomes of interest to ask whether there are known general conditions under which we can be sure that a given equation in the form of

$$F(y, x_1, \dots, x_m) = 0 \quad (36)$$

does indeed define an implicit function

$$y = f(x_1, \dots, x_m). \quad (37)$$

Implicit Function Theorem

Given (36),

(A) if the function F has continuous partial derivatives F_y, F_1, \dots, F_m ,

and

(B) if F_y is non zero at a point $(y_0, x_{10}, \dots, x_{m0})$ satisfying the equation (36),

there exists an m -dimensional neighborhood of $(y_0, x_{10}, \dots, x_{m0})$, N , in which y is an implicitly defined function of the variables x_1, \dots, x_m in terms of (37). The function (37) has the following properties:

(i) $y_0 = f(x_{10}, \dots, x_{m0})$

(ii) for every m -tuple (x_1, \dots, x_m) in the neighborhood N , the equation (36) is satisfied,

$$F(\underbrace{f(x_1, \dots, x_m)}_y, x_1, \dots, x_m) = 0. \quad (38)$$

The above equation has the status of an *identity* in that neighborhood.

and

(iii) the implicit function f is continuous and has continuous partial derivatives f_1, \dots, f_m .

Let us apply this theorem to the equation (33).

Verify that the conditions (A) and (B) are satisfied.

(A) $F_y = 2y$ and $F_x = 2x$ are continuous.

(B) the condition (B) is satisfied except at $(-3, 0)$ and $(3, 0)$ because F_y is nonzero except when $y = 0$.

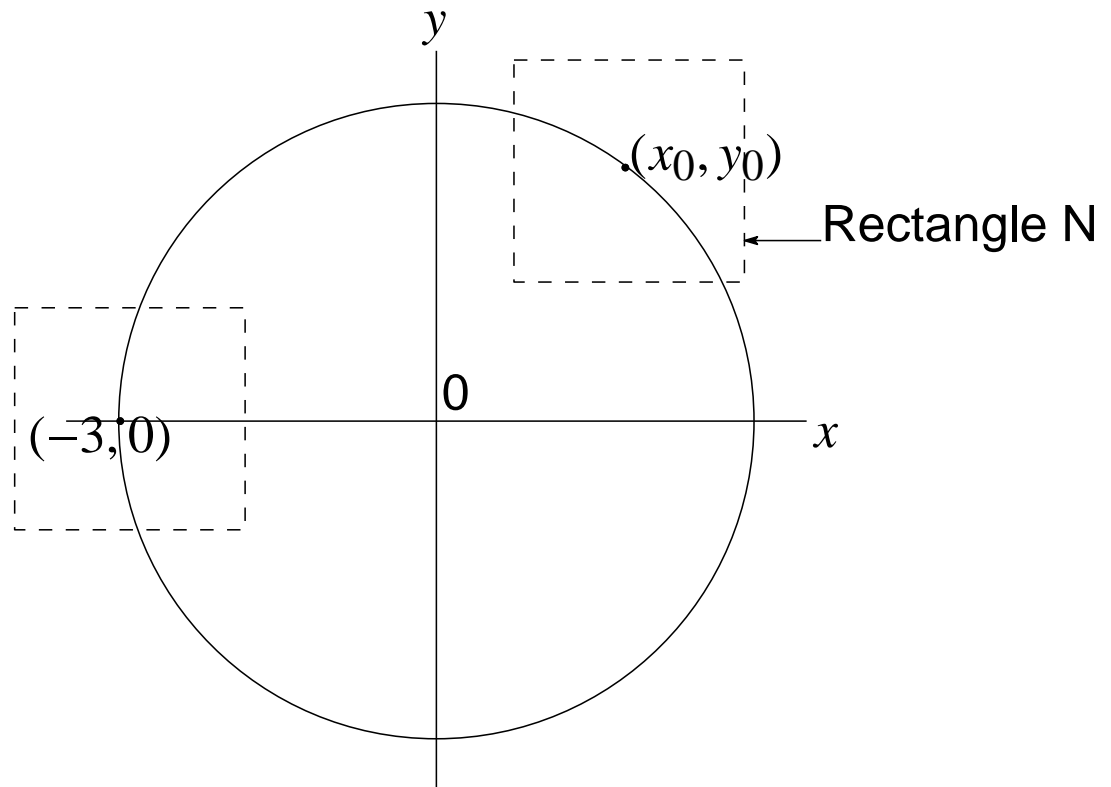


Figure 8.6b

Around any point on the circle except $(-3, 0)$ and $(3, 0)$, we can construct a neighborhood in which the equation (33) defines an implicit function $y = f(x)$. In the rectangle N , a unique value of y corresponds to each value of x . Hence, the implicit function is defined. However, two values of y correspond to each value of x around $(-3, 0)$ and $(3, 0)$.

Several things should be noted about the implicit-function theorem.

1. The conditions cited in the theorem are sufficient (but not necessary) conditions.

If we happen to find $F_y = 0$ at a point satisfying (36), this does not mean that an implicit function does not exist around that point.

2. Even if the existence of an implicit function f is assured, the theorem gives no clue as to the specific form the function f . Nor, does it tell us the exact size of the neighborhood.

- Derivatives of Implicit Functions

If the equation $F(y, x_1, \dots, x_m) = 0$ can be solved for y , we can explicitly write out the function $y = f(x_1, \dots, x_m)$, and find its derivatives. But what if the given equation $F(y, x_1, \dots, x_m) = 0$ cannot be solved for y explicitly? In this case, if under the terms of the implicit function theorem an implicit function is known to exist, we can still obtain the desired derivatives without solving for y first. To do this, we make use of the following basic facts:

1. if two expressions are *identically* equal, their respective total differentials must be equal;
2. if we divide dy by dx_1 and let all the other differentials (dx_2, \dots, dx_m) be zero, the quotient can be interpreted as the partial derivative $\partial y / \partial x_1$.

$F(y, x_1, \dots, x_m) = 0$ has the status of an *identity* in the neighborhood where the implicit function is defined. Thus, the total differentials of both sides must be equal.

$$dF = d0 \quad (39)$$

or

$$F_y dy + F_{x_1} dx_1 + \dots + F_{x_m} dx_m = 0. \quad (40)$$

Suppose that only y and x_i are allowed to vary.

$$F_y dy + F_{x_i} dx_i = 0. \quad (41)$$

Dividing both sides by dx_i and solving for dy/dx_i , we get

$$\left. \frac{dy}{dx_i} \right|_{\text{other variables constant}} \equiv \frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y} \quad (42)$$

Consider an equation $F(y, x_1, x_2) = 0$. Geometrically, the partial derivative $\partial y / \partial x_1$ corresponds to the slope of the tangent line of the contour line as shown in Figure 8.7.

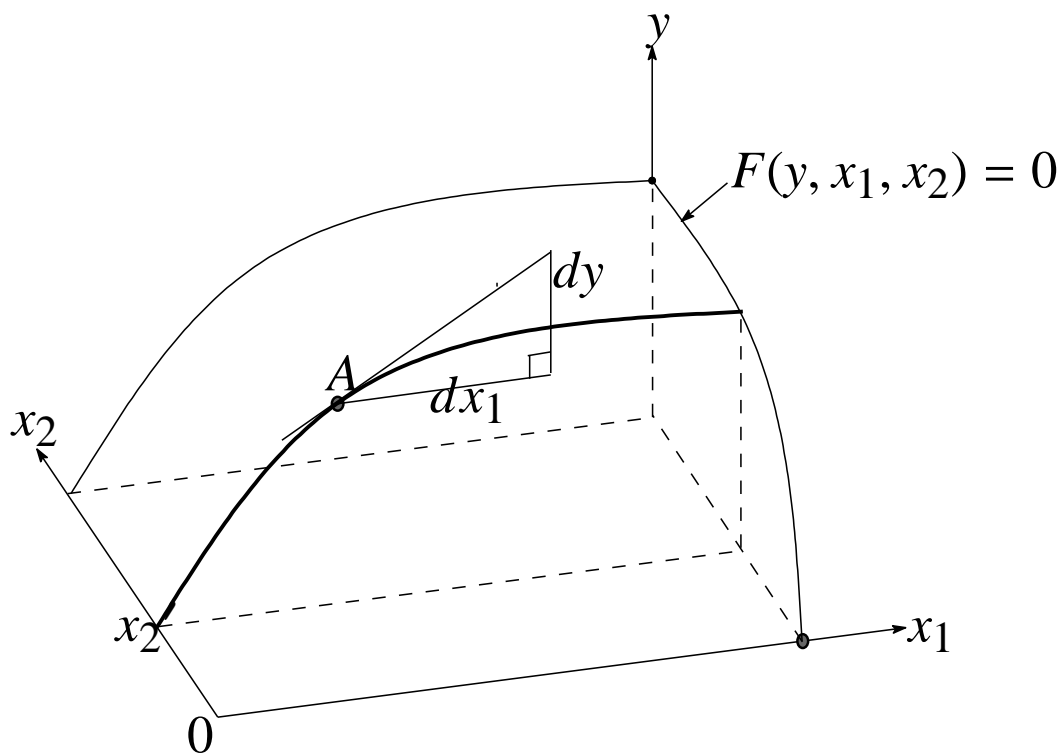


Figure 8.7

Example 1

Find $\partial y/\partial x$ for any implicit function that may be defined by the equation

$$F(y, x, w) = y^3 x^2 + w^3 + yxw - 3 = 0. \quad (43)$$

F_y , F_x , and F_w are all obviously continuous.

$$F_y = 3y^2 x^2 + xw, \quad (44)$$

$$F_x = 2y^3 x + yw, \quad (45)$$

$$F_w = 3w^2 + yx. \quad (46)$$

At a point such as $(1, 1, 1)$, F_y is nonzero. The existence of an implicit function $y = f(x, w)$ is assured around that point at least. Since the total differentials of both sides are equal.

$$F_y dy + F_x dx + F_w dw = 0. \quad (47)$$

Setting $dw = 0$ and rearranging the above equation, we obtain

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3 x + yw}{3y^2 x^2 + xw}. \quad (48)$$

At the point $(1, 1, 1)$, this derivative has the value $-\frac{3}{4}$.