9. Optimization: A Special Variety of Equilibrium Analysis

9.1 Optimum Values and Extreme Values

A business firm may seek to maximize profit π, that is, to maximize the difference between total revenue R and total cost C. Since R and C are both functions of the output level Q, π is also a function of Q.

$$\pi(Q) = R(Q) - C(Q). \tag{1}$$

The optimization problem is to choose the level of Q such that π will be a maximum.

In the following discussion, let us consider the general function

$$y = f(x) \tag{2}$$

and attempt to develop a procedure for finding the level of x that will maximize or minimize the value of y.

9.2 Relative Maximum and Minimum: First-Derivative Test

• Relative versus Absolute Extreme

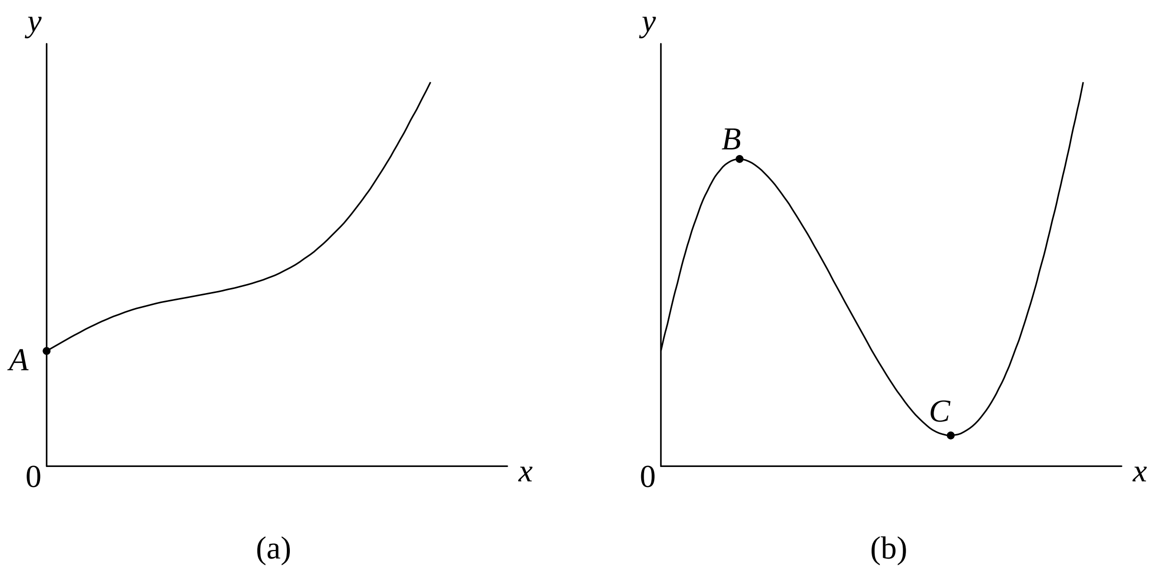


Figure 9.1

In Figure 9.1(a), the point A is the *absolute* minimum in the range of the function. In Figure 9.1(b), the points B and C are examples of a *relative* extreme. The fact that point B (C) is a relative maximum (minimum) is no guarantee that it is also the global maximum (minimum) of the function, although this may happen to be the case. We continue our discussion mainly with reference to the search for *relative* extrema such as points B and C.

• First-Derivative Test

We assume that $y = f(x)$ is continuous and possesses a continuous derivative. For smooth functions, relative extreme values can occur only where the first derivative has a zero value. We must add, however, that a zero slope is *not sufficient* to establish a relative extreme while it is *necessary*.

First-derivative test for relative extreme

If the first derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.

b. A relative *minimum* if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right.

c. Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of point x_0.

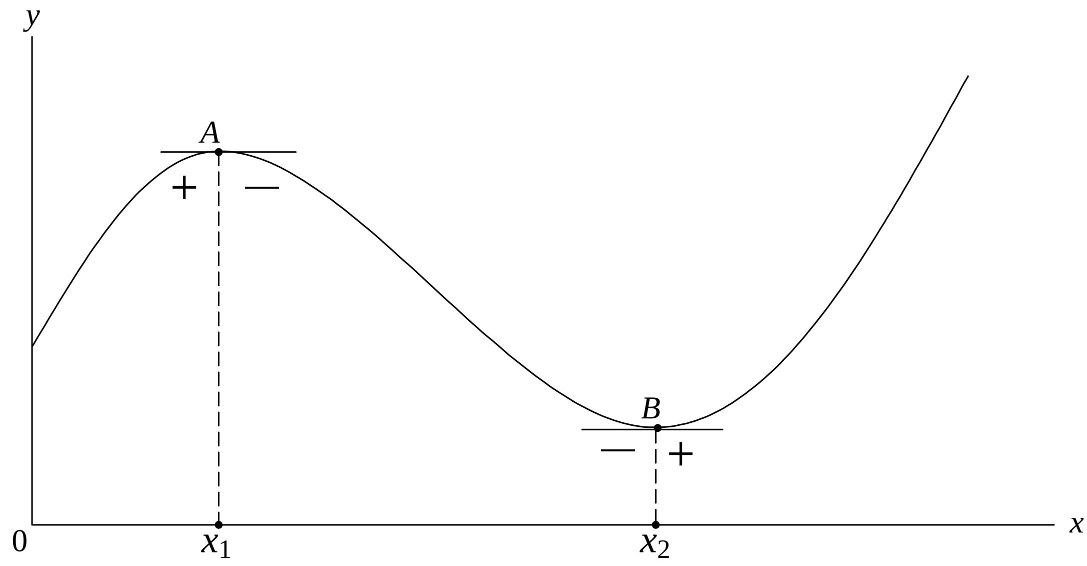


Figure 9.2

Let us call the value x_0 a *critical value* of x if $f'(x_0) = 0$, and refer to $f(x_0)$ as a *stationary value* of y. The point $(x_0, f(x_0))$ can be called a *stationary point*.

The first possibility listed in the first-derivative test will establish the stationary point as the peak of a hill, such as point A in Figure 9.2. We can easily verify that the derivative $f'(x)$ changes its sign from positive to negative at point A. The second possibility will establish the stationary point as the bottom of a valley, such as point B where the derivative $f'(x)$ changes its sign from negative to positive. Note that in view of the existence of a third possibility, we are unable to regard the condition $f'(x) = 0$ as a *sufficient condition* for a relative maximum.

Let us explain the third possibility. See Figure 9.3(a). The function f attains a zero slope at point J when $x = j$. Even though $f'(c)$, the derivative does not change its sign from one side of $x = j$ to the other. According to the first-derivative test, point J gives neither a maximum nor a minimum. Rather, it exemplifies what is known as an *inflection point*. The characteristic feature of an inflection point is that the derivative function reaches an extreme value at that point. The other type of inflection point is shown in Figure 9.3(b), where the slope of the function $g(x)$ increases till the point k is reached and decreases thereafter.

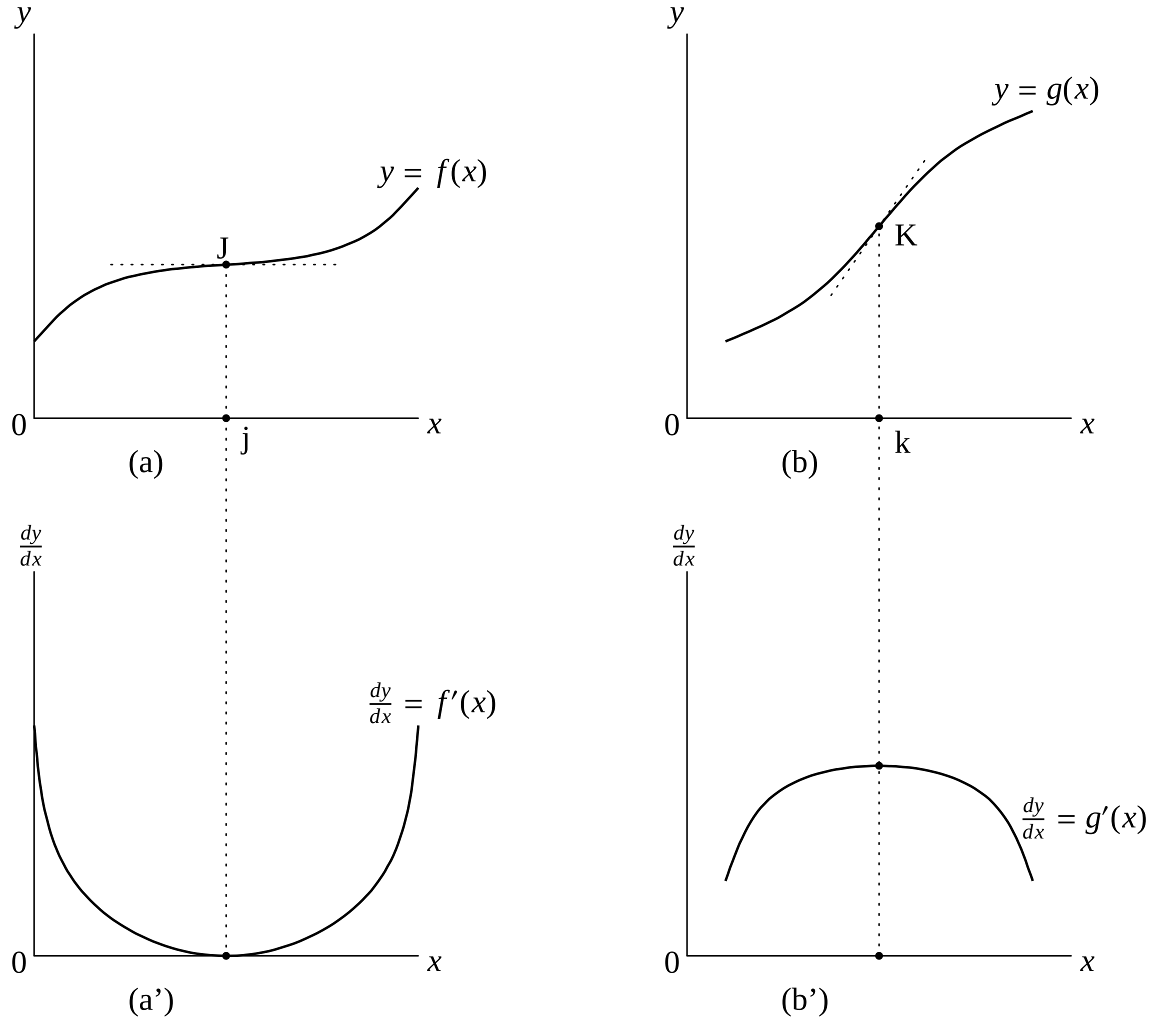


Figure 9.3

Example 1 Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8. \tag{3}$$

The derivative function is

$$f'(x) = 3x^2 - 24x + 36. \tag{4}$$

To get the critical values, we set the derivative function equal to zero.

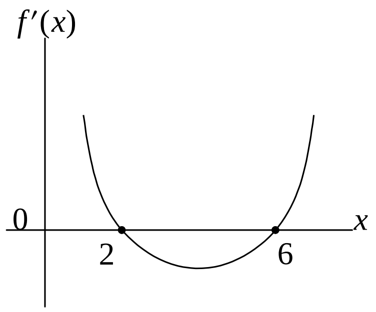
$$3x^2 - 24x + 36 = 0, \quad \text{or} \quad 3(x-2)(x-6) = 0. \tag{5}$$

Then we obtain the following pair of roots:

$$\bar{x}_1 = 2, \quad \text{and} \quad \bar{x}_2 = 6. \tag{6}$$

Since $f'(2) = f'(6) = 0$, these two values of x are the critical values.

It is easy to verify that $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$ in the immediate neighborhood of $x = 2$(See the above graph). Thus, the corresponding value of the function $f(2) = 40$ is established as a relative maximum. Similarly, since $f'(x) < 0$ for $x < 6$ and $f'(x) > 0$ for $x > 6$ in the immediate neighborhood of $x = 6$, the value must be a relative minimum.



9.3 Second and Higher Derivatives

• Derivative of a Derivative

Since the first derivative $f'(x)$ is itself a function of x, it should be differentiable with respect to x, provided that it is continuous and smooth. The second derivative is denoted by

$f''(x)$ where the double prime indicates that $f(x)$ has been differentiated with respect to x twice. The second derivative is again a function of x.

or

$\frac{d^2y}{dx^2}$ where the notation stems from the consideration that the second derivative means $\frac{d}{dx}\left(\frac{dy}{dx}\right)$.

• Interpretation of the Second Derivative

The first derivative function $f'(x)$ measures the rate of change of the function f. The second derivative function $f''(x)$ measures the rate of change of the first derivative f'.

To sum up: with a given infinitesimal increase in the independent variable x from a point $x = x_0$,

$$\left.\begin{array}{ll} f'(x_0) & > 0 \\ f'(x_0) & < 0 \end{array}\right\} \text{means that the } value\ of\ the\ function \text{ tends to} \left\{\begin{array}{l} \text{increase} \\ \text{decrease} \end{array}\right.$$

and

$$\left.\begin{array}{ll} f''(x_0) & > 0 \\ f''(x_0) & < 0 \end{array}\right\} \text{means that the } slope\ of\ the\ curve \text{ tends to} \left\{\begin{array}{l} \text{increase} \\ \text{decrease.} \end{array}\right.$$

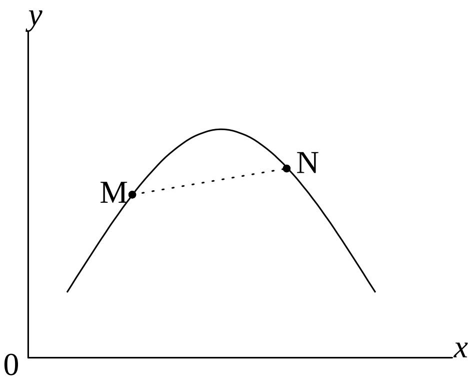
From the above, we know that a positive first derivative coupled with a positive second derivative at $x = x_0$ implies that the slope of the curve at that point is positive and increasing.

• Concavity and Convexity

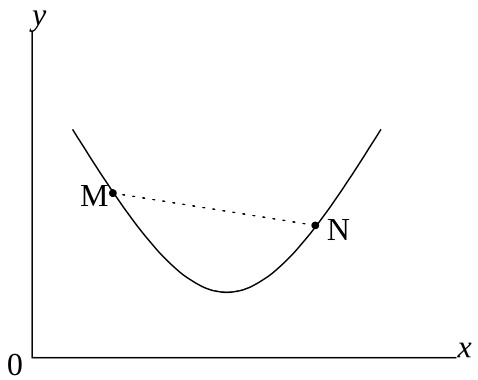
Viewing the two graphs in Figure 9.4, from the standpoint of the horizontal axis, we find the one in diagram (a) to be concave throughout, whereas the one in diagram (b) is convex throughout. Concavity and convexity are descriptions of how the curve to bends. The second derivative of a function informs us about the curvature of its graph, just as the first derivative tells us about its slope.

We pick up any pair of points M and N on its curve and join them by as straight line.

	the line segment MN	the second derivative $f''(x)$
Strictly concave function	lies entirely below the curve	negative for all x
Strictly convex function	lies entirely above the curve	positive for all x



(a) Concave Function



(b) Convex Function

Figure 9.4

9.4 Second-Derivative Test

• Second-Derivative Test

Second-derivative test for relative extremum If the first derivative of a function f at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the second-derivative value at $x = x_0$ is $f''(x) < 0$.

b. A relative *minimum* if the second-derivative value at $x = x_0$ is $f''(x) > 0$.

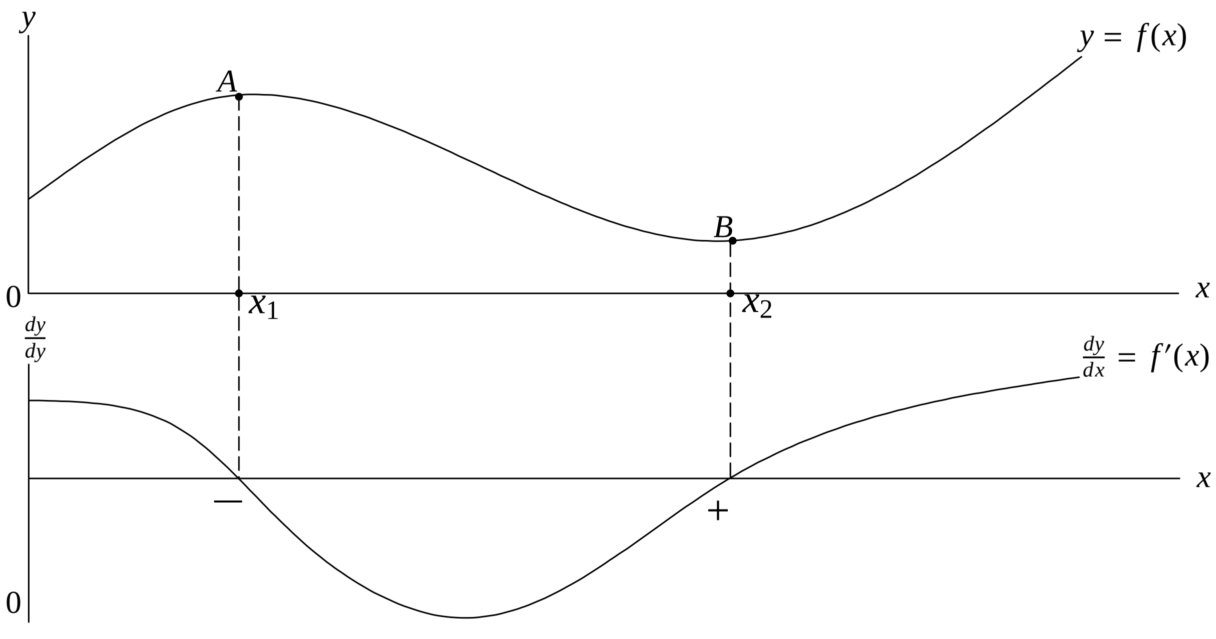


Figure 9.5

In figure 9.5, there exists the two stationary points A and B. The point A is a relative maximum. We can easily verify that the derivative $f'(x)$ has a negative slope, thus, the second-derivative $f''(x)$ has a negative sign, at $x = x_1$. Thus, the corresponding value of the function $f(x_1)$ is established as a relative maximum. The point B is a relative minimum. At $x = x_2$, the second-derivative $f''(x)$ has a positive sign.

• Necessary versus Sufficient Conditions

As was the case with the first-derivative test, the zero-slope condition $f'(x) = 0$ plays the role of a *necessary* condition in the second-derivative test. Since this condition is based on the first-order derivative, it is often referred to as the *first-order condition*. Once we find the first-order condition, the negative (positive) sign of $f''(x)$ is *sufficient* to establish the stationary value in question as a relative maximum (minimum). These sufficient conditions are often referred to as *second-order conditions*.

Example 2 Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8. \tag{7}$$

The derivative function is

$$f'(x) = 3x^2 - 24x + 36. \tag{8}$$

To get the critical values, we set the derivative function equal to zero.

$$3x^2 - 24x + 36 = 0, \quad \text{or} \quad 3(x-2)(x-6) = 0. \tag{9}$$

Then we obtain the following pair of roots:

$$\bar{x}_1 = 2, \quad \text{and} \quad \bar{x}_2 = 6. \tag{10}$$

The second-derivative function is

$$f''(x) = 6x - 24. \tag{11}$$

At $x = 2$, the second-derivative value, $f''(2) = -12$, is negative. the corresponding value of the function $f(2) = 40$ is established as a relative maximum. Similarly, since the second-derivative value, $f''(6) = 12$, is positive at $x = 6$, the value must be a relative minimum.

9.5 Maclaurin and Taylor Series

In this section, we discuss the so-called expansion of a function $y = f(x)$ into what are known as a *Maclaurin series* (expansion around the point $x = 0$) and a *Taylor series* (expansion around any point $x = x_0$).

• Maclaurin Series of a Polynomial Function

Consider the expansion of a polynomial function of the nth degree,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n, \tag{12}$$

into an equivalent nth-degree polynomial where the coefficients are expressed instead in terms of the derivative values $f'(0)$, $f''(0)$, etc.

We get the derivatives as follows

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1}, \\
f''(x) &= 2a_2 + 3\times 2a_3x + 4\times 3a_4x^2 + \cdots + n(n-1)a_nx^{n-2}, \\
f'''(x) &= 3\times 2a_3 + 4\times 3\times 2a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3}, \\
f^{(4)}(x) &= 4\times 3\times 2a_4 + 5\times 4\times 3\times 2a_5x + \cdots + n(n-1)(n-2)(n-3)a_nx^{n-4}, \\
&\vdots \\
f^{(n)}(x) &= n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n.
\end{aligned}
$$

We evaluate these derivatives at $x = 0$.

$$
\begin{aligned}
&f'(0) = a_1, \quad f''(0) = 2a_2, \quad f'''(0) = 3\times 2a_3, \quad f^{(4)}(0) = 4\times 3\times 2a_4, \\
&\cdots \quad f^{(n)}(0) = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)a_n. \qquad (13)
\end{aligned}
$$

The result in (13) can be written as

$$
a_1 = \frac{f'(0)}{1!}, \; a_2 = \frac{f''(0)}{2!}, \; a_3 = \frac{f'''(0)}{3!}, \; a_4 = \frac{f^{(4)}(0)}{4!}, \; \cdots \; a_n = \frac{f^{(n)}(0)}{n!}. \qquad (14)
$$

Substituting these into (12) and utilizing $f(0) = a_0$, we can express the given function $y = f(x)$ as a new same-degree polynomial in which the coefficients are expressed in terms of derivatives evaluated at $x = 0$.

$$
f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \; \cdots \; \frac{f^{(n)}(0)}{n!}x^n. \qquad (15)
$$

Example 1
Find the Maclaurin series for the function

$$
f(x) = 2 + 4x + 3x^2. \qquad (16)
$$

The derivatives are

$$
f'(x) = 4 + 6x, \quad \text{and} \quad f''(x) = 6. \qquad (17)
$$

Hence,

$$
f'(0) = 4, \quad \text{and} \quad f''(0) = 6. \qquad (18)
$$

The Maclaurin series is

$$
\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 \qquad (19) \\
&= 2 + 4x + 3x^2. \qquad (20)
\end{aligned}
$$

• Taylor Series of a Polynomial Function

More generally, the polynomial function in (12) can be expanded around any point x_0, not necessarily zero. We interpret any given point x as a deviation from x_0. Let $x = x_0 + \delta$, where δ represents the deviation from x_0. The given function (16) and its derivatives become

$$
\begin{aligned}
f(x) &= 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2 \\
f'(x) &= 4 + 6(x_0 + \delta), \quad \text{and} \quad f''(x) = 6. \qquad (21)
\end{aligned}
$$

Consider the following function of δ.

$$g(\delta) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2 [\equiv f(x)] \tag{22}$$

The derivatives are

$$\begin{aligned} g'(\delta) &= 4 + 6(x_0 + \delta)[\equiv f'(x)] \\ g''(\delta) &= 6[\equiv f''(x)] \end{aligned}$$

The Maclaurin series of $g(\delta)$ is

$$g(\delta) = \frac{g(0)}{0!} + \frac{g'(0)}{1!}\delta + \frac{g''(0)}{2!}\delta^2. \tag{23}$$

Since $x = x_0 + \delta$, the fact that $\delta = 0$ implies $x = x_0$. So, we have

$$g(0) = f(x_0), \quad g'(0) = f'(x_0), \quad g''(0) = f''(x_0). \tag{24}$$

Substituting these into (23), we find the expansion of $f(x)$ around the point x_0.

$$f(x)[= g(\delta)] = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2. \tag{25}$$

For the specific function (16), we have

$$f(x_0) = 2 + 4x_0 + 3x_0{}^2, \quad f'(x_0) = 4 + 6x_0, \quad f''(x_0) = 6. \tag{26}$$

The Taylor polynomial in (25) becomes

$$f(x) = (2 + 4x_0 + 3x_0{}^2) + (4 + 6x_0)(x - x_0) + \frac{6}{2}(x - x_0)^2 \tag{27}$$

$$= 2 + 4x + 3x^2. \tag{28}$$

The expansion formula in (25) can be generalized to apply to the nth-degree polynomial of (12). The generalized formula is

$$\begin{aligned} f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots \\ + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned} \tag{29}$$

• Expansion of an Arbitrary Function

In general, it is also possible to express any arbitrary function $\phi(x)$ in a polynomial function similar to (29), provided $\phi(x)$ has finite, continuous derivatives up to the desired order at the expansion point x_0.

Taylor's Theorem

Given an arbitrary function $\phi(x)$, if we know the value of the function at $x = x_0$ and the values of its derivatives at x_0, then this function can be expanded around the point x_0 as follows (n = a fixed positive integer arbitrary chosen):

$$\begin{aligned} f(x) &= \left[\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n \\ &= P_n + R_n \end{aligned} \tag{30}$$

where P_n represents the nth-degree polynomial, and R_n denotes a *remainder*.

The presence of R_n is what distinguishes (30) from (29). The appearance of R_n is due to the fact that we are dealing with an arbitrary function ϕ which cannot always be transformed *exactly* into, but can only be approximated by, the polynomial form shown in (29). Therefore, a remainder term is included as a supplement to the P_n part, to represent the discrepancy between $\phi(x)$ and P_n. Thus, P_n constitutes a *polynomial approximation* to $\phi(x)$, with the term R_n as a measure of the error of approximation.

If we choose $n = 1$, for example, we have

$$\phi(x) = [\phi(x_0) + \phi'(x_0)(x - x_0)] + R_1 = P_1 + R_1 \tag{31}$$

where P_1 consists of 2 terms and constitutes a *linear* approximation to $\phi(x)$. If we choose $n = 2$, a second-power term will appear, so that

$$\phi(x) = [\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2] + R_2 = P_2 + R_2 \tag{32}$$

where P_2, consisting of 3 terms, is a *quadratic* approximation to $\phi(x)$. As n increases, the polynomial approximation P_n to $\phi(x)$ increases its approximation accuracy. Thus, the quadratic approximation ($n = 2$) is a better approximation to $\phi(x)$ than the linear approximation ($n = 1$), as shown in Figure 9.6.

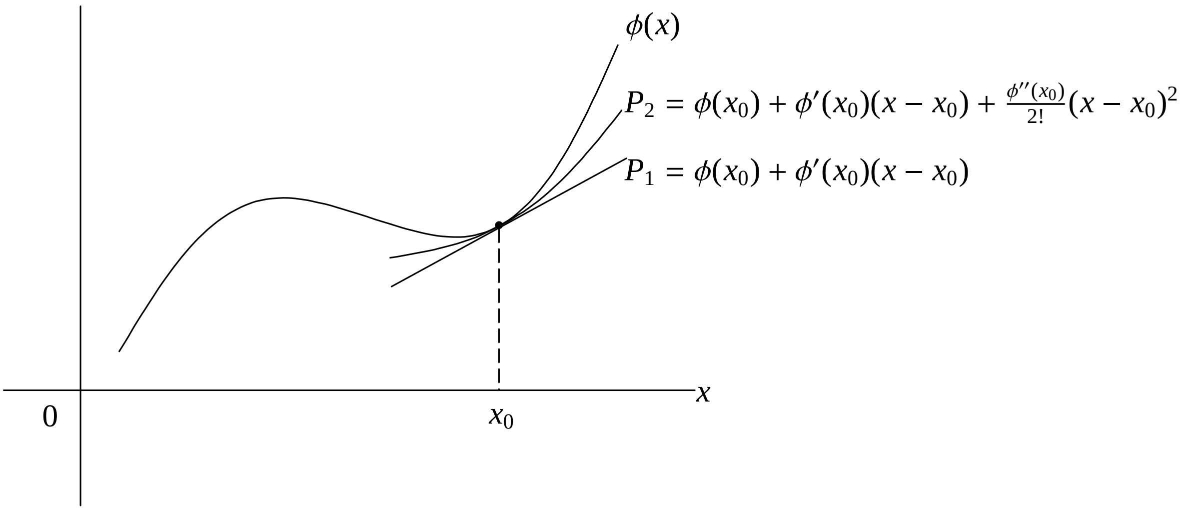


Figure 9.6

Example 2

Expand the nonpolynomial function

$$\phi(x) = \frac{1}{1 + x} \tag{33}$$

around the point $x_0 = 1$, with $n = 4$. The derivatives are

$$\phi'(x) = -(1 + x)^{-2}, \quad \phi''(x) = 2(1 + x)^{-3}, \quad \phi'''(x) = -6(1 + x)^{-4}, \quad \text{and} \quad \phi^{(4)}(x) = 24(1 + x)^{-5}.$$

So, we have

$$\phi(1) = \frac{1}{2}, \quad \phi'(1) = -\frac{1}{4}, \quad \phi''(1) = \frac{1}{4}, \quad \phi'''(1) = -\frac{3}{8}, \quad \text{and} \quad \phi^{(4)}(1) = \frac{3}{4}.$$

Taylor series with remainder is given by

$$\begin{aligned}\phi(x) &= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + \frac{1}{32}(x-1)^4 + R_4 \\ &= \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4 + R_4. \end{aligned} \tag{34}$$

• Lagrange Form of the Remainder

According to the *Lagrange form of the remainder*, we can express R_n as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1} \tag{35}$$

where p is some number between x and x_0.

If we find that

$$R_n \to 0 \text{ as } n \to \infty \quad \text{so that} \quad P_n \to \phi(x) \text{ as } n \to \infty \tag{36}$$

then the Taylor series converges to $\phi(x)$ at the point of approximation. The Taylor series can be written as a *convergent infinite series* as follows:

$$\phi(x) = \frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots \tag{37}$$

It will be possible to make P_n as an accurate approximation to $\phi(x)$ by choosing a large enough value for n.