4. Linear Models and Matrix Algebra

Matrix algebra can enable us to do many things. In the first place, it provides a compact way of writing an equation system, even an extremely large one. Second, it leads to a way of testing existence of a solution by evaluation of a *determinant*. Third, it gives a method of finding that solution. Since equation systems are encountered not only in static analysis but also in comparative static and dynamic analyses and in optimization problems, you will find ample application of matrix algebra in almost every chapter that is to follow. This is why it is desirable to introduce matrix algebra early.

4.1 Matrices and Vectors

- Matrices as Arrays

In general, a system of m linear equations in n variables $(x_1, x_2, \dots, x_n)$ can arranged into a following format:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m. \end{aligned} \tag{1}$$

In the equation system (1), there are essentially three types of ingredients. The first is the set of coefficients a_{ij}; the second is the set of variables $x_1, x_2, \dots, x_n$; and the last is the set of constant terms $d_1, \dots, d_m$. If we arrange the three sets as three rectangular arrays and label them, respectively, as A (the *coefficient matrix*), x, and d (without subscripts), then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}. \tag{2}$$

Each of the three array in (2) constitutes a *matrix*.

- Vectors as Special Matrices

The number of rows and the number of columns in a matrix together define the *dimension* of the matrix. Since matrix A in (2) contains m rows and n columns, it is said to be of dimension $m \times n$ (read "m by n"). In the special case where $m = n$, the matrix is called a *square matrix*.

4.2 Matrix Operations

- Addition and Subtraction of Matrix

Example 1

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}.$$

Example 2

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}.$$

Example 3

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 19-6 & 3-8 \\ 2-1 & 0-3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}.$$

- Scalar Multiplication

Example 4

$$7\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & 35 \end{bmatrix}.$$

Example 5

$$-1\begin{bmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -d_1 \\ -a_{21} & -a_{22} & -d_2 \end{bmatrix}.$$

- Multiplication of Matrices

Suppose that, given two matrices A and B, we want to find the product AB. In general, if A is of dimension $m \times n$ and B is of dimension $p \times q$, the matrix product AB will be defined if and only if $n = p$. If defined, moreover, the product matrix AB will have the dimension $m \times q$.[*1] For instance, if

$$A = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}, \tag{3}$$

the product AB then is defined and will be 1×3, since A has *two column* and B has *two rows*–precisely the same number.[*2]

It remains to define the exact procedure of multiplication. For this purpose, let us take the matrices A and B in (3) for illustration. Since the product AB is defined and is expected to be of dimension 1×3, we may write in general that

$$AB = C = [c_{11} \;\; c_{12} \;\; c_{13}].$$

Each element in the row matrix C, denoted by c_{ij}, is defined as a sum of products, to be computed from the elements in the ith *row* of the lead matrix A, and those in the jth *column* of the lag matrix B. To find c_{11}, for instance, we should take the *first row* in A (since $i = 1$) and the *first column* in B (since $j = 1$)—as shown in the top panel of Figure 4.1—and then pair the elements together sequentially, multiply out each pair, and take the sum of the resulting products, to get

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}. \tag{4}$$

[*1] That is, the conformability condition for multiplication is that the *column* dimension of A (the "lead" matrix in the expression AB) must be equal to the row dimension of B (the "lag" matrix).

[*2] On the other hand, the reverse product AB is not defined in this case, because B (now the lead matrix) has *three* columns while A has only *one* row; hence conformability condition is violated.

Similarly, for c_{12}, we take the *first row* in A (since $i = 1$) and the *second column* in B (since $j = 2$), and calculate the indicated sum of products—in accordance with the lower panel of Figure 4.1— as follows:

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}. \tag{5}$$

By the same token, we should also have

$$c_{13} = a_{11}b_{13} + a_{12}b_{23}. \tag{6}$$

It is the particular pairing requirement in this process which necessitates the matching of the column dimension of the lead matrix and the row dimension of the lag matrix before multiplication can be performed.

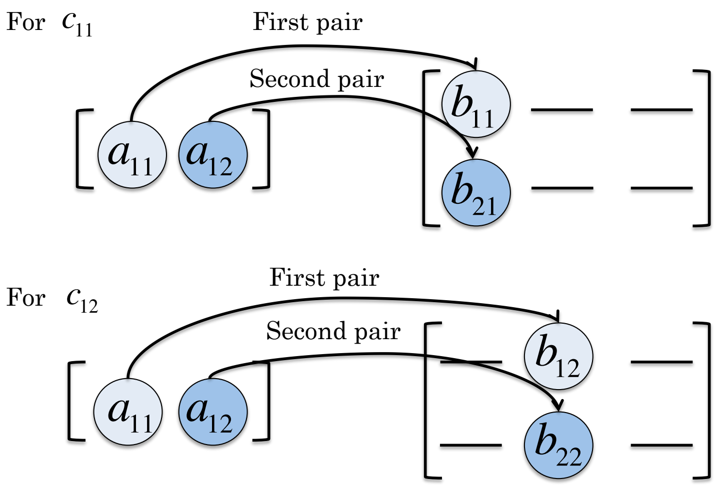

Figure 4.1

Example 6

Given

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 9 \end{bmatrix},$$

find AB. The product AB is indeed defined because A has two columns and B has two rows. Their product matrix should be 3×1, a column vector;

$$AB = \begin{bmatrix} 1 \times 5 + 3 \times 9 \\ 2 \times 5 + 8 \times 9 \\ 4 \times 5 + 0 \times 9 \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}.$$

Example 7

Given

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{3}{10} \\ -1 & \frac{1}{5} & \frac{7}{10} \\ 0 & \frac{2}{5} & -\frac{1}{10} \end{bmatrix},$$

find AB. The same rule of multiplication now yields a very special product matrix:

$$AB = \begin{bmatrix} 0+1+0 & -\frac{3}{5}-\frac{1}{5}+\frac{4}{5} & \frac{9}{10}-\frac{7}{10}-\frac{2}{10} \\ 0+0+0 & -\frac{1}{5}+0+\frac{6}{5} & \frac{3}{10}+0-\frac{3}{10} \\ 0+0+0 & -\frac{4}{5}+0+\frac{4}{5} & \frac{12}{10}+0-\frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This last matrix—a square matrix with 1s in its *principal* diagonal (the diagonal running from northwest to southeast) and 0s everyewhere else—exemplifies the important type of matrix known as the *identity matrix*.

4.3 Commutative Associative, and Distributive Laws

In ordinary scalar algebra, the additive and multiplicative operations obey the commutative, associative, and distributive laws as follow:

Commutative law of addition:	$a + b = b + a$
Commutative law of multiplication:	$ab = ba$
Associative law of addition:	$(a + b) + c = a + (b + c)$
Associative law of multiplication:	$(ab)c = a(bc)$
Distributive law:	$a(b + c) = ab + ac$

- Matrix Addition

Matrix addition is commutative as well as associative. Their lows can be stated as follow:

Commutative law of addition:	$A + B = B + A$
Associative law of addition:	$(A + B) + C = A + (B + C)$

Example 8

Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$, we find that

$$A + B = B + A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}.$$

Example 9

Given $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we find that

$$(v_1 + v_2) - v_3 = \begin{bmatrix} 12 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

which is equal to

$$v_1 + (v_2 - v_3) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

- Matrix Multiplication

  Matrix multiplication is *not* commutative, that is,

$$AB \neq BA.$$

Example 10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$; then

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}.$$

Although it is not in general commutative, matrix multiplication is associative and distributive:

Associative law of multiplication:	$(AB)C = A(BC)$
Distributive law:	$A(B + C) = AB + AC$
	$(B + C)A = BA + CA$

Example 11

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix}$; then

$$(AB)C = \begin{bmatrix} 45 & 45 \\ 30 & 30 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 180 & 585 \\ 120 & 390 \end{bmatrix}$$
$$A(BC) = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 59 \\ 28 & 68 \end{bmatrix} = \begin{bmatrix} 180 & 585 \\ 120 & 390 \end{bmatrix}.$$

Moreover,

$$A(B + C) = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 11 \\ 9 & 13 \end{bmatrix} = \begin{bmatrix} 60 & 111 \\ 40 & 74 \end{bmatrix}$$
$$AB + AC = \begin{bmatrix} 45 & 45 \\ 30 & 30 \end{bmatrix} + \begin{bmatrix} 15 & 66 \\ 10 & 44 \end{bmatrix} = \begin{bmatrix} 60 & 111 \\ 40 & 74 \end{bmatrix}.$$

4.4 Identity Matrices and Null Matrices

- Identity Matrices

  Identity matrix is defined as a *square* matrix with 1s in its principal diagonal and 0s everywhere else. It is denoted by the symbol I, or I_n, in which the subscript n serves to indicate its row (as well as column) dimension. Thus,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For any matrix A, we have

$$IA = AI = A \tag{7}$$

Example 12

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$, then

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A.$$

- Null Matrices

A *null matrix* is simply a matrix whose elements are all zero. Unlike I, the zero matrix is not restricted to being square. Thus it is possible to write

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and so forth.

- Idiosyncrasies of Matrix Algebra

Despite the apparent similarities between matrix algebra and scalar algebra, the case or matrices does display certain idiosyncrasies that serve to warn us not to ”borrow” from scalar algebra too unquestioningly. We have already seen that, in general, $AB \neq BA$ in matrix algebra. Let us look at two more such idiosyncrasies of matrix algebra.

For one thing, in the case of scalars, the equation $ab = 0$ always implies that either a or b is zero, but this is not so in matrix multiplication. Thus, we have

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

although neither A nor B is itself a zero matrix.

As another illustration, for scalars, the equation $cd = ce$ (with $c \neq 0$) implies that $d = e$. The same does not hold for matrices. Thus, given

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$$

we find that

$$CD = CE = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix}$$

even though $D \neq E$.

These strange results actually pertain only to the special class of matrices known as *singular matrices*, of which the matrices A, B, and C are examples.[*3] Nevertheless, such examples do reveal the pitfalls of unwarranted extension of algebraic theorems to matrix operations.

[*3] Roughly, these matrices contain a row which is a multiple of another row. See also 4.5.

4.5 Transposes and Inverses

- Transposes

When the rows and columns of a matrix A are interchanged—so that first row becomes the first column, and vice versa—we obtain the *transpose* of A, which is denoted by A' or A^T.

Example 13

Given $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$, we can interchange the rows and columns and write

$$A' = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}.$$

By definition, if a matrix A is $m \times n$, then its transpose A' must be $n \times m$. An $n \times n$ square matrix, however, possesses a transpose with the same dimension.

Example 14

If $C = \begin{bmatrix} 9 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$, then

$$C' = \begin{bmatrix} 9 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}.$$

Here, the dimension of each transpose is identical with that of the original matrix. In D', we also note the remarkable result that D' inherits not only the dimension of D but also the original array of element. The fact that $D' = D$ is the result of the symmetry of the elements with reference to the principal diagonal. The matrix D exemplifies the special cases of square matrices known as *symmetric matrix*.

- Properties of Transposes

$$(A')' = A \tag{8}$$

$$(A + B)' = A' + B' \tag{9}$$

$$(AB)' = B'A' \tag{10}$$

Example 15

If $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$, then

$$(A + B)' = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}' = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

and

$$A' + B' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}.$$

Example 16

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$, we have

$$(AB)' = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}' = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

and

$$B'A' = \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}.$$

- Inverses and Their Properties

The inverse of matrix A, denoted by A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition

$$AA^{-1} = A^{-1}A = I. \tag{11}$$

The following points are worth noting:

1. Not every square matrix has an inverse—squareness is a *necessary* condition, but *not* a *sufficient* condition, for the existence of an inverse.[*4] If a square matrix A has an inverse, A is said to be *nonsingular*; if A possesses no inverse, it is called *singular* matrix.
2. If A^{-1} does exist, A and A^{-1} are inverse of each other.
3. If A is $n \times n$, then A^{-1} must also be $n \times n$.
4. If an inverse exists, then it is unique.
5. The two parts of condition (11) actually imply each other, so that satisfying either equation is sufficient to establish the inverse relationship between A and A^{-1}.

Example 17

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \frac{1}{6}\begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$; then, since the scalar multiplier $\frac{1}{6}$ in B can be moved to the rear (commutative law), we can write

$$AB = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} \frac{1}{6} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This establishes B as the inverse of A, and vice versa. The reverse multiplication, as expected, also yields the same identity matrix:

$$BA = \frac{1}{6} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Three properties of inverse of and inverse If A and B are nonsingular matrices with dimension $n \times n$, then

$$(A^{-1})^{-1} = A \tag{12}$$

$$(AB)^{-1} = B^{-1}A^{-1} \tag{13}$$

$$(A')^{-1} = (A^{-1})' \tag{14}$$

[*4] With regard to definitions of necessary condition and sufficient condition, see 5.1.

5. Linear Models and Matrix Algebra (Continued)

5.1 Conditions for Nonsingularity of a Matrix

A given coefficient matrix A can have an inverse (i.e., can be "nonsingular") only if it is square. As was pointed out earlier, however, the squareness condition is necessary but not sufficient for the existence of the inverse A^{-1}. A matrix can be square, but singular nonetheless.

- Necessary versus Sufficient Conditions

A necessary condition is in the nature of a prerequisite: Suppose that a statement p is true *only if* another statement q is true; then q constitutes a necessary condition of p. Symbolically, we express this as follows:

$$p \Rightarrow q \tag{15}$$

which is read as "*p only if q*", or alternatively, "*if p, then q*". It is also logically correct to interpret (15) to mean "*p implies q*". It may happen, of course, that we also have $p \Rightarrow w$ at the same time. Then both q and w are necessary conditions for p.

A different types of situation is one in which a statement p is true if q is true, but p can also be true when q is not true. In this case, q is said to be a sufficient condition for p. The truth of q suffices to establish the truth of p, but it is not a necessary condition for p. This case is expressed symbolically by

$$p \Leftarrow q \tag{16}$$

which is read "*p if q*", or alternatively, "*if q, then p*". It can also be interpreted to mean "*qimplies p*. In a third possible situation, q is *both* necessary and sufficient for p. In such an event, we write

$$p \Leftrightarrow q \tag{17}$$

which is read: "*p if and only if q*" (also written as "p iff q"). Note that (17) states not only that p implies q but also that q implies p.

5.2 Test of Nonsingularity by Use of Determinant

- Determinants and Nonsingularity

The determinant of a square matrix A, denoted by $|A|$, is a uniquely defined scalar (number) associated with that matrix. Determinant are defined only for *square* matrices. The smallest possible matrix is, of course, the 1×1 matrix $A = [a_{11}]$. By definition, its determinant is equal to the single element a_{11} itself: $|A| = |a_{11}| = a_{11}$.[*5]

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, its determinant is defined to be the sum of two terms as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \qquad [= \text{a scalar}] \tag{18}$$

which is obtained by multiplying the two elements in the principal diagonal of A and the subtracting the product of the two remaining elements. In view of the dimension of matrix A,

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[*5] The symbol $|a_{11}|$ here must not be confused with the look-alike symbol for absolute value of a number. the determinant symbol preserves the sign of the element, so while $|5| = 5$ (positive number), we have $|-5| = -5$(negative number).

the determinant $|A|$ given (18) is called a *second-order determinant*.

Example 18

Given $A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$, their determinants are

$$|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 10 \times 5 - 8 \times 4 = 18$$

$$|B| = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = 3 \times (-1) - 0 \times 5 = -3.$$

- Evaluating a Third-Order Determinant

  A determinant of order 3 is associated with a 3×3 matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

its determinant has the value

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \quad (19)$$

In the determinant shown in Figure 5.1, each element in the top row has been linked with two other elements via two *solid* arrows as follows: $a_{11} \to a_{22} \to a_{33}$, $a_{21} \to a_{23} \to a_{31}$, and $a_{13} \to a_{32} \to a_{21}$. Each triplet of elements so linked can be multiplied out, and their product taken as one of the six product terms in (19). The solid-arrow product terms are to be prefixed with plus signs.

On the other hand, each top-row element has also been connected with two other elements via two *broken* arrows as follows: $a_{11} \to a_{32} \to a_{23}$, $a_{12} \to a_{21} \to a_{33}$, and $a_{13} \to a_{22} \to a_{31}$. Each triplet of elements so connected can also be multiplied out, and their product taken as one of the six terms in (19). Such products are prefixed by minus signs. The sum of all the six products will then be the values of the determinant.

Example 19

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \times 5 \times 9 + 1 \times 6 \times 7 \times +3 \times 8 \times 4 - 2 \times 8 \times 6 - 1 \times 4 \times 9 - 3 \times 5 \times 7 = -9.$$

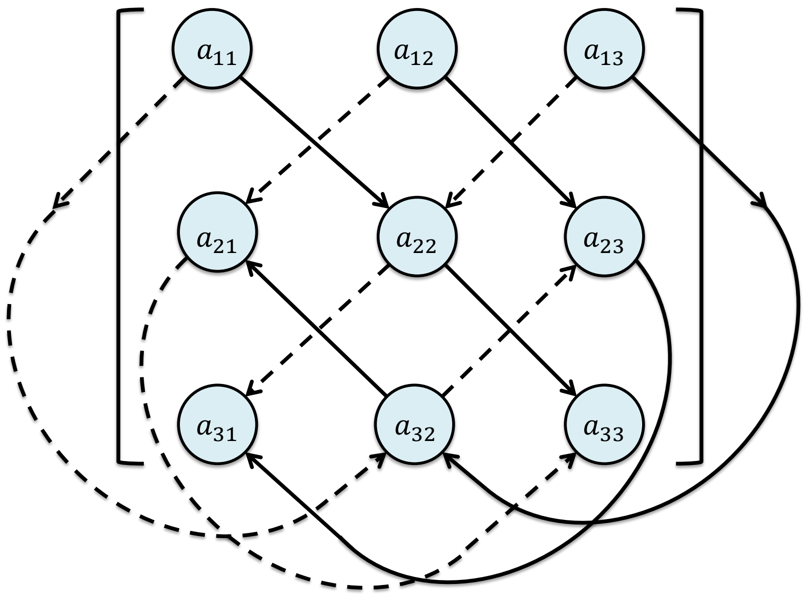

Figure 5.1

- Evaluating an *n*th-Order Determinant by Laplace Expansion

Let us first explain the *Laplace-expansion* process for a third-order determinant. Returning to the first line of (19), we see that the value of $|A|$ can also be regarded as a sum of *three* terms, each of which is a product of a first-low element and a particular *second*-order determinant. This latter process of evaluating $|A|$ —by means of certain lower-order determinants— illustrates the Laplace expansion of the determinant.

The three second-order determinants in (19) are not arbitrarily determined, but are specified by means of a definite rule. The first one, $\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, is a *subdeterminant* of $|A|$ obtained by deleting the *first* row and *first* column of $|A|$. This is called the *minor* of the element a_{11} (the element at the intersection of the deleted row and column) and is denoted by $|M_{11}|$ (See also Figure 5.2). In general, the symbol $|M_{ij}|$ can be used to represent the minor obtained by deleting the ith row the jth column of a given determinant. As the reader can verify, the other two second-order determinant in (19) are, respectively, the minors $|M_{12}|$ and $|M_{13}|$; that is,

$$M_{11} \equiv \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad M_{12} \equiv \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad M_{13} \equiv \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}$$

A concept closely related to the minor is that of the *cofactor*. A cofactor, denoted by $|C_{ij}|$, is a minor with a prescribed algebraic sign attached to it. The rule of sign is as follows. If the sum of the two subscripts i and j in the minor $|M_{ij}|$ is even, then the cofactor takes the same sign as the minor; that is, $|C_{ij}| \equiv |M_{ij}|$. If it is odd, then the cofactor takes the opposite sign to the minor; that is, $|C_{ij}| \equiv -|M_{ij}|$. In short, we have

$$|C_{ij}| \equiv (-1)^{i+j}|M_{ij}|$$

where it is obvious that the expression $(-1)^{i+j}$ can be positive if and only if $(i + j)$ is even. The fact that a cofactor has a specific sign is of extreme importance and should always be borne in mind.

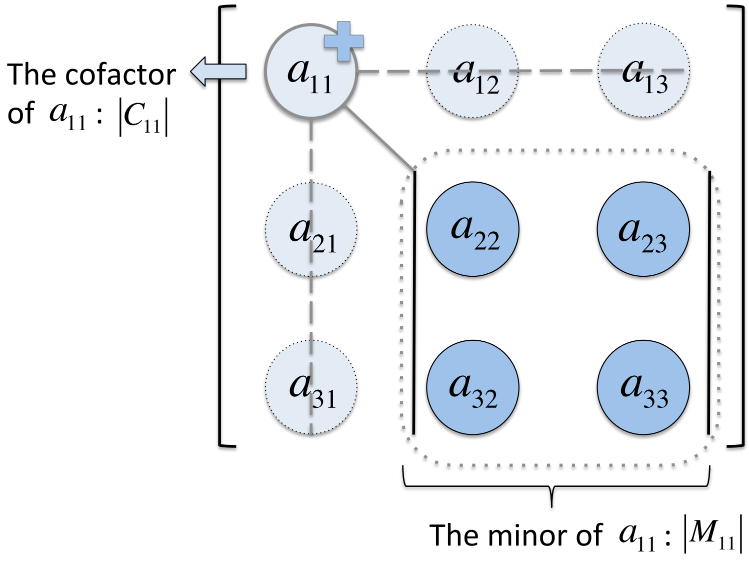

Figure 5.2 : $|M_{11}|$ and C_{11}.

Example 20

In the determinant $\begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$, the minor of the element 8 is

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = -6$$

but the cofactor of the same element is

$$|C_{12}| = -|M_{12}| = 6$$

because $i + j = 1 + 2 = 3$ is odd. Similarly, the cofactor of the element 4 is

$$|C_{23}| = -|M_{23}| = -\begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} = 6.$$

Using these new concepts, we can express a third-order determinant as

$$\begin{aligned} |A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\ &= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{1j}|C_{1j}| \end{aligned} \tag{20}$$

i.e., as a sum of three terms, each of which is the product of a first-row element and its corresponding cofactor. Note that difference in the signs of the $a_{12}|M_{12}|$ and $a_{12}|C_{12}|$ terms in (20). This is because 1 + 2 gives an odd number.

The Laplace expansion of a *third*-order determinant serves to reduce the evaluation problem to one of evaluating only certain *second*-order determinants. A similar reduction is achieved in the Laplace expansion of higher-order determinants. In a fourth-order determinant $|B|$, for instance, the top row will contain four elements, $b_{11} \ldots b_{14}$; thus, in the spirit of (20), we may write

$$|B| = \sum_{j=1}^{4} b_{1j}|C_{1j}|$$

where the cofactors $|C_{1j}|$ are of order 3. Each third-order cofactor can then be evaluated as in (19). In general, the Laplace expansion of an nth-order determinant will reduce the problem to one of evaluating n cofactors, each of which is of the $(n-1)$st order, and the repeated application of the process will methodically lead to lower and lower orders of determinants, eventually culminating in the basic second-order determinants as defined in (18). Then the value of the original determinant can be easily calculated.

Although the process of Laplace expansion has been couched in terms of the cofactors of the first-row elements, it is also feasible to expand a determinant by the cofactor of any row or, for that matter, of any column. For instance, if the first column of a third-order determinant $|A|$ consists of the elements a_{11}, a_{21}, and a_{31}, expansion by the cofactors of these elements will also yield the value of $|A|$:

$$|A| = a_{11}|C_{11}| + a_{21}|C_{21}| + a_{31}|C_{31}| = \sum_{i=1}^{3} a_{i1}|C_{i1}|.$$

Example 21

Given $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$, expansion by the first *row* products the result

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 6\begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = 0 + 0 - 27 = -27$$

But expansion by the first *column* yields the identical answer:

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 2\begin{vmatrix} 6 & 1 \\ -3 & 0 \end{vmatrix} + 7\begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 0 - 6 - 21 = -27.$$

Insofar as numerical calculation is concerned, this fact affords us an opportunity to choose some "easy" row or column for expansion. A row or column with the largest number of 0s or 1s is always preferable for this purpose, because a 0 times its cofactor is simply 0, so that the term will drop out, and a 1 times its cofactor is simply the cofactor itself, so that at least one multiplication step can saved. In Example 18, the easiest way to expand the determinant is by the third column, which consists of elements 1, 0, and 0. We could therefore have evaluated it thus:

$$|A| = 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -6 - 21 = -27$$

To sum up, the value of a determinant $|A|$ of order n can be found by the Laplace expansion of *any row* or *any column* as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } i\text{th row]}$$
$$= \sum_{i=1}^{n} a_{ij}|C_{ij}| \quad \text{[expansion by the } i\text{th column]}$$

5.3 Basic Properties of Determinants

Property 1 The interchange of row and columns does not affect the value of a determinant. In other words, the determinant of a matrix A has the same value as that of its transpose A', that is, $|A| = |A'|$.

Example 22

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Property 2 The interchange of any *two* row (or any *two* columns) will alter the sign, but not the numerical value, of the determinant.

Example 23

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, but the interchange of the two rows yields

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc).$$

Property 3 The multiplication of any *one* row (or *one* column) by scalar k will change the value of the determinant k-fold.

Example 24

By multiplying the top row of the determinant in Example 20 by k, we get

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Property 4 The addition (subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered. The same holds true if we replace the word *row* by *column* in the previous statement.

Example 25

Adding k times the top row of the determinant in Example 20 to its second row, we end up with there original determinant:

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Property 5 If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero. As a special case of this, when two rows (or two columns) are *identical*, the determinant will vanish.

Example 26

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0, \qquad \begin{vmatrix} c & d \\ c & d \end{vmatrix} = cd - cd = 0.$$

- Determinantal Criterion for Nonsigularity

Consider and equation system $Ax = d$:

$$\begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

This system can have a unique solution if and only if the rows in the coefficient matrix A are linearly independent, so that A is nonsingular.[*6] But the second row is five times the first; the rows are indeed *dependent*, and hence no unique solution exists. The detection of this row dependence was by visual inspection, but by virtue of Property 5 we could also have discovered it through the fact that $|A| = 0$.

Although we have tied the non singularity of a matrix principally to the linear independence among *rows*, we have made the claim that, for a *square* matrix A, row independence $\Leftrightarrow$ column independence. We are now equipped to prove that claim:

According to Property 1, we know that $|A| = |A'|$. Since row independence in $A \Leftrightarrow |A| \neq 0$, we may also state that row independence in $A \Leftrightarrow |A'| \neq 0$. But $|A'| \neq 0 \Leftrightarrow$ row independence in the transpose $A' \Leftrightarrow$ column independence in A. Therefore, *row* independence in $A \Leftrightarrow$ *column* independence in A.

Our discussion of the test to non singularity can now be summarized. Given a linear equation system $Ax = d$, Where is an $n \times n$ coefficient matrix,

$$\begin{aligned} |A| \neq 0 &\Leftrightarrow \text{there is row (column) independence in matrix} \\ &\Leftrightarrow A \text{ is nonsingular} \\ &\Leftrightarrow A^{-1} \text{ exists} \\ &\Leftrightarrow \text{a unique solution } x^* = A^{-1}d \text{ exists.} \end{aligned}$$

Example 27

Does the equation system

$$\begin{aligned} 7x_1 - 3x_2 - 3x_3 &= 7 \\ 2x_1 + 4x_2 + x_3 &= 0 \\ -2x_2 - x_3 &= 2 \end{aligned}$$

[*6] A set of vectors $v_1, \ldots, v_n$ is said to be *linearly dependent* if (and only if) any one of them can be expressed as a linear combination of the remaining vectors; otherwise they are *linearly independent*.

possess a unique solution? The determinant $|A|$ is

$$\begin{vmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & -1 \end{vmatrix} = -8 \neq 0$$

Therefore a unique solution does exist.

5.4 Finding the Inverse Matrix

Property 6 The expansion of a determinant by *alien cofactors* (the cofactors of a "wrong" row or column) always yields a value of zero.

Example 28

If we expand the determinant $\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$ by using its *first-row* elements but the cofactors of the *second-row* elements

$$|C_{21}| = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \qquad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \qquad |C_{23}| = -\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

we get $a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}| = -12 + 10 + 2 = 0$.

- Matrix Inversion

Assume that an $n \times n$ nonsingular matrix A is given:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (|A| \neq 0) \tag{21}$$

Since each element of A has a cofactor $|C_{ij}|$, it is possible to form a matrix of cofactors by replacing each element a_{ij} in (21) with its cofactor $|c_{ij}|$. Such a cofactor matrix, denoted by $C = [|C_{ij}|]$, must also be $n \times n$. For our present purpose, however, the transpose of C is of more interest. This transpose C' is commonly referred to as the *adjoint* of A and is symbolized by *adj* A. Written out, the adjoint takes the form

$$C' \equiv adjA \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \tag{22}$$

The matrices A and C' are conformable for multiplication, and AC' is another $n \times n$ matrix in which each element is a sum of products. By utilizing the formula for Laplace expansion as well as Property 6 of determinants, the product AC' may be expressed as follows:

$$AC' = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}|C_{1j}| & \sum_{j=1}^{n} a_{1j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{1j}|C_{nj}| \\ \sum_{j=1}^{n} a_{2j}|C_{1j}| & \sum_{j=1}^{n} a_{2j}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{2j}|C_{nj}| \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{n} a_{nj}|C_{1j}| & \sum_{j=1}^{n} a_{nj}|C_{2j}| & \cdots & \sum_{j=1}^{n} a_{nj}|C_{nj}| \end{bmatrix}$$

$$= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = |A| I_n$$

As the determinant $|A|$ is a nonzero scalar, it is permissible to divide both sides of the equation $AC' = |A|I$ by $|A|$. The result is

$$\frac{AC'}{|A|} = I \quad \text{or} \quad A\frac{C'}{|A|} = I$$

Premultiplying both sides of the last equation by A^{-1}, and using the result that $A^{-1}A = I$, we can get $\frac{C'}{|A|} = A^{-1}$, or

$$A^{-1} = \frac{1}{|A|} adj A \tag{23}$$

Now, we have found a way to invert the matrix A.

The general procedure for finding the inverse of a square matrix A

1. Find $|A|$.
2. Find the cofactors of all the elements of A, and arrange them as a matrix $C = [|C_{ij}|]$.
3. Take the transpose of C to get *adj* A.
4. divide adj A by the determinant $|A|$.

Example 29

Find the inverse of $B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$. Since $|B| = 99 \neq 0$, the inverse B^{-1} also exists. The cofactor matrix is

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

Therefore,

$$\text{adj}B = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

and the desired inverse matrix is

$$B^{-1} = \frac{1}{|B|}\text{adj}B = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}.$$

5.5 Cramer's Rule

- Derivation of the Rule

Given an equation system $Ax = d$, where A is $n \times n$, the solution can be written as

$$x^* = A^{-1}d = \frac{1}{|A|}\,(adjA)\,d$$

provided A is nonsingular. This means that

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \frac{1}{|A|}\begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} d_1|C_{11}| + d_2|C_{21}| + \cdots + d_n|C_{n1}| \\ d_1|C_{12}| + d_2|C_{22}| + \cdots + d_n|C_{n2}| \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ d_1|C_{1n}| + d_2|C_{2n}| + \cdots + d_n|C_{nn}| \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} \sum_{i=1}^{n} d_i|C_{i1}| \\ \sum_{i=1}^{n} d_i|C_{i2}| \\ \vdots \\ \sum_{i=1}^{n} d_i|C_{in}| \end{bmatrix}$$

Equating the corresponding elements on the two sides of the equation, we obtain the solution values

$$x_1^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i1}| \qquad x_2^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i2}| \quad \text{(etc.)} \tag{24}$$

Recall $|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}|$. Form this, we see that the Laplace expansion of a determinant $|A|$ by its first column can expressed in the form $\sum_{j=1}^{n} a_{i1}|C_{i1}|$. If we replace their columns of $|A|$ by the column vector d but keep all the other columns intact, then a new determinant will result, which we can call $|A_1|$—the subscript 1 indicating that the first column has been replaced by d. The expansion of $|A_1|$ by its first column (the d column) will yield the expression $\sum_{i=1}^{n} d_i|C_{i1}|$, because the elements d_i now take the place of the elements a_{i1}. Returning to (24), we see therefore that

$$x^* = \frac{1}{|A|}|A_1|$$

This procedure can now be generalized. To find the solution value of the jth variable x_j^*, we can merely replace the jth column of the determinant$|A|$ by the constant terms $d_1 \cdots d_n$ to get a new determinant $|A_j|$ and then divide $|A_j|$ by the original determinant $|A|$. Thus, the solution of the system $Ax = d$ can be expressed as

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|} \begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix} \tag{25}$$

This result in (25) is the statement of Cramer's rule.

Example 30

Find the solution of the equation system

$$\begin{cases} 5x_1 + 3x_2 = 30 \\ 6x_1 - 2x_2 = 8 \end{cases}$$

The coefficients and the constant terms give the following determinants:

$$|A| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28 \qquad |A_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84 \qquad |A_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

Therefore, we can immediately write

$$x_1^* = \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \quad \text{and} \quad x_2^* = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5.$$