4. Linear Models and Matrix Algebra

4.1 Matrices and Vectors

- Matrices as Arrays

A system of m linear equations in n variables $(x_1, x_2, \ldots, x_n)$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= d_m. \end{aligned} \tag{1}$$

In the equation system (1),

$$\begin{cases} A : \text{the set of coefficients} \quad a_{ij}; \\ x : \text{the set of variables} \quad x_1, x_2, \ldots, x_n; \\ d : \text{the set of constant terms} \quad d_1, \ldots, d_m. \end{cases}$$

Then we have

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad d = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{bmatrix}. \tag{2}$$

Each of the three array in (2) constitutes a *matrix*.

- Vectors as Special Matrices

The *dimension* of the matrix :
the number of *rows* and the number of *columns* in a matrix.
→ A in (2) is of dimension $m \times n$ (read "m by n").
→ If $m = n$, the matrix is called a *square matrix*.

4.2 Matrix Operations

- Addition and Subtraction of Matrix

Example 1

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \\ a_{31}+b_{31} & a_{32}+b_{32} \end{bmatrix}.$$

Example 2

$$\begin{bmatrix} 4 & 9 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 4+2 & 9+0 \\ 2+0 & 1+7 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 2 & 8 \end{bmatrix}.$$

Example 3

$$\begin{bmatrix} 19 & 3 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 8 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 19-6 & 3-8 \\ 2-1 & 0-3 \end{bmatrix} = \begin{bmatrix} 13 & -5 \\ 1 & -3 \end{bmatrix}.$$

- Scalar Multiplication

Example 4

$$7\begin{bmatrix} 3 & -1 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 7\times 3 & 7\times -1 \\ 7\times 0 & 7\times -5 \end{bmatrix} = \begin{bmatrix} 21 & -7 \\ 0 & -35 \end{bmatrix}.$$

Example 5

$$-1\begin{bmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \end{bmatrix} = \begin{bmatrix} -a_{11} & -a_{12} & -d_1 \\ -a_{21} & -a_{22} & -d_2 \end{bmatrix}.$$

• Multiplication of Matrices

The product AB

$$\begin{cases} A : \mathbf{m} \times \mathbf{n} \\ \\ B : \mathbf{p} \times \mathbf{q} \end{cases} \Rightarrow AB : \mathbf{m} \times \mathbf{q} \quad \text{if and only if} \quad \mathbf{n} = \mathbf{p}.$$

For instance, if

$$A = \begin{bmatrix} a_{11} & a_{12} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}, \tag{3}$$

the product AB then is defined and will be 1×3.

Now we may write in general that

$$AB = C \equiv [c_{11} \quad c_{12} \quad c_{13}]$$

where

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}, \tag{4}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}, \tag{5}$$

and

$$c_{13} = a_{11}b_{13} + a_{12}b_{23}. \tag{6}$$

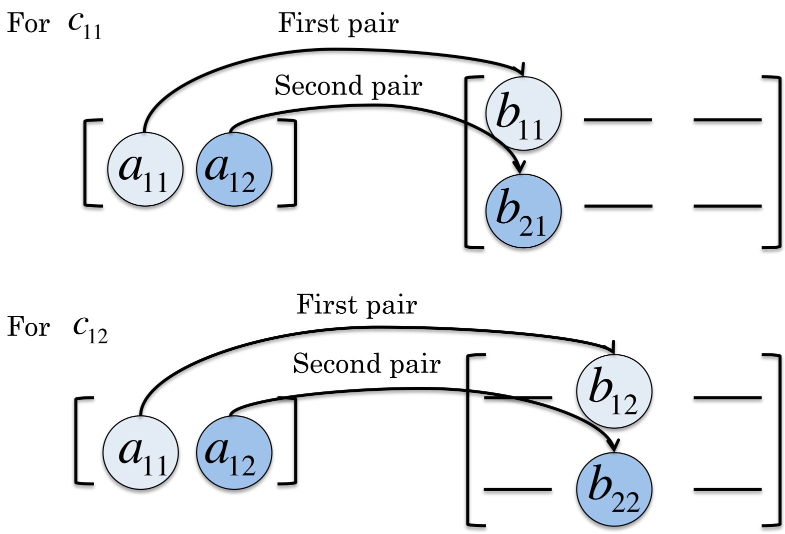
For c_{11}
First pair
Second pair
a_{11}
a_{12}
b_{11}
b_{21}
For c_{12}
First pair
Second pair
a_{11}
a_{12}
b_{12}
b_{22}

Figure 4.1

Example 6

Given

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 \\ 9 \end{bmatrix},$$

find AB.

$$AB = \begin{bmatrix} 1 \times 5 + 3 \times 9 \\ 2 \times 5 + 8 \times 9 \\ 4 \times 5 + 0 \times 9 \end{bmatrix} = \begin{bmatrix} 32 \\ 82 \\ 20 \end{bmatrix}.$$

Example 7

Given

$$A = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 3 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -\frac{1}{5} & \frac{3}{10} \\ -1 & \frac{1}{5} & \frac{7}{10} \\ 0 & \frac{2}{5} & -\frac{1}{10} \end{bmatrix},$$

find AB.

$$AB = \begin{bmatrix} 0+1+0 & -\frac{3}{5} - \frac{1}{5} + \frac{4}{5} & \frac{9}{10} - \frac{7}{10} - \frac{2}{10} \\ 0+0+0 & -\frac{1}{5} + 0 + \frac{6}{5} & \frac{3}{10} + 0 - \frac{3}{10} \\ 0+0+0 & -\frac{4}{5} + 0 + \frac{4}{5} & \frac{12}{10} + 0 - \frac{2}{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

→ The *identity matrix*.

4.3 Commutative Associative, and Distributive Laws

- Matrix Addition

  Matrix addition is commutative as well as associative:

Commutative law of addition: $A + B = B + A$

Associative law of addition: $(A + B) + C = A + (B + C)$

Example 8

Given $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$, we find that

$$A + B = B + A = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}.$$

Example 9

Given $v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, we find that

$$(v_1 + v_2) - v_3 = \begin{bmatrix} 12 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

which is equal to

$$v_1 + (v_2 - v_3) = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}.$$

- Matrix Multiplication

Matrix multiplication is NOT commutative, that is,

$$AB \neq BA.$$

Example 10

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$; then

$$AB = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix} \quad \text{but} \quad BA = \begin{bmatrix} -3 & -4 \\ 27 & 40 \end{bmatrix}.$$

Matrix multiplication is associative and distributive:

Associative law of multiplication: $(AB)C = A(BC)$

Distributive law: $A(B + C) = AB + AC$

$(B + C)A = BA + CA$

Example 11

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 7 \\ 8 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix}$; then

$$(AB)C = \begin{bmatrix} 45 & 45 \\ 30 & 30 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 1 & 9 \end{bmatrix} = \begin{bmatrix} 180 & 585 \\ 120 & 390 \end{bmatrix}$$

$$A(BC) = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 4 & 59 \\ 28 & 68 \end{bmatrix} = \begin{bmatrix} 180 & 585 \\ 120 & 390 \end{bmatrix}.$$

Moreover,

$$A(B + C) = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 2 & 11 \\ 9 & 13 \end{bmatrix} = \begin{bmatrix} 60 & 111 \\ 40 & 74 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 45 & 45 \\ 30 & 30 \end{bmatrix} + \begin{bmatrix} 15 & 66 \\ 10 & 44 \end{bmatrix} = \begin{bmatrix} 60 & 111 \\ 40 & 74 \end{bmatrix}.$$

4.4 Identity Matrices and Null Matrices

- Identity Matrices

Identity matrix, I_n :
A *square* matrix with 1s in its principal diagonal and 0s everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For any matrix A, we have

$$IA = AI = A \tag{7}$$

Example 12

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix}$, then

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \end{bmatrix} = A.$$

- Null Matrices

  A *null matrix* : a matrix whose elements are all zero.

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Idiosyncrasies of Matrix Algebra

  1. Scalar Algebra : $ab = 0$. $\Rightarrow$ Either a or b is zero.
     →But this is NOT so in matrix multiplication.

$$AB = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0,$$

     although neither A nor B is itself a zero matrix.

  2. Scalar Algebra : $cd = ce$ (with $c \neq 0$) $\Rightarrow d = e$.
     →The same DOES NOT hold for matrices. Given

$$C = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad \text{and} \quad E = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix},$$

     we find that

$$CD = CE = \begin{bmatrix} 5 & 8 \\ 15 & 24 \end{bmatrix}$$

     even though $D \neq E$.

→ A, B, and C are known as *singular matrices*.

4.5 Transposes and Inverses

- Transposes

$A : \boldsymbol{m} \times \boldsymbol{n} \rightarrow A' : \boldsymbol{n} \times \boldsymbol{m}$

$\rightarrow$ An $n \times n$ square matrix, however, possesses a transpose with the same dimension.

Example 13

Given $A = \begin{bmatrix} 3 & 8 & -9 \\ 1 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 1 & 7 \end{bmatrix}$, we can interchange the rows and columns and write

$$A' = \begin{bmatrix} 3 & 1 \\ 8 & 0 \\ -9 & 4 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 3 & 1 \\ 4 & 7 \end{bmatrix}.$$

Example 14

If $C = \begin{bmatrix} 9 & -1 \\ 2 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}$, then

$$C' = \begin{bmatrix} 9 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad D' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 7 \\ 4 & 7 & 2 \end{bmatrix}.$$

$\rightarrow D' = D$ is known as *symmetric matrix*.

- Properties of Transposes

$$(A')' = A \tag{8}$$

$$(A + B)' = A' + B' \tag{9}$$

$$(AB)' = B'A' \tag{10}$$

Example 15

If $A = \begin{bmatrix} 4 & 1 \\ 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 7 & 1 \end{bmatrix}$, then

$$(A + B)' = \begin{bmatrix} 6 & 1 \\ 16 & 1 \end{bmatrix}' = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}$$

and

$$A' + B' = \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 16 \\ 1 & 1 \end{bmatrix}.$$

Example 16

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 6 & 7 \end{bmatrix}$, we have

$$(AB)' = \begin{bmatrix} 12 & 13 \\ 24 & 25 \end{bmatrix}' = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}$$

and

$$B'A' = \begin{bmatrix} 0 & 6 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 12 & 24 \\ 13 & 25 \end{bmatrix}.$$

- Inverses

The inverse of matrix A, A^{-1}, is defined only if A is a square matrix, in which case the inverse is the matrix that satisfies the condition

$$AA^{-1} = A^{-1}A = I. \tag{11}$$

Some points:

1. Not every square matrix has an inverse.
   →squareness is a *necessary* condition for the existence of an inverse.
   →If a square matrix A has an inverse, A is said *nonsingular*. Otherwise, it is *singular* matrix.
2. If A^{-1} does exist, A and A^{-1} are inverse of each other.
3. If A is $n \times n$, then A^{-1} must also be $n \times n$.
4. If an inverse exists, then it is unique.
5. (11) imply that it is sufficient to establish the inverse relationship between A and A^{-1}.

- Three properties of inverses

If A and B are nonsingular matrices with dimension $n \times n$,

$$(A^{-1})^{-1} = A \tag{12}$$

$$(AB)^{-1} = B^{-1}A^{-1} \tag{13}$$

$$(A')^{-1} = (A^{-1})' \tag{14}$$

5. Linear Models and Matrix Algebra (Continued)

5.1 Conditions for Nonsingularity of a Matrix

- Necessary versus Sufficient Conditions

Suppose that a statement p is true *only if* another statement q is true; then q constitutes a necessary condition of p:

$$p \Rightarrow q \qquad [\text{“}p \textit{ only if } q\text{” or “}\textit{if } p\textit{, then } q\text{”}]. \tag{15}$$

→ It is logically correct to interpret (15) to mean "*p implies q*".

A different types of situation is one in which a statement p is true if q is true, but p can also be true when q is not true. In this case, q is said to be a sufficient condition for p:

$$p \Leftarrow q \qquad [\text{“}p \textit{ if } q\text{” or “}\textit{if } q\textit{, then } p\text{”}]. \tag{16}$$

→ It can be interpreted to mean "*q implies p*.

In a third possible situation, q is *both* necessary and sufficient for p:

$$p \Leftrightarrow q \qquad [\text{“}p \textit{ if and only if } q\text{” (also written as “}p \textit{ iff } q\text{”)}]. \tag{17}$$

5.2 Test of Nonsingularity by Use of Determinant

- Determinants and Nonsingularity

$|A|$: The *determinant* of a square matrix A.
$\rightarrow$ $|A|$ is a uniquely defined scalar associated with that matrix and are defined only for *square* matrices.
$\rightarrow$ If $A = [a_{11}]$, $|A| = |a_{11}| = a_{11}$.

Second-order determinant

For a 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, its determinant is defined as follows:

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \qquad [= \text{a scalar}] \qquad (18)$$

Example 18

Given $A = \begin{bmatrix} 10 & 4 \\ 8 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix}$, their determinants are

$$|A| = \begin{vmatrix} 10 & 4 \\ 8 & 5 \end{vmatrix} = 10 \times 5 - 8 \times 4 = 18$$

$$|B| = \begin{vmatrix} 3 & 5 \\ 0 & -1 \end{vmatrix} = 3 \times (-1) - 0 \times 5 = -3.$$

- Evaluating a Third-Order Determinant

Given

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \quad (19)$$

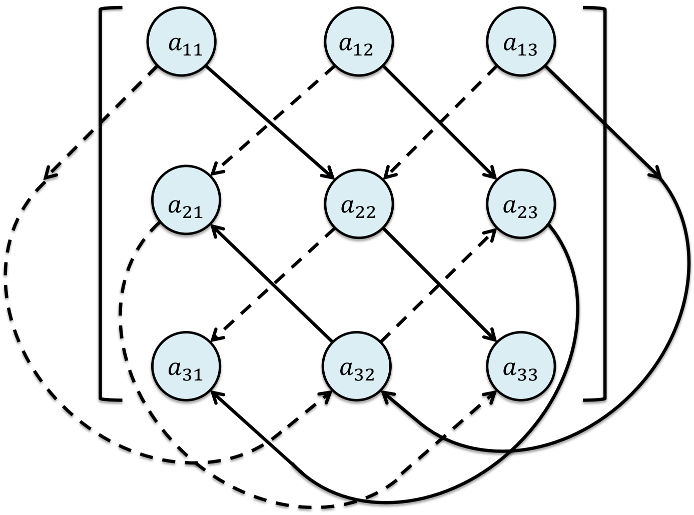

Figure 5.1

① *Solid* arrows (*plus* signs):

$a_{11} \to a_{22} \to a_{33}$, $a_{21} \to a_{23} \to a_{31}$, and $a_{13} \to a_{32} \to a_{21}$.

② *Broken* arrows (*minus* signs):

$a_{11} \to a_{32} \to a_{23}$, $a_{12} \to a_{21} \to a_{33}$, and $a_{13} \to a_{22} \to a_{31}$.

Example 19

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2\times5\times9+1\times6\times7\times+3\times8\times4-2\times8\times6-1\times4\times9-3\times5\times7 = -9.$$

• Evaluating an nth-Order Determinant by Laplace Expansion

The minor : $|M_{ij}|$
→ $|M_{ij}|$ is obtained by deleting the ith row and the jth column of a given determinant.

Example : The three second-order determinants in (19)

$$M_{11} \equiv \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \quad M_{12} \equiv \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \quad M_{13} \equiv \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix}.$$

The *cofactor* : $|C_{ij}|$
→ It is a minor with a prescribed algebraic sign attached to it.

$$\begin{cases} |C_{ij}| \equiv |M_{ij}| & \text{if the sum of } i \text{ and } j \text{ in } |M_{ij}| \text{ is even;} \\ |C_{ij}| \equiv -|M_{ij}| & \text{otherwise.} \end{cases}$$

In short, we have

$$|C_{ij}| \equiv (-1)^{i+j}|M_{ij}|$$

where it is obvious that the expression $(-1)^{i+j}$ can be positive if and only if $(i + j)$ is even.

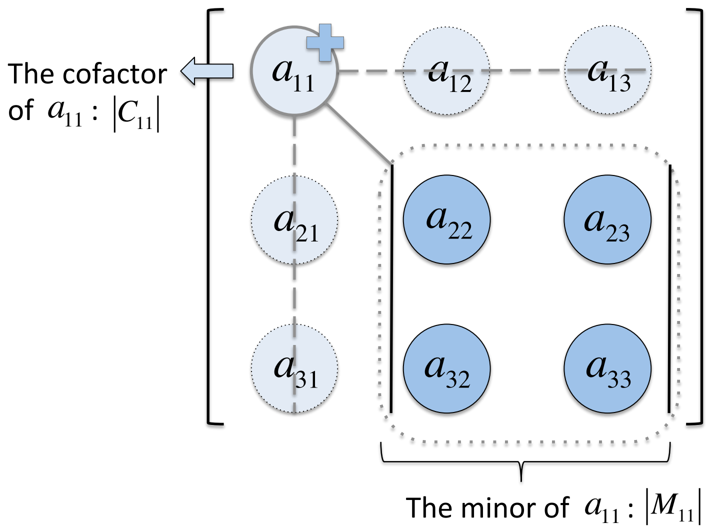

Figure 5.2 : The minor and the cofactor of the a_{11}

Example 20

In the determinant $\begin{vmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$, the minor of the element 8 is

$$M_{12} = \begin{vmatrix} 6 & 4 \\ 3 & 1 \end{vmatrix} = -6$$

but the cofactor of the same element is

$$|C_{12}| = -|M_{12}| = 6$$

because $i + j = 1 + 2 = 3$ is odd. Similarly, the cofactor of the element 4 is

$$|C_{23}| = -|M_{23}| = -\begin{vmatrix} 9 & 8 \\ 3 & 2 \end{vmatrix} = 6.$$

Using these new concepts, we can express a third-order determinant as

$$
\begin{aligned}
|A| &= a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}| \\
&= a_{11}|C_{11}| + a_{12}|C_{12}| + a_{13}|C_{13}| = \sum_{j=1}^{3} a_{1j}|C_{1j}| \qquad (20)
\end{aligned}
$$

Note that difference in the signs of the $a_{12}|M_{12}|$ and $a_{12}|C_{12}|$ terms in (20). This is because 1 + 2 gives an odd number.

Example 21

Given $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 2 & 3 & 0 \\ 7 & -3 & 0 \end{vmatrix}$, expansion by the first *row* products the result

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 6\begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = 0 + 0 - 27 = -27$$

But expansion by the first *column* yields the identical answer:

$$|A| = 5\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix} - 2\begin{vmatrix} 6 & 1 \\ -3 & 0 \end{vmatrix} + 7\begin{vmatrix} 6 & 1 \\ 3 & 0 \end{vmatrix} = 0 - 6 - 21 = -27.$$

∗ A row or column with the largest number of 0s or 1s is always preferable.

We could therefore have evaluated it thus:

$$|A| = 1\begin{vmatrix} 2 & 3 \\ 7 & -3 \end{vmatrix} = -6 - 21 = -27.$$

To sum up, the value of a determinant $|A|$ of order n can be found by the Laplace expansion of *any* row or *any* column as follows:

$$|A| = \sum_{j=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } i\text{th row]}$$

$$= \sum_{i=1}^{n} a_{ij}|C_{ij}| \qquad \text{[expansion by the } j\text{th column]}$$

5.3 Basic Properties of Determinants

Property 1 The determinant of a matrix A has the same value as that of its transpose A', that is, $|A| = |A'|$.

Example 22

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Property 2 The interchange of any *two* row (or any *two* columns) will alter the sign, but not the numerical value, of the determinant.

Example 23

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, but the interchange of the two rows yields

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -(ad - bc).$$

Property 3 The multiplication of any *one* row (or *one* column) by scalar k will change the value of the determinant k-fold.

Example 24

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = kad - kbc = k(ad - bc) = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Property 4 The addition (subtraction) of a multiple of any row to (from) another row will leave the value of the determinant unaltered. The same holds true if we replace the word *row* by *column* in the previous statement.

Example 25

$$\begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix} = a(d + kb) - b(c + ka) = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

Property 5 If one row (or column) is a multiple of another row (or column), the value of the determinant will be zero. As a special case of this, when two rows (or two columns) are *identical*, the determinant will vanish.

Example 26

$$\begin{vmatrix} 2a & 2b \\ a & b \end{vmatrix} = 2ab - 2ab = 0, \qquad \begin{vmatrix} c & d \\ c & d \end{vmatrix} = cd - cd = 0.$$

- Determinantal Criterion for Nonsingularity

Given a $Ax = d(A : n \times n)$,

$$
\begin{aligned}
|A| \neq 0 &\Leftrightarrow \text{there is row (column) independence} \\
&\Leftrightarrow A \text{ is nonsingular} \\
&\Leftrightarrow A^{-1} \text{ exists} \\
&\Leftrightarrow \text{a unique solution } x^* = A^{-1}d \text{ exists.}
\end{aligned}
$$

For instance, consider the following equation system $Ax = d$:

$$
\begin{bmatrix} 3 & 4 & 2 \\ 15 & 20 & 10 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}
$$

→ The second row is five times the first; the rows are *dependent*, and hence no unique solution exists (Property 5, $|A| = 0$).

Example 27

Does the equation system

$$
\begin{aligned}
7x_1 - 3x_2 - 3x_3 &= 7 \\
2x_1 + 4x_2 + x_3 &= 0 \\
-2x_2 - x_3 &= 2
\end{aligned}
$$

possess a unique solution? The determinant $|A|$ is

$$
\begin{vmatrix} 7 & -3 & -3 \\ 2 & 4 & 1 \\ 0 & -2 & -1 \end{vmatrix} = -8 \neq 0
$$

Therefore a unique solution does exist.

5.4 Finding the Inverse Matrix

Property 6 The expansion of a determinant by *alien cofactors* (the cofactors of a "wrong" row or column) always yields a value of zero.

Example 28

If we expand the determinant $\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix}$ by using its *first-row* elements but the cofactors of the *second-row* elements

$$|C_{21}| = -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} = -3 \quad |C_{22}| = \begin{vmatrix} 4 & 2 \\ 1 & 3 \end{vmatrix} = 10 \quad |C_{23}| = -\begin{vmatrix} 4 & 1 \\ 1 & 0 \end{vmatrix} = 1$$

we get $a_{11}|C_{21}| + a_{12}|C_{22}| + a_{13}|C_{23}| = -12 + 10 + 2 = 0$.

- Matrix Inversion

Assume that an $n \times n$ nonsingular matrix A is given:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad (|A| \neq 0) \qquad (21)$$

$C := [|C_{ij}|](\leftarrow n \times n)$
$\rightarrow$ This is a matrix of cofactors by replacing each element a_{ij} in (21) with its cofactor $|C_{ij}|$.

C' : the *adjoint* of A (*adj* A).

$$C' \equiv adjA \equiv \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \cdots & \cdots & \cdots & \cdots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \tag{22}$$

By utilizing the formula for Laplace expansion as well as Property 6 of determinants, the product AC' may be expressed as follows:

$$\begin{aligned} AC' &= \begin{bmatrix} \sum_{j=1}^n a_{1j}|C_{1j}| & \sum_{j=1}^n a_{1j}|C_{2j}| & \cdots & \sum_{j=1}^n a_{1j}|C_{nj}| \\ \sum_{j=1}^n a_{2j}|C_{1j}| & \sum_{j=1}^n a_{2j}|C_{2j}| & \cdots & \sum_{j=1}^n a_{2j}|C_{nj}| \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^n a_{nj}|C_{1j}| & \sum_{j=1}^n a_{nj}|C_{2j}| & \cdots & \sum_{j=1}^n a_{nj}|C_{nj}| \end{bmatrix} \\ &= \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix} \\ &= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = |A|I_n \end{aligned}$$

As the determinant $|A|$ is a nonzero scalar, it is permissible to divide both sides of the equation $AC' = |A|I$ by $|A|$. The result is

$$\frac{AC'}{|A|} = I \quad \text{or} \quad A\frac{C'}{|A|} = I$$

Premultiplying both sides of the last equation by A^{-1}, and using the result that $A^{-1}A = I$, we can get $\frac{C'}{|A|} = A^{-1}$, or

$$A^{-1} = \frac{1}{|A|} adjA \tag{23}$$

The general procedure for finding the A^{-1}

1. Find $|A|$.
2. Find the cofactors of all the elements of A, and arrange them as a matrix $C = [|C_{ij}|]$.
3. Take the transpose of C to get *adj* A.
4. divide adj A by the determinant $|A|$.

Example 29

Find the inverse of $B = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 3 & 2 \\ 3 & 0 & 7 \end{bmatrix}$. Since $|B| = 99 \neq 0$, the inverse B^{-1} also exists. The cofactor matrix is

$$\begin{bmatrix} \begin{vmatrix} 3 & 2 \\ 0 & 7 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 3 & 7 \end{vmatrix} & \begin{vmatrix} 0 & 3 \\ 3 & 0 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 0 & 7 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ 3 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} 21 & 6 & -9 \\ -7 & 31 & 3 \\ 5 & -8 & 12 \end{bmatrix}$$

Therefore,

$$\text{adj}B = \begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}$$

and the desired inverse matrix is

$$B^{-1} = \frac{1}{|B|}\text{adj}B = \frac{1}{99}\begin{bmatrix} 21 & -7 & 5 \\ 6 & 31 & -8 \\ -9 & 3 & 12 \end{bmatrix}.$$

5.5 Cramer's Rule

- Derivation of the Rule

Given an equation system $Ax = d$, where A is $n \times n$, the solution can be written as

$$x^* = A^{-1}d = \frac{1}{|A|}\ (adjA)\ d$$

provided A is nonsingular. This means that

$$\begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} |C_{11}| & |C_{21}| & \cdots & |C_{n1}| \\ |C_{12}| & |C_{22}| & \cdots & |C_{n2}| \\ \ldots & \ldots & \ldots & \ldots \\ |C_{1n}| & |C_{2n}| & \cdots & |C_{nn}| \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} d_1|C_{11}| + d_2|C_{21}| + \cdots + d_n|C_{n1}| \\ d_1|C_{12}| + d_2|C_{22}| + \cdots + d_n|C_{n2}| \\ \ldots \\ d_1|C_{1n}| + d_2|C_{2n}| + \cdots + d_n|C_{nn}| \end{bmatrix}$$

$$= \frac{1}{|A|} \begin{bmatrix} \sum_{i=1}^{n} d_i|C_{i1}| \\ \sum_{i=1}^{n} d_i|C_{i2}| \\ \vdots \\ \sum_{i=1}^{n} d_i|C_{in}| \end{bmatrix}$$

Equating the corresponding elements on the two sides of the equation, we obtain the solution values

$$x_1^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i1}| \qquad x_2^* = \frac{1}{|A|}\sum_{i=1}^{n} d_i|C_{i2}| \quad \text{(etc.)} \qquad (24)$$

If we replace their columns of $|A|$ by the column vector d but keep all the other columns intact, then a new determinant will be obtained :

$$|A_1| \equiv \sum_{i=1}^{n} d_i|C_{i1}| \quad (\text{Recall } |A| = \sum_{j=1}^{n} a_{ij}|C_{ij}|).$$

→ The subscript 1 indicating that the first column has been replaced by d.

Returning to (24), we see therefore that

$$x^* = \frac{1}{|A|}|A_1|$$

In general, to find the solution value of the jth variable x_j^*,

① replace the jth column of the determinant$|A|$ by the constant terms $d_1 \cdots d_n$ to get $|A_j|$;

② divide $|A_j|$ by the original determinant $|A|$.

Thus, the solution of the system $Ax = d$ can be expressed as

$$x_j^* = \frac{|A_j|}{|A|} = \frac{1}{|A|}\begin{vmatrix} a_{11} & a_{12} & \cdots & d_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & d_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & d_n & \cdots & a_{nn} \end{vmatrix} \tag{25}$$

This result in (25) is the statement of Cramer's rule.

Example 30

Find the solution of the equation system

$$\begin{cases} 5x_1 + 3x_2 = 30 \\ 6x_1 - 2x_2 = 8 \end{cases}$$

The coefficients and the constant terms give the following determinants:

$$|A| = \begin{vmatrix} 5 & 3 \\ 6 & -2 \end{vmatrix} = -28 \quad |A_1| = \begin{vmatrix} 30 & 3 \\ 8 & -2 \end{vmatrix} = -84 \quad |A_2| = \begin{vmatrix} 5 & 30 \\ 6 & 8 \end{vmatrix} = -140$$

Therefore, we can immediately write

$$x_1^* = \frac{|A_1|}{|A|} = \frac{-84}{-28} = 3 \quad \text{and} \quad x_2^* = \frac{|A_2|}{|A|} = \frac{-140}{-28} = 5.$$