10. Exponential and Logarithmic Functions

10.1 The Nature of Exponential Functions

In power expressions such as x^3 or x^5, the exponents are constants.

A function whose independent variable appears in the role of an exponent such as 3^x is called an exponential function.

- Simple exponential function

$$y = f(t) = b^t \tag{1}$$

y : the dependent variables, t : the independent variable,

b : a fixed base of the exponent.

- Generalized Exponential Function

$$y = f(t) = ab^{ct} \tag{2}$$

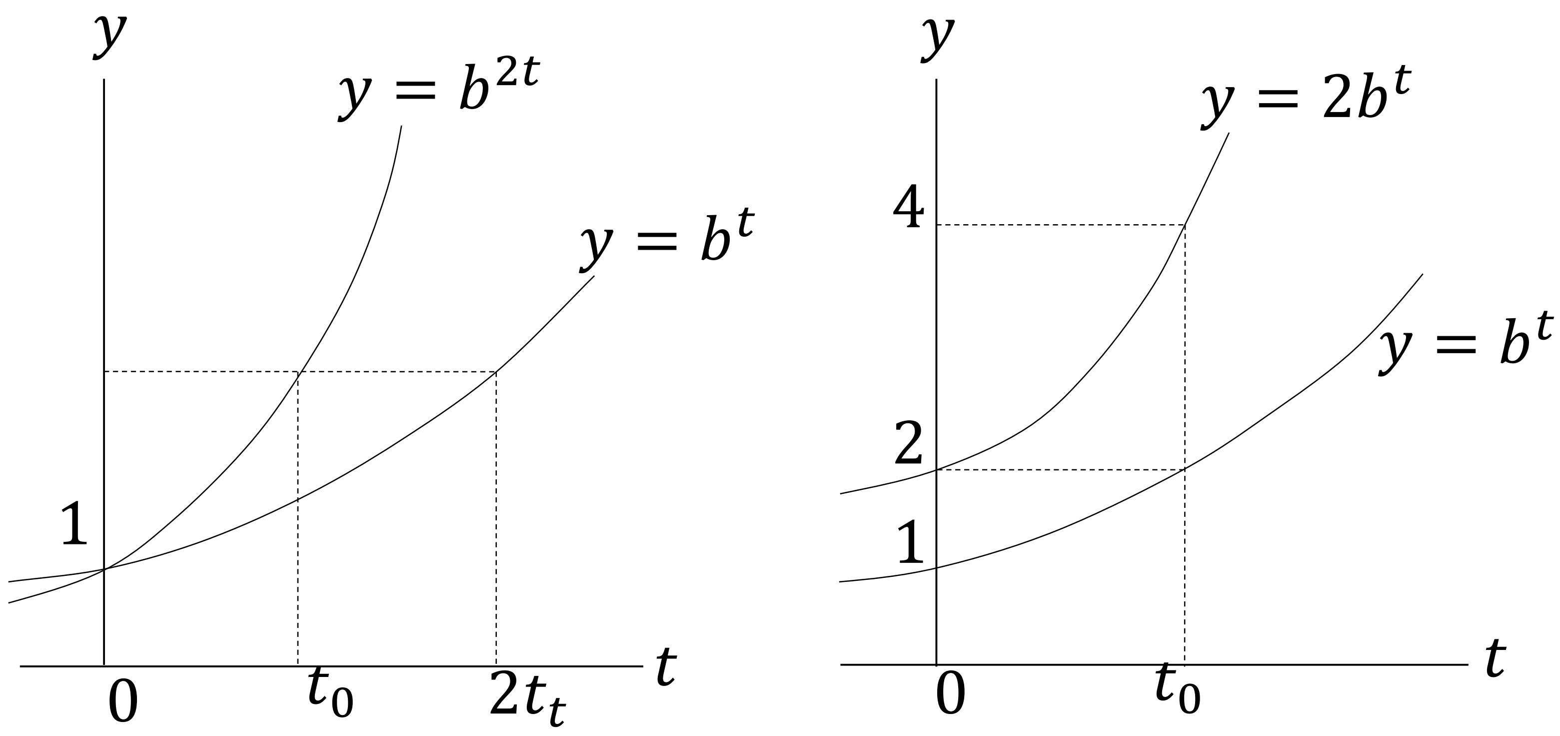


Figure 10.2

- A Preferred Base (Napier's constant)

$$e = 2.71828\ldots$$

- Natural Exponential Function

$$y = e^{t}, \quad y = e^{3t}, \quad y = Ae^{rt}$$
$$y = \exp(t), \quad y = \exp(3t), \quad y = A\exp(rt),$$

- The Derivative of Natural Exponential Function

$$\frac{d}{dt} b^{t} = b^{t} \ln b$$

$$\frac{d}{dt} e^{t} = e^{t}, \quad \frac{d}{dt} Ae^{rt} = rAe^{rt}$$

10.2 Natural Exponential Functions and the Problem of Growth

- The Number e

Let

$$f(m) = \left(1 + \frac{1}{m}\right)^m. \tag{3}$$

➢ The function $f(m)$ is increasing in m.

$$f(1) = \left(1 + \frac{1}{1}\right)^1 = 2$$

$$f(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25$$

$$f(3) = \left(1 + \frac{1}{3}\right)^3 = 2.37037 \ldots$$

$$f(4) = \left(1 + \frac{1}{4}\right)^4 = 2.44141 \ldots$$

$$\vdots$$

➢The function of $f(m)$ is bounded from above.

$$
\begin{aligned}
f(m) &= 1 + \binom{m}{1}\frac{1}{m} + \binom{m}{2}\frac{1}{m^2} + \cdots + \binom{m}{m}\frac{1}{m^m} \\
&= 1 + \frac{1}{1!} + \frac{1}{2!}\left(1 - \frac{1}{m}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{m}\right) \cdots \left(1 - \frac{m-1}{m}\right) \\
&\le 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} \\
&\le 1 + 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{m-1}} \\
&= 1 + \frac{1 - \frac{1}{2^m}}{1 - \frac{1}{2}} \\
&< 1 + \frac{1}{1 - \frac{1}{2}} \\
&= 1 + 2 = 3 \qquad (4)
\end{aligned}
$$

- $f(m)$ is bounded from above ($f(m) < 3$)
- $f(m)$ is monotonically increasing in m

$\Rightarrow f(m) \to$ a certain number as $m \to \infty$.

Definition of *e*: $$e \equiv \lim_{m\to\infty} f(m) = 2.71828\ldots \qquad (5)$$

- The approximation value of e

Consider the Maclaurin series of $\phi(x) = e^x$.

$$\phi(x) = \phi(0) + \frac{\phi'(0)}{1!}x + \frac{\phi''(0)}{2!}x^2 + \cdots + \frac{\phi^{(n)}(0)}{n!}x^n + R_n$$

$$= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n, \qquad (6)$$

where $R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}x^{n+1} = \frac{e^p}{(n+1)!}x^{n+1}$ $(0 < p < x)$. (7)

Since $R_n \to 0$ as $n \to \infty$, we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots \qquad (8)$$

Substituting $x = 1$, we find that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

$$= 2 + 0.5 + 0.1666667 + \cdots$$

$$\cong 2.7182819 \qquad (9)$$

- An Economic Interpretation of e

The number e can be interpreted as the result of a special process of interest compounding.

➢ Suppose that, starting out with a principal of \$1, we find a banker to offer us the interest rate of 100% per annum. If interest is to be compounded once a year, the value of our asset at the end of the year will be \$2.

$$V(1) = \text{initial principal} \times (1 + \text{interest rate})$$

$$= 1 \times \left(1 + \frac{100\%}{1}\right)^{1} = 2 \qquad (10)$$

- Suppose that interest is compounded semiannually. Then, we have

$$V(2) = \left(1 + \frac{100\%}{2}\right) \times \left(1 + \frac{100\%}{2}\right) = \left(1 + \frac{1}{2}\right)^2 \quad (11)$$

- If the frequency of compounding in 1 year is m, our year end asset value is

$$V(m) = \left(1 + \frac{1}{m}\right)^m \quad (12)$$

- When $m \rightarrow \infty$, the value of the asset at the end of 1 year will be

$$\lim_{m \rightarrow \infty} V(m) = e \quad (13)$$

The number of e can be interpreted as the year-end value to which a principal of \$1 will grow if interest at the rate of 100% per annum is compounded continuously.

10.3 Logarithms

- The Meaning of Logarithm

The log of y to the base b is the power to which the base b must be raised to attain the value y.

➢ $$y = b^t \quad \Leftrightarrow \quad t = \log_b y \qquad (14)$$

➢ $$\ln x \quad \Leftrightarrow \quad \log_e x.$$

Examples

$\log_4 16 = \log_4 4^2 = 2$ $\qquad \log_{10} 1000 = \log_{10} 10^3 = 3$

$\log_{10} 0.01 = \log_{10} 10^{-2} = -2$

$\ln e^2 = \log_e e^2 = 2$ $\qquad \ln 1 = \log_e e^0 = 0$

$\ln \frac{1}{e} = \log_e e^{-1} = -1$

- Rules of Logarithms

Rule I : $\ln(uv) = \ln u + \ln v \quad (u, v > 0)$
Rule II : $\ln(u/v) = \ln u - \ln v \quad (u, v > 0)$
Rule III : $\ln u^a = a \ln u \quad (u > 0)$
Rule IV : $\log_b u = (\log_b e)(\log_e u) \quad (u > 0)$
Rule V : $\log_b e = \frac{1}{\log_e b}$

Proof of Rule I.

$$uv = e^{\ln u} e^{\ln v} = e^{\ln u + \ln v} \text{ and } uv = e^{\ln uv}$$

$$\Rightarrow \ln uv = \ln u + \ln v$$

10.4 Logarithm Functions

- Log Functions and Exponential Functions

  Log functions are inverse functions of certain exponential functions.

$$t = \log_b y \qquad \text{and} \qquad t = \ln y \tag{15}$$

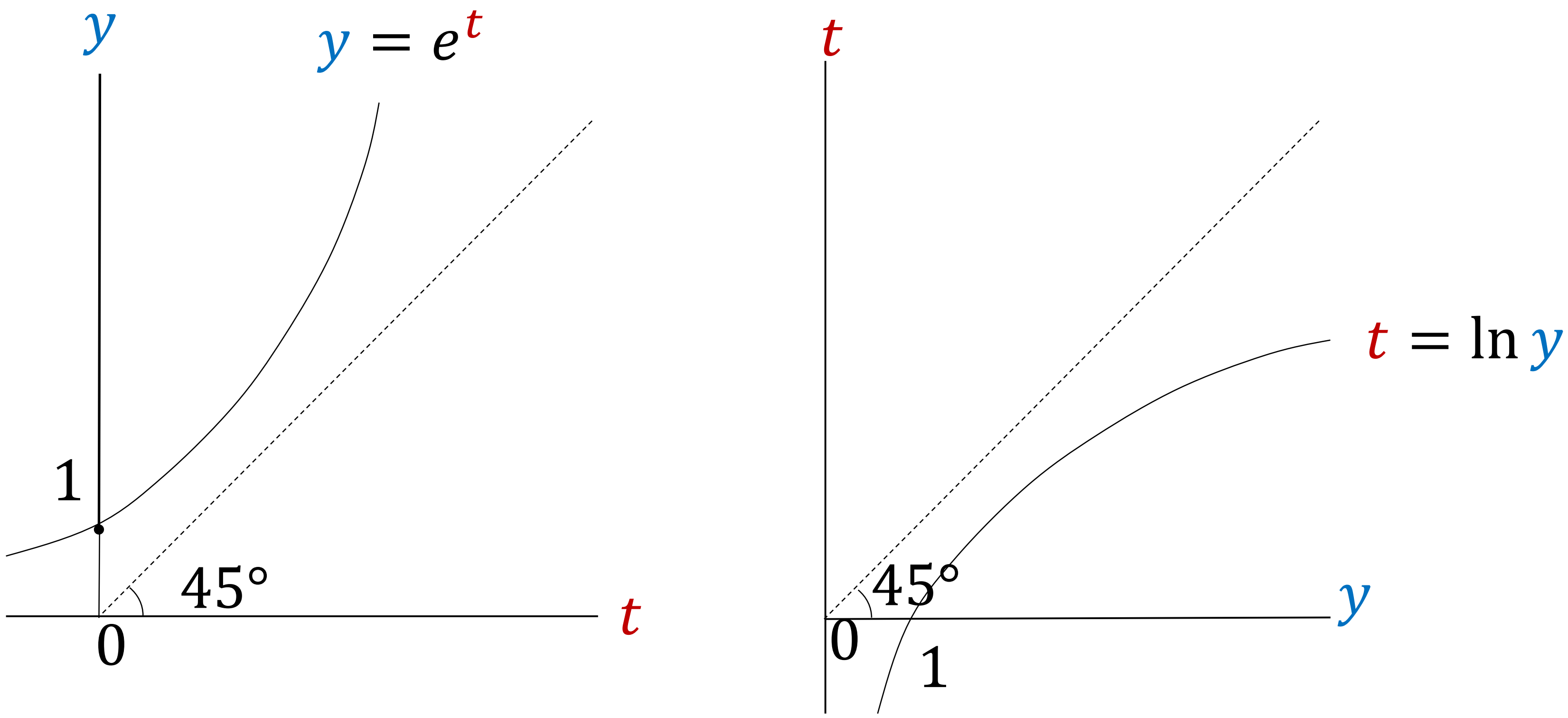
↓ Inverse function

$$y = b^t \qquad \text{and} \qquad y = e^t \tag{16}$$

- The Graphical Form

  $y = e^t$ and $t = \ln y$ are drawn as follows.

Figure 10.3 mirror-relationship



- We consider the inverse of $y = Ae^{rt}$.

Taking the natural log of both sides of this exponential function,

$$\ln y = \ln(Ae^{rt}) = \ln A + rt \ln e = \ln A + rt, \tag{17}$$

Solving for t,

$$t = \frac{\ln y - \ln A}{r}. \tag{18}$$

- As inverse function of monotonically increasing functions, logarithmic functions must also be monotonically increasing.

$$\ln y_1 = \ln y_2 \iff y_1 = y_2$$

$$\ln y_1 > \ln y_2 \iff y_1 > y_2 \tag{19}$$

➢ For any base $b > 1$,

$$\left.\begin{array}{c} 0 < y < 1 \\ y = 1 \\ y > 1 \end{array}\right\} \quad \Leftrightarrow \quad \left\{\begin{array}{l} \log_b y < 0 \\ \log_b y = 0 \\ \log_b y > 0 \end{array}\right. \tag{20}$$

- Base Conversion

Let us consider the conversion of Ab^{ct} into Ae^{rt}.

$$e^r = b^c \quad \Longrightarrow \quad \ln e^r = \ln b^c$$

$$\Longrightarrow \quad r = c \ln b \tag{21}$$

Thus,

$$Ab^{ct} = Ae^{(c \ln b)t}$$

10.5 Derivative of Exponential and Logarithmic Functions

- Log-Function Rule

$$\frac{d}{dt}\ln t = \frac{1}{t} \tag{23}$$

- Exponential-Function Rule

$$\frac{d}{dt}e^{t} = e^{t} \tag{24}$$

- The Rules Generalized

$$\frac{d}{dt}e^{f(t)} = f'(t)e^{f(t)}$$

$$\frac{d}{dt}\ln f(t) = \frac{f'(t)}{f(t)} \tag{25}$$