11. The Case of More than One Choice Variable

11.1 The Differential Version of Optimization Conditions

Consider the following function

$$z = f(x).$$

The differential of z is given by,

$$dz = f'(x)dx$$

- *Review* : First-*derivative* condition (Chp. 9)

  a necessary condition for an extremum of z : $f'(x) = 0$

To expand the discussion for the case of more than one choice variable, we first consider how this condition can equivalently be expressed in terms of *differential*.

- **First-Order differential Condition (F.O.C.)**

In terms of differentials, a ***necessary*** condition for an extremum of z:

$$dz = 0 \text{ for an arbitrary nonzero } dx \qquad (1)$$

Clearly, the F.O.C. $dz = 0$ is equivalent to the derivative version of F.O.C. $f'(x) = 0$.

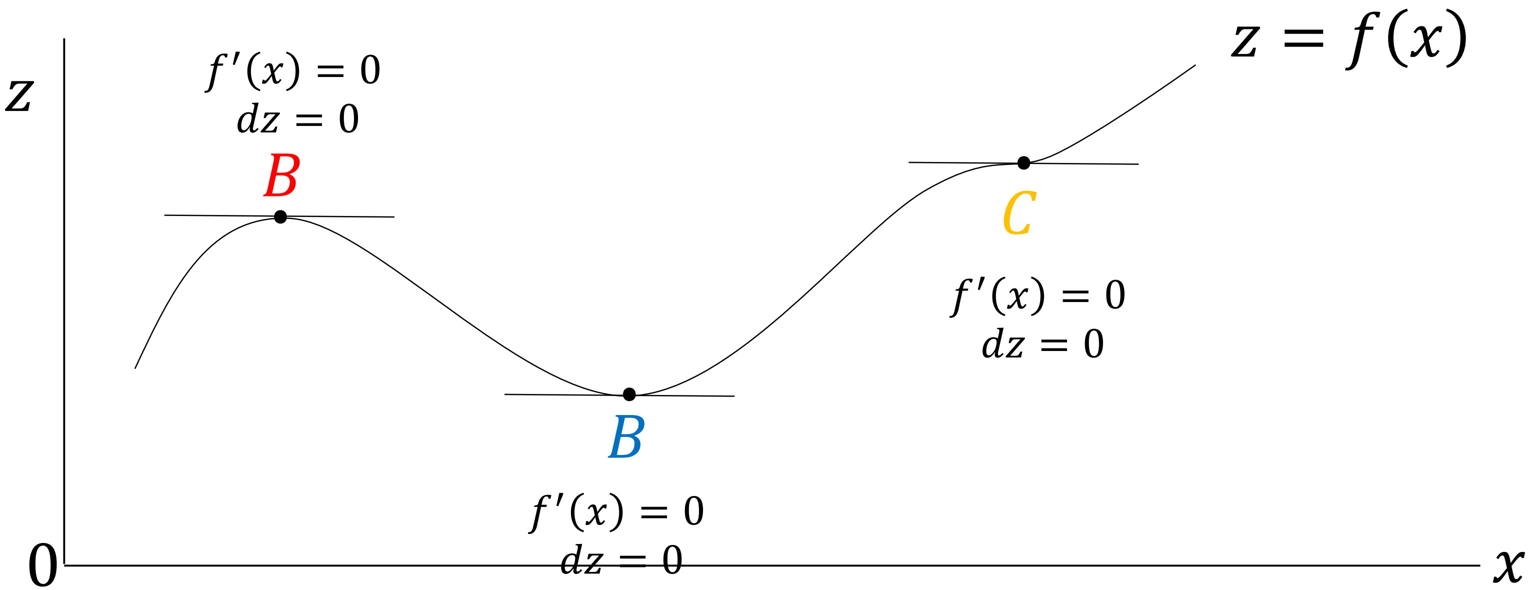

Figure 11.1

Question: Why this differential condition is *not* sufficient?

- *Review* : Second-derivative condition (Chp. 9)

  a *sufficient* condition for a maximum (minimum) :

$$f''(x) < 0 \; (> 0)$$

- **Second-order differential condition (S.O.C.)**

  In terms of differentials, a *sufficient* condition for a maximum (minimum) is,

$$d^2z < 0 \; (> 0) \text{ for an arbitrary nonzero } dx, \qquad (2)$$

where d^2z is the differential of a differential, i.e., $d(dz)$.

Question: Why this condition is *not* necessary ?

11.2 Extreme Values of A Function of Two Variables

A graph of two choice variables function $z = f(x, y)$ becomes a surface in a 3-space

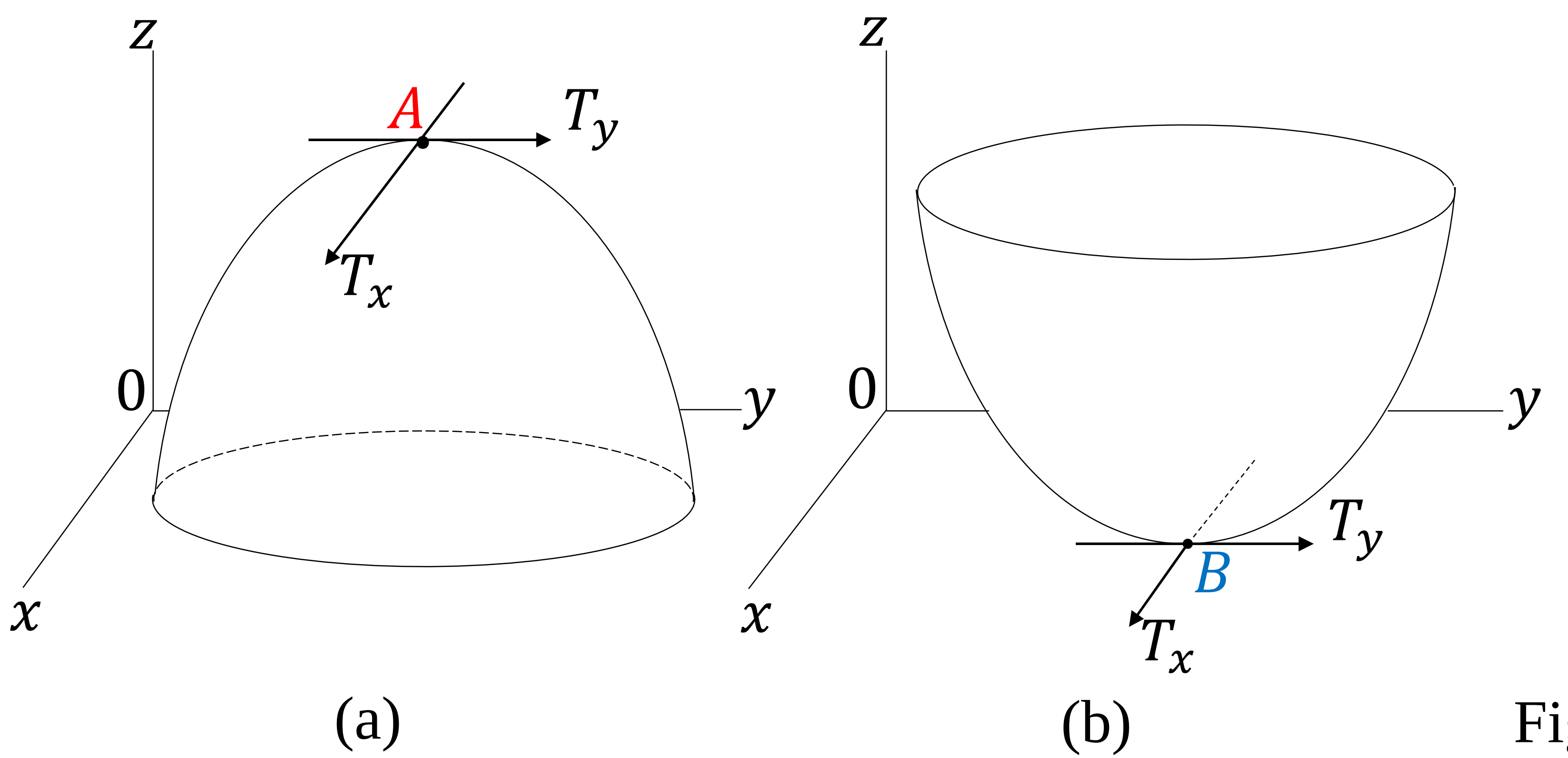

Figure 11.2

Point A constitutes a maximum and point B constitutes a minimum.

- F.O.C. for two choice variables function

For the function $z = f(x, y)$, the first-order differential condition $dz = 0$, should be modified to the form:

$$dz = f_x dx + f_y dy = 0$$ for arbitrary values of dx and dy, not both zero,

where $f_x \equiv \frac{\partial z}{\partial x}$ and $f_y \equiv \frac{\partial z}{\partial y}$.

In order to satisfy the above condition, the two partial derivatives f_x and f_y must be simultaneously equal to zero:

$$f_x = f_y = 0 \qquad \left[\text{or } \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0\right]$$

Note that the first-order condition is necessary but *not* sufficient.

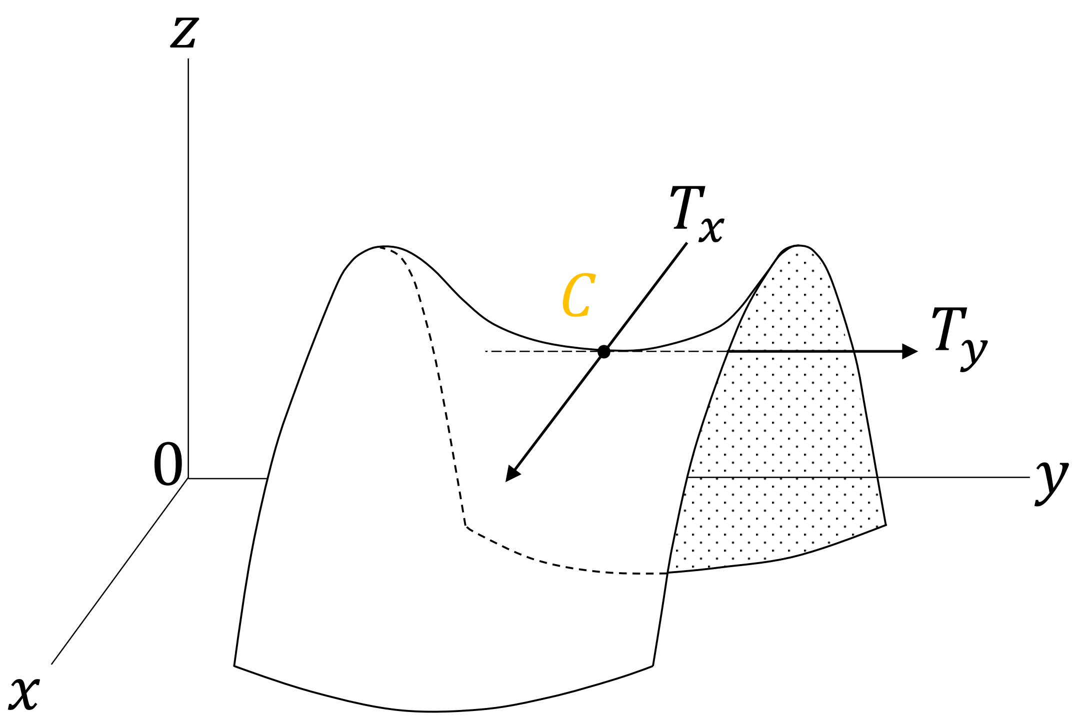

Figure 11.3

Point C is a minimum on yz plane, but be a maximum on xz plane (saddle point(鞍点)).

- Second-Order Partial Derivatives

A particular second-order partial derivative:

$$f_{xx} \equiv \frac{\partial}{\partial x}(f_x) \quad \text{or} \quad \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$f_{yy} \equiv \frac{\partial}{\partial y}(f_y) \quad \text{or} \quad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Since f_x (f_y) is also a function of y (x), we have the following cross partial derivatives,

$$f_{xy} \equiv \frac{\partial^2 z}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \quad \text{and} \quad f_{yx} \equiv \frac{\partial^2 z}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

As long as the two cross partial derivatives are both continuous, $f_{xy} = f_{yx}$ ($Young's\ theorem$)

Example 1

Find the four second-order partial derivatives of

$$z(x, y) = x^3 + 5xy - y^2$$

Since $f_x \equiv \frac{\partial z}{\partial x}$ and $f_y \equiv \frac{\partial z}{\partial y}$, we have,

$$f_x(x, y) = 3x^2 + 5y \quad \text{and} \quad f_y(x, y) = 5x - 2y$$

Furthermore,

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2$$

- Second-Order Total Differential

A second-order total differential of $z = f(x)$ is given by,

$$d^2z \equiv d(dz) = \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy$$

$$= \frac{\partial}{\partial x}\left(f_x dx + f_y dy\right)dx + \frac{\partial}{\partial y}\left(f_x dx + f_y dy\right)dy$$

$$= \left(f_{xx}dx + f_{xy}dy\right)dx + \left(f_{yx}dx + f_{yy}dy\right)dy$$

$$= f_{xx}dx^2 + f_{xy}dydx + f_{yx}dxdy + f_{yy}dy^2$$

$$= \boldsymbol{f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2} \qquad [f_{xy}=f_{yx}]$$

Example 2

Find dz and d^2z of

$$z = x^3 + 5xy - y^2$$

The first-order derivatives are:

$$f_x = 3x^2 + 5y \quad \text{and} \quad f_y = 5x - 2y$$

Substituting into $dz = f_x dx + f_y dy$, we have,

$$dz = (3x^2 + 5y)dx + (5x - 2y)dy$$

Furthermore,

$$f_{xx} = 6x \quad f_{yx} = 5 \quad f_{xy} = 5 \quad f_{yy} = -2$$

Substituting into $d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$, we have,

$$d^2z = 6xdx^2 + 10dxdy - 2dy^2$$

- S.O.C. for two choice variables function

Once the first-order necessary condition is satisfied, the second-order *sufficient* condition for a maximum (minimum) of $z = f(x, y)$ is,

$$d^2z < 0 \ (> 0) \text{ for arbitrary value of } dx \text{ and } dy, \text{ not both zero}$$

Question: What the second-order *necessary* conditions are?

- In the two-variable case, for any values of dx and dy, not both zero,

$$d^2z \begin{cases} < 0 & \Leftrightarrow \quad f_{xx} < 0;\ \ f_{yy} < 0;\ \text{and}\ \ f_{xx}f_{yy} > {f_{xy}}^2 \\ > 0 & \Leftrightarrow \quad f_{xx} > 0;\ \ f_{yy} > 0;\ \text{and}\ \ f_{xx}f_{yy} > {f_{xy}}^2 \end{cases}$$

(This translation will be discussed in the next subsection)

- For operational convenience, we often refer these second-order derivatives conditions instead of the second-order differential conditions.

Table 11.1 Conditions for relative extremum: $z = f(x, y)$

Condition	Maximum	Minimum
First-order necessary condition	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order *sufficient* condition	$f_{xx}, f_{yy} < 0$ $f_{xx} f_{yy} > {f_{xy}}^2$	$f_{xx}, f_{yy} > 0$ $f_{xx} f_{yy} > {f_{xy}}^2$

Note: in the case of $f_{xx} f_{yy} = {f_{xy}}^2$, the stationary value may nevertheless turn out to be an extremum. On the other hand, when $f_{xx} f_{yy} < {f_{xy}}^2$, the point is a saddle point.

Review: (a) Point A constitutes a maximum ($f_x = f_y = 0$, $f_{xx}, f_{yy} < 0$, $f_{xx}f_{yy} > {f_{xy}}^2$).

(b) Point B constitutes a minimum ($f_x = f_y = 0$, $f_{xx}, f_{yy} > 0$, $f_{xx}f_{yy} > {f_{xy}}^2$).

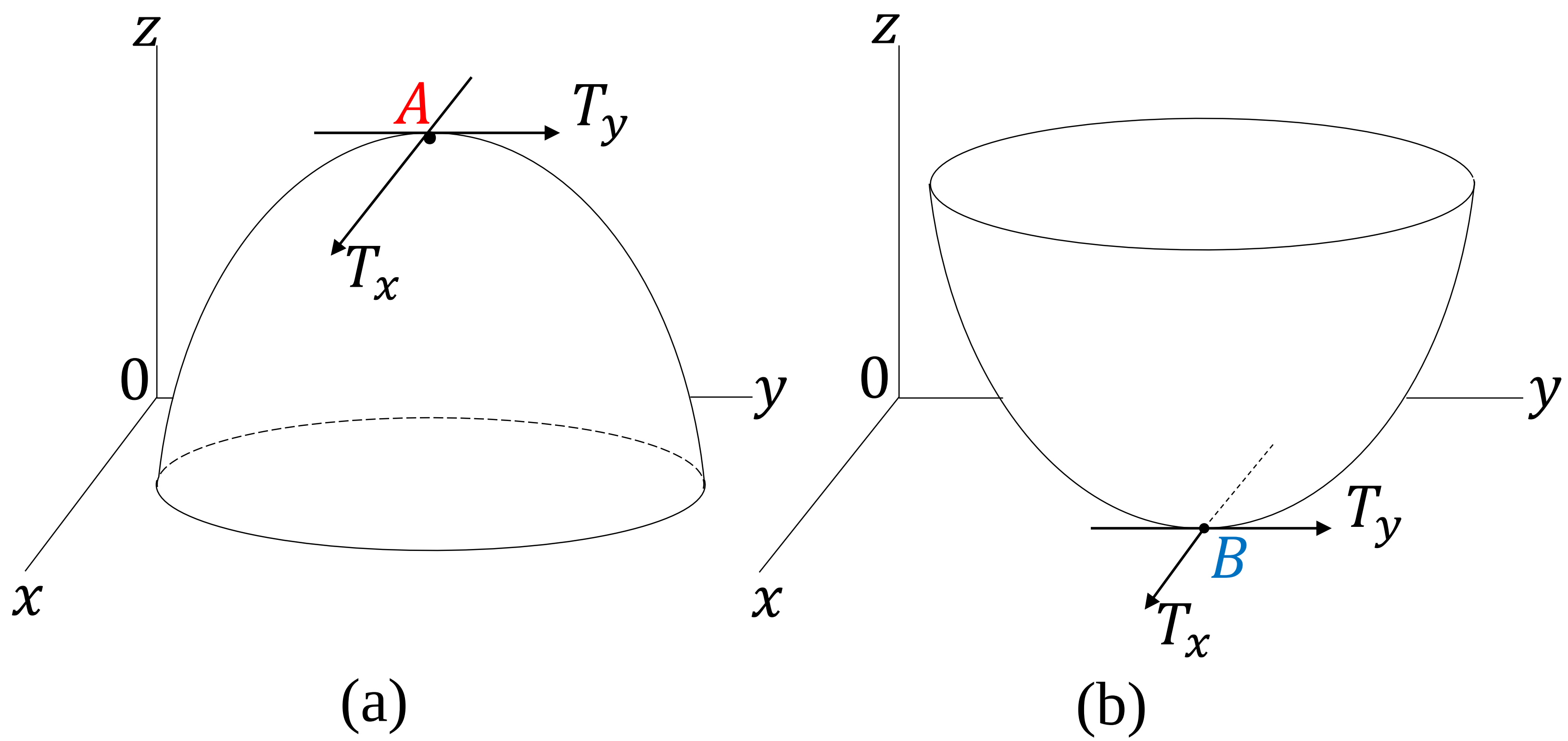

Figure 11.2

Example 3

Find the extreme value(s) of $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

$$f_x = 24x^2 + 2y - 6x, \qquad f_y = 2x + 2y,$$

$$f_{xx} = 48x - 6,\ f_{yy} = 2,\ \text{and}\ f_{xy} = 2$$

The first-order conditions are:

$$f_x = 24x^2 + 2y - 6x = 0$$

$$f_y = 2x + 2 = 0$$

The solutions for the above simultaneous equations are

$$x^* = y^* = 0, \text{and } x^* = \frac{1}{3},\ y^* = -\frac{1}{3}$$

Case 1. $x^* = y^* = 0$

$$f_{xx} = -6, \quad f_{yy} = 2 \quad \text{so} \quad f_{xx}f_{yy} \leq {f_{xy}}^2 \qquad \text{(fails S.O.C.)}$$

Case 2. $x^* = \frac{1}{3}, y^* = -\frac{1}{3}$

$$f_{xx} = 10, \; f_{yy} = f_{xy} = 2 \; \text{so} \; f_{xx}f_{yy} > {f_{xy}}^2 \qquad \text{(satisfies S.O.C.)}$$

There exists only one *minimum* which is characterized by following triplet:

$$(x^*, y^*, z^*) = (\frac{1}{3}, -\frac{1}{3}, \frac{23}{27})$$

11.3 Quadratic Forms

Let us define a *form* as a polynomial expression in which each component term has a uniform degree.

Linear form *(first degree)*	*Quadratic form* *(second degree)*
$x - 9y + z$	$4x^2 - xy + 3y^2$ $x^2 + 2xy - yw + 7w^2$

- Second-Order Total Differential as a Quadratic Form

Recall that the second-order total differential of z is given by:

$$d^2z = f_{xx}dx^2 + 2f_{xy}dxdy + f_{yy}dy^2$$

Let us define,

$$q \equiv d^2z, \qquad u \equiv dx, \qquad v \equiv dy, \qquad a \equiv f_{xx}, \qquad b \equiv f_{yy},$$

and $h \equiv f_{xy}[= f_{yx}]$

Then we can restate the above second-order total differential as:

$$q = au^2 + 2huv + bv^2$$

Note that in this quadratic form, $u \equiv dx$ and $v \equiv dy$ are cast in the role of *variables*, whereas the second partial derivatives are treated as constants.

- Positive and Negative Definiteness (定値性)

$q = au^2 + 2huv + bv^2$

If q is *invariably* $\left\{\begin{array}{ll} \text{negative} & (< 0) \\ \text{nonpositive} & (\leq 0) \\ \text{nonnegative} & (\geq 0) \\ \text{positive} & (> 0) \end{array}\right\}$ then q is said to be $\left\{\begin{array}{l} \text{negative definite} \\ \text{negative semidefinite} \\ \text{positive semidefinite} \\ \text{positive definite} \end{array}\right\}$

regardless of any values of variables in the quadratic form (*e.g.*, $u(\equiv dx)$ and $v(\equiv dy)$), not all zero.

- Determinantal Test for Sign Definiteness

Applying *completing the square* (平方完成) to q , we have,

$$q = au^2 + 2huv + \frac{h^2}{a}v^2 + bv^2 - \frac{h^2}{a}v^2$$
$$= a\left(u^2 + 2\frac{h}{a}uv + \frac{h^2}{a^2}v^2\right) + \left(b - \frac{h^2}{a}\right)v^2$$
$$= \boldsymbol{a}\left(u + \frac{h}{a}v\right)^2 + \frac{\boldsymbol{ab - h^2}}{a}v^2$$

We can predicate the sign of q on the values of coefficients $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{h}$ as follows:

$$q \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ iff } \begin{Bmatrix} \boldsymbol{a < 0} \\ \boldsymbol{a > 0} \end{Bmatrix} \text{ and } \boldsymbol{ab - h^2} > 0$$

- Matrix representation

$q = au^2 + 2huv + bv^2$

$$q = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

The condition can be alternatively expressed as:

$$q \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ iff } \begin{Bmatrix} |a| < 0 \\ |a| > 0 \end{Bmatrix} \text{ and } \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$$

$$\left(\det |a| = a, \det \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2\right)$$

Recall that $u \equiv dx,\ v \equiv dy,\ a \equiv f_{xx},\ b \equiv f_{yy},$ and $h \equiv f_{xy}[= f_{yx}]$, it yields,

$$q \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ iff } \begin{Bmatrix} |a| < 0 \\ |a| > 0 \end{Bmatrix} \text{ and } \begin{vmatrix} a & h \\ h & b \end{vmatrix} > 0$$

$$\Updownarrow$$

$$d^2z \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ iff } \begin{Bmatrix} f_{xx} < 0 \\ f_{xx} > 0 \end{Bmatrix} \text{ and } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0$$

The determinant with the second-order partial derivatives as its elements $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = |H|$ is called a *Hessian determinant* (or simply a *Hessian*).

Example 4

Is $q = 5u^2 + 3uv + 2v^2$ either positive or negative definite?

$$|a| = 5 > 0$$

$$|H| = \begin{vmatrix} 5 & 1.5 \\ 1.5 & 2 \end{vmatrix} = 5 \times 2 - 1.5^2 = 10 - 2.25 = 7.75 > 0$$

$\rightarrow q$ is positive definite

11.4 Objective Functions with More than Two Variables

- Let us consider a function of three choice variables:

$$z = f(x_1, x_2, x_3)$$

- F.O.C. for extremum

As our earlier discussion suggests, to have a maximum or minimum of z, it is necessary that $dz = 0$ for arbitrary value of dx_1, dx_2 and dx_3, not all zero.

$$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0$$

The only way to guarantee $dz = 0$ is to have,

$$f_1 = f_2 = f_3 = 0$$

- Second-Order Condition

$$d^2z = d(dz) = \frac{\partial(dz)}{\partial x_1}dx_1 + \frac{\partial(dz)}{\partial x_2}dx_2 + \frac{\partial(dz)}{\partial x_3}dx_3$$

$$= \frac{\partial}{\partial x_1}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_1 + \frac{\partial}{\partial x_2}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_2$$

$$+ \frac{\partial}{\partial x_3}(f_1dx_1 + f_2dx_2 + f_3dx_3)dx_3$$

$$= f_{11}dx_1{}^2 + f_{12}dx_1dx_2 + f_{13}dx_1dx_3 + f_{21}dx_2dx_1 + f_{22}dx_2{}^2 + f_{23}dx_2dx_3$$

$$+ f_{31}dx_3dx_1 + f_{32}dx_3dx_2 + f_{33}dx_3{}^2$$

Then we have the symmetric Hessian determinant:

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

whose leading principal minors may be denoted by

$$|H_1| = |\ f_{11}\ |, \quad |H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}, \quad |H_3| = |H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

What are the determinantal criteria for positive and negative definiteness?

- Quadratic form (Rough sketch)

A quadratic form with three variables can be converted into the following expression:

$$q = d_{11}{u_1}^2 + d_{12}u_1u_2 + d_{13}u_1u_3 + d_{21}u_2u_1 + d_{22}{u_2}^2 + d_{23}u_2u_3 + d_{31}u_3u_1 + d_{32}u_3u_2 + d_{33}{u_3}^2$$

$$= d_{11}\left(u_1 + \frac{d_{12}}{d_{11}}u_2 + \frac{d_{13}}{d_{11}}u_3\right)^2$$

$$+ \frac{d_{11}d_{12} - {d_{12}}^2}{d_{11}}\left(u_2 + \frac{d_{11}d_{23} - d_{12}d_{13}}{d_{11}d_{22} - {d_{12}}^2}u_3\right)^2$$

$$+ \frac{d_{11}d_{22}d_{33} - d_{11}{d_{23}}^2 - d_{22}{d_{13}}^2 - d_{33}{d_{12}}^2 + 2d_{12}\,d_{13}d_{23}}{d_{11}d_{22} - {d_{12}}^2}{u_3}^2$$

Hence, for positive definiteness (positive q), we have following three conditions,

$$|d_{11}| = |D_1| > 0, \quad \frac{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}}{|d_{11}|} = \frac{|D_2|}{|D_1|} > 0, \text{ and } \frac{\begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}}{\begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}} = \frac{|D_3|}{|D_2|} > 0$$

or,

$$|D_1| > 0 \qquad \text{(i)}$$

$$|D_2| > 0 \quad \text{(given that (i) already)} \quad \text{(ii)}$$

$$|D_3| > 0 \quad \text{(given that (ii) already)} \quad \text{(iii)}$$

To sum up, the second-order sufficient condition for an extremum of z is as follows:

$$z \text{ is } \begin{Bmatrix} \text{maximum} \\ \text{minimum} \end{Bmatrix} \text{ if } \begin{cases} |H_1| < 0;\ |H_2| > 0;\ \ |H_3| < 0\ \ (d^2z \text{ negative definite}) \\ |H_1| > 0;\ |H_2| > 0;\ \ |H_3| > 0\ \ (d^2z \text{ positive definite}) \end{cases}$$

In using this condition, we must evaluate all the leading principal minors at the stationary point where $f_1 = f_2 = f_3 = 0$.

Example 5

Find the extreme value(s) of

$$z = 2{x_1}^2 + x_1 x_2 + 4{x_2}^2 + x_1 x_3 + {x_3}^2 + 2$$

The first-order conditions for extremum are:

$$f_1 = 4x_1 + x_2 + x_3 = 0$$

$$f_2 = x_1 + 8x_2 = 0$$

$$f_3 = x_1 + 2x_3 = 0$$

This homogeneous linear-equation system has the single solution $x_1^* = x_2^* = x_3^* = 0$. This means that there is only one stationary value, $z^* = 2$.

The Hessian determinant of this function is

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

whose leading principal minors are all positive:

$$|H_1| = 4 > 0\,, \quad |H_2| = 31 > 0, \quad |H_3| = 54 > 0 \;\; (d^2z \text{ positive definite})$$

Thus we can conclude that $z^* = 2$ is a minimum.