12. Optimization with Equality Constraints

- In Chp. 11, all the choice variables are independent (no constraint).
- In this chapter, we consider a *constrained optimum*, in which there are dependence between the choice variables.

Example : production quota constraint: $Q_1 + Q_2 = 950$

12.1 Effects of a Constraint

Let us consider a consumer with the following utility function:

$$U = x_1 x_2 + 2x_1 \qquad (1)$$

Suppose that the consumer intends to spend a given sum, \$60, on the two goods and that the current prices are $P_{10} = 4$ and $P_{20} = 2$.

Then, the *budget constraint* can be expressed by the linear equation:

$$4x_1 + 2x_2 = 60 \qquad (2)$$

The problem now is to maximize (1) subject to the constraint (2).

After the budget constraint is added, the domain is immediately reduced to the set of points lying on the budget line.

Only that subsets of the utility surface lying directly above the budget-constraint line will now be relevant.

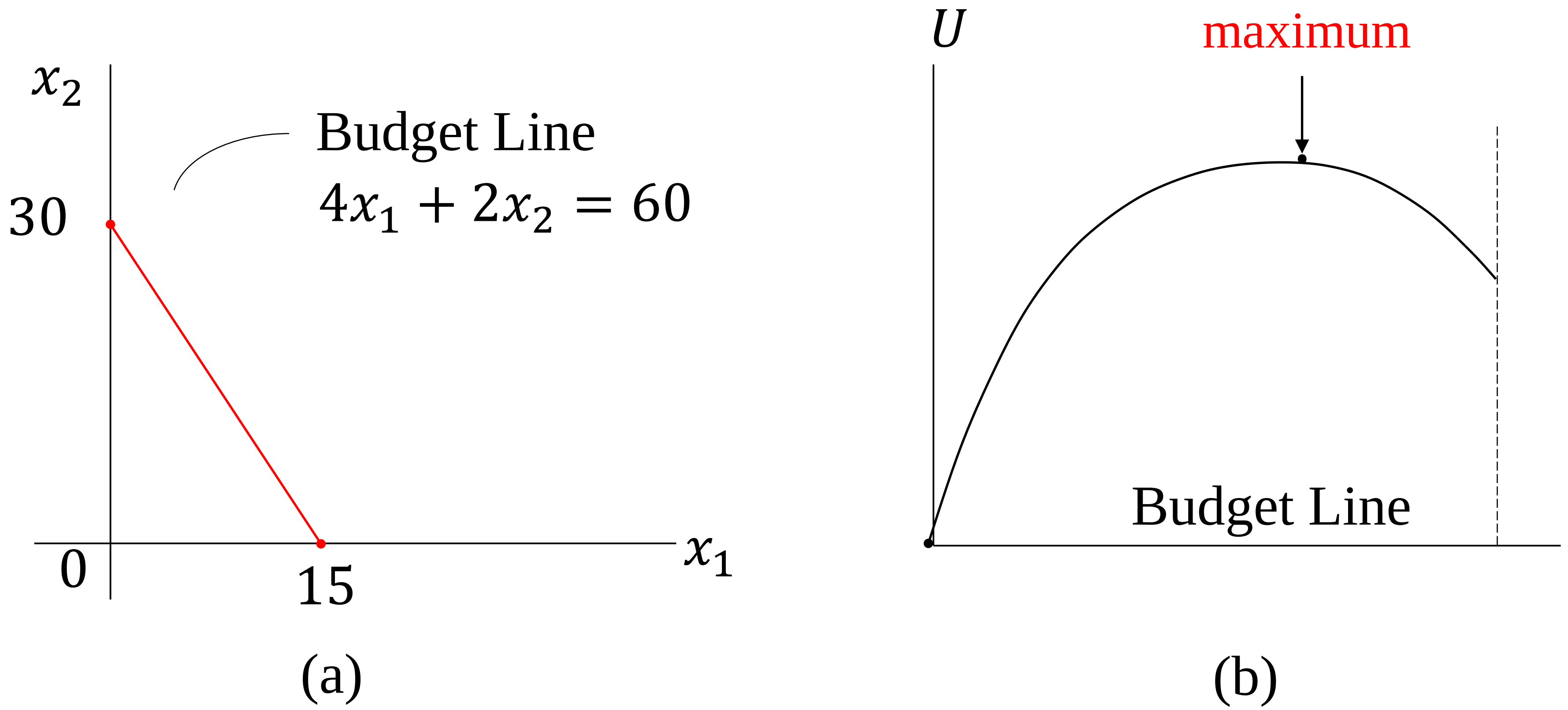

Figure 12.1

The difference between a constrained extremum and a free extremum in the three-dimensional graph of Figure 12.2

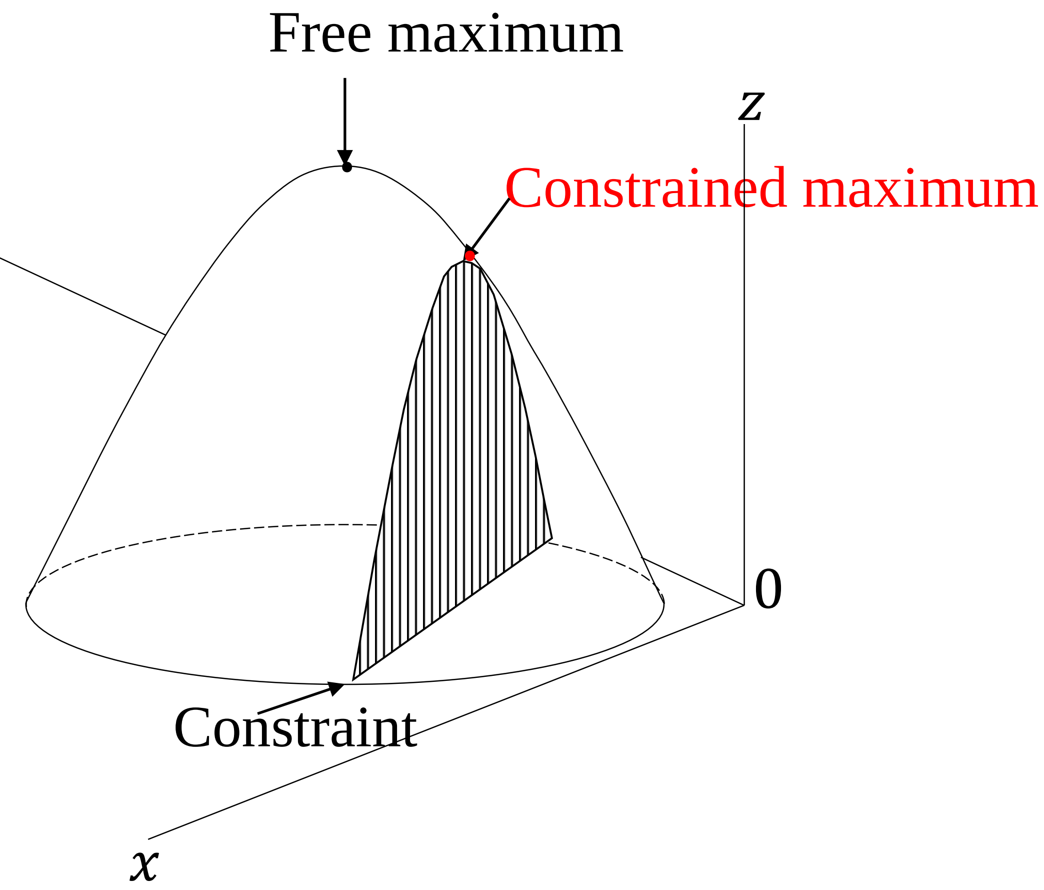

Figure 12.2

12.2 Finding the Stationary Values

- Technique of substitution and elimination of variables

The constraint (2) implies,

$$x_2 = 30 - 2x_1 \qquad (3)$$

Substituting the above equation into (1), it yields,

$$U = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2{x_1}^2$$

By setting the first derivative equal to zero:

$$\frac{dU}{dx_1} = 32 - 4x_1 = 0$$

We get the solution $x_1 = 8$, which by virtue of (3) leads to $x_2 = 14$. Then, we can find the stationary value $U^* = 128$.

Since the second derivative is,

$$\frac{d^2U}{dx_1{}^2} = -4 < 0$$

That stationary value constitutes a maximum of U.

However, when the constraint is a complicated function, or there are several constraints, this technique of substitution could become a burdensome task.

→ **Lagrange-Multiplier Method**

- Lagrange-Multiplier Method

$$Z \equiv x_1 x_2 + 2x_1 + \lambda(\mathbf{60 - 4x_1 - 2x_2})$$

Define the *Lagrangian function*, which is a modified version of the objective function that incorporates the constraint ($4x_1 + 2x_2 = 60$).

The symbol λ is called a *Lagrange multiplier*.

If the constraint is satisfied, then the last term in the above equation will vanish regardless of the value of λ. So, Z will be identical with U. Moreover, we only have to seek the free maximum of Z, instead of the constrained maximum of U.

We simply treat λ as an additional choice variable in the Lagrangian function.

The F.O.C. for free maximum of Z are:

$$\boldsymbol{Z_{\lambda}(\equiv \frac{\partial Z}{\partial \lambda}) = 60 - 4x_1 - 2x_2 = 0}$$

$$Z_{x_1}(\equiv \frac{\partial Z}{\partial x_1}) = x_2 + 2 - 4\lambda = 0$$

$$Z_{x_2}(\equiv \frac{\partial Z}{\partial x_2}) = x_1 - 2\lambda = 0$$

The first equation guarantees the satisfaction of the constraint.

Solving these equations for the critical values of the variables, we find,

$$x_1^* = 8,\ x_2^* = 14, \text{and } Z^* = 128$$

This is identical with the value of U^* found earlier.

Example 1

Find the extremum of

$$z = xy \quad \text{s.t. (subject to)} \quad x + y = 6$$

The Lagrangian function is,

$$Z \equiv xy + \lambda(6 - x - y)$$

For a stationary value of Z, it is necessary that,

$$Z_\lambda = 6 - x - y = 0,\ Z_x = y - \lambda = 0,\ Z_y = x - \lambda = 0$$

Thus, we find,

$$\lambda^* = 3 \quad x^* = 3 \quad y^* = 3$$

The stationary value is $Z^* = z^* = 9$ (For the moment, ignore a S.O.C.)

Example 2

Find the extremum of

$$z = {x_1}^2 + {x_2}^2 \quad s.t.\ x_1 + 4x_2 = 2$$

The Lagrangian function is

$$Z \equiv {x_1}^2 + {x_2}^2 + \lambda(2 - x_1 + 4x_2)$$

The F.O.C. for a stationary value is:

$$Z_\lambda = 6 - x_1 + 4x_2 = 0, \;\; Z_1 = 2x_1 - \lambda = 0, \;\; Z_2 = 2x_2 - 4\lambda = 0$$

The stationary value of Z, defined by the solution

$$\lambda^* = \frac{4}{17} \quad x_1^* = \frac{2}{17} \quad x_2^* = \frac{8}{17}$$

Therefore, $Z^* = z^* = \frac{4}{17}$.

In general, given an objective function:

$$z = f(x, y) \qquad (4)$$

subject to the constraint (c is constant):

$$g(x, y) = c, \qquad (5)$$

We can write the Lagrangian function as:

$$Z \equiv f(x, y) + \lambda[c - g(x, y)] \qquad (6)$$

For stationary value of Z, the first-order necessary condition is:

$$\begin{aligned} Z_\lambda &= c - g(x, y) = 0 \\ Z_x &= f_x(x, y) - \lambda g_x(x, y) = 0 \qquad (7) \\ Z_y &= f_y(x, y) - \lambda g_y(x, y) = 0 \end{aligned}$$

Since the first equation in (7) is simply a restatement of (5), the stationary value of the Lagrangian function Z will satisfy the constraint of the original function z.

And Since the expression $\lambda[c - g(x, y)]$ is now assuredly zero, the stationary values of Z in (6) must be identical with those of (4) subject to (5).

- Total-Differential Approach

In the discussion of the free extremum of $z = f(x, y)$, the first-order necessary condition is:

$$dz = f_x dx + f_y dy = 0 \qquad (8)$$

This statement remains valid after a constraint $g(x, y) = c$ is added. For if $g(x, y) = c$ is added, then dg must be equal to dc, which is zero since c is a constant. Hence,

$$(dg =) g_x dx + g_y dy = 0 \qquad (9)$$

From (8) and (9), it should be clear that in order to satisfy this necessary condition, we must have,

$$\frac{f_x}{g_x} = \frac{f_y}{g_y} \qquad (10)$$

We can find the critical values of x and y by using the above condition and the constraint $g(x, y) = c$.

- Total-differential approach and Lagrange-multiplier method

Consider the Lagrange-multiplier method again.

Recall that $Z_x = f_x(x, y) - \lambda g_x(x, y) = 0$ and $Z_y = f_y(x, y) - \lambda g_y(x, y) = 0$, these necessary conditions can be written as:

$$\frac{f_x}{g_x} = \lambda \quad \text{and} \quad \frac{f_y}{g_y} = \lambda \qquad (11)$$

These conditions convey precisely the same information as (10), in which is derived by Total-differential approach.

- An Interpretation of the Lagrange Multiplier

λ^* provides a measure of the sensitivity Z^*(and z^*) to changes in the constant *c*, i.e.,

$$\frac{dZ^*}{dc} = \lambda$$

To show this, let us perform the comparative-static analysis. (Recall *Implicit function theorem*).

To apply Implicit function theorem, taking the three equation in (7) to be in the form of $F^j(\lambda, x, y; c) = 0$ (with $j = 1,2,3$) as follows:

$$F^1 (= Z_\lambda) = c - g(x, y) = 0$$

$$F^2 = f_x(x, y) - \lambda g_x(x, y) = 0 \qquad (7)'$$

$$F^3 = f_y(x, y) - \lambda g_y(x, y) = 0$$

Review: Jacobian Determinants and Implicit function theorem (Rough sketch)

For n variables function, if we get all n^2 partial derivatives and arrange them into a square matrix, we obtain the following *Jacobian determinant* and *Jacobian matrix*:

$$\underbrace{|J|}_{Jacobian} \equiv \underbrace{\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}}_{Jacobian\ matrix}$$

If *Jacobian determinant* is nonzero, then we can apply the implicit function theorem.

(cont'd) Assuming that them to have continuous partial derivatives, we have,

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial \lambda} & \frac{\partial F^1}{\partial x} & \frac{\partial F^1}{\partial y} \\ \frac{\partial F^2}{\partial \lambda} & \frac{\partial F^2}{\partial x} & \frac{\partial F^2}{\partial y} \\ \frac{\partial F^3}{\partial \lambda} & \frac{\partial F^3}{\partial x} & \frac{\partial F^3}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{vmatrix}$$

For the moment, we assume that $|J| \neq 0$ (we will see in the next section).

If so, we can express λ^*, x^* and y^* as implicit functions of the parameter c:

$$\lambda^* = \lambda^*(c) \quad x^* = x^*(c) \quad \text{and} \quad y^* = y^*(c)$$

(from Implicit function theorem), we also have the equilibrium *identities*:

$$\begin{aligned} c - g(x^*, y^*) &\equiv 0 \\ f_x(x^*, y^*) - \lambda^* g_x(x^*, y^*) &\equiv 0 \\ f_y(x^*, y^*) - \lambda^* g_y(x^*, y^*) &\equiv 0 \end{aligned} \qquad (12)$$

Now since the optimal value of Z depends on λ^*, x^* and y^*, that is,

$$Z^* = f(x^*(c), y^*(c)) + \lambda^*(c)[c - g(x^*(c), y^*(c))]$$

Thus we can consider Z^* to be a function of c alone.

Differentiating Z^* totally with respect to c, we find,

$$\frac{dZ^*}{dc} = f_x \frac{dx^*}{dc} + f_y \frac{dy^*}{dc} + [c - g(x^*, y^*)] \frac{d\lambda^*}{dc} + \lambda^* \left(1 - g_x \frac{dx^*}{dc} - g_y \frac{dy^*}{dc}\right)$$

$$= \underbrace{(f_x - \lambda^* g_x)}_{=0 \text{ (from (12))}} \frac{dx^*}{dc} + \underbrace{(f_y - \lambda^* g_y)}_{=0} \frac{dy^*}{dc} + \underbrace{[c - g(x^*, y^*)]}_{=0} \frac{d\lambda^*}{dc} + \lambda^*$$

$$= \lambda^*$$

It yields,

$$\frac{dZ^*}{dc} = \lambda$$

The above expression shows that the solution value of the Lagrangian multiplier constitutes a measure of the effect of a change in the constraint via the parameter c on the optimal value of the objective function.

12.3 Second-Order Conditions

- Second-Order Total Differential

Since the constraint $g(x, y) = c$ means $dg = g_x dx + g_y dy$ as in (9) (pp.13), dx and dy no longer are both arbitrary.

If we take dx as an arbitrary change, then dy must satisfy,

$$dy = -\frac{g_x}{g_y} dx$$

To find an appropriate new expression for d^2z, we must treat dy as a variable dependent on x and y during differentiation. Thus,

$$d^2z = d(dz) = \frac{\partial(dz)}{\partial x}dx + \frac{\partial(dz)}{\partial y}dy$$

$$= \frac{\partial}{\partial x}\left(f_x dx + f_y dy\right)dx + \frac{\partial}{\partial y}\left(f_x dx + f_y dy\right)dy$$

$$= \left[f_{xx}dx + \left(f_{xy}dy + f_y\frac{\partial(dy)}{\partial x}\right)\right]dx + \left[f_{yx}dx + \left(f_{yy}dy + f_y\frac{\partial(dy)}{\partial y}\right)\right]dy$$

$$= f_{xx}dx^2 + f_{xy}dydx + f_y\frac{\partial(dy)}{\partial x}dx + f_{yx}dxdy + f_{yy}dy^2 + f_y\frac{\partial(dy)}{\partial y}dy$$

Since,

$$f_y\left[\frac{\partial(dy)}{\partial x}dx+\frac{\partial(dy)}{\partial y}dy\right]=f_y d(dy)=f_y d^2y,$$

the desired expression for d^2z is:

$$d^2z=f_{xx}dx^2+2f_{xy}dxdy+f_{yy}dy^2+f_y d^2y \quad (13)$$

By totally differentiating the constraint $g(x, y) = c$ twice, we have,

$$(d^2 g =) g_{xx} dx^2 + 2g_{xy} + g_{yy} dy^2 + g_y d^2 y = 0,$$

since $dg = 0$.

Solving this last equation for $d^2 y$ and substituting into (13), we obtain,

$$d^2 z = \left(f_{xx} - \frac{f_y}{g_y} g_{xx} \right) dx^2 + 2 \left(f_{xy} - \frac{f_y}{f_y} g_{xy} \right) dxdy + \left(f_{yy} - \frac{f_y}{g_y} g_{yy} \right) dy^2$$

Recall the first-order necessary condition:

$$Z_\lambda = c - g(x, y) = 0$$

$$Z_x = f_x(x, y) - \lambda g_x(x, y) = 0 \qquad (7)$$

$$Z_y = f_y(x, y) - \lambda g_y(x, y) = 0$$

By partially differentiating, we find the following second derivatives:

$$Z_{xx} = f_{xx} - \lambda g_{xx}$$

$$Z_{xy} = f_{xy} - \lambda g_{xy} = Z_{yx}$$

$$Z_{yy} = f_{yy} - \lambda g_{yy}$$

Then, we can finally express d^2z more neatly as follows:

$$d^2z = \underbrace{\left(f_{xx} - \frac{f_y}{g_y} g_{xx}\right)}_{=Z_{xx}} dx^2 + 2\underbrace{\left(f_{xy} - \frac{f_y}{f_y} g_{xy}\right)}_{=Z_{xy}} dxdy + \underbrace{\left(f_{yy} - \frac{f_y}{g_y} g_{yy}\right)}_{=Z_{yy}} dy^2$$

$$= Z_{xx}dx^2 + 2Z_{xy}dxdy + Z_{yy}dy^2$$

$$= Z_{xx}dx^2 + Z_{xy}dxdy + Z_{yx}dydx + Z_{yy}dy^2 \qquad (14)$$

It can give rise to a *Hessian* determinant $\begin{vmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{vmatrix}$.

- Second-Order Conditions

The second-order sufficient conditions are:

For maximum of z: d^2z negative definite, **subject to** $\boldsymbol{dg = 0}$

For minimum of z: d^2z positive definite, **subject to** $\boldsymbol{dg = 0}$**,**

Inasmuch as the dx, dy pairs satisfying the constraint $dg = g_x dx + g_y dy = 0$ constitute merely a subset of the set of all possible dx and dy. (*easier* to satisfy!)

- The Bordered Hessian

Let us first analyze the conditions for the sign definiteness of a two-variable quadratic form subject to a linear constraint:

$$q = au^2 + 2huv + bv^2 \quad s.t.\ \alpha u + \beta v = 0$$

Since the constraint implies $v = -(\alpha/\beta)u$, we can rewrite q as a function of one variable only:

$$q = au^2 - 2h\frac{\alpha}{\beta}u^2 + b\frac{\alpha^2}{\beta^2}u^2 = \left(\boldsymbol{a\beta^2 - 2h\alpha\beta + b\alpha^2}\right)\frac{u^2}{\beta^2}$$

q is negative (positive) definite *if and only if* the **expression in parentheses** is negative (positive) .

Notice that,

$$-\left(\alpha\beta^2 - 2h\alpha\beta + b\alpha^2\right) = \begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix}$$

Consequently, we can state that,

$$q \text{ is} \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ iff } \begin{vmatrix} 0 & \alpha & \beta \\ \alpha & a & h \\ \beta & h & b \end{vmatrix} \begin{cases} > 0 \\ < 0 \end{cases}$$

Notice that the original quadratic form determinant is $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$. In this sense, the determinant is called "*Bordered*". Moreover, the border is merely composed of the two coefficient α and β from the constraint, plus zero.

When applied to the quadratic form,

$$d^2z = Z_{xx}dx^2 + Z_{xy}dxdy + Z_{yx}dydx + Z_{yy}dy^2 \quad (14),$$

we have the following determinantal criterion for the sign definiteness of d^2z:

$$d^2z \text{ is } \begin{Bmatrix} \text{negative definite} \\ \text{positive definite} \end{Bmatrix} \text{ subject to } dg = 0 \text{ iff } \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix} \begin{cases} > 0 \\ < 0 \end{cases}$$

The determinant to the right, often referred to as a "*Bordered Hessian(フチ付きヘシアン)*", shall be denoted by $|\bar{H}|$. A positive (negative) $|\bar{H}|$ is *sufficient* to establish it as a relative maximum (minimum) of z.

Remark : Recall that we assumed the Jacobian determinant is not zero:

$$|J| = \begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda g_{xy} & f_{yy} - \lambda g_{yy} \end{vmatrix} \neq 0$$

Multiplying both the first column and the first row of this Jacobian by -1 (which will leave the value of the determinant unaltered), we have:

$$|J| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & Z_{xx} & Z_{xy} \\ g_y & Z_{yx} & Z_{yy} \end{vmatrix} = |\bar{H}|$$

Thus, in applying the implicit function theorem, we can use the condition $|\bar{H}| \neq 0$.

Example 4

Let us return to Example 1 ($z = xy \quad \text{s.t.}\ x + y = 6$) and ascertain whether the stationary value found there gives a maximum or a minimum.

The first and second-order partial derivatives are:

$$Z_x = y - \lambda,\ Z_y = x - \lambda,\ Z_{xx} = 0,\ Z_{xy} = Z_{yx} = 1, \text{ and } Z_{yy} = 0$$

The border elements are $g_x = 1$ and $g_y = 1$ (since $g = x + y = 6$). Thus we find,

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2 > 0$$

which establishes the value $z^* = 9$ as a maximum.

Example 5 (Example 2)

$$z = {x_1}^2 + {x_2}^2 \quad s.t.\ x_1 + 4x_2 = 2$$

We have,

$$Z_1 = 2x_1 - \lambda,\ Z_2 = 2x_2 - 4\lambda,\ Z_{11} = 2,\ Z_{12} = Z_{21} = 0, \text{ and } \ Z_{22} = 2$$

From the constraint $x_1 + 4x_2 = 2$, we obtain $g_1 = 1$ and $g_2 = 4$.

The bordered Hessian is,

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & 4 \\ 1 & 2 & 0 \\ 4 & 0 & 2 \end{vmatrix} = -34 < 0$$

Hence, the value $z^* = \frac{4}{17}$ is a minimum.

12.5 Utility Maximization and Consumer Demand

Let us reexamine the maximization of a utility function cited in Section 12.1. The problem is to maximize a smooth utility function,

$$U = U(x, y) \quad \left(U_x, U_y > 0\right)$$

$$\text{s.t.} \quad xP_x + yP_y = B$$

- First-Order Condition

The Lagrangian function of this problem is:

$$Z = U(x, y) + \lambda\left(B - xP_x - yP_y\right)$$

As the first-order condition, we have the following set of simultaneous equations:

$$Z_\lambda = B - xP_x - yP_y = 0$$
$$Z_x = U_x - \lambda P_x = 0$$
$$Z_y = U_y - \lambda P_y = 0$$

The last two equations are equivalent to:

$$\frac{U_x}{P_x} = \frac{U_y}{P_y} = \lambda$$

This equality means that in order to maximize utility, consumers must allocate their budgets so as to equalize the ratio of *marginal utility to price* for every goods. And in the optimum, these ratios should have the common value λ^*.

The optimum value of this Lagrangian multiplier can be interpreted as the ***marginal utility of money*** when the consumer’s utility is maximized.

If we restate the condition $\frac{U_x}{P_x} = \frac{U_y}{P_y}$ in the form,

$$\frac{U_x}{U_y} = \frac{P_x}{P_y},$$

the first-order condition can be given an alternative interpretation, in terms of *indifference curves*.

An *indifference curve* is defined as the locus of the combinations of x and y that will yield a **constant** level of U.

On an indifference curve, we must have,

$$dU = U_x dx + U_y dy = 0,$$

with the implication that,

$$\frac{dy}{dx} = -\frac{U_x}{U_y}$$

An indifference curve has a slope which is equal to the negative of the marginal-utility ratio U_x/U_y. Note that U_x/U_y, the negative of the indifference-curve *slope*, is called the *marginal rate of substitution (MRS)* between the two goods.

What about the meaning of P_x/P_y?

The budget constraint, $xP_x + yP_y = B$, can be written alternatively as:

$$y = \frac{B}{P_y} - \frac{P_x}{P_y}x$$

So, the negative of P_x/P_y is the *slope* of the budget constraint.

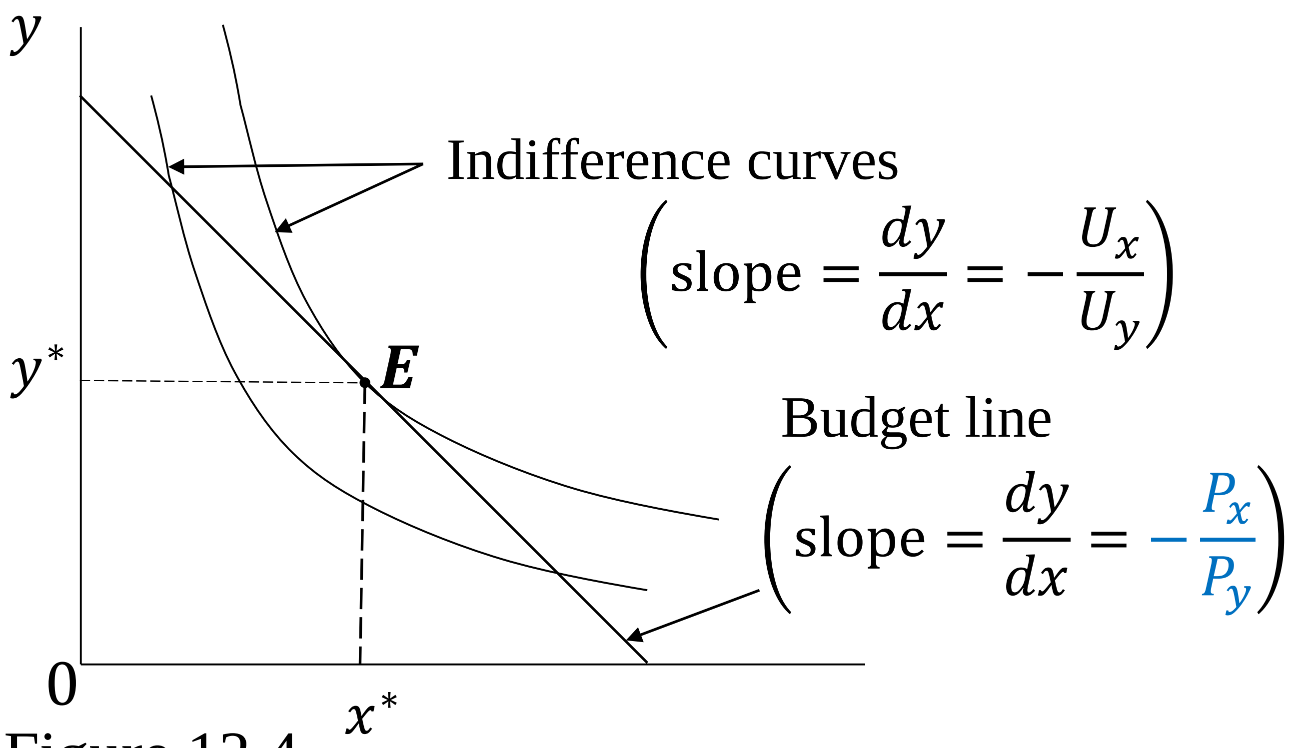

Figure 12.4

To maximize utility, a consumer must allocate the budget such that the slope of the budget line is equal to the slope of some indifference curve. This condition is met at point ***E*** in Figure 12.4.

- Second-Order Condition

If the bordered Hessian in the present problem is positive:

$$|\bar{H}| = \begin{vmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{vmatrix} = 2P_xP_yU_{xy} - P_y{}^2U_{xx} - P_x{}^2U_{yy} > 0,$$

then the stationary value of U will assuredly be a maximum.

A positive $|\bar{H}|$ means the strict convexity of the indifference curve at the point of tangency $\boldsymbol{E}$.

Question: Is the indifference curve *actually* strict convexity at the point of tangency $\boldsymbol{E}$?

The strict convexity would be ensured by a positive d^2y/dx^2. Recall that $\frac{dy}{dx} = -\frac{U_x}{U_y}$.

By totally differentiating this equation, we have the expression for d^2y/dx^2 such as:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{U_x}{U_y}\right) = -\frac{1}{{U_y}^2}\left(U_y\frac{dU_x}{dx} - U_x\frac{dU_y}{dx}\right)$$

As shown in Figure 12.4, y is itself a function of x along an indifference curve.

Then, we have,

$$\frac{dU_x}{dx} = U_{xx} + U_{yx}\frac{dy}{dx} \qquad \frac{dU_y}{dx} = U_{xy} + U_{yy}\frac{dy}{dx},$$

where dy/dx refers to the slope of the indifference curve.

At the point of tangency E, this slope is identical with that of the budget constraint ($\frac{dy}{dx} = -\frac{P_x}{P_y}$), it yields,

$$\frac{dU_x}{dx} = U_{xx} - U_{yx}\frac{P_x}{P_y} \qquad \frac{dU_y}{dx} = U_{xy} - U_{yy}\frac{P_x}{P_y} \qquad (15)$$

Since $\frac{U_x}{U_y} = \frac{P_x}{P_y}$, we have,

$$U_x = \frac{U_y P_x}{P_y} \qquad (16)$$

Substituting (15) and (16) into $\frac{d^2y}{dx^2}$, we finally obtain,

$$\frac{d^2y}{dx^2} = -\frac{1}{{U_y}^2}\left(U_y\left(U_{xx} - U_{yx}\frac{P_x}{P_y}\right) - \left(\frac{U_yP_x}{P_y}\right)\left(U_{xy} - U_{yy}\frac{P_x}{P_y}\right)\right)$$

$$= \frac{\boldsymbol{2P_xP_yU_{xy} - {P_y}^2U_{xx} - {P_x}^2U_{yy}}}{U_y{P_y}^2} = \frac{1}{U_y{P_y}^2}|\boldsymbol{\bar{H}}|$$

Hence, when the second-order sufficient condition ($|\bar{H}| > 0$) is satisfied, the second derivative $\frac{d^2y}{dx^2}$ is positive, and the relevant indifferent curve is actually strictly convex at the point of tangency $\boldsymbol{E}$.

12.6 Homogeneous Functions

Definition: A function $f(x)$ is *homogeneous* of degree r, if

$$f(jx_1,\dots,jx_n) = j^r f(x_1,\dots,x_n)$$

Example 1

$$f(x,y,z) = \frac{x}{y} + \frac{2z}{3x}$$

Multiplying j each variables, we obtain,

$$f(jx,jy,jz) = \frac{jx}{jy} + \frac{2jz}{3jx} = \frac{x}{y} + \frac{2z}{3x} = j^0 f(x,y,z)$$

The function f is homogeneous of degree 0.

Example 2

$$g(x,y,z) = \frac{x^2}{y} + \frac{2z^2}{3x}$$

$$g(jx,jy,jz) = \frac{(jx)^2}{jy} + \frac{2(jz)^2}{3jx} = j\left(\frac{x}{y} + \frac{2z}{3x}\right) = j^1 g(x,y,z)$$

The function g is homogeneous of degree 1.

- Linear Homogeneity

In the discussion of production functions, wide use is made of homogeneous functions of the first degree. These functions are often referred to as *linearly homogeneous* functions.

Let us adopt as the framework of our discussion a production function in the form, say,

$$Q = f(K, L) \qquad (17)$$

The mathematical assumption of linear homogeneity would amount to the economic assumption of *constant returns to scale*.

What unique properties characterize this linearly homogeneous production function?

- **Property I** The average physical product of labor (APP_L) and of capital (APP_K) can be expressed as functions of the capital-labor ratio, $k \equiv K/L$, alone.

Proof: Since $f(K, L)$ is linearly homogeneous, we have,

$$f(jK, jL) = jf(K, L) = jQ$$

Substituting $1/L$ into j, we obtain,

$$f\left(\frac{K}{L}, \frac{L}{L}\right) = f\left(\frac{K}{L}, 1\right) = f(k, 1)\big(\equiv \phi(k)\big) = \frac{Q}{L}$$

It implies $APP_L \equiv \frac{Q}{L} = \phi(k)$. Likewise, $APP_K \equiv \frac{Q}{K} = \frac{Q}{L}\frac{L}{K} = \frac{\phi(k)}{k}$.

Since both average products depend on k alone, both APP_L and APP_K are homogeneous of degree zero in the variables K and L.

- **Property II** The marginal physical products MPP_K and MPP_L can be expressed as functions of k alone.

Proof : We first differentiate k w.r.t. (with respect to) K and L. It yields:

$$\frac{\partial k}{\partial K} = \frac{\partial}{\partial K}\left(\frac{K}{L}\right) = \frac{1}{L} \qquad \frac{\partial k}{\partial L} = \frac{\partial}{\partial L}\left(\frac{K}{L}\right) = -\frac{K}{L^2}$$

Since $APP_L \equiv \frac{Q}{L} = \phi(k)$, the total product can be expressed as:

$$Q = L\phi(k)$$

Then differentiate Q with respect to K and L, we obtain MPP_K and MPP_L as follows:

$$MPP_K \equiv \frac{\partial Q}{\partial K} = \frac{\partial L\phi(k)}{\partial K} = L\frac{\partial \phi(k)}{\partial K} = L\frac{d\phi(k)}{dk}\frac{\partial k}{\partial K} \quad \text{(Chain rule)}$$

$$= L\phi'(k)\left(\frac{1}{L}\right) = \phi'^{(k)}$$

$$MPP_L \equiv \frac{\partial Q}{\partial L} = \frac{\partial L\phi(k)}{\partial L} = \phi(k) + L\frac{\partial \phi(k)}{\partial L} = \phi(k) + L\frac{d\phi(k)}{dk}\frac{\partial k}{\partial L}$$

$$= \phi(k) + L\phi'(k)\frac{\partial k}{\partial L} = \phi(k) + L\phi'(k)\frac{-K}{L^2} = \phi(k) - k\phi'(k)$$

Since both marginal products depend on k alone, both MPP_K and MPP_L are homogeneous of degree zero in the variables K and L.

- **Property III (Euler's Theorem)** If $Q = f(K, L)$ is linearly homogeneous, then,

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} \equiv Q$$

Proof : Recall that $MPP_K = \phi'^{(k)}$ and $MPP_L = \phi(k) - k\phi'(k)$. It yields:

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = K\phi'(k) + L[\phi(k) - k\phi'(k)]$$

$$= K\phi'(k) + L\phi(k) - K\phi'(k) \quad \text{(Since } kL = K\text{)}$$

$$= L\phi(k)$$

$$= Q$$

Note that this result is valid for any values K and L. (You may use these properties soon)

- Cobb-Douglas Production Function

One specific production function widely used in economic analysis is the *Cobb-Douglas production function*:

$$Q = AK^{\alpha}L^{1-\alpha}$$

where A is a positive constant, and $\alpha \in [0, 1]$. What we shall consider here first is a generalized version of this function.

$$Q = AK^{\alpha}L^{\beta}$$

Some of the major features of this function are:

1. it is homogeneous of degree $\alpha + \beta$;

$$A(jK)^{\alpha}(jL)^{\beta} = j^{\alpha+\beta}AK^{\alpha}L^{\beta} = j^{\alpha+\beta}Q$$

2. in the special case of $\alpha + \beta = 1$, it is linearly homogeneous;

$$A(jK)^{\alpha}(jL)^{\beta} = j^{\alpha+\beta}Q = jQ$$

3. its isoquants(等量曲線) are negatively sloped throughout and strictly convex for positive values of K and L.

For any positive output Q_0, $Q = AK^{\alpha}L^{\beta}$ can be written as:

$$Q_0 = AK^{\alpha}L^{\beta}$$

Taking the natural log of both sides, we have,

$$\ln A + \alpha \ln K + \beta \ln L - \ln Q_0 = 0,$$

which implicitly defines K as a function of L.

By the implicit function rule and the log rule,

$$\frac{dK}{dL} = -\frac{\partial F/\partial L}{\partial F/\partial K} = -\frac{\beta/L}{\alpha/K} = -\frac{\beta K}{\alpha L} < 0$$

Then,

$$\frac{d^2K}{dL^2} = \frac{d}{dL}\left(-\frac{\beta K}{\alpha L}\right) = -\frac{\beta}{\alpha}\frac{1}{L^2}\left(L\frac{dK}{dL} - K\right) > 0$$

The strict convexity would be ensured by a positive $\frac{d^2K}{dL^2}$.

Thus, the isoquant is downward-slopping throughout and strictly convex in the LK plane for positive values of K and L.

- $\alpha + \beta = 1$ case

The total product in this special case is expressed as:

$$Q = AK^{\alpha}L^{1-\alpha} = A\left(\frac{K}{L}\right)^{\alpha} L = LAk^{\alpha}$$

Therefore, the average products are:

$$APP_L = \frac{Q}{L} = Ak^{\alpha}$$

$$APP_K = \frac{Q}{K} = \frac{LAk^{\alpha}}{K} = \frac{Ak^{\alpha}}{k} = Ak^{\alpha-1}$$

And the marginal products are

$$\frac{\partial Q}{\partial K} = A\alpha K^{\alpha-1}L^{-(\alpha-1)} = A\alpha\left(\frac{K}{L}\right)^{\alpha-1} = A\alpha k^{\alpha-1}$$

$$\frac{\partial Q}{\partial L} = AK^{\alpha}(1-\alpha)L^{-\alpha} = A(1-\alpha)\left(\frac{K}{L}\right)^{\alpha} = A(1-\alpha)k^{\alpha}$$

We can verify Euler's theorem.

$$K\frac{\partial Q}{\partial K} + L\frac{\partial Q}{\partial L} = KA\alpha k^{\alpha-1} + LA(1-\alpha)k^{\alpha}$$

$$= LAk^{\alpha}\left[\frac{K\alpha}{Lk} + 1 - \alpha\right]$$

$$= LAk^{\alpha}[\alpha + 1 - \alpha] = LAk^{\alpha} = Q$$

If each input is assumed to be paid by the amount of its marginal product, the relative share of total product accruing to capital will be

$$\frac{K}{Q}(\partial Q/\partial K) = \frac{KA\alpha k^{\alpha-1}}{LAk^{\alpha}} = \alpha$$

Similarly, labor's relative share will be

$$\frac{L}{Q}(\partial Q/\partial L) = \frac{LA(1-\alpha)k^{\alpha}}{LAk^{\alpha}} = 1-\alpha$$

Thus the exponent of each input variable (α and $1-\alpha$) indicates the relative share of that input in the total product.