8. Comparative-Static Analysis of General-Function Models

8.1 Differentials

Difference (差商): $\frac{\Delta y}{\Delta x} = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$

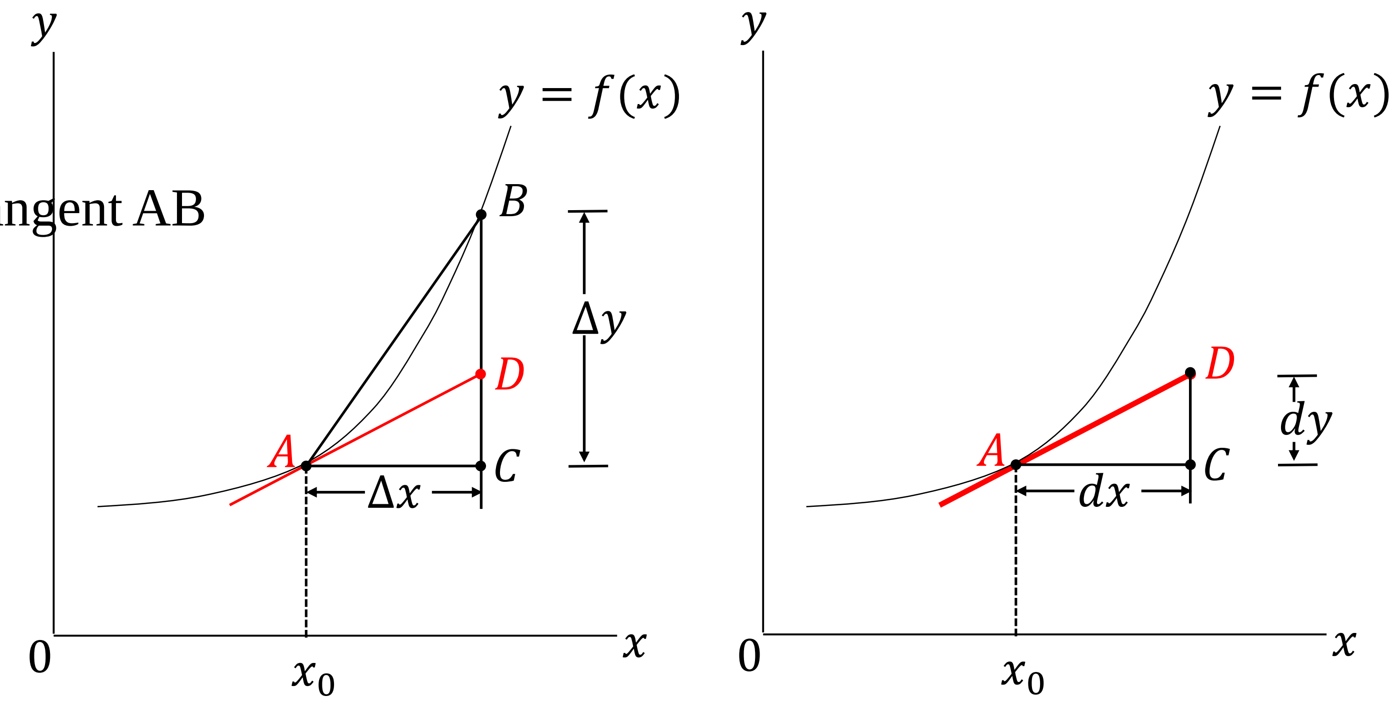
- Differentials and Derivatives

The **derivative** : $dy/dx = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ (1)

In Figure 8.1,

$$\frac{dy}{dx} = \text{slope of tangent } AD = f'(x). \qquad (2)$$

$\frac{CB}{AC} = \frac{\Delta y}{\Delta x}$: slope of tangent AB



(a) (b)

Figure 8.1

In that sense, the slope of tangent AB closes to the slope of tangent AD as $\Delta x \to 0$.

Multiplying the **derivative** $dy/dx = f'(x)$ by dx, we have,

$$dy = f'(x)dx. \qquad (3)$$

The process of finding the differential dy from a given function $y = f(x)$ is called ***differentiation.***

Example 1

Given $y = 3x^2 + 7x - 5$, find differential dy.

$$dy = \left(\frac{dy}{dx}\right)dx = (6x + 7)dx \qquad (4)$$

- Differentials and Point Elasticity (omit)
  - For a demand function $Q = f(P)$, the *point elasticity* of demand is defined as:

$$\varepsilon_d \equiv \frac{dQ/dP}{Q/P} \tag{5}$$

  - In general, for any function $y = f(x)$, the point elasticity of y w. r. t. x is:

$$\varepsilon_{yx} \equiv \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}} \tag{6}$$

Example 2

Find ε_d if the demand function is $Q = 100 - 2P$.

$$\frac{Q}{P} = \frac{100-2P}{P} \quad \text{and} \quad \frac{dQ}{dP} = -2. \tag{7}$$

Thus,

$$\varepsilon_d = \frac{-2}{(100-2P)/(P)} = \frac{-P}{50-P} \tag{8}$$

Review: Partial Differentiation

- Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n). \tag{9}$$

If the variable x_1 change Δx_1 while $x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely Δy.

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1}. \tag{10}$$

More generally, if we take the limit of $\frac{\Delta y}{\Delta x_i}$ as $\Delta x_i \to 0$, we have the partial derivative of y with respect to x_i:

$$f_{x_i} \equiv \frac{\partial y}{\partial x_i} \equiv \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}. \tag{11}$$

- Geometric Interpretation of Partial Derivatives

Consider a production function $Q = Q(K, L)$. Let us hold capital fixed at the level K_0 and consider *only* variations on the input L. The *slope* of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L.

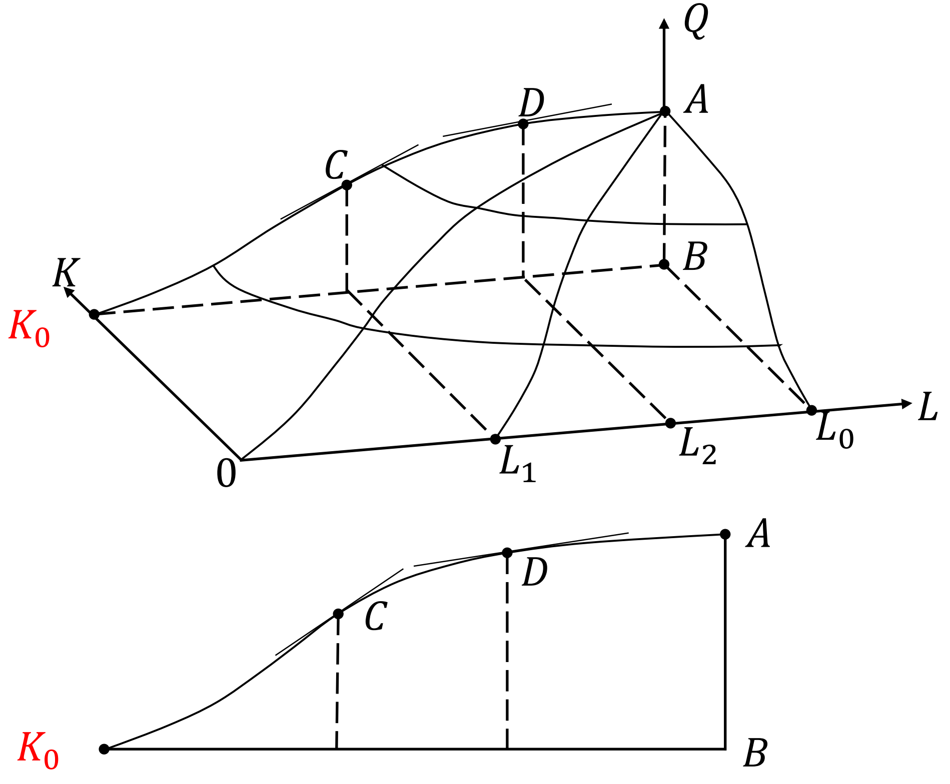


Figure 8.2

8.2 Total Differentials

Consider saving function

$$S = S(Y, i). \tag{12}$$

(S: savings, Y: national income, and i: interest rate)

- Total differential of the saving function

The **total change** in S with respect to infinitesimal changes in Y and i:

$$dS = \underbrace{\frac{\partial S}{\partial Y} dY}_{\text{changes in Y}} + \underbrace{\frac{\partial S}{\partial i} di}_{\text{changes in } i} \quad \text{or} \quad dS = S_Y dY + S_i di \tag{13}$$

Geometrically, total differential corresponds to the tangential plane as shown in Figure 8.3,

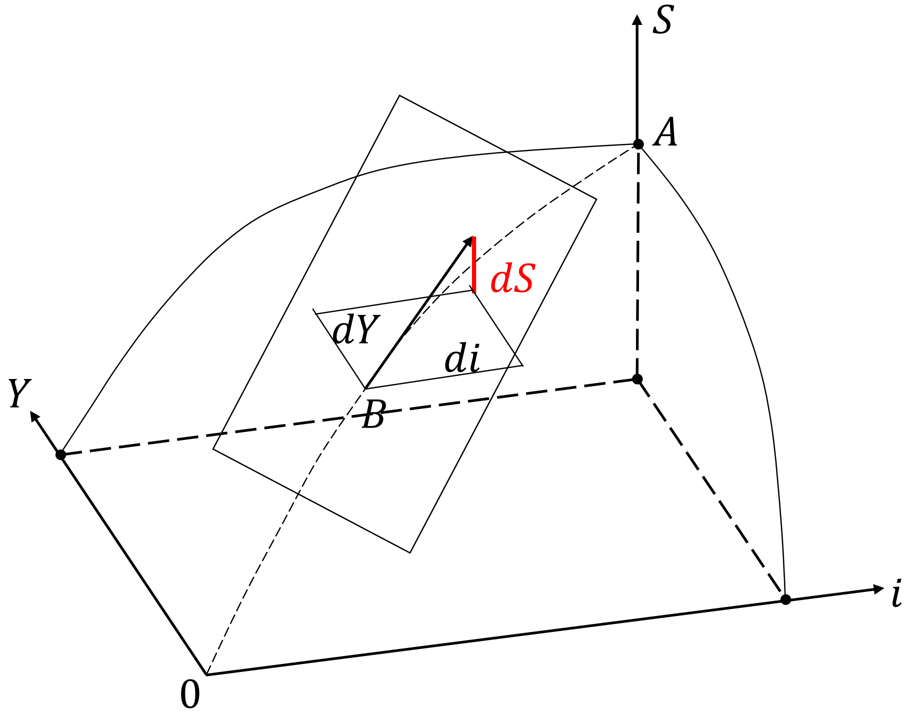


Figure 8.3

- For more general case such as

$$U = U(x_1, x_2, \ldots, x_n), \tag{14}$$

the total differential of U can be written as

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \cdots + \frac{\partial U}{\partial x_n} dx_n \tag{15}$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \cdots + U_n dx_n = \sum_{i=1}^{n} U_i dx_i \,. \tag{16}$$

Example 1

(a) $U(x_1, x_2) = ax_1 + bx_2$

(b) $U(x_1, x_2) = x_1^2 + x_2^3 + x_1x_2$

The total differentials are as follows:

(a)
$$\frac{\partial U}{\partial x_1} = U_1 = a, \quad \frac{\partial U}{\partial x_2} = U_2 = b. \tag{17}$$

Thus,
$$dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2. \tag{18}$$

(b)
$$\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2, \quad \frac{\partial U}{\partial x_2} = U_2 = 3x_2^2 + x_1. \tag{19}$$

Thus,
$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + (3x_2^2 + x_1)dx_2. \tag{20}$$

8.3 Rules of Total Differentials (omit)

c: constant u and v: functions

Rule I	$dc = 0$	(constant-function rule)
Rule II	$d(cu^n) = cnu^{n-1}du$	(power-function rule)
Rule III	$d(u \pm v) = du \pm dv$	(sum-difference rule)
Rule IV	$d(uv) = v\,du + u\,dv$	(product rule)
Rule V	$d\left(\frac{u}{v}\right) = \frac{1}{v^2}(v\,du - u\,dv)$	(quotient rule)

8.4 Total Derivatives (全導関数)

- Finding the Total Derivative

Let us consider any function

$$y = f(x, w) \quad \text{where} \quad x = g(w). \tag{21}$$

Thus, $y = f(g(w), w)$.

The variable w can affect y through two channels:

(1) directly, via the function f.

(2) indirectly, via the function g and then f.

Chain Rule !

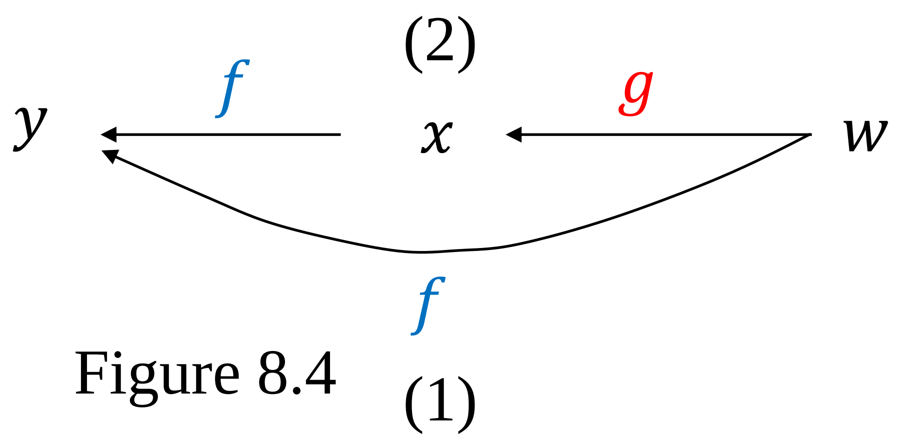


Figure 8.4

- To obtain this total derivative, we first differentiate y totally to get the total differential

$$dy = f_x dx + f_w dw.$$

- Dividing both sides of this equation by the differential dw,

$$\frac{dy}{dw} = \underbrace{f_x \frac{dx}{dw}}_{indirect\ effect} + \underbrace{f_w \frac{dw}{dw}}_{direct\ effect}$$

$$= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \qquad (22)$$

Example 1

Find the total derivative dy/dw, given the function

$$y = f(x, w) = 3x - w^2 \quad \text{where} \quad x = g(w) = 2w^2 + w + 4. \quad (23)$$

By virtue of (22), the total derivative is

$$\frac{dx}{dw} = g'(w) = 4w + 1$$

$$\frac{dy}{dw} = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$$

$$= 3(4w + 1) + (-2w) = 10w + 3. \quad (24)$$

- A Variation on the Theme

Let us consider any function

$$y = f(x_1, x_2, w) \qquad \text{where} \quad \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases} \tag{25}$$

The variable w can affect y through three channels:

(1) indirectly, via the function g and then f.

(2) indirectly, via the function h and then f.

(3) directly, via the function f.

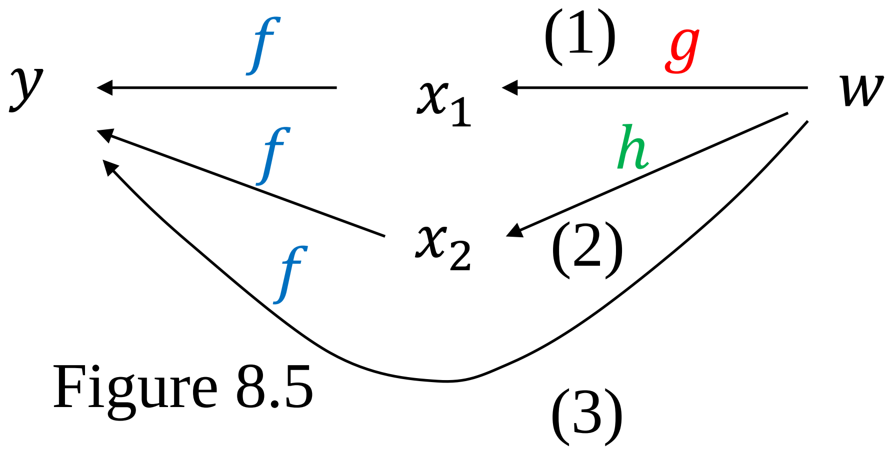


Figure 8.5

The total derivative of y with respect to w is given by

$$\frac{dy}{dw} = \frac{\partial y}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial y}{\partial w}$$

$$= \underbrace{f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw}}_{\text{indirect via } x_1 \text{ and } x_2} + \underbrace{f_w}_{\text{direct}} \qquad (26)$$

Example 2

Let the production function be

$$Q = Q(K, L, t) \tag{27}$$

(K: Captal, L: Labor, t: time)

Let

$$K = K(t) \qquad \text{and} \qquad L = L(t). \tag{28}$$

Then, the rate of change of output with respect to time can be expressed as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t} \tag{29.1}$$

or

$$\frac{dQ}{dt} = Q_K K'(t) + Q_L L'(t) + Q_t. \tag{29.2}$$

8.5 Derivatives of Implicit Functions

Implicit function theorem (Rough sketch):

Given a *relation* of the form $F(y, x_1, \dots, x_m) = 0$,

(A) if the function F has continuous partial derivatives $F_y, F_1, \dots, F_m$. And,

(B) if $F_y(y_0, x_{10}, \dots, x_{m0}) \neq 0$, at a point $(y_0, x_{10}, \dots, x_{m0})$ which satisfies $F(y_0, x_{10}, \dots, x_{m0}) = 0$.

Then we can find a function $y_0 = f(x_{10}, \dots, x_{m0})$ in which is defined by $F(y, x_1, \dots, x_m) = 0$ at the m-dimensional neighborhood of $(x_{10}, \dots, x_{m0})$ N.

Furthermore, ***implicit function theorem*** implies,

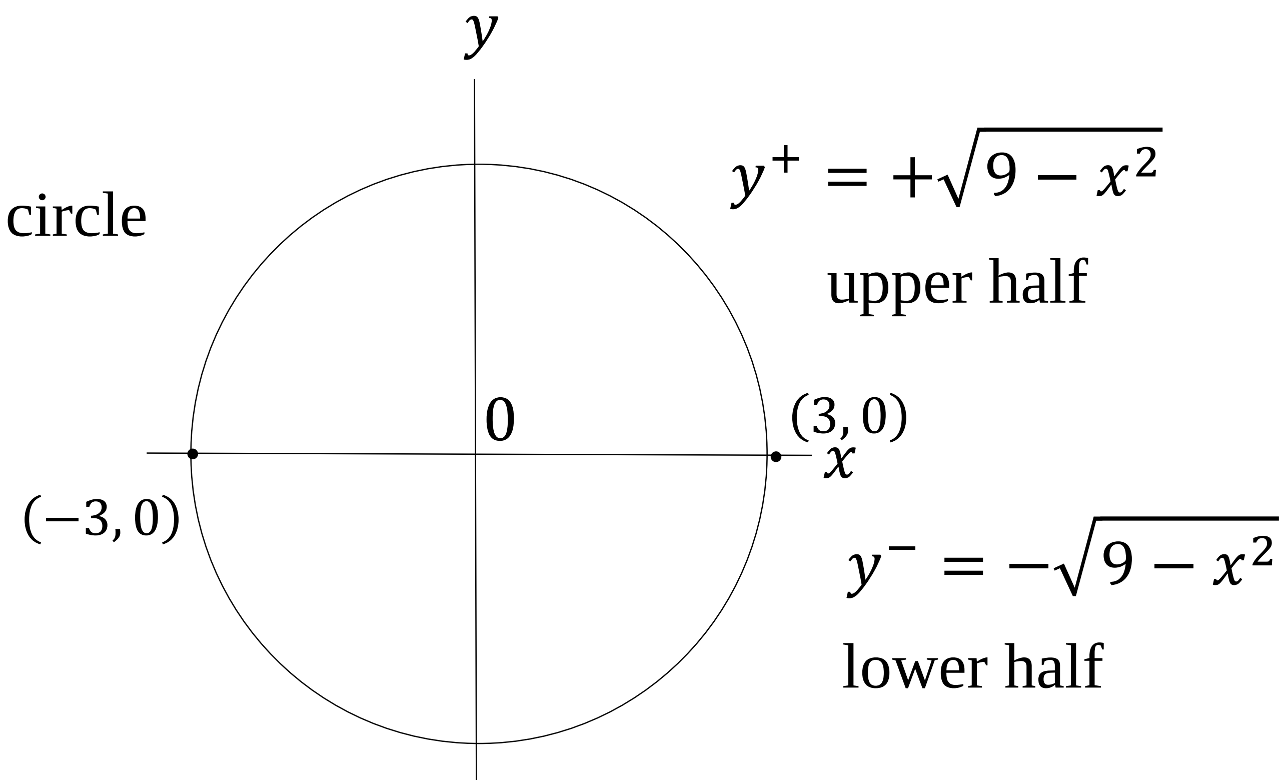
$$\left.\frac{dy}{dx_i}\right|_{other\ variables\ constant} = -\frac{F_{x_i}}{F_y}$$

Example 1

Suppose that variables *x* and *y* have the following relation

$$F(y, x) = x^2 + y^2 - 9 = 0.$$

Find $\partial y/\partial x$ by using the implicit function theorem. (For the moment, you can ignore the conditions which must be satisfied to apply the theorem).



First, we derive the partial derivatives as follows:

$$F_y = 2y, \quad \text{and} \quad F_x = 2x.$$

Recall that the implicit function theorem implies $\frac{dy}{dx} = -\frac{F_x}{F_y}$. It yields,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y},$$

Thus, $\frac{dy}{dx} = -\frac{x}{y}$.

Note : we can derive $\frac{dy}{dx}$ by differentiating $y^+ = +\sqrt{9 - x^2}$ directly as follows:

$$\frac{dy^+}{dx} = \frac{d}{dx}(9 - x^2)^{1/2} = \frac{-x}{\sqrt{9-x^2}} = -\frac{x}{y^+}$$

- Implicit Function (*Advanced*)

Consider a *relation* of the form $F(y, x) = 0$. To satisfy this relation, x and y no longer take arbitrary values each other. Thus, with some abusing of terminology, we can regard y as a *function* of x (say, $y = f(x)$). By substituting $y = f(x)$ into the above form, it yields a relation of form $F(f(x), x) = 0$.

If this relation holds for a neighborhood $\boldsymbol{N}$ of a point x, $f(x)$ is *implicitly* defined by a relation $F(y, x) = 0$, and refer such a $f(x)$ as an ***implicit function***.

$y^2 = 1 - x^2$

Note that in some cases, we can find more than one explicit function (e.g., circle). In that sense, we often refer to y as the "*multivalued function*"(多価関数) of x, rather than the "*function*".

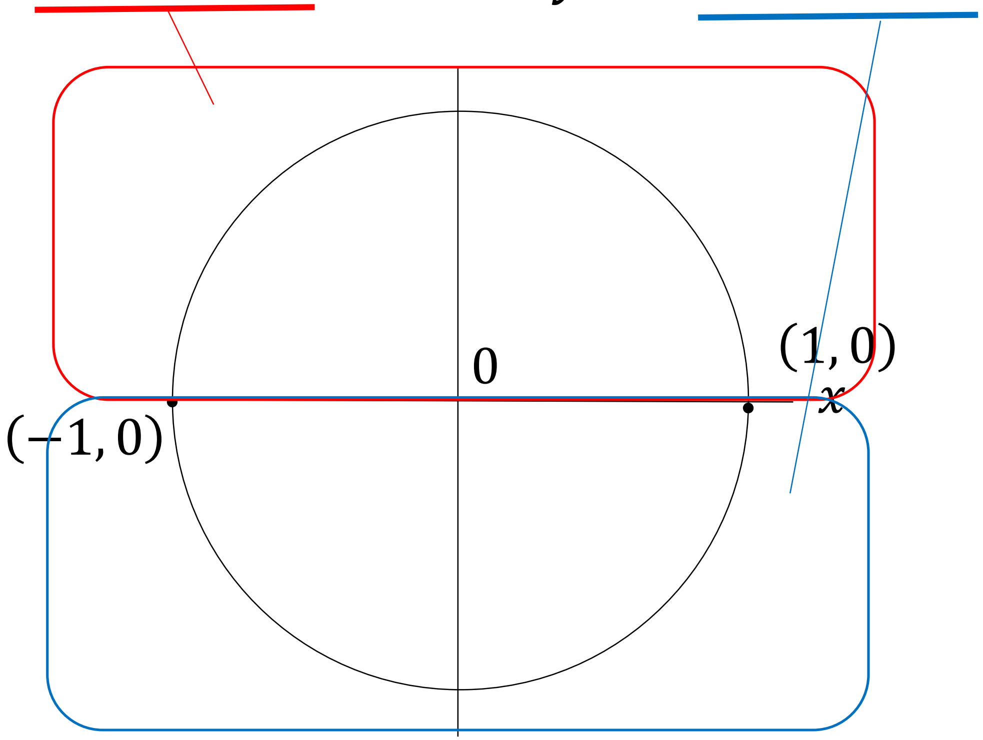
Example: $x^2 + y^2 = 1$ (*unit circle*).

A relation of form $x^2 + y^2 = 1$ defines the unit circle. If we solve this for y:

$$y^2 = 1 - x^2,$$

thus we obtain ***two*** functions:

$$y^+ = +\sqrt{1 - x^2} \quad \text{or} \quad y^- = -\sqrt{1 - x^2}$$



In general, there may not be a single function whose graph can represent the entire relation, but there may be such a function on a restriction of the domain of the relation.

For example, when we remove the lower half of the unit circle, we have

$$y = \sqrt{1 - x^2}.$$

and this function represents the entire relation of *y* and *x*.

Roughly speaking, the *implicit function theorem* is a existence theorem in which gives a *sufficient* condition to ensure that there is such a function.

Implicit Function Theorem

Given

$$F(y, x_1, \dots, x_m) = 0,$$

(A) if the function F has continuous partial derivatives $F_y, F_1, \dots, F_m$,

and

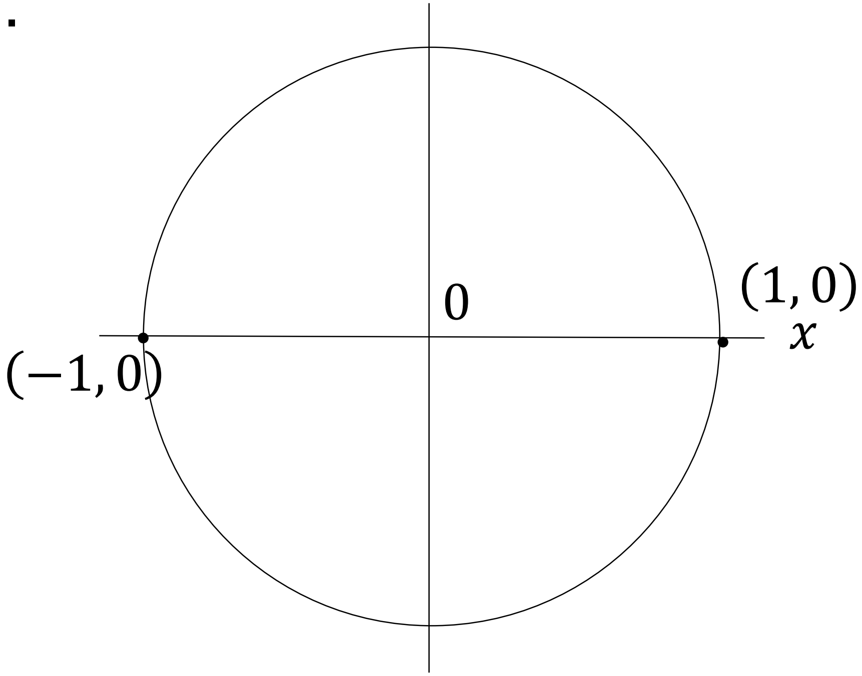
(B) if $F_y(y_0, x_{10}, \dots, x_{m0}) \neq 0$ at a point $(y_0, x_{10}, \dots, x_{m0})$ which satisfies $F(y_0, x_{10}, \dots, x_{m0}) = 0$,

then there exists an m-dimensional neighborhood of $(x_{10}, \dots, x_{m0})$ N, in which y is an implicitly defined function of the variables $x_1, \dots, x_m$, i.e.,

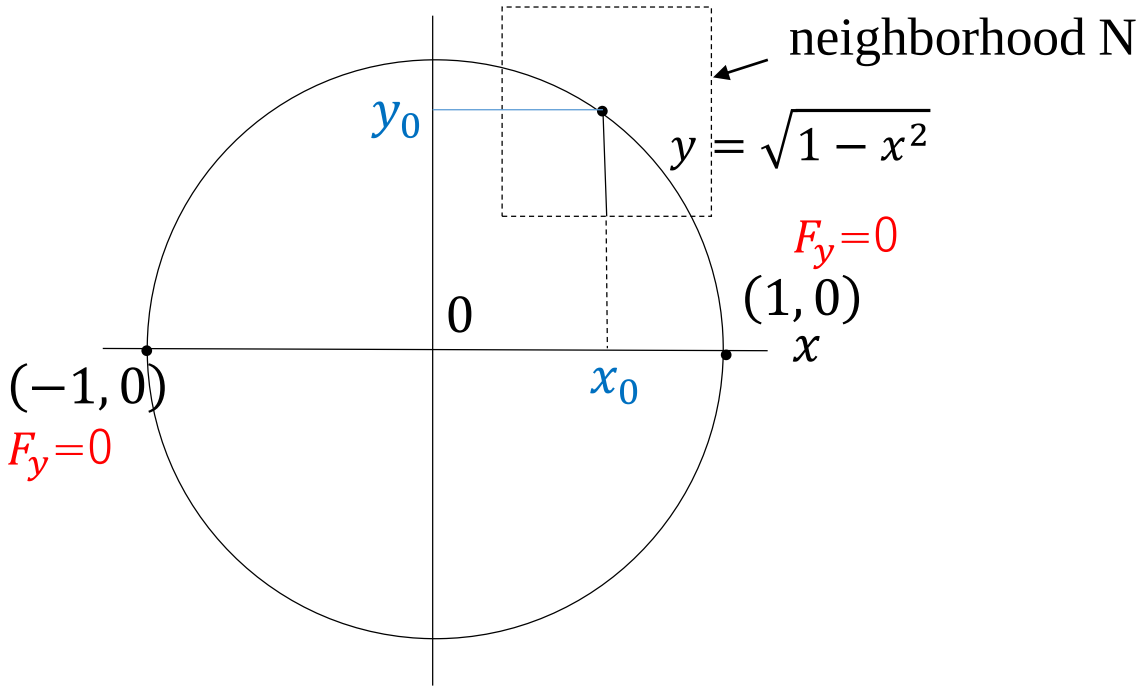
$$y_0 = f(x_{10}, \dots, x_{m0}).$$

We often ignore (A) by an assumption such that F has continuous partial derivatives. Condition (B) $F_y(y_0, x_{10}, \dots, x_{m0}) \neq 0$ would rather an important (and critical).

For example, in the case of the (unit) circle, for any points except for $(-1, 0)$ and $(1, 0)$ (fails in (B)), we can construct a neighborhood $\boldsymbol{N}$ in which a relation $F(y, x) = 0$ defines *implicitly* $f(x)$.



In a neighborhood ***N*** of any points except for $(-1,0)$ and $(1,0)$, a unique value of y corresponds to each value of x (e.g., $y=\sqrt{1-x^2}$ for positive y).



Note that the implicit function theorem doesn’t gives any clue as to about the size of the neighborhood N.

The function $y_0 = f(x_{10}, \ldots, x_{m0})$ in which is defined in the m-dimensional neighborhood of $(x_{10}, \ldots, x_{m0})$ N has the following properties:

(i) for every m-tuple $(x_1, \ldots, x_m)$ in the neighborhood N, the following equation satisfied.

$$F\left(\underbrace{f(x_1, \ldots, x_m)}_{y}, x_1, \ldots, x_m\right) = 0.$$

(ii) the implicit function f is continuous and has continuous partial derivatives $f_1, \ldots, f_m$.

Property (ii) enables us to derive *total differential* of dy.

By the property (i), the total differentials of both sides of $F(y, x_1, \ldots, x_m) = 0$ must be equal:

$$dF(y, x_1, \ldots, x_m) = d0,$$

or,

$$F_y dy + F_{x_1} dx_1 + \cdots + F_{x_m} dx_m = 0.$$

Suppose that only y and x_i are allowed to vary, we have,

$$F_y dy + F_{x_i} dx_i = 0.$$

Dividing both side by dx_i and solving for dy/dx_i, we get

$$\left.\frac{dy}{dx_i}\right|_{other\ variables\ constant} = -\frac{F_{x_i}}{F_y}$$

Notice that the denominator of RHS F_y also requires that $F_y \neq 0$.

Example 1 (again)

Find $\partial y/\partial x$ for any implicit function that may be defined by the relation

$$F(y,x) = x^2 + y^2 - 9 = 0.$$

The partial derivatives are given as follows:

$$F_y = 2y, \quad \text{and} \quad F_x = 2x.$$

Since F_y and F_x are continuous (satisfies (A)), and $F_y \neq 0$ expect for the points $y \neq 0$ (B), there exists an implicit function $y = f(x)$. Since,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y},$$

we have $\frac{dy}{dx} = -\frac{x}{y}$.

Example 2

Find $\partial y/\partial x$ for any implicit function that may be defined by the relation

$$F(y, x, w) = y^3x^2 + w^3 + yxw - 3 = 0,$$

and is defined around the point $(x, y, z) = (1, 1, 1)$.

The partial derivatives are given as follows:

$$F_y = 3y^2x^2 + xw,$$

$$F_x = 2y^3x + yw,$$

$$F_w = 3w^2 + yx.$$

Since F_y, F_x and F_w are all obviously continuous (satisfies (A)), and $F_y \neq 0$ at the point $(1, 1, 1)$, there exists an implicit function $y = f(x, w)$.

Since the total differentials of both sides are equal:

$$F_y dy + F_x dx + F_w dw = 0.$$

To derive $\partial y/\partial x$, setting $dw = 0$ and rearranging the above equation,

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x + yw}{3y^2x^2 + xw}.$$

At the point $(1, 1, 1)$, this derivative has the value $-\frac{3}{4}$.

Example 3 (Indifferent curve) :

Consider the utility function $U(x_1, x_2)$. Find $\partial x_1/\partial x_2$ (MRS: 限界代替率).

Suppose u^* represents a some fixed utility level (scalar). Then we can write down the following relation:

$$U(x_1, x_2) = u^*,$$

Assume that the utility function is differentiable and it has differentiable derivatives. Then we have,

$$U_{x_1} = \frac{\partial u}{\partial x_1} \quad \text{and} \quad U_{x_2} = \frac{\partial u}{\partial x_2}.$$

The implicit function theorem implies,

$$\frac{\partial x_1}{\partial x_2} = -\frac{\frac{\partial u}{\partial x_2}}{\frac{\partial u}{\partial x_1}}$$