9. Optimization: A Special Variety of Equilibrium Analysis

9.1 Optimum Values and Extreme Values

A business firm may seek to maximize profit π, that is, to maximize the difference between total revenue R and total cost C.

Since R and C are both functions of the output level Q, π is also a function of Q:

$$\pi(\underline{Q}) = R(Q) - C(Q).$$

The optimization problem is to choose the level of Q such that π will be a maximum.

In the following discussion, let us consider the general function

$$y = f(x)$$

and attempt to develop a procedure for finding the level of x that will maximize or minimize the value of y.

9.2 Relative Maximum and Minimum: First-Derivative Test

- Relative versus Absolute Extreme

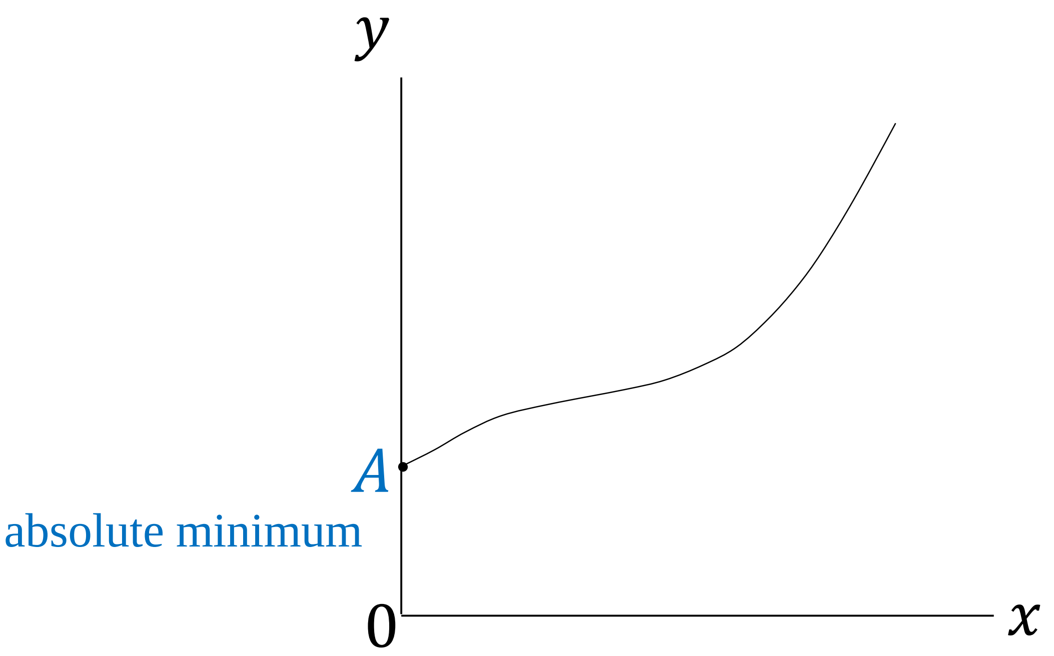


Figure 9.1 (a)

If the set of nonnegative real numbers is taken to be its domain, the point A is the *absolute* minimum in the range of the function (but no finite maximum).

The points B and C are examples of a *relative* extreme.

The fact that point $B(C)$ is a relative maximum (minimum) is no guarantee that it is also the global maximum (minimum) of the function, although this may happen to be the case.

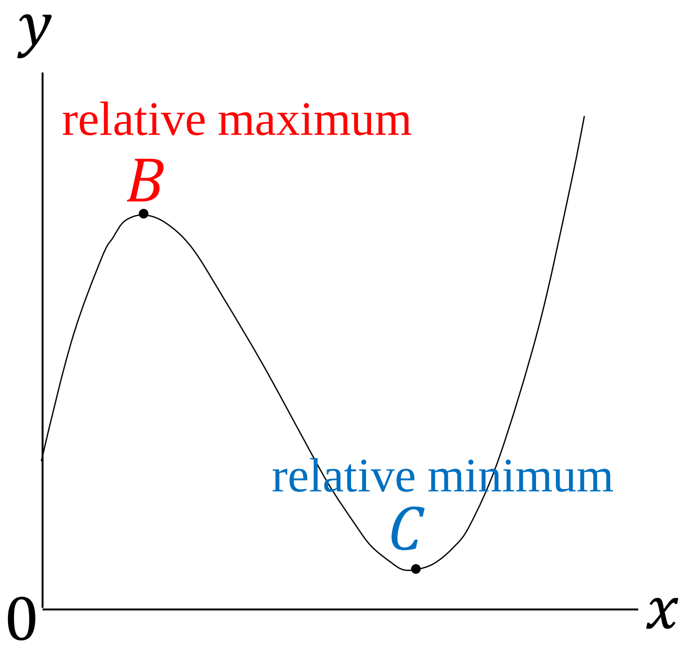


Figure 9.1 (b)

We continue our discussion mainly with reference to the search for *relative* extrema.

- **First-order derivative test for relative extremum**

In the following context, assume that $y = f(x)$ is continuous and possesses a continuous derivative.

Let us call the value x_0 a *critical value* (臨界値) of x if $f'(x_0) = 0$, and refer to $f'(x_0)$ as a *stationary value* (定常値 or 停留値) of y. The point $(x_0, f'(x_0))$ can be called a *stationary point* (定常点 or 停留点).

For a relative extremum (either maximum or minimum), in the context of smooth functions, take the condition $f'(x_0) = 0$ to be a ***necessary*** condition.

If the first-order derivative of a function $f(x)$ at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.

b. A relative *minimum* if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right.

c. Neither a relative maximum nor a relative minimum if $f'(x)$ has the same sign on both the immediate left and right of point x_0.

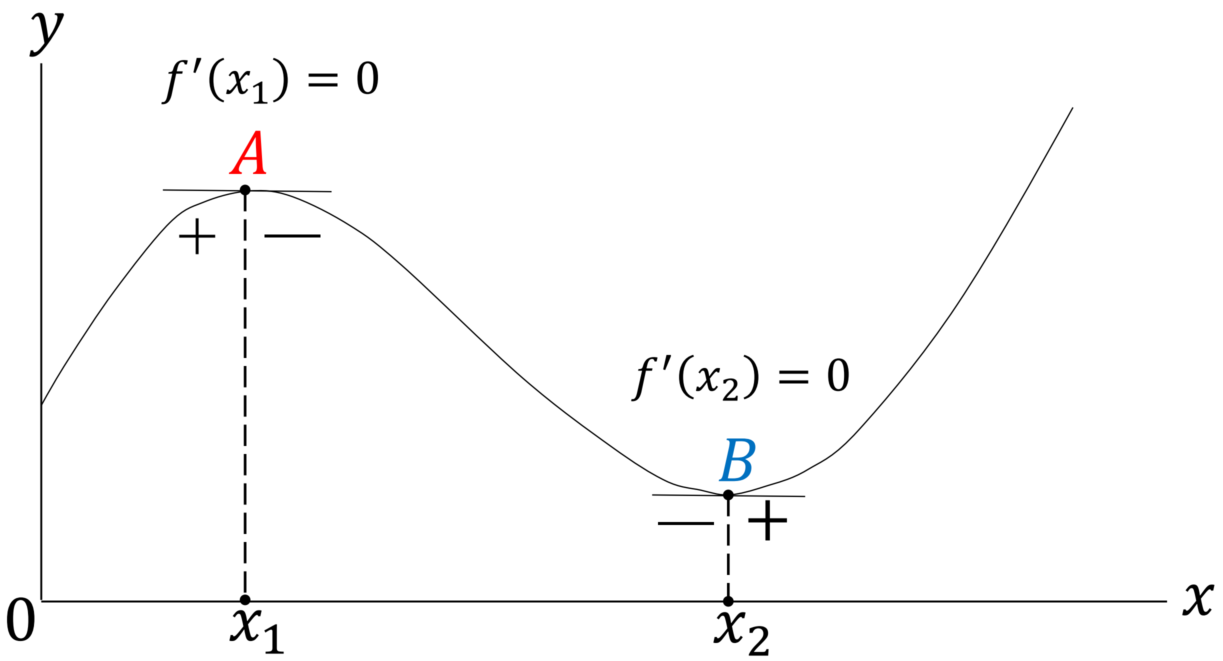


Figure 9.2

The first possibility: point A where the derivative $f'(x)$ changes its sign from positive to negative.

The second possibility: point B where the derivative $f'(x)$ changes its sign from negative to positive.

Note that in view of the existence of a third possibility, we are **unable to** regard the condition $f'(x) = 0$ as a *sufficient condition* for a relative extremum (*Why?*)

The third possibility: point J gives neither a maximum nor a minimum. Rather, it exemplifies what is know as an *inflection poi*nt. (see Figure 9.3(a) and 9.3(a'))

inflection point : the derivative function $f'(x)$ reaches an extreme value at that point.

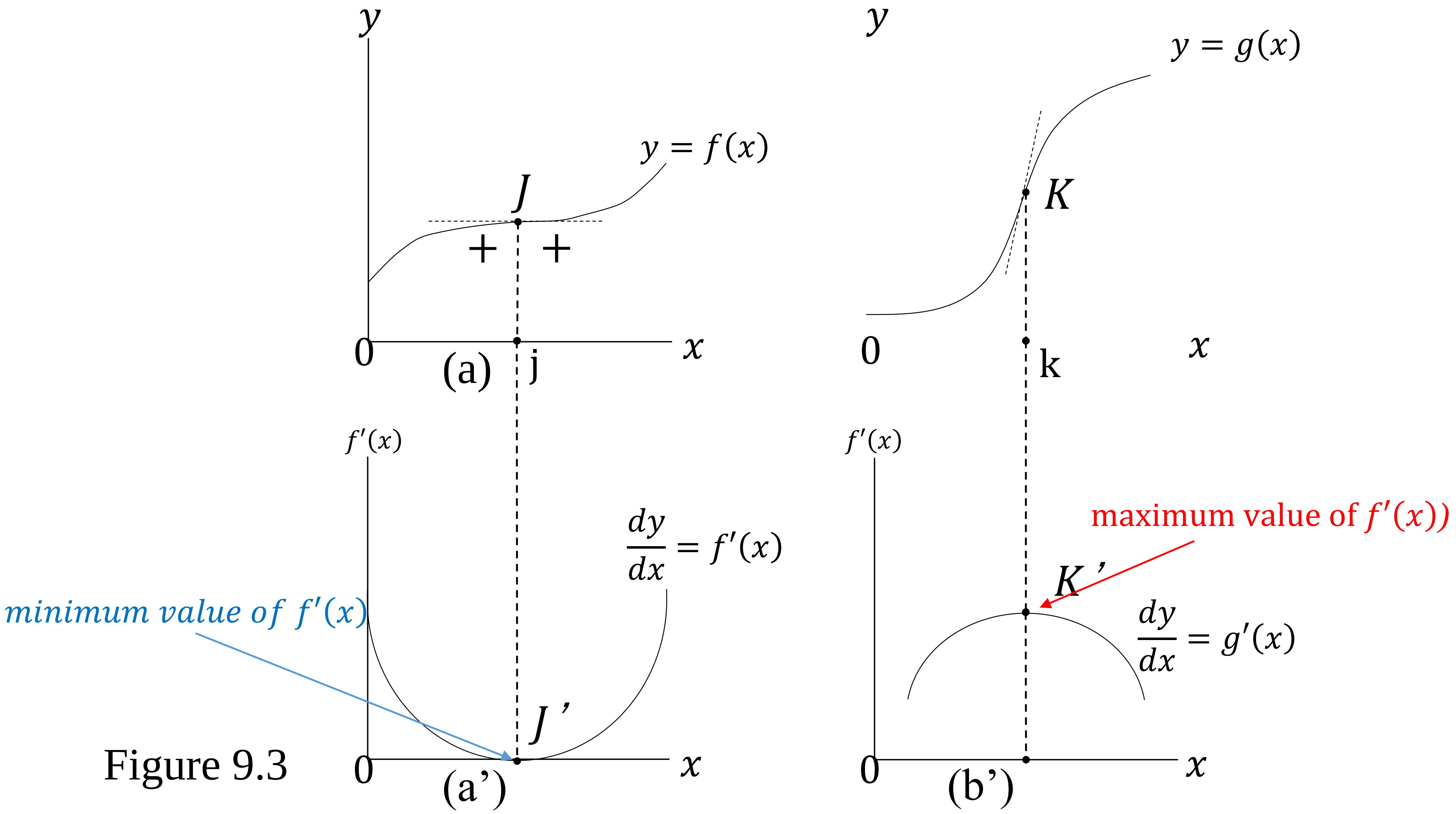


Figure 9.3

The derivative function $f'(x)$ reaches an extreme value at point *J’* and *K’*.

Example 1

Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8.$$

The derivative function is:

$$f'(x) = 3x^2 - 24x + 36.$$

From $3x^2 - 24x + 36 = 0$ or $3(x-2)(x-6) = 0$, critical values are:

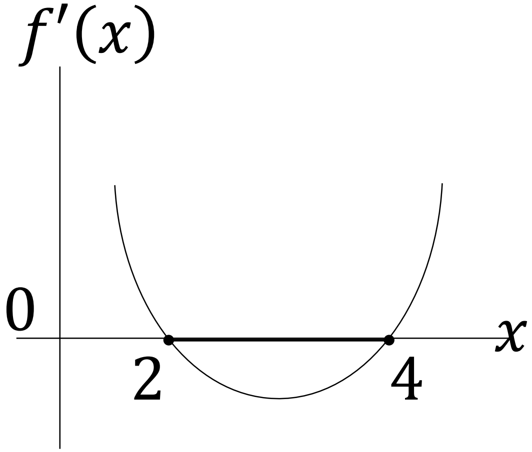
$$\bar{x}_1 = 2, \text{ and } \bar{x}_2 = 6.$$

Case 1 : x= 2

In the immediate neighborhood of $x = 2$,

$f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$ (from positive to negative).

Thus, the corresponding value $f(2) = 40$ is a relative maximum.



Case 2 : x= 6

In the immediate neighborhood of $x = 6$,

$f'(x) < 0$ for $x < 6$ and $f'(x) > 0$ for $x > 6$ (from negative to positive).

Thus, the corresponding value $f(6) = 8$ is a relative minimum.

9.3 Second and Higher Derivatives

- Derivative of a Derivative

Since the first derivative $f'(x)$ is itself a function of x, it should be differentiable with respect to x, provided that it is continuous and smooth.

The second derivative is denoted by,

$$f''(x) \quad \text{or} \quad \frac{d^2y}{dx^2}.$$

- Interpretation of the Second Derivative

The first derivative function $f'(x)$ measures the rate of change of the function *f*. The second derivative function $f''(x)$ measures the rate of change of the first derivative $f'(x)$, i.e., it measures the *rate of change of the rate of change*.

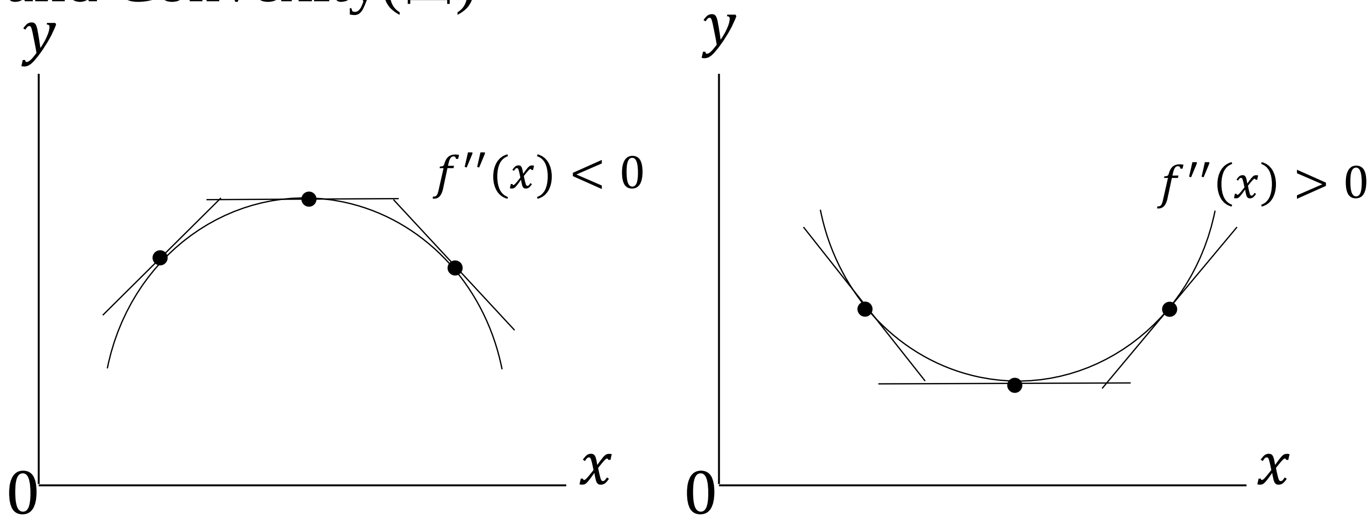
With a given infinitesimal increase in the independent variable x from a point $x = x_0$,

$f'(x_0) > 0(< 0)$ means that the value of the function ends to increase (decrease)

and

$f''(x_0) > 0(< 0)$ means that the slope of the curve tends to increase (decrease).

- Concavity(凹) and Convexity(凸)



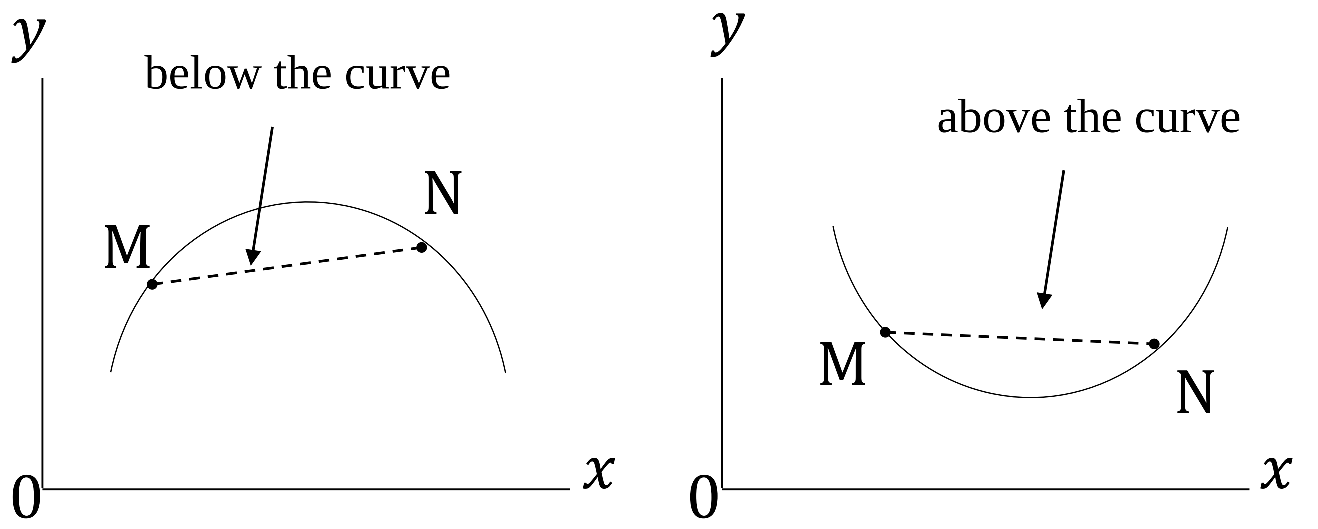
(a) Concave Function (b) Convex Function

Figure 9.4

From the standpoint of the horizontal axis, the graph in diagram (a) to be concave throughout, whereas the graph in diagram (b) is convex throughout.

Concavity and convexity are descriptions of how the curve to bends. The second derivative of a function informs us about the curvature of its graph.

We pick up ***any*** pair of points M and N on its curve and join them by as straight line.



(a) Concave Function $f''(x) < 0$

(b) Convex Function $f''(x) > 0$

Figure 9.4

	the line segment MN	$f''(x)$
Strictly concave function	entirely below the curve	negative for all x
Strictly convex function	entirely above the curve	positive for all x

9.4 Second-Derivative Test

- **Second-derivative test for relative extremum**

If the first derivative of a function f at $x = x_0$ is $f'(x_0) = 0$, then the value of the function at x_0, $f(x_0)$, will be

a. A relative *maximum* if the second-derivative value at $x = x_0$ is $f''(x) < 0$.

b. A relative *minimum* if the second-derivative value at $x = x_0$ is $f''(x) > 0$.

There exists the two stationary points A and B.

At the point A ($x = x_1$), the second-derivative $f''(x)$ has a negative sign. Hence, A is a relative maximum.

Likewise, at the point B, $f''(x)$ has a positive sign (relative minimum).

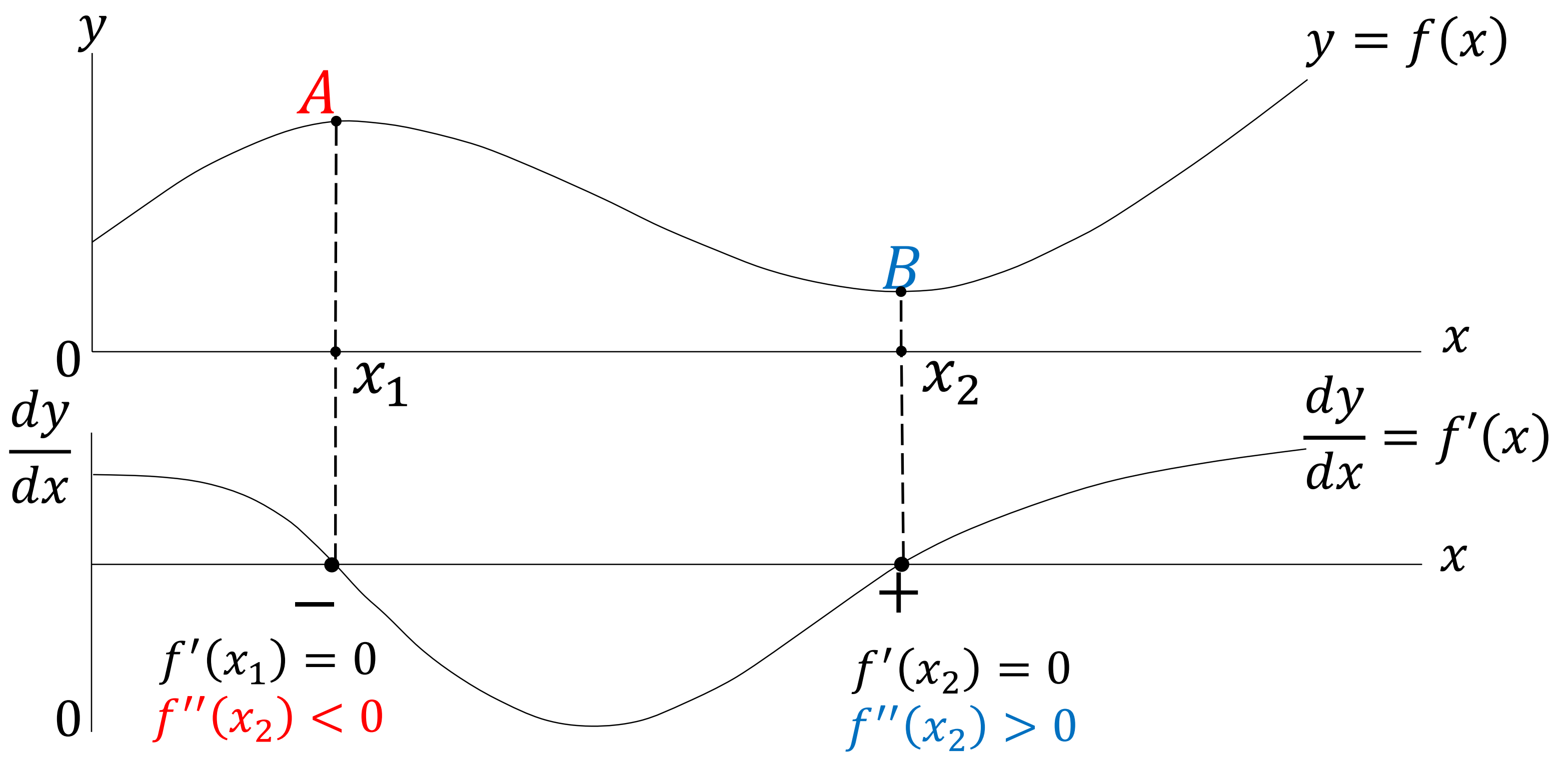


Figure 9.5

- Necessary versus Sufficient Conditions

As was the case with the first-derivative test, the zero-slope condition $f'(x) = 0$ plays the role of a *necessary* condition in the second-derivative test. Since this condition is based on the first-order derivative, it is often referred to as the ***F.O.C.*** (short for *first-order condition*).

Once we find the first-order condition satisfied at $x_0 = 0$, the negative (positive) sign of $f''(x)$ is *sufficient* to establish the stationary value in question as a relative maximum (minimum). These sufficient conditions are often referred to as ***S.O.C.*** (short for *second-order conditions*).

Example 2

Find the relative extrema of the function

$$y = f(x) = x^3 - 12x^2 + 36x + 8.$$

The derivative function is

$$f'(x) = 3x^2 - 24x + 36.$$

From $f'(x) = 0$, we obtain the following critical values:

$$\bar{x}_1 = 2, \text{ and } \bar{x}_2 = 6.$$

The second-derivative function is:

$$f''(x) = 6x - 24.$$

Case 1: $f''(2) = -12 < 0 \rightarrow$ the corresponding value of the function $f(2) = 40$ is established as a relative maximum.

Case 2: $f''(6) = 12 > 0 \rightarrow$ the value must be a relative minimum.

9.5 Maclaurin and Taylor Series

- Maclaurin Series of a Polynomial Function

In the Maclaurin series expansion, we expand a function $y = f(x)$ around the point $x = 0$.

Consider the expansion of a polynomial function of the *n*th degree,

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n,$$

into an equivalent *n*th-degree polynomial where the coefficients (a_0, a_1, etc.) are

expressed instead in terms of the derivative values $f'(0)$, $f''(0)$, etc.

We get the derivatives as follows

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots + na_nx^{n-1},$$

$$f''(x) = 2a_2 + 3 \times 2a_3x + 4 \times 3a_4x^2 + \cdots + n(n-1)a_nx^{n-2},$$

$$f'''(x) = 3 \times 2a_3 + 4 \times 3 \times 2a_4x + \cdots + n(n-1)(n-2)a_nx^{n-3},$$

$$\vdots$$

$$f^{(n)}(x) = n(n-1)(n-2)(n-3) \cdots (3)(2)(1)a_n.$$

Notice that each successive differentiation reduces the number of term by one

(the additive constant in front drops out).

We evaluate these derivatives at $x = 0$, with the result that all terms involving x will drop out. It yields,

$$f'(0) = a_1, \qquad f''(0) = 2a_2, \qquad f'''(0) = 3 \times 2a_3,$$

$$\cdots\ f^{(n)}(0) = n(n-1)(n-2)(n-3)\ \cdots\ (3)(2)(1)a_n.$$

If we adopt a factorial symbol ($n!$), the result can be written as:

$$a_1 = \frac{f'(0)}{1!},\ a_2 = \frac{f''(0)}{2!},\ a_3 = \frac{f'''(0)}{3!},\ \cdots\ a_n = \frac{f^n(0)}{n!}.$$

Review: $0! = 1$ and $1! = 1$.

Substituting these into the original function ($f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$), and utilizing $f(0) = a_0$, we can express the given function $y = f(x)$ as a new same-degree polynomial in which the coefficients are expressed in terms of derivatives evaluated at $x = 0$:

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \frac{f^n(0)}{n!}x^n.$$

0!=1!=1

This is called "*Maclaurin's formula*".

Example 1

Find the Maclaurin series for the function

$$f(x) = 2 + 4x + 3x^2.$$

The derivatives are

$$f'(x) = 4 + 6x, \text{ and } \quad f''(x) = 6.$$

Hence,

$$f'(0) = 4, \quad \text{and} \quad f''(0) = 6.$$

The Maclaurin series is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 2 + 4x + 3x^2.$$

The Maclaurin series does indeed correctly represent the given function.

- Taylor Series of a Polynomial Function

More generally, the polynomial function can be expanded around ***any*** point x_0, not necessarily zero. We interpret any given point x as a deviation from x_0.

Let $x = x_0 + \delta$, where δ represents the deviation from x_0. The given function $f(x) = 2 + 4x + 3x^2$ $(Example\ 1)$ and its derivatives become,

$$f(x) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2$$

$$f'(x) = 4 + 6(x_0 + \delta), \text{ and } f''(x) = 6.$$

Since x_0 in the present context is a *fixed* number, only δ can be regarded as a variable. Thus, consider the following function of δ:

$$g(\delta) = 2 + 4(x_0 + \delta) + 3(x_0 + \delta)^2 \qquad [\equiv f(x)]$$

The derivatives are:

$$g'(\delta) = 4 + 6(x_0 + \delta)[\equiv f'(x)]$$

$$g''(\delta) = 6[\equiv f''(x)]$$

The Maclaurin series of $g(\delta)$ is:

$$g(\delta) = \frac{g(0)}{0!} + \frac{g'(0)}{1!}\delta + \frac{g''(0)}{2!}\delta^2. \qquad (1)$$

Since $x = x_0 + \delta$, the fact such that $\delta = 0$ implies $x = x_0$. So, we have

$$g(0) = f(x_0),\ \ g'(0) = f'(0),\ \ g''(0) = f''(0),$$

for the case of $\delta = 0$.

Substituting these into $g(\delta) = \frac{g(0)}{0!} + \frac{g'(0)}{1!}\delta + \frac{g''(0)}{2!}\delta^2$, we find the expansion of $f(x)$ around the point x_0.

$$f(x)[= g(\delta)] = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2. \qquad (2)$$

E.g., for the specific function $f(x) = 2 + 4x + 3x^2$, we have,

$$f(x_0) = 2 + 4x_0 + 3{x_0}^3, f'(x_0) = 4 + 6x_0, f''(x_0) = 6.$$

The Taylor polynomial becomes

$$f(x) = \left(2 + 4x_0 + 3{x_0}^3\right) + (4 + 6x_0)(x - x_0) + \frac{6}{2}(x - x_0)^2$$

$$= 2 + 4x + 3x^2.$$

(Compare this result with the Maclaurin polynomial)

The expansion formula in (2) can be generalized to apply to the *n*th-degree polynomial.

The generalized formula is,

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{3}$$

Which is called “Taylor’s formula”.

- Expansion of an Arbitrary Function

Taylor's Theorem

Given an arbitrary n differentiable function $\phi(x)$,
if we know $\phi(x_0)$ and the value of its derivatives at x_0,
then this function can be expanded around the point x_0 as follows ($n =$ a fixed positive integer arbitrary chosen):

$$f(x) = \left[\frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{\phi^{(n)}(x_0)}{n!}(x - x_0)^n\right] + R_n$$

$$= P_n + R_n. \tag{30}$$

P_n: the nth-degree polynomial R_n: a *remainder*.

- The appearance of R_n is due to the fact that we are dealing with an arbitrary function ϕ which cannot always be transformed *exactly* into, but can only be approximated by, the polynomial form shown in (29).

- A remainder term R_n represent the discrepancy between $\phi(x)$ and P_n.

- P_n constitutes a *polynomial approximation* to $\phi(x)$, with the term R_n as a measure of the error of approximation.

If we choose $n = 1$, for example, we have

$$\phi(x) = [\phi(x_0) + \phi'(x_0)(x - x_0)] + R_1 = P_1 + R_1 \tag{31}$$

where P_1 constitutes a *linear* approximation to $\phi(x)$.

If we choose $n = 2$, a second-power term will appear, so that

$$\phi(x) = \left[\phi(x_0) + \phi'(x_0)(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2\right] + R_2$$

$$= P_2 + R_2 \tag{32}$$

where P_2 is a *quadratic* approximation to $\phi(x)$.

As n increases, the polynomial approximation P_n to $\phi(x)$ increases its approximation accuracy.

P_2 is a better approximation to $\phi(x)$ than P_1, as shown in Figure 9.6.

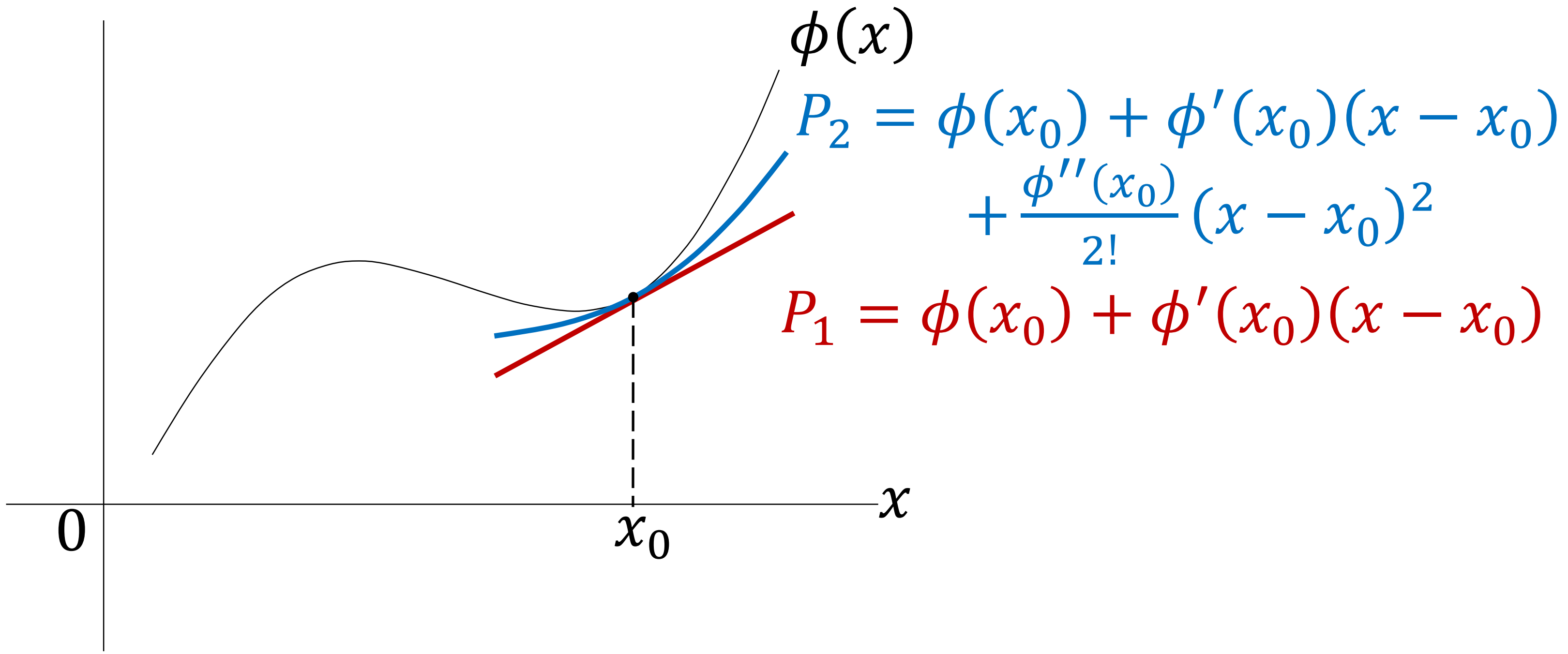


Figure 9.6

Example 2

Expand the nonpolynomial function

$$\phi(x) = \frac{1}{1+x} \tag{33}$$

around the point $x_0 = 1$, with $n = 4$. The derivatives are

$$\phi'(x) = -(1+x)^{-2},\ \phi''(x) = 2(1+x)^{-3},$$

$$\phi'''(x) = -6(1+x)^{-4},\ \text{and}\ \ \phi^{(4)}(x) = 24(1+x)^{-5}.$$

So,

$$\phi(1) = \frac{1}{2},\ \phi'(1) = -\frac{1}{4}, \quad \phi''(1) = \frac{1}{4},$$

$$\phi'''(1) = -\frac{3}{8},\ \text{and}\ \phi^{(4)}(1) = \frac{3}{4}.$$

Taylor series with remainder is given by

$$\phi(x) = \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}\ (x-1)^3 + \frac{1}{32}(x-1)^4 + R_4$$

$$= \frac{31}{32} - \frac{13}{16}x + \frac{1}{2}x^2 - \frac{3}{16}x^3 + \frac{1}{32}x^4 + R_4. \tag{34}$$

- Lagrange Form of the Remainder

According to the *Lagrange form of the remainder*, we can express R_n as

$$R_n = \frac{\phi^{(n+1)}(p)}{(n+1)!}(x - x_0)^{n+1} \qquad (35)$$

where p is some number between x and x_0.

If we find that

$$R_n \to 0 \text{ as } n \to \infty \quad \text{so that} \quad P_n \to \phi(x) \; as \; n \to \infty \tag{36}$$

then the Taylor series converges to $\phi(x)$ at the point of approximation.

The Taylor series can be written as a *convergent infinite series* as follows:

$$\phi(x) = \frac{\phi(x_0)}{0!} + \frac{\phi'(x_0)}{1!}(x - x_0) + \frac{\phi''(x_0)}{2!}(x - x_0)^2 + \cdots \tag{37}$$

It will be possible to make P_n as an accurate approximation to $\phi(x)$ by choosing a large enough value of n.