8. Comparative-Static Analysis of General-Function Models

8.1 Differentials

- Differentials and Derivatives

The derivative $dy/dx=f'(x)$ is the limit of a difference quotient:

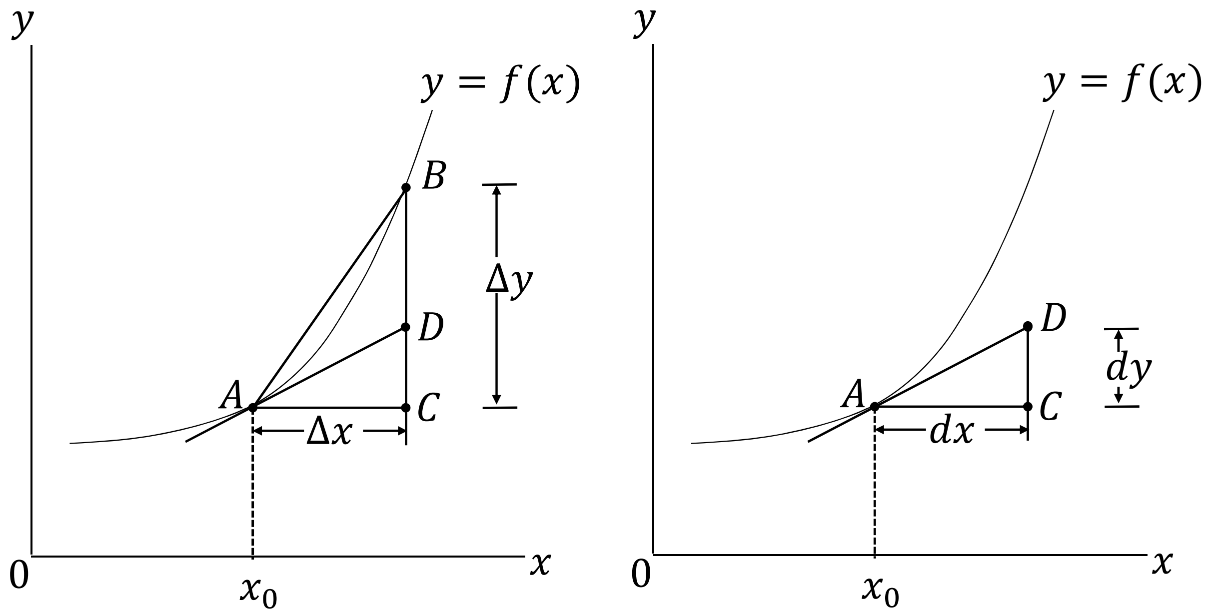
$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \tag{1}$$

As in Figure 8.1, we have

$$\frac{dy}{dx} = \text{ slope of tangent } AD = f'(x). \tag{2}$$

After multiplying through by dx, we get the change of y with respect to an infinitesimal change in x.

$$dy \equiv \left(\frac{dy}{dx}\right) dx \qquad \text{or} \qquad dy \equiv f'(x)dx. \tag{3}$$



(a) (b)

Figure 8.1

Example 1

Given $y = 3x^2 + 7x - 5$, find dy.

$$dy = \left(\frac{dy}{dx}\right) dx = (6x + 7)dx \tag{4}$$

- Differentials and Point Elasticity

For a demand function $Q = f(P)$, the *point elasticity* of demand is defined as:

$$\epsilon_d \equiv \frac{dQ/dP}{Q/P} \qquad (5)$$

In general, for any function $y = f(x)$, the point elasticity of y with respect to x is defined as

$$\epsilon_{yx} \equiv \frac{dy/dx}{y/x} = \frac{\text{marginal function}}{\text{average function}} \qquad (6)$$

Example 2

Find ϵ_d if the demand function is $Q = 100 - 2P$.

$$\frac{Q}{P} = \frac{100-2P}{P} \quad \text{and} \quad \frac{dQ}{dP} = -2 \tag{7}$$

Thus, we obtain

$$\epsilon_d = \frac{-2}{(100-2P)/(P)} = \frac{-P}{50-P} \tag{8}$$

Partial Differentiation

- Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \dots, x_n) \tag{9}$$

If the variable x_1 undergoes a change Δx_1 while $x_2, \dots, x_n$ all remain fixed, there will be a corresponding change in y, namely Δy.

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\Delta x_1} \tag{10}$$

Partial derivative of y with respect to x_i:

$$f_{x_1} \equiv \frac{\partial y}{\partial x_i} \equiv \lim_{\Delta x_i \to 0} \frac{\Delta y}{\Delta x_i}. \tag{11}$$

- Geometric Interpretation of Partial Derivatives

Consider a production function $Q = Q(K, L)$. Let us hold capital fixed at the level K_0 and consider only variations on the input L. The slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L.

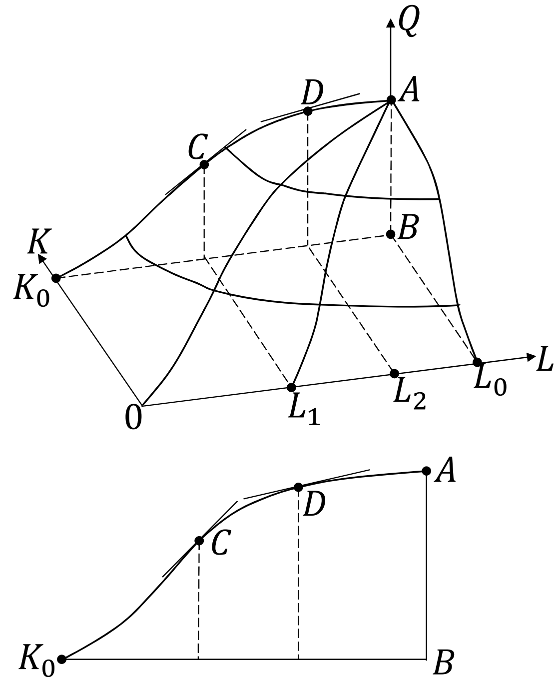


Figure 8.2

8.2 Total Differentials

Consider saving function

$$S = S(Y, i) \tag{12}$$

S: savings, Y: national income, and i: interest rate.

〈Total differential of the saving function〉

The total change in S with respect to infinitesimal changes in Y and i:

$$dS = \frac{\partial S}{\partial Y} dY + \frac{\partial S}{\partial i} di \quad \text{or} \quad dS = S_Y dY + S_i di \tag{13}$$

Geometrically, total differential corresponds to the tangential plane as shown in Figure 8.3,

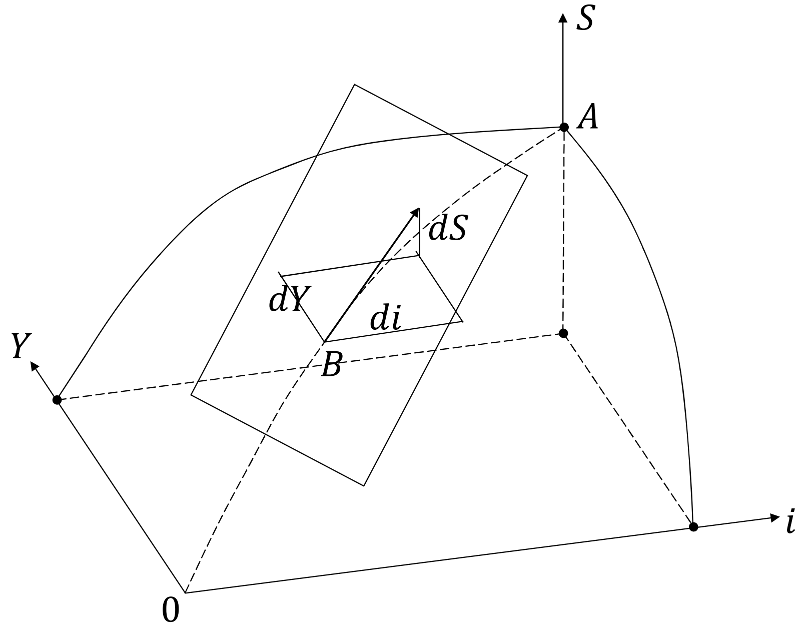


Figure 8.3

For the more general case of a function of n independent variables such as

$$U = U(x_1, x_2, \dots, x_n), \tag{14}$$

the total differential of U can be written as

$$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \cdots + \frac{\partial U}{\partial x_n} dx_n \tag{15}$$

or

$$dU = U_1 dx_1 + U_2 dx_2 + \cdots + U_n dx_n = \sum_{i=1}^{n} U_i dx_i \,. \tag{16}$$

Example 1

(a) $U(x_1, x_2) = ax_1 + bx_2$

(b) $U(x_1, x_2) = {x_1}^2 + {x_2}^3 + x_1 x_2$

The total differentials are as follows:

(a) $$\frac{\partial U}{\partial x_1} = U_1 = a, \quad \frac{\partial U}{\partial x_2} = U_2 = b \tag{17}$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = adx_1 + bdx_2. \tag{18}$$

(b) $$\frac{\partial U}{\partial x_1} = U_1 = 2x_1 + x_2, \quad \frac{\partial U}{\partial x_2} = U_2 = 3{x_2}^2 + x_1 \tag{19}$$

and

$$dU = U_1 dx_1 + U_2 dx_2 = (2x_1 + x_2)dx_1 + \left(3{x_2}^2 + x_1\right)dx_2. \tag{20}$$

8.3 Rules of Differentials

Let c be a constant and u and v be two functions.

Rule I	$dc = 0$	(constant-function rule)
Rule II	$d(cu^n) = cnu^{n-1}du$	(power-function rule)
Rule III	$d(u \pm v) = du \pm dv$	(sum-difference rule)
Rule IV	$d(uv) = v\,du + u\,dv$	(product rule)
Rule V	$d\left(\frac{u}{v}\right) = \frac{1}{v^2}(v\,du - u\,dv)$	(quotient rule)

8.4 Total Derivatives

- Finding the Total Derivative

Let us consider any function

$$y = f(x, w) \quad \text{where} \quad x = g(w) \tag{21}$$

The three variables y, x and w related to one another as Figure 8.4.

The variable w can affect y through two channels:

(1) directly, via the function f.

(2) indirectly, via the function g and then f.

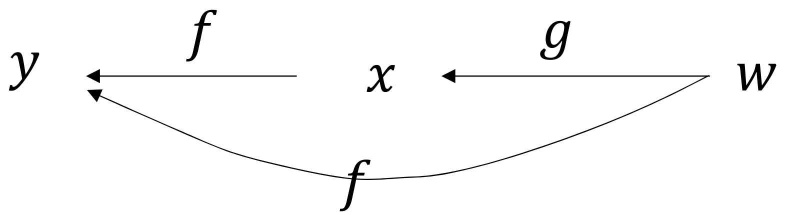


Figure 8.4

To obtain this total derivative, we first differentiate y totally to get the total differential $dy = f_x dx + f_w dw$. Then we divide both sides of this equation by the differential dw.

$$\frac{dy}{dw} = \underbrace{f_x \frac{dx}{dw}}_{\textit{indirect effect}} + \underbrace{f_w \frac{dw}{dw}}_{\textit{direct effect}}$$

$$= \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w} \qquad (22)$$

Example 1

Find the total derivative dy/dw, given the function

$$y = f(x, w) = 3x - w^2 \quad \text{where} \quad x = g(w) = 2w^2 + w + 4 \quad (23)$$

By virtue of (22), the total derivative should be

$$\frac{dy}{dw} = \frac{\partial y}{\partial x}\frac{dx}{dw} + \frac{\partial y}{\partial w}$$

$$= 3(4w + 1) + (-2w) = 10w + 3 \quad (24)$$

- A Variation on the Theme

Let us consider any function

$$y = f(x_1, x_2, w) \qquad \text{where} \qquad \begin{cases} x_1 = g(w) \\ x_2 = h(w) \end{cases} \qquad (25)$$

The variable w can affect y through three channels:

(1) indirectly, via the function g and then f.

(2) indirectly, via the function h and then f.

(3) directly, via the function f.

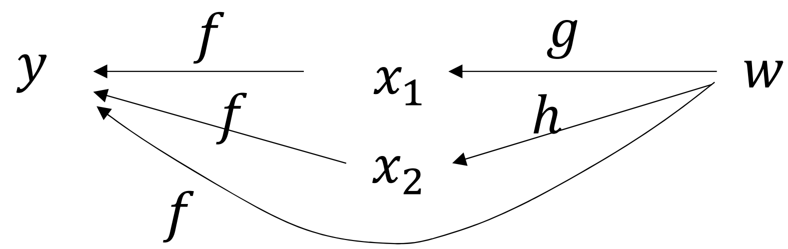


Figure 8.5

The total derivative of y with respect to w is given by

$$\frac{dy}{dw} = f_1 \frac{dx_1}{dw} + f_2 \frac{dx_2}{dw} + f_w \frac{dw}{dw}$$

$$= \frac{\partial y}{\partial x_1}\frac{dx_1}{dw} + \frac{\partial y}{\partial x_2}\frac{dx_2}{dw} + \frac{\partial y}{\partial w} \qquad (26)$$

Example 2

Let the production function be

$$Q = Q(K, L, t) \qquad (27)$$

where, aside from the two inputs K and L, there is a third argument t, denoting time. Since capital and labor, too, can change over time, we may write

$$K = K(t) \qquad \text{and} \qquad L = L(t). \qquad (28)$$

Then, the rate of change of output with respect to time can be expressed as

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial K}\frac{dK}{dt} + \frac{\partial Q}{\partial L}\frac{dL}{dt} + \frac{\partial Q}{\partial t} \qquad (29.1)$$

or

$$\frac{dQ}{dt} = Q_K K'(t) + Q_L L'(t) + Q_t. \qquad (29.2)$$

8.5 Derivatives of Implicit Functions

The concept of total differentials can also enable us to find the derivatives of so-called "implicit-functions".

- Implicit Functions

  A function given in the form of $y = f(x)$, say,

$$y = f(x) = 3x^4 \qquad (30)$$

is called an *explicit function* because the variable y is explicitly expressed as a function of x. If this function is written alternatively in the equivalent form

$$y - 3x^4 = 0, \qquad (31)$$

we no longer have an explicit function. The function (30) is then only *implicitly* defined by the equation (31). The function $y = f(x)$ implied by (31) is called an *implicit function*.

(31) can be denoted in general by

$$F(y, x) = 0. \tag{32}$$

In case where there are more than two arguments in the F function:

$$F(y, x_1, x_2, \ldots, x_m) = 0$$

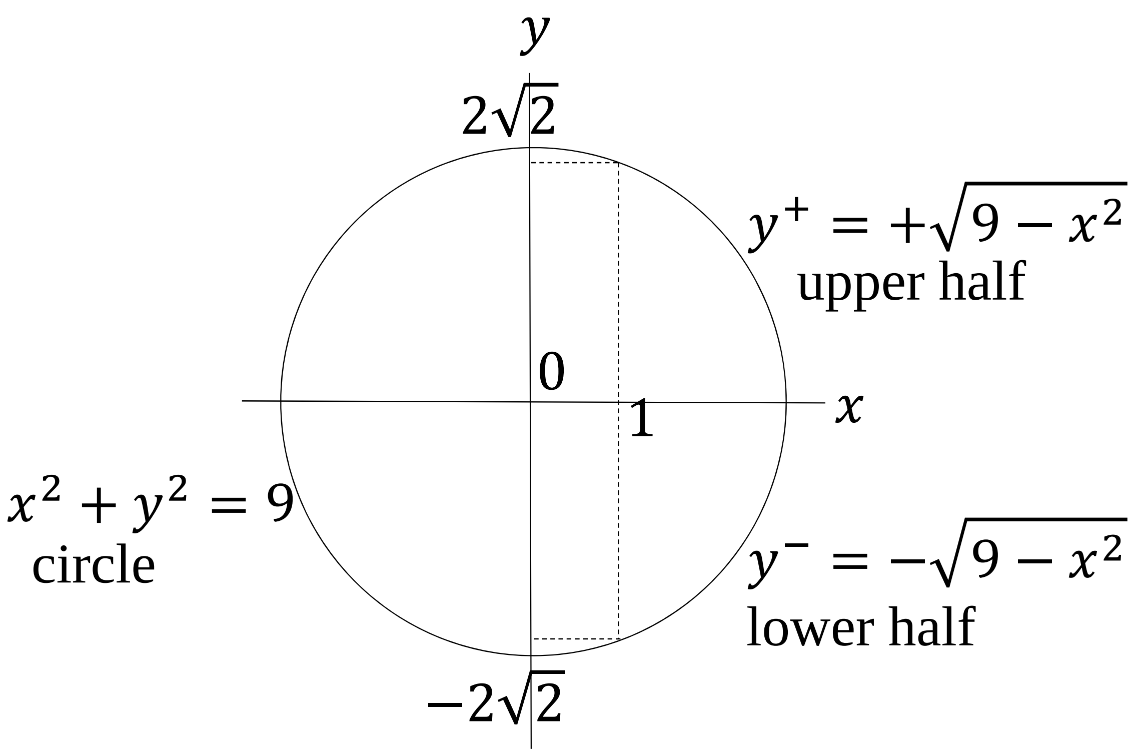
Such an equation may also define an implicit function $y = f(x_1, x_2, \ldots, x_m)$.

- An explicit function, say, $y = f(x)$, can always be transformed into an equation $F(y, x) = 0$.
- The reverse transformation is not always possible.

For instance, the equation

$$F(y, x) = x^2 + y^2 - 9 = 0 \tag{33}$$

implies not a function, but a relation, because (33) plots as a circle as shown in Figure 8.6. Hence, no unique value of y corresponds to each value of x.



y
$2\sqrt{2}$
$y^+ = +\sqrt{9 - x^2}$
upper half
0
1
x
$x^2 + y^2 = 9$
circle
$y^- = -\sqrt{9 - x^2}$
lower half
$-2\sqrt{2}$

Figure 8.6

If we restrict y to nonnegative values, then we will have the upper half of the circle that does constitute a function

$$y = +\sqrt{9 - x^2}. \tag{34}$$

Similarly, the lower half of the circle constitutes another function

$$y = -\sqrt{9 - x^2}. \tag{35}$$

In view of this uncertainty, it becomes of interest to ask whether there are known general conditions under which we can be sure that a given equation in the form of

$$F(y, x_1, \ldots, x_m) = 0 \tag{36}$$

does indeed define an implicit function

$$y = f(x_1, \ldots, x_m). \tag{37}$$

Implicit Function Theorem

Given (36),

(A) if the function F has continuous partial derivatives $F_y, F_1, \ldots, F_m$,

and

(B) if F_y is non zero at a point $(y_0, x_{10}, \ldots, x_{m0})$ satisfying the equation (36),

there exists an m-dimensional neighborhood of $(y_0, x_{10}, \ldots, x_{m0})$, N, in which y is an implicitly defined function of the variables $x_1, \ldots, x_m$ in terms of (37). The function (37) has the following properties:

(i) $y_0 = f(x_{10}, \dots, x_{m0})$

(ii) for every m-tuple $(x_1, \dots, x_m)$ in the neighborhood N, the equation (36) is satisfied,

$$F(\underbrace{f(x_1, \dots, x_m)}_{y}, x_1, \dots, x_m) = 0. \tag{38}$$

The above equation has the status of an *identity* in that neighborhood.

and

(iii) the implicit function f is continuous and has continuous partial derivatives $f_1, \dots, f_m$.

Let us apply this theorem to the equation (33).

Verify that the conditions (A) and (B) are sarisfied.

(A) $F_y = 2y$ and $F_x = 2x$ are continuous.

(B) the condition (B) is satisfied except at $(-3, 0)$ and $(3, 0)$ because F_y is nonzero except when $y = 0$.

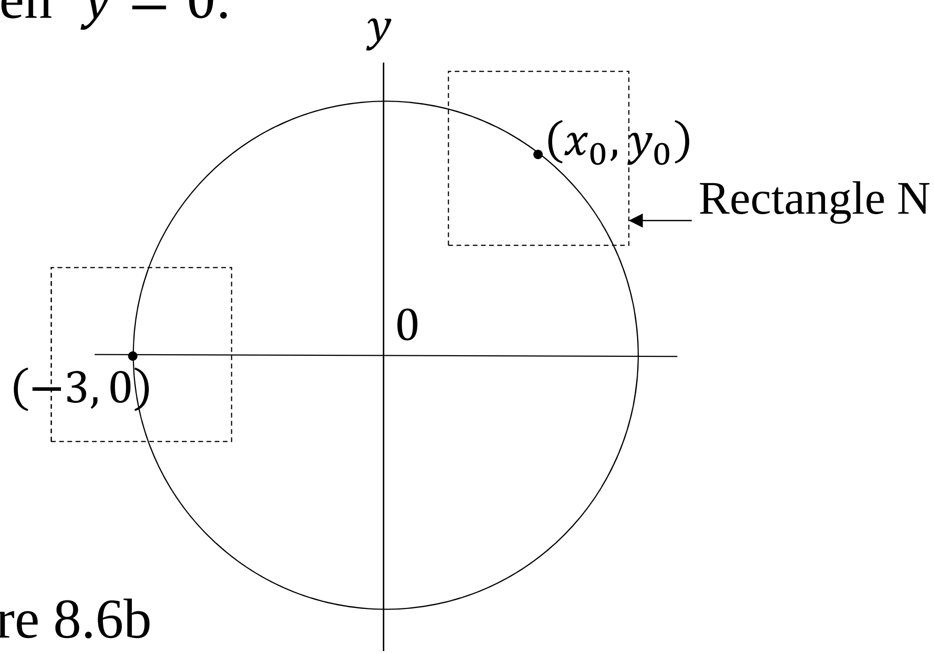


Figure 8.6b

Around any point on the circle except $(-3, 0)$ and $(3, 0)$, we can construct a neighborhood in which the equation (33) defines an implicit function $y = f(x)$. In the rectangle N, an unique value of y corresponds to each value of x. Hence , the implicit function is defined. However, two values of y correspond to each value of x around $(-3, 0)$ and $(3, 0)$.

Several things should be noted about the implicit-function theorem.

1. The conditions cited in the theorem are sufficient (but not necessary) conditions. If we happen to find $F_y = 0$ at point satisfying (36), this does not mean that an implicit function does not exist around that point.

2. Even if the existence of an implicit function f is assured, the theorem gives no clue as to the specific form the function f. Nor, does it tell us the exact size of the neighborhood.

- Derivatives of Implicit Functions

If the equation $F(y, x_1, \dots, x_m) = 0$ can be solved for y, we can explicitly write out the function $y = f(x_1, \dots, x_m)$, and find its derivatives. But what if the given equation $F(y, x_1, \dots, x_m) = 0$ cannot be solved for y explicitly? In this case, if under the terms of the implicit function theorem an implicit function is known to exist, we can still obtain the desired derivatives without solving for y first. To do this, we make use of the following basic facts:

1. if two expressions are *identically* equal, their respective total differentials must be equal;

2. if we divide dy by dx_1 and let all the other differential $(dx_2, \dots, dx_m)$ be zero, the quotient can be interpreted as the partial derivative $\partial y/\partial x_1$.

$F(y, x_1, \ldots, x_m) = 0$ has the status of an *identity* in the neighborhood where the implicit function is defined. Thus, the total differentials of both sides must be equal.

$$dF = d0 \tag{39}$$

or

$$F_y dy + F_{x_1} dx_1 + \cdots + F_{x_m} dx_m = 0. \tag{40}$$

Suppose that only y and x_i are allowed to vary.

$$F_y dy + F_{x_i} dx_i = 0. \tag{41}$$

Dividing both side by dx_i and solving for dy/dx_i, we get

$$\left.\frac{dy}{dx_i}\right|_{other\ variables\ constant} \equiv \frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y} \tag{42}$$

Consider an equation $F(y, x_1, x_2) = 0$. Geometrically, the partial derivative $\partial y / \partial x_1$ corresponds to the slope of the tangent line of the contour line as shown in Figure 8.7.

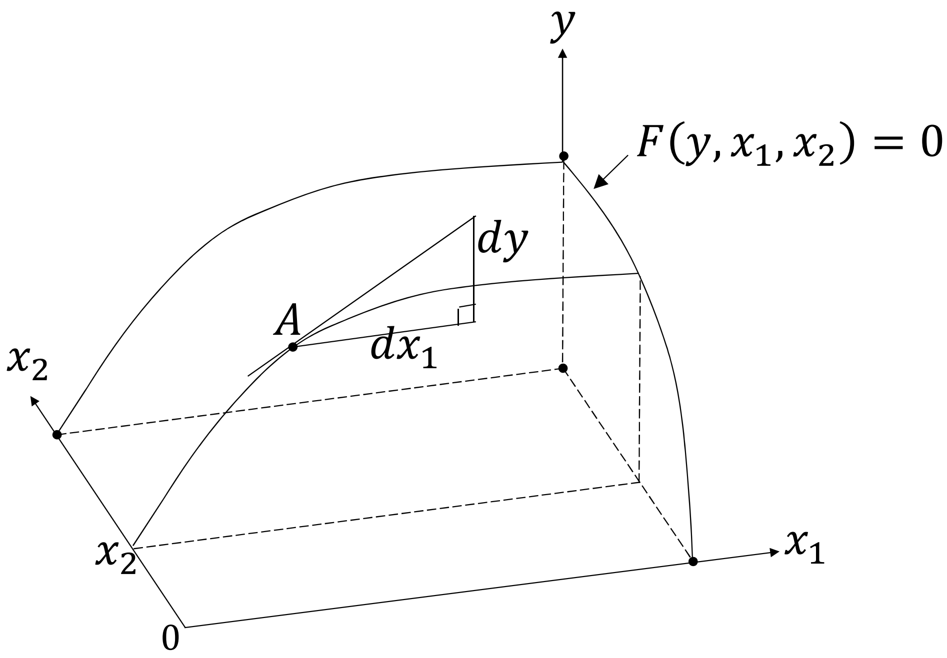


Figure 8.7

Example 1

Find $\partial y/\partial x$ for any implicit function that may be defined by the equation

$$F(y,x,w) = y^3x^2 + w^3 + yxw - 3 = 0 \quad (43)$$

F_y, F_x and F_w are all obviously continuous.

$$F_y = 3y^2x^2 + xw, \quad (44)$$

$$F_x = 2y^3x + yw, \quad (45)$$

$$F_w = 3w^2 + yx. \quad (46)$$

At a point such as $(1,1,1)$, F_y is nonzero. The existence of an implicit function $y = f(x,w)$ is assured around that point at least. Since the total differentials of both sides are equal.

$$F_y dy + F_x dx + F_w dw = 0. \tag{47}$$

Setting $dw = 0$ and rearranging the above equation, we obtain

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{2y^3x+yw}{3y^2x^2+xw}. \tag{48}$$

At the point $(1,1,1)$, this derivative has the value $-\frac{3}{4}$.