Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update

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Abstract

A group of agents have claims on a resource, but there is not enough of it to honor all of the claims. How should it be divided? A group of agents decide to undertake a public project that they can jointly afford. How much should each of them contribute? This essay is an update of Thomson (2003), a survey of the literature devoted to the study of such problems.

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1 Introduction

A group of agents have claims on a resource, but there is not enough of it to honor all of the claims. How should it be divided? Although societies have had to deal with situations of this type since time immemorial, their formal study began in earnest with O'Neill (1982). O'Neill describes a number of fascinating historical examples dating to antiquity and medieval times, together with the resolutions proposed for them then. He suggests a mathematical representation of the problem of adjudicating conflicting claims, develops several methodologies to handle it, axiomatic and game-theoretic, and applies these methodologies to derive a number of rules. A survey covering the literature that this seminal paper generated is Thomson (2003), hereafter referred to as T2003.[1]

The model has other interpretations. It covers in particular the problem faced by a group of agents undertaking a public project that they can jointly afford, and having to decide how much each of them should contribute (hence the reference to taxation in our title), but for simplicity, we will use language that is appropriate for the adjudication of conflicting claims.

It is remarkable how quickly the literature developed. An important reason is undoubtedly that researchers could take advantage of the conceptual apparatus and of the techniques developed in other branches of the axiomatics of resource allocation. The study of how to adjudicate conflicting claims is quite rewarding, and it has a unique place in this program. Indeed, the model is one for which many interesting rules can easily be defined. Also, several central principles that are often too strong to be met in other contexts are satisfied by many rules here.

The field has kept growing. Since the publication of T2003, a number of gaps were closed. The implications of various axiom systems are much better understood today and new axiomatic perspectives have been explored. Particularly significant are advances in the study of consistency and of the distributional implications of various rules. Also, O'Neill's model has been enriched in a variety of ways. Thus, an update appeared useful. We take up in turn each of the topics covered then, and in each case, we describe what we have learned in the last ten years. We also discuss new trends.

Section 2 introduces the model and the rules that have come up in its analysis. Section 3 reports on the progress made on the axiomatic front. Sec-

[1] Pedagogical expositions are by Malkevitch (undated, 2008, 2009).

tion 4 focuses on its game-theoretic modeling, both cooperative and strategic. Section 5 discusses experimental work. Section 6 presents the various ways in which O'Neill's model has been adapted to accommodate broader classes of environments. We have also updated the references of a number of papers that we had discussed in T2003 but were not yet published.

2 The base model of adjudication of conflicting claims and an inventory of rules

The notation and most of the language we use are as in T2003, except for a few terms, which we have replaced by ones that we feel are more informative. We keep the overlap with T2003 to the minimum necessary for a self-contained exposition. Readers familiar with the literature can skip this section, devoted to definitions.

2.1 The model

In order to distinguish the model introduced by O'Neill (1982) from the various enriched models that have been formulated more recently (Section 6), we refer to it as the **base** model. It is as follows: Let $N \equiv \{1, \ldots, n\}$ be a set of **claimants**. Each claimant $i \in N$ has a **claim** $c_i \in \mathbb{R}_+$ on an **endowment** $E \in \mathbb{R}_+$. The endowment is insufficient to honor all of the claims. Altogether, a claims problem, or simply a **problem**, is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum c_i \geq E$. Let $\boldsymbol{\mathcal{C}^N}$ be the class of all problems. An **awards vector of $(\boldsymbol{c}, \boldsymbol{E})$** is a vector $x \in \mathbb{R}^N$ satisfying **non-negativity** (no claimant should be asked to pay: $x \geqq 0$), **claims boundedness** (no claimant should be awarded more than his claim: $x \leqq c$), and **balance** (the sum of the awards should be equal to the endowment: $\sum x_i = E$). A **rule** is a function that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ a unique awards vector of (c, E). Our generic notation for a rule is the letter S. The **path of awards of a rule for a claims vector $\boldsymbol{c}$** is the locus of the choice it makes as the endowment ranges from 0 to $\sum c_i$.

We will also consider the generalization of the model obtained by letting the population of claimants vary. Then, there is an infinite set of "potential" claimants, indexed by the natural numbers $\mathbb{N}$, but in each problem, only finitely many of them are present. Let $\mathcal{N}$ be the family of all finite subsets of $\mathbb{N}$. Still using the notation $\boldsymbol{\mathcal{C}^N}$ for the class of problems with claimant

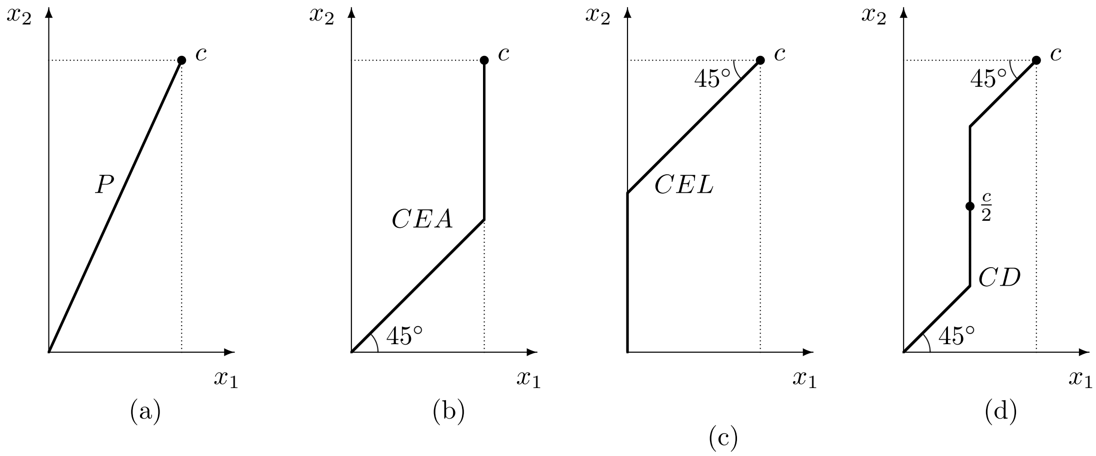

Figure 1: Paths of awards of four central rules. Typical paths of awards of four central rules for $N \equiv \{1, 2\}$ and $c \in \mathbb{R}^N_+$. (a) Proportional rule. (b) Constrained equal awards rule (c) Constrained equal losses rule. (b) Concede-and-divide.

set N, a rule is now defined on the union $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$: it associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, an awards vector of (c, E).

Given $a, b \in \mathbb{R}^N$, let $\mathbf{seg}[\boldsymbol{a}, \boldsymbol{b}]$ denote the segment connecting these two points.

2.2 Rules

All of the following rules will come up at some point. We define them for a fixed N. Let $(c, E) \in \mathcal{C}^N$. It will simplify some definitions to assume that no two claims are equal. The adjustments necessary to cover possible equality of claims are straightforward. The first four definitions are illustrated in Figure 1.

For the **proportional rule,** $\boldsymbol{P}$, (Aristotle, 1985), for each $i \in N$, agent i's award is λc_i, λ being chosen, as in the next two definitions, so that awards add up to E.

For the **constrained equal awards rule,** $\boldsymbol{CEA}$, (Maimonides, 12th Century) agent i's award is $\min\{c_i, \lambda\}$. An algorithmic definition will be useful, keeping $c \in \mathbb{R}^N_+$ fixed and letting the endowment grow from 0 to $\sum c_i$. At first, equal division takes place until each claimant receives an amount equal to the smallest claim. The smallest claimant drops out, and the next increments of the endowment are divided equally among the others until each of them receives an amount equal to the second smallest claim. The second smallest claimant drops out, and so on.

For the **constrained equal losses rule,** $\boldsymbol{CEL}$, (Maimonides, 12th Century) each claimant $i \in N$ receives $\max\{c_i - \lambda, 0\}$. A symmetric algorithm to that underlying the constrained equal awards rule can be defined, this time letting the endowment decrease from $\sum c_i$—then each claimant is fully compensated—to 0. At first, equal losses are imposed on all claimants until their common loss is equal to the smallest claim. The smallest claimant receives 0 then and he drops out. As the endowment continues to decrease, equality of losses is maintained for the others until their common loss is equal to the second smallest claim. The second smallest claimant drops out, and so on.

Concede-and-divide (Aumann and Maschler, 1985) is the two-claimant rule that first assigns to each claimant the difference between the endowment and the other agent's claim (or 0 if this difference is negative), and divides the remainder equally.

The **Talmud rule, *T***, (Aumann and Maschler, 1985) can be seen as a hybrid of the constrained equal awards and constrained equal losses rules: it selects $CEA(\frac{c}{2}, E)$ if $E \leq \frac{\sum c_i}{2}$, and $\frac{c}{2} + CEL(\frac{c}{2}, E - \frac{\sum c_i}{2})$ otherwise. An algorithm producing it is obtained by applying in succession the algorithms generating the constrained equal awards and constrained equal losses rules, but using as switchpoints the half-claims instead of the claims themselves. The **reverse Talmud rule** (Chun, Schummer, and Thomson, 2001) is derived from this definition by exchanging the roles played by the constrained equal awards and constrained equal losses rules.

Letting once again the endowment grow from 0 to $\sum c_i$, the **constrained egalitarian rule** (Chun, Schummer, and Thomson, 2001) selects $CEA(\frac{c}{2}, E)$ until the endowment reaches $\frac{\sum c_i}{2}$. The next increments go to the smallest claimant until he receives the maximum of his claim and half of the second smallest claim. The next increments are divided equally between the two smallest claimants until the smallest claimant receives his claim, in which case the second smallest claimant receives each additional unit until he receives the maximum of his claim and half of the third smallest claim, or they reach half of the third smallest claim; and so on.

Piniles' rule, *Pin*, (Piniles, 1861) results from a "double" application of the constrained equal awards rule, using as in the Talmud rule the half-claims instead of the claims themselves: it selects $CEA(\frac{c}{2}, E)$ if $E \leq \frac{\sum c_i}{2}$, and $\frac{c}{2} + CEA(\frac{c}{2}, E - \frac{\sum c_i}{2})$ otherwise.

The **random arrival rule**[2]**, *RA***, (O'Neill, 1982) selects the average of the awards vectors obtained by specifying an order on the claimant set and fully compensating each claimant, in that order, until the endowment runs out, all orders being given equal probabilities.

[2]Some authors refer to it as the "run-to-the-bank" rule.

The **minimal overlap rule, *MO***, (O'Neill, 1982) is defined by imagining the endowment to consist of individual "units", and distributing the various claims over the endowment in such a way that the number of units that are claimed exactly once is maximized; among the distributions solving this maximization exercise, identifying those at which the number of units that are claimed exactly twice is maximized, and so on; finally, dividing each unit equally among all agents claiming it. This rule can be seen as an extension to the entire domain of problems of an incompletely specified rule—it is only defined for problems in which no claim exceeds the endowment—proposed by Rabad. We will refer to it as "Rabad's proposal".

2.3 Families of rules

We introduce next several families of rules that will help organize our inventory as well as offer additional choices. First, given an order on the agent set, $\prec$, the **sequential priority rule associated with $\prec$, $SP^{\prec}$,** assigns to each agent in turn, in that order, the minimum of his claim and what remains of the endowment. (The random arrival rule is the simple average of these rules.)

Second is an important family introduced by Young (1987). Let $\mathbf{\Phi}$ be the family of functions $f\colon \mathbb{R}_+ \times [\underline{\lambda}, \bar{\lambda}] \to \mathbb{R}_+$, where $-\infty \leq \underline{\lambda} < \bar{\lambda} \leq \infty$, that are continuous, nowhere decreasing with respect to their second argument, and such that for each $c_0 \in \mathbb{R}_+$, we have $f(c_0, \underline{\lambda}) = 0$ and $f(c_0, \bar{\lambda}) = c_0$. The **parametric rule of representation $f \in \Phi$, S^f,** is defined as follows: for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $S^f(c, E)$ is the awards vector x such that there is $\lambda \in [\underline{\lambda}, \bar{\lambda}]$, for which, for each $i \in N$, $x_i = f(c_i, \lambda)$.

In addition to the proportional rule, the family contains the constrained equal awards, constrained equal losses, Talmud and Piniles' rules.

Last is a family, the **ICI family** (Thomson, 2000, 2008b), that generalizes the Talmud rule. The pattern of distribution is the same but the definition allows the critical values of the endowment at which claimants come in and out of the distribution to differ from the half-claims, and moreover, to depend on the claims vector. To specify a rule in the family, we need lists $F \equiv (F_k)_{k=1}^{k=n-1}$ and $G \equiv (G_k)_{k=1}^{k=n-1}$ (where $n \equiv |N|$) of functions from $\mathbb{R}^N_+$ to $\mathbb{R}_+$ such that for each pair $k, k' \in \{1, \ldots, n-1\}$ with $k < k'$, $F_k \leq F_{k'}$ and $G_{k'} \leq G_k$. Let $c \in \mathbb{R}^N_+$ be given, and let E grow from 0 to $\sum c_i$. The distribution is as follows. The first units are divided equally until E reaches $F_1(c)$, at which point the smallest claimant drops out for a while. The next units are divided equally among the others until E reaches $F_2(c)$, at which point the second smallest claimant also drops out for a while. This goes on until E reaches $F_{n-1}(c)$, at which point only the largest claimant is left; he receives each additional unit until E reaches $G_{n-1}(c)$. The other claimants return for more, one at a time, in the reverse order of their departure. As E increases from $G_{n-1}(c)$ to $G_{n-2}(c)$, each increment is divided equally between the two largest claimants, and so on. The process continues until E reaches $G_1(c)$, at which point each increment is divided equally among all claimants, and until the end. To guarantee that then, each agent receives exactly his claim, the lists $F(c) \equiv (F_k(c))_{k=1}^{k=n-1}$ and $G(c) \equiv (G_k(c))_{k=1}^{k=n-1}$ have to satisfy certain linear relations, the **ICI relations**, which we omit.[3]

The family contains the constrained equal awards, constrained equal losses, and Talmud rules. Surprisingly, given the completely different scenario on which it is based, the minimal overlap rule is included. A parallel family, the **CIC** family, can also be defined by exchanging the roles played by the ideas of equal awards and equal losses. The **TAL family** (Moreno-Ternero and Villar, 2006a) is the subfamily of the ICI family defined as follows. Let $\theta \in [0,1]$ and $T^\theta(c,E) = CEA(\theta\frac{c}{2}, E)$ if $E \leq \theta\frac{\sum c_i}{2}$, and $T(c,E) = \theta c + CEL((1-\theta)c, E - \theta\sum c_i)$ otherwise. This subfamily still contains the constrained equal awards, constrained equal losses, and Talmud rules, but the minimal overlap rule is not in it. Another family that contains the TAL family, but is not a subfamily of the ICI family, is defined by Moreno-Ternero (2011a).

3 Axiomatic studies

We will discuss properties of rules both in the context of a fixed population of claimants, but also when the population of claimants is allowed to vary.

3.1 Duality and consistency

The space of rules is highly structured, and one of the most useful concepts to bring out this structure is duality. Thus, we introduce the concept right away. We apply it in succession to problems, rules, properties of rules, and

[3]The acronym ICI stands for increasing-constant-increasing, reflecting the evolution of each claimant's award as the endowment increases.

mappings from the space of rules into itself, which we call operators (Subsection 3.11). We also define two properties that have played a key role in relating rules across populations of claimants, as well as in extending rules defined for two claimants to arbitrarily many claimants. Many other properties will be invoked below, but we will introduce them as needed. For motivation, we refer the reader to the primary sources, listed in T2003.

Two problems (c, E) and $(c', E') \in \mathcal{C}^N$ are dual if $c = c'$ and $E = \sum c_i - E'$. **Two rules S and S' are dual** if for each problem, S divides what is available in the same way as S' divides what is missing (the difference between the sum of the claims and the endowment): for each $(c, E) \in \mathcal{C}^N$, $S(c, E) = c - S'(c, \sum c_i - E)$. **Two properties are dual** if whenever a rule satisfies one of them, its dual satisfies the other. (For a number of relational properties pertaining to a change in a single parameter, the dual property pertains to changes in two parameters that are linked in a particular way.) **Two operators are dual** if whenever two rules S and S' are dual, the rule obtained by applying one operator to S and the rule obtained by applying the other operator to S', are dual too. An object (a problem, a rule, a property, an operator) is **self-dual** if it coincides with its dual.

The proportional, Talmud (thus, concede-and-divide as well), and random arrival rules are *self-dual*. The constrained equal awards and constrained equal losses rules are dual. The sequential priority, ICI, CIC, TAL, and parametric families are closed under duality.

We can also speak of **two theorems being dual**: the dual of a characterization, for example, is obtained from it by replacing each axiom by its dual, and each rule (or each family of rules) by its dual. We will state a large number of results and, in order to save space, we will let the reader deduce the results that follow by duality. For example, each of the characterizations of the constrained equal awards rule stated in Theorem 4 has a counterpart yielding the constrained equal losses rule, these two rules being dual.

Much use is made of duality notions by Herrero and Villar (2001), who organize their survey around them.

Consistency, a property of a rule defined over arbitrary populations, says that the choice the rule makes for each problem should always be "in agreement" with the choice it makes for each "reduced" problem obtained by imagining some agents leaving with their awards, and reassessing the opportunities open to the remaining agents at that point: for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, the restriction of $S(c, E)$ to $\mathbb{R}^{N'}$ should be the choice S makes for the **reduced problem associated with N'**

and $\boldsymbol{S(c,E)}$, namely the problem with agent set N' in which these agents' claims are $(c_i)_{i\in N'}$ and the endowment is what remains after the members of $N\setminus N'$ have received their awards, $(S_i(c,E))_{i\in N\setminus N'}$, and left; alternatively, the endowment in this reduced problem is the sum $\sum_N S_i(c,E)$ of the awards intended for the members of N'.[4]

Bilateral consistency is the version of the property obtained by requiring that all but two agents leave. **Null claims consistency** is the considerably weaker form of *consistency* obtained by limiting its application to the departure of agents whose claims are 0 (and whose awards, by definition of a rule, are 0 as well).

Converse consistency of a rule says the following: suppose that an awards vector for a problem is such that the rule chooses its restriction to each two-claimant subset of the claimants it involves for the associated reduced problem these claimants face. Then the rule should choose the awards vector for the initial problem: for each $N\in\mathcal{N}$, each $(c,E)\in\mathcal{C}^N$, and each awards vector x of (c,E), if for each $N'\subset N$ with $|N'|=2$, $x_{N'}=S(c_{N'},E-\sum_{N\setminus N'}x_i)$, then $x=S(c,E)$.

The **Elevator Lemma** (Thomson, 2006; 2011) states that if a rule S is *bilaterally consistent*, a rule S' is *conversely consistent*, and $S=S'$ for two claimants, then in fact, $S=S'$ for arbitrarily many claimants. We will invoke this lemma on multiple occasions to extend characterizations from the two-claimant case to arbitrary populations.

3.2 Order preservation properties

The next requirement can be seen as a generalization of **equal treatment of equals**, which says that, for each problem, two agents with equal claims should be awarded equal amounts. **Order preservation of awards** says that, for each problem, awards should be ordered as claims are, and **order preservation of losses** that so should losses. We refer to the conjunction of these two properties as **order preservation** (Aumann and Maschler, 1985).

The next requirements are relational. They express the same sort of idea in situations in which some parameter(s) of the problem changes (change). **Order preservation under endowment variation** (Dagan, Serrano, and

[4] *Consistency* is often described as an "operational" principle. An interpretation as a fairness property can be given (Thomson, 2012b).

Volij, 1997)[5] is the requirement that if the endowment increases from some initial value to some final value, given two claimants i and j, if $c_i \geq c_j$, the difference between claimant i's final and initial awards should be at least as large as the corresponding difference for claimant j.

Order preservation under claim variation (Thomson, 2006) pertains to an increase in some agent k's claim from some initial to final values: given two claimants i and j, if $c_i \geq c_j$, the difference between claimant i's initial and final awards should be at least as large as the corresponding difference for claimant j.

When we allow variations in populations, two further applications of the order preservation idea are possible. **Order preservation under population variation** (Thomson, 2006) pertains to the departure of some claimants: given two remaining claimants i and j, if $c_i \geq c_j$, the difference between claimant i's new and initial awards should be at least as large as the corresponding difference for claimant j.

Finally, **order preservation under the reduction operation** (Thomson, 2006) pertains to the departure of some claimants with their awards (as opposed to empty-handed, as imagined in the previous definition), when we consider the problem of dividing what remains among the remaining claimants: given two such claimants i and j, if $c_i \geq c_j$, the difference between claimant i's new and initial awards should be at least as large as the corresponding difference for claimant j.[6]

All of these properties are met very generally.

3.3 Lower and upper bounds

Defining and imposing lower and upper bounds on assignments, utilities, welfare, and so on, is an essential part of most approaches to the problem of fair allocation. The theory concerning the adjudication of conflicting claims is no exception and, in fact, a range of such requirements have been proposed.

3.3.1 Defining lower bounds

Various lower bounds can be placed on an agent's award as a function of the parameters of a problem, and T2003 lists several. The **minimal right of**

[5] These authors discuss it under the name of "super-modularity".

[6] The reduction operation is the operation underlying the definition of *consistency* in Subsection 3.1. *Consistency* says that all these differences should be equal to 0.

claimant i in (c, E) is whatever is left after every one else has been fully reimbursed, or 0 if that is not possible, altogether $\max\{E - \sum_{N\setminus\{i\}} c_j, 0\}$. A rule satisfies the **minimal rights lower bound** if in each problem, it assigns to each claimant at least his minimal right. The **reasonable lower bound for claimant $i \in N$ in (c, E)** is $\min\{\frac{c_i}{|N|}, \frac{E}{|N|}\}$ (Moreno-Ternero and Villar, 2004).[7] The **conditional equal division lower bound for claimant $i \in N$ in (c, E)** is $\min\{c_i, \frac{E}{|N|}\}$ (Moulin, 2000).

It follows directly from the definitions that any rule satisfies the *minimal rights lower bound*. Many rules satisfy the *reasonable lower bound* but not all; for instance, the proportional and constrained equal losses rules do not. For $|N| = 2$, the constrained equal awards rule is the only rule satisfying the *conditional equal division lower bound*.

For $|N| = 2$, the **low-claim lower bound** says that if an agent's claim is at most as large as the endowment, his award should be at least half of his claim. Also, for $|N| = 2$, the **high-claim lower bound** says that if an agent's claim is at least as large as the endowment, his award should be at least half of the endowment (Moreno-Ternero and Villar, 2006c). When $|N|$ is arbitrary, replace half by $\frac{1}{|N|}$ in these expressions.[8]

In each of these definitions, the focus is on what claimants receive. To each of them can be associated a lower bound on losses by duality. For instance, the **reasonable lower bound on losses for claimant i in (c, E)** is $\frac{1}{|N|}\min\{c_i, \sum c_j - E\}$.

A rule satisfies **conditional full compensation** if for each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, if by substituting c_i for the claim of each other agent whose claim is greater, there is now enough to compensate everyone, then agent i should be fully compensated (Herrero and Villar, 2002). The dual property, **conditional null compensation**, says that if a claim is "small enough" (in relation to the endowment), its owner should not be assigned anything.[9]

Parameterized versions of these last two properties are proposed by van den Brink, Funaki, and van den Laan (2013). They come in dual pairs. For $|N| = 2$, there is a strongest dual pair that is compatible with *order preservation under endowment variation*, and only one rule satisfies all three

[7]Moreno-Ternero and Villar refer to this property as "securement".

[8]Moreno-Ternero and Villar refer to these properties as "lower securement" and "upper securement".

[9]Herrero and Villar refer to these properties as "sustainability" and "independence of residual claims".

properties, the reverse Talmud rule.

3.3.2 Recursive applications of lower bounds

When a lower bound for a problem is interpreted as an undisputable amount that a claimant should get, it is natural to solve the problem by first assigning to each agent this amount, and in a second step, to distribute what remains of the endowment. In the second step, claims should of course be revised down by the amounts awarded in the first step. It is natural to want to repeat the process, namely to calculate the lower bounds in the revised problem and to assign these amounts; to revise claims again, and so on.

Two possibilities emerge. One of them is that at some step, the lower bounds are in fact zero: no further awards can be made. This is the case already at the second step for the minimal rights of Subsection 3.3.1.

Another possibility is that the recursive assignment of the lower bounds exhausts the total endowment at some step or in the limit. If this phenomenon occurs for each problem, a rule has been defined. That is actually what happens for the reasonable lower bound of Subsection 3.3.1, as shown by Dominguez and Thomson (2006), who study the resulting rule (under the name of "recursive rule"), and establish a number of properties it has. In fact, a weak property of lower bounds guarantees this type of result:

Theorem 1 (Dominguez, 2013) *Consider a lower bound function that is continuous and, for each non-zero claims vector and positive endowment, specifies a positive amount for at least one claimant. Then, its recursive application exhausts the total endowment in the limit, thereby defining a rule.*

A generalization of the concept of a lower bound is due to Hougaard, Moreno-Ternero, and Østerdal (2012): a **baseline** is a function that associates with each problem a vector satisfying all the properties of an awards vector except possibly for the balance condition. If the sum of the coordinates of this vector is smaller than the endowment, the vector can be interpreted and used as a lower bound on awards as just discussed, and if the opposite holds, as an upper bound. Truncation and duality operations (see below) can also be applied to baselines. Now, consider a baseline that is continuous and, for each non-zero claims vector, specifies a vector with at least one positive coordinate and at least one coordinate that is smaller than the corresponding coordinate of the claims vector. Then, recursively assigning the baseline amounts when they are lower bounds and replacing claims by these amounts when they are upper bounds exhausts the total endowment in the limit, thereby defining a rule.

3.3.3 Deriving lower bounds by applying rules in a family

The idea of dividing the endowment in a progressive manner can be exploited in a different way. Suppose that the decision has been made to use a rule in a certain family, but that no particular member of the family has been identified as being most desirable. The family may be given explicitly, or through a list of properties its members should have. Given a problem, for each claimant, let us calculate the smallest amount awarded to him by any of the rules in the family. The decision to use only these rules should certainly imply that the claimant is entitled to at least this amount. So, let us award it to him, revise his claim down accordingly, perform the same operation for each agent, revise the endowment down by the sum of the amounts awarded, and let us repeat, that is, calculate for each claimant the smallest amount awarded to him by any of the rules when applied to the revised problem. If the sequence of residual endowments converges to zero, this process yields an awards vector for the problem under consideration; if this is the case for each problem, we have defined a rule. (The process is similar to that underlying a game proposed by Herrero, 2003).

This is the proposal made by Giménez-Gómez and Marco (2012) and we will refer to the process just defined as the **GGM process**. These authors work with families of rules defined by means of lists of properties they are required to satisfy. They ask whether the residual endowment converges to zero, and if that is the case, whether the rule defined in the limit can be identified. To describe their results, we need two additional properties of rules.

Endowment monotonicity says that if the endowment increases, each claimant should be awarded at least as much as initially. The **midpoint property**, an obvious implication of *self-duality*, says that if the endowment is equal to the half-sum of the claims, each agent should be awarded half of his claim.

Theorem 2 (Giménez-Gómez and Marco, 2012) *Under the GGM process,*

(a) for the list of properties consisting of order preservation, *and for each problem, the residual endowment converges to zero. The rule defined in the limit is the constrained equal losses rule.*

(b) for $|N| = 2$, and for the list of properties consisting of order preservation, endowment monotonicity, and the midpoint property, and for each problem, the residual endowment converges to zero. The rule defined in the limit is the dual of the constrained egalitarian rule.

For $|N| > 2$, and for the list of properties of Theorem 2b, the rule defined in the limit is not *endowment monotonic*. Thus, interestingly but of course it is a disappointment, a rule obtained by the GGM process for a particular list of properties does not necessarily satisfy all of these properties.

A further study along the same lines is due to Giménez-Gómez and Peris (2013c). These authors used both lower and upper bounds in restricting the set of awards vectors at each step.

We conclude our discussion by noting that awards and changes in awards can be related to lower bounds and to the changes in lower bounds caused by changes in the endowment. The constrained equal awards rule has been derived in this manner (Giménez-Gómez and Peris, 2013b).

3.4 Monotonicity properties

For this model, and as opposed to what is the case for many other models of resource allocation, monotonicity properties are easily met. A central requirement in various literatures is that as opportunities expand, each agent should end up at least as well off as he was initially. Here, opportunities are defined through the endowment and we obtain *endowment monotonicity* (Subsubsection 3.3.3). Monotonicity with respect to claims is also a meaningful requirement: **claims monotonicity** says that if an agent's claim increases, his award should not decrease.

One can say more about what should happen when some agent i's claim increases. First, we may consider the impact that the increase has on the others: **others-oriented claims monotonicity** (Thomson, 2003) is the requirement that none of these claimants' awards should increase. We may also impose an upper bound on the increase in claimant i's award. A most natural one is the amount by which his claim increases—let us call it δ. An appealing upper bound on the decrease in each of the other claimants' awards is also δ. Finally, we may formulate versions of these properties pertaining to simultaneous increases in the claims of several agents.

Kasajima and Thomson (2011) propose these and other properties of this type. They identify their duals (the hypotheses of the dual properties often involve simultaneous but linked changes in several components of the problem). They study whether the properties are preserved by operators (Subsection 3.11), and whether they are lifted by *consistency* (Subsection 3.10).

3.5 Invariance properties

The following properties will appear repeatedly in what follows.

Claims truncation invariance (Dagan and Volij, 1993) says that truncating claims at the endowment should not affect which awards vector is chosen. **Minimal rights first** (Curiel, Maschler, and Tijs, 1987), its dual, says that one should be able to calculate the awards vector chosen in either one of the following ways: (i) directly, or (ii) by first assigning minimal rights, and in a second step, after revising claims down by these amounts, applying the rule to allocate what remains of the endowment. (The minimal rights of the revised problem are zero.)

A change in a parameter of a problem can often be looked at from several perspectives, all equally legitimate, and a robustness requirement on a rule is that these perspectives should lead to the same awards vector. The idea has been applied to changes in the endowment, as follows. **Composition down** (Moulin, 2000) says that if it decreases, one should be able to calculate the awards vector chosen for the smaller endowment in either one of the following two ways, (i) directly, that is, ignoring the awards vector chosen for the initial endowment, or (ii) by using as claims vector this awards vector. **Composition up** (Young, 1988), its dual, pertains to increases in the endowment. It says that one should be able to calculate the awards vector for the larger endowment in either one of the following two ways: (i) directly, or (ii) by first assigning the awards obtained by applying the rule to the initial endowment, and in a second step, after revising claims down by these amounts, applying the rule to allocate the increment in the endowment.[10]

A rule is **homogeneous** if multiplying claims and endowment by any $\lambda >$

[10] Martínez (2008) proposes "strong composition down": starting from some initial problem, suppose that the endowment decreases. Then the requirement is that one should be able to calculate the awards vector chosen for the resulting problem in either one of the following two ways: (i) directly, or (ii) by using as claims vector any vector obtained from the initial one by replacing any of its coordinates by the corresponding coordinate of the awards vector chosen for the initial endowment. This is undoubtedly a very strong requirement, but it is met by the constrained equal awards rule as well as by versions of this rule that do not necessarily satisfy *equal treatment of equals*.

0 results in a new problem for which it chooses the awards vector obtained by multiplying by λ the awards vector it chooses for the initial problem. Thus, a problem with low stakes is perceived as essentially the same thing as a problem with high stakes. The axiom has been erroneously interpreted as meaning that the units of measurement do not matter (whether the problem is expressed in dollars say, as opposed to euros), but it is a substantial requirement. (The fallacy is exposed in Thomson, 2006, and Marchant, 2008).

Given a point $x \in \mathbb{R}^N_+$, axioms can also be formulated on the shape of the set of claims vectors $c \in \mathbb{R}^N$ for which $x = S(c, \sum x_i)$: this set is the **inverse set of S for x**. The concept has been found useful in the theory of bargaining, and a variety of axioms have been proposed pertaining to the shape of inverse sets (they are reviewed in Thomson, 2001). One requirement is that they should be convex (if two claims vectors lead to the same awards vector, so should any average of them). A second is that they should be star-shaped with x as a center of the star. A third is that they should be cone-shaped with x as the vertex of the cone. A fourth is that they should be "broom-shaped with respect to x" (if c is in the set, then so is any c' such that c is a convex combination of x and c').

3.6 Some characterizations

The following characterizations involve axioms introduced in several previous sections.

Theorem 3 *For $|N| = 2$. Concede-and-divide is the only rule satisfying*

(a) the reasonable lower bound and self-duality, or the reasonable lower bound and its dual (Moreno-Ternero and Villar, 2004).

(b) the low-claim lower bound, claims monotonicity, and self-duality, or the low-claim lower bound, its dual, claims monotonicity, and its dual, (Moreno-Ternero and Villar, 2006c).

(c) the high-claim lower bound, its dual, and endowment monotonicity, or the high-claim lower bound, endowment monotonicity, and self-duality (Moreno-Ternero and Villar, 2006c).

(e) the high-claim lower bound and minimal rights first (Moreno-Ternero and Villar, 2006c).

(f) the reasonable lower bound and minimal rights first (Yeh, 2008).

Theorem 4 *For $|N| = 2$. The constrained equal awards rule is the only rule satisfying*

(a) conditional equal division full compensation and composition down (Herrero and Villar, 2002; see also Yeh, 2004).

(b) conditional full compensation and claims monotonicity (Yeh, 2006).

(c) conditional equal division full compensation and order preservation under endowment variation (Yeh, 2006).

(d) the reasonable lower bound and composition up (Yeh, 2008).

(e) equal treatment of equals and strong composition up (Martínez, 2008).

Conditional equal division full compensation is a weaker requirement than *conditional full compensation*, but in the two-claimant case, they are equivalent. Thus, Theorem 4(a) follows from the fact that—the result holds for arbitrary populations—the constrained equal awards rule is the only rule satisfying *conditional full compensation* and *composition down* (Herrero and Villar, 2002). Part (d) also exploits this logical relation.

The next two theorems are derived from the previous two by means of the Elevator Lemma.

Theorem 5 *[Moreno-Ternero and Villar, 2006c] Each of the statements of Theorem 3 yields a characterization of the Talmud rule for any number of claimants if consistency is added to the list of required axioms.*

Theorem 6 *[Herrero and Villar, 2002; Yeh, 2004, 2006, 2008; Martínez, 2008]. Each of the statements of Theorem 4 yields a characterization of the constrained equal awards rule for any number of claimants if consistency is added to the list of required axioms.*[11]

The uniqueness part of the extension of Theorem 4(d) still holds if *null claims consistency* is imposed instead of *consistency* (Yeh, 2008).

A characterization of the reverse Talmud rule follows from an earlier characterization of the rule for two claimants (end of Subsection 3.3.1) if *consistency* is additionally imposed. It too results from an application of the Elevator Lemma.

[11] We stated the *low-claim lower bound* for two claimants, and this extension holds if this two-claimant axiom is imposed. However the bound can be formulated for arbitrary populations: an agent whose claim is at least as large as the endowment should be assigned at least $\frac{1}{|N|}E$. The Talmud rule satisfies it.

The entire class of rules satisfying *composition down* can also be given a transparent characterization (Thomson, 2006). Consider a network of (a) weakly monotone and continuous curves in $\mathbb{R}^N_+$ emanating from the origin such that (b) given any point in $\mathbb{R}^N_+$, there is at least one curve passing through it, and (c) following any curve up from the origin, if one encounters a point at which the curve splits into branches, these branches never meet again. A network satisfying (a)-(c) constitutes a **weakly monotone space-filling tree in $\mathbb{R}^{\boldsymbol{N}}_+$.**

Theorem 7 (Thomson, 2006) *A rule on $\mathcal{C}^N$ satisfies composition down if and only if there is a weakly monotone space-filling tree in $\mathbb{R}^N_+$ such that, for each $c \in \mathbb{R}^N_+$, the path of awards of the rule for c is obtained, from any branch emanating from the origin and passing through c, as the part of it that lies in the box $\{x \in \mathbb{R}^N_+ \colon 0 \leqq x \leqq c\}$. If in addition the rule is claims continuous, each branch of the tree is unbounded above.*

A characterization of the class of two-claimant rules satisfying *homogeneity*, *composition down*, and *composition up* is given by Moulin (2000) and described in T2003 (see also Thomson, 2013b). What if *homogeneity* is dropped? Answers are the following (Chambers, 2006).

For $|N| = 2$, consider a continuous and monotone curve C in $\mathbb{R}^N$ that is unbounded in both directions and is concave either to the northwest or to the southeast. Such a curve has exactly two asymptotic directions, one in $\mathbb{R}^N_+$—let us call it b^1 –and the other in $\mathbb{R}^N_-$—let us call it b^2. For each claims vector c in the convex cone whose boundary rays are b_1 and $-b_2$, we identify two points x^1 and x^2 on C such that $x^2 - x^1 = c$ (such a pair exists). We translate the part of C that lies between x^1 and x^2 by the vector $-x^1$. This brings x^1 to the origin. Now, we select that part of C as the path of awards for c. (The pair $\{x^1, x^2\}$ may not be unique, but the path is uniquely defined.) Given a second curve C' with the same properties, we use it to similarly define paths of awards for all claims vectors in the cone associated with it in the same way. The two cones should not intersect, but they may have a boundary ray in common. More generally, the characterization involves a family of curves satisfying the properties listed above for C, such that the cones associated with them in the manner just described, together with their boundary rays if needed, "cover" awards space. To obtain the path of awards of a particular claims vector, we identify the cone in the partition to which it belongs, and proceed as explained above. Thereby, we associate a rule with this family of curves. Let us refer to it as a **relative-claims difference rule**. We omit the definition of the extension of this family to arbitrarily many claimants proposed by Chambers (2006), the **difference family**.

Theorem 8 (Chambers, 2006) *(a) For $|N| = 2$. The relative-claims difference rules are the only rules satisfying* composition down *and* composition up.

(b) For arbitrary $|N|$. The difference rules are the only rules satisfying these two axioms.

Our next theorem is a characterization of a large family of rules based mainly on *claims truncation invariance*:

Theorem 9 (Thomson, 2006) *For $|N| = 2$, say $N \equiv \{1, 2\}$. A rule S satisfies* equal treatment of equals *and* claims truncation invariance *if and only if, for each $c_2 \in \mathbb{R}_+$, there is a path $G^S(c_2) \subset \mathbb{R}^N_+$ with the following three properties:* (i) *it contains* $\text{seg}[(0,0), (\frac{c_2}{2}, \frac{c_2}{2})]$; (ii) *for each $E \geq c_2$, it meets the line of equation $\sum x_i = E$ exactly once;* (iii) *it is bounded above by the line of equation $x_2 = c_2$; and, for each $c_1 \in [c_2, \infty[$, the path of awards of S for (c_1, c_2) contains $G^S(c_2)$ up to its intersection with the line of equation $\sum x_i = c_1$.*

For each $c_1 \in \mathbb{R}_+$, there is a path $G^S(c_1) \subset \mathbb{R}^N_+$ with similar properties whose statements we omit, and, for each $c_2 \in [c_1, \infty[$, the path of awards of S for (c_1, c_2) contains $G^S(c_1)$ up to its intersection with the line of equation $\sum x_i = c_2$.

The calculation of the inverse sets of the rules that have been central in the literature can be found in Hokari (2000), but the implications of axioms pertaining to the shapes of these sets have been studied in greatest detail by Juarez (2005) to whom the results listed next are due. The most natural of the requirements is probably that **inverse sets be convex**. However, for two claimants, it turns out that all four requirements formulated at the end of Section 3.5 pertaining to the shapes of inverse sets are equivalent. Let us then focus on the requirement that *inverse sets be cones*. We omit the details as the classes of rules identified in the next theorems are somewhat complex. First is the class of two-claimant *homogeneous* rules whose *inverse sets are cones*; second is the class of rules that satisfy *composition up* and whose *inverse sets are cones*; third is the subclass of n-claimant rules that are *homogeneous*. Finally, recall the characterization of the two-claimant rules satisfying *homogeneity* and the two composition properties and *homogeneity* listed above (Moulin, 2000). In this characterization, the requirement that *inverse sets be convex* can (essentially) be substituted for either one of the *composition* properties.

3.7 Additivity properties

Ever since Shapley (1953) made it the keystone of his characterization of the Shapley value, additivity properties have played an important role in various branches of game theory and resource allocation theory. Several expressions of the idea have been proposed in the context of claims problems. The most natural one, let us call it **full additivity**, says that if two problems are added (the class of claims problems is closed under this operation), the awards vector chosen for the sum problem should be the sum of the awards vectors chosen for the component problems. Bergantiños and Méndez-Naya (2001) had noted that no rule is *fully additive*. This is because of the non-negativity requirement imposed on awards vectors. Indeed, if this requirement is dropped, it can be met, and a characterization is available.

Several restricted versions of the property have been formulated, as well as variants. The first restriction is obtained by limiting attention to problems in which no claim is greater than the endowment. Let us call it **claims-bounded-by-endowment additivity**. Then for $|N| = 2$, concede-and-divide is the only rule satisfying *equal treatment of equals*, *claims truncation invariance*, and *claims-bounded-by-endowment additivity* (Salonen, 2004).

A conditional version of *additivity*—it applies to pairs of problems that are not "too different" from each other—has been proposed, and a characterization of the minimal overlap rule has been based on this requirement, together with *anonymity* and *continuity* (Alcalde, Marco and Silva, 2013).

The next properties are obtained from *full additivity* not by limiting its scope but by generalizing the required relationship between claims and endowment. **Endowment additivity** (Chun, 1988) says that if, keeping the claims vector fixed, the endowment comes in two installments E and E', the awards vector selected for the sum of these installments should be the sum of the awards vectors selected for each of them. **Addition invariance 1** (Marchant, 2008) says that if each claim increases by δ and so does the endowment, then each claimant's award should increase by $\frac{1}{|N|}\delta$. **Addition invariance 2** (Marchant, 2008) says that if each claim increased by δ, the awards vector should not change. The dual says that if each claim increases by δ and the endowment increases by $n\delta$, each award should increase by δ.[12]

Theorem 10 *(a) The proportional rule is the only rule satisfying* endowment additivity *(Chun, 1988; Bergantiños and Vidal-Puga, 2004).*

(b) For $|N| = 2$. Concede-and-divide is the only rule satisfying additive invariance 1. *On $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, the minimal overlap rule is the only rule that also satisfies* null claims consistency *(Marchant, 2008).*

(c) For $|N| = 2$. The constrained equal losses rule is the only rule satisfying additive invariance 2. *On $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$, it is the only rule that also satisfies* null claims consistency *(Marchant, 2008).*

3.8 Rationalizability of rules by binary relations

Here, we ask a question analogous to ones that have been the object of extensive literature in demand theory: Given a rule, is there a binary relation defined on $\mathbb{R}^N_+$ such that, for each problem, the awards vector chosen by the rule for that problem is the unique maximizer of the relation over its set of awards vectors (Kıbrıs, 2012)? If so, we say that the rule is **rationalizable**.[13] The relation can be interpreted as representing society's ranking of income distributions.

Let us associate with a rule S a binary relation R^S on $\mathbb{R}^N_+$ as follows: $\boldsymbol{x}\ \boldsymbol{R^S}\ \boldsymbol{y}$ if there is a problem admitting both x and y as awards vectors and for which S chooses x. Let P^S denote the strict relation associated with R^S. *Rationalizability* is equivalent to the **Weak Axiom of Revealed Preference**, abbreviated as *WARP*, which says that if $x\ P^S\ y$, then it should not be the case that $y\ P^S\ x$: the relation P^S is asymmetric. *Rationalizability by a transitive relation* is equivalent to the **Strong Axiom of Revealed Preference**, abbreviated as *SARP*, which says that the relation P^S is acyclic. Also, a rule is **representable by a numerical function** if there is a function $\mathbb{R}^N_+ \to \mathbb{R}$ such that, for each problem, the awards vector it selects is the maximizer of the function over its set of awards vectors. Finally, the counterpart for this model of Nash's (1950) **contraction independence** (usually referred to as "independence of irrelevant alternatives") says that if a rule chooses some awards vector x for some problem and the problem changes in

[12]Marchant (2008) proposes two other invariance properties.

[13]In demand theory, the question pertains to a demand function or correspondence instead of rules and to bundles of commodities instead of awards vectors.

such a way that the feasible set contracts but x remains feasible, then the rule should still choose x.

A relevant property in describing the results pertaining to these properties is *others-oriented claims monotonicity.*

Theorem 11 (Kıbrıs, 2012) *(a) A rule is rationalizable (equivalently satisfies WARP) if and only if it is contraction independent.*

(b) If a rule is contraction independent and others-oriented claims monotonic, then it satisfies SARP.

(c) If a rule is contraction independent, others-oriented claims monotonic, and continuous, then it is representable by a numerical function.

Further results are available. The main one is a characterization of a family of rules—let us name them **K-rules**[14]—as follows. We first need to specify two objects, a function that associates with each problem a non-negative vector whose coordinates add up to the endowment (thus, the *claims boundedness* requirement on awards vectors is relaxed) and an "adjustment function", which associates with each such vector a new one. This adjustment function, which can be understood as mimicking the decision process of a decision maker, is required to satisfy certain monotonicity requirements and to produce a sequence of vectors that converges in $|N|$ steps to an awards vector.

Theorem 12 (Kıbrıs, 2012) *The K-rules are the only rules that are claims continuous, others-oriented claims monotonic, and rationalizable.*

3.9 Claims problems with a large population of claimants.

We now turn to variable-population issues. As announced in Section 2, we imagine an infinite set of potential agents, indexed by the natural numbers, $\mathcal{N}$ designating the family of all finite subsets of $\mathbb{N}$. For each $N \in \mathcal{N}$, $\mathcal{C}^N$ designates the class of problems that N may face. A **rule** is a function defined over $\bigcup_{N\in\mathcal{N}} \mathcal{C}^N$ and taking its values in $\bigcup_{N\in\mathcal{N}} \mathbb{R}^N_+$ in the natural way.

[14]Kıbrıs (2012) refers to them as "recursive rules". We avoid this term here because a number of other rules have been defined through a recursion.

3.9.1 Replicating problems

An interesting question concerns situations in which the number of claimants is large and each claimant's importance is negligible. In general equilibrium theory, various ways of modeling large economies, economies in which each trader is "small", have been formulated, the goal being to study the behavior of solutions, in particular the core and the Walrasian solution, when there is perfect competition. The simplest way to do so is through replication of a "model" economy.

This is the approach followed by Chun and Thomson (2005). Given $k \in \mathbb{N}$, the notion of a ***k*-replica** of a problem is straightforward: each claimant has $k-1$ clones—their claims are equal to his—and the endowment is multiplied by k. A rule is **replication invariant** if for each problem and each $k \in \mathbb{N}$, each claimant's award in the problem is equal to the award to each of his clones in the k-replica of the problem.

It is obvious that all parametric rules are *replication invariant*. For some rules that violate the property, interesting statements can be made about the manner in which they do so. Indeed, a rule that satisfies *equal treatment of equals* assigns equal amounts to all the clones of each claimant in the problem that is replicated. Thus, one can study the behavior of the vector of awards to each initial claimant. It is an awards vector of the initial problem. It turns out that for the random arrival rule, this awards vector approaches the proportional awards vector, and that for the minimal overlap rule, it approaches the constrained equal losses awards vector (Chun and Thomson, 2005).

Here is another convergence result, this time to the proportional rule.

Theorem 13 *(Dominguez and Thomson, 2006). Consider the rule defined by recursive assignments of the reasonable lower bounds (Subsubsection 3.3.1). The awards vector the rule selects for a replicated problem is the replica of an awards vector it selects for the problem subjected to the replication that, as the order of replication increases without bound, converges to its proportional awards vector.*

3.9.2 More general sequences of problems.

The same sort of questions can be asked about other types of sequences of problems. Chun and Lee (2007) consider sequences for which (i) the endowment is "not too small" as compared to the aggregate claim, and (ii) each claim is "not too large" as compared to any other. However, an agent's claim is not required to be constant, and in fact, an agent's claim along a sequence need not be bounded. Theorem 14 generalizes Chun and Thomson (2005).

Formally, consider a sequence $(N^k)_{k\in\mathbb{N}}$ of populations facing problems $(c^k, E^k)_{k\in\mathbb{N}} \in \mathcal{C}^{N^k}$ such that, for each $k \in \mathbb{N}$, $|N^k| = k$, there is $r > 0$ such that $\lim_{k\to\infty} \frac{E^k}{\sum_{i\in N^k} c_i^k} > r$, and $\lim_{k\to\infty} \frac{m^k}{k} = 0$, where $m^k \equiv \max\{\ell' - \ell + 1\colon$ for each $i \in N^k$, $c_i^k > 0$ and $\sum_{i=\ell}^{\ell'} c_i^k \leq c_k^k\}$. Let $\mathcal{S}$ be the class of all such sequences.

We say that two rules ***S* and *S'* converge** if the following holds: for each $k \in \mathbb{N}$, each $(c^k, E^k) \in \mathcal{C}^{N^k}$, and each $i \in N^k$, if either $\lim_{k\to\infty} S_i(c^k, E^k)$ or $\lim_{k\to\infty} S'_i(c^k, E^k)$ are bounded, then $\lim_{k\to\infty}(S_i(c^k, E^k) - S'_i(c^k, E^k)) = 0$, or $\lim_{k\to\infty} \frac{S_i(c^k,E^k)-S'_i(c^k,E^k)}{c_i^k} = 0$.

Theorem 14 (Chun and Lee, 2007) *For each sequence in $\mathcal{S}$, the corresponding sequences of random arrival awards vectors and proportional awards vectors converge.*

3.9.3 Computational issues.

With the recent increased interest in resource allocation problems among computer scientists, issues are being addressed that economists had traditionally ignored. A first paper along these lines in the context of claims problem is Aziz (2013), who establishes that the random arrival rule is in general at least as hard to compute as the hardest counting problems.

3.10 Consistency

The study of *consistency* and its *converse* (Subsection 3.1) has proceeded along several fronts.

3.10.1 Consistent extensions: existence and construction

In the search for well-behaved rules, it is natural to start from the two-claimant case, which is conceptually and mathematically simpler, and in a second step to invoke *consistency* to deal with arbitrary populations. Indeed, given a two-claimant rule S, there might be a rule defined for populations of any size that is *consistent* and in the two-claimant case, coincides with S.

Such a rule is a **consistent extension of *S***. (It follows from Aumann and Maschler, 1985, that if S is *endowment monotonic* and has a *consistent* extension, the extension is unique.) We have already seen that Theorems 5 and 6 exploit the Elevator Lemma to provide answers to the question of existence of *consistent* extensions of several important two-claimant rules.

Other questions can be asked about *consistent* extensions. First, we may have decided to apply one of several *anonymous* rules in the two-claimant case, without having narrowed our choice down to a single rule. Some of these rules may have *consistent* extensions and others not. The issue then is to identify which ones do admit such extensions.

Second, when *anonymity* is not imposed, we may have decided, for each two-claimant population, to apply one of several rules, the family these rules constitute possibly depending on the two-claimant population. Then, we can ask whether, for each population, there is a rule in the family specified for that population such that the constellation of these selections (now, we have a way of solving all two-claimant problems), admits a *consistent* extension.

Third is the question of constructing *consistent* extensions when they exist.

The following observation has been key to answering these questions. Let S be a *consistent* rule defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. Let $N \in \mathcal{N}$ with $|N| = 3$ (it will suffice to focus on that case), $c \in \mathbb{R}^N_+$, and $N' \subset N$. It follows directly from the definition of *consistency* that the projection onto $\mathbb{R}^{N'}$ of the path of awards of S for c is a subpath of its path of awards for $c_{N'}$. If S is *endowment continuous*, a very mild property (it has never been found to be in conflict with other properties of interest), *consistency* is in fact equivalent to the coincidence of this projected path with the path of S for $c_{N'}$. Thus, for a two-claimant rule to have a *consistent* extension, the converse of the previous statement should hold: given $c \in \mathbb{R}^N_+$, there should be a path for c whose projection on each two-dimensional subspace corresponding to a two-claimant subpopulation N' of N coincides with the path specified for $c_{N'}$. A general constructive technique to solve the question of existence of *consistent* extensions that exploits this observation is developed in Thomson (2007).

The construction is straightforward if the two-claimant rule is in fact **strictly endowment monotonic**, which says that if the endowment increases, each claimant whose claim is positive is assigned more. Then, a unique candidate path for c can be recovered from only two—any two—of its projections onto two-dimensional subspaces. Then, it suffices to check whether the projection of this candidate path on the third two-dimensional subspace is the path specified for the projection of c on that subspace.

The focus on the geometric properties of entire paths of awards is important. For rules whose paths are piecewise linear, the argument is particularly fruitful, because kinks in the paths of awards for two agents typically generate kinks in the paths for more agents, and under strict monotonicity, the kinks in the three-dimensional paths can only come from kinks in the paths of these two-agent problems. These facts are used by Thomson (2013b) in developing an alternate proof of the characterization mentioned earlier of the class of rules satisfying *homogeneity*, *composition down*, *composition up*, and *consistency* (Moulin, 2000). It has also been used to answer the following type of questions.

First is whether "compromising" between different rules so as to accommodate conflicting views about the best way of solving problems is compatible with *consistency*. Depending upon the manner in which the compromise is defined, the answer is negative or positive.

(a) Compromising between two rules may be through averaging. (Averaging is meaningful because the set of awards vectors of each problem is convex). We will consider weighted averages of the constrained equal awards and constrained equal losses rules.

(b) It can be done by "combining" paths of awards in other ways, as we illustrate next, in two different ways, for the constrained equal awards and proportional rules. First, for each $N \in \mathcal{N}$ with $|N| = 2$, and each $c \in \mathbb{R}^N_+$, let $a^N(c) \in [0, \min c_i]$. Then, designating by e_N the vector in $\mathbb{R}^N$ whose coordinates are all 1's, consider the rule on $\mathcal{C}^N$ whose path of awards is $\text{seg}[0, a^N(c)e_N] \cup \text{seg}[a^N(c)e_N, c]$. Let $\boldsymbol{\mathcal{H}}$ be the family of rules on $\bigcup_{N \in \mathcal{N}, |N|=2} \mathcal{C}^N$ associated in this way with a list $(a^N)_{N \in \mathcal{N}, |N|=2}$ of functions as just defined.[15]

(c) Alternatively, for each $N \in \mathcal{N}$ with $|N| = 2$, and each $c \in \mathbb{R}^N_+$, let $g^N(c) \in [\min c_k, \max c_k]$. Then, consider the rule on $\mathcal{C}^N$ whose path of awards, assuming $c_i \leq c_j$, is $\text{seg}[0, (c_i, g^N(c))] \cup \text{seg}[(c_i, g^N(c)), c]$.[16] Let $\boldsymbol{\mathcal{F}}$ be the family of rules on $\bigcup_{N \in \mathcal{N}, |N|=2} \mathcal{C}^N$ associated in this way with a list $(g^N)_{N \in \mathcal{N}, |N|=2}$ of functions as just defined.

[15] The properties of the member of the family obtained by setting $a^N(c) \equiv \min c_i$ are studied by Giménez-Gómez and Peris (2013a).

[16] Thomson (2013c) identifies the various subfamilies of this family consisting of all rules satisfying one or the other of the most central properties.

Theorem 15 *(a) Consider a two-claimant weighted average of the constrained equal awards and constrained equal losses rules. The rule has a* consistent *extension if and only if either (i) all the weight is always placed on the former, or (ii) all the weight is always placed on the latter (Thomson, 2007). [So, in fact, no* consistent *compromise can be reached here.]*

(b) Consider a rule in $\mathcal{H}$, and let $(a^N)_{N\in\mathcal{N},|N|=2}$ be the list of functions with which it is associated. The rule has a consistent *extension if and only if there is $\alpha \in \mathbb{R}_+$ such that, for each $N \in \mathcal{N}$ with $|N| = 2$ and each $c \in \mathbb{R}^N_+$, $a^N(c) = \min\{\alpha, \min c_i\}$. Then, for each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, the path of awards of the rule follows that of the constrained equal awards rule until all claimants whose claims are at least α have received α, and it concludes with a segment to c (Thomson, 2007).*

(c) Consider a rule in $\mathcal{F}$, and let $(g^N)_{N\in\mathcal{N},|N|=2}$ be the list of functions with which it is associated. The rule has a consistent *extension if and only if there is a function $G\colon \mathbb{R}_+ \to \mathbb{R}_{++}$ that is nowhere decreasing and such that the function $c_0 \in \mathbb{R}_{++} \to \frac{G(c_0)}{c_0}$ is also nowhere increasing (these properties imply that G is continuous) such that, for each $N \in \mathcal{N}$ with $|N| = 2$, and each $c \in \mathbb{R}^N_+$, $g^N(c) = (\max c_k)\frac{G(\min c_k)}{G(\max c_k)}$ (Thomson, 2013c).*

A second type of questions is whether the two-claimant version of a rule that is not *consistent* nevertheless has a *consistent* extension. We reported in T2003 that the two-claimant versions of the random arrival and minimal overlap rules (recall that in that case, both rules coincide with concede-and-divide), neither one of which is *consistent*, has such an extension. The extension is the Talmud rule. The next theorem identifies two situations where no *consistent* extension exists.

Theorem 16 *(a) The version of the proportional rule obtained by truncating claims at the endowment is not* consistent*. Moreover, its two-claimant version has no* consistent *extension (Thomson, 2008a; this result is first proved by Dagan and Volij, 1997, by means of a different technique).*

(b) The rule defined by recursively assigning the reasonable lower bounds (Subsubsection 3.3.2) is not consistent*. Moreover, its two-claimant version has no* consistent *extension (Dominguez and Thomson, 2006).*

A third type of questions is whether selections from a family of two-claimant rules can be made so as to obtain *consistency*. The next theorem provides an answer for the ICI family (Subsection 2.3). A parallel answer can be given for the CIC family. (We omit the details for this second family.)

Theorem 17 (Thomson, 2008b) *Consider a two-claimant ICI rule and let (F, G) be the pair of "breakpoint" functions with with which it is associated. The rule has a consistent extension if and only if there is a nowhere decreasing function $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\gamma(0) = 0$ and the function $t \in \mathbb{R}_+ \to t - \gamma(t)$ is also non-negative and nowhere decreasing, such that, for each $N \in \mathcal{N}$ and each $c \in \mathbb{R}^N_+$, the breakpoints $F_1(c)$, $F_2(c)$, $\ldots$, $F_{n-1}(c)$ are, denoting by $\tilde{c}_1, \ldots, \tilde{c}_n$ the coordinates of c written in increasing order, $n\gamma(\tilde{c}_1)$, $\gamma(\tilde{c}_1) + (n-1)\gamma(\tilde{c}_2)$, $\ldots$, $\gamma(\tilde{c}_1) + \gamma(\tilde{c}_2) + \cdots + 2\gamma(\tilde{c}_{n-1})$. (The function G is then defined by invoking the ICI relations.) Among the resulting rules, the only homogeneous ones are the TAL rules (Section 2).*

Finally, we can report that a complete characterization of the class of rules satisfying *equal treatment of equals*, *claims truncation invariance*, and *consistency* is available (Hokari and Thomson, 2007). We know that the Talmud rule satisfies all three properties, but there are many others. The class is somewhat complex and we omit its description. We simply note that the proof builds on Theorem 9, a characterization of the class of rules satisfying the first two properties.

3.10.2 Generalizing Young's characterization of the parametric rules

The central result concerning *consistency* is Young's (1987) theorem: the parametric rules are the only rules satisfying *equal treatment of equals*, *continuity*, and *consistency*.

An interesting question is what additional rules become available if *equal treatment of equals* is dropped. Generalizations of the parametric rules can easily be defined that allow treating agents differently even if they have equal claims: instead of specifying a single function f, as explained in Subsection 2.3 where we defined the parametric family, simply select, for each potential agent $i \in \mathbb{N}$, a function f_i as defined there. These functions should of course all have the same domain, but otherwise, they can be chosen independently.

Sequential priority rules can be obtained as a special case. We introduced these rules in the fixed-population model (Subsection 2.3). To accommodate variable populations, select, for each $N \in \mathcal{N}$ an order on N to be invoked for that population; let it be denoted by $\prec^N$. In order to achieve *consistency*, the orders $(\prec^N)_{N\in\mathcal{N}}$ should be related across populations, however:

specifically, there should be a "reference order" $\prec$ on the entire set of potential claimants such that, for each $N \in \mathcal{N}$, the order $\prec^N$ is the order on N induced from $\prec$. To show that the sequential priority rule that results is a generalized parametric rule, it suffices for instance to take the real line as common domain of definition of the functions $(f_i)_{i \in N}$, partition it into countably many non-degenerate intervals, label these intervals according to $\prec$—let $I_i = [a_i, b_i]$ be the interval assigned to each potential claimant $i \in \mathbb{N}$—and for each $i \in \mathbb{N}$ and each $c_i \in \mathbb{R}_+$, define f_i to be the function whose graph consists of (a) the horizontal half-line $\{x \in \mathbb{R}^2_+ \colon$ there is $t \leq a_i$ such that $x = (t, 0)\}$, (b) $\text{seg}[(a_i, 0), (b_i, c_i)]$, and (c) the horizontal half-line $\{x \in \mathbb{R}^2_+ \colon$ there is $t \in \mathbb{R}_+$ such that $x = (b_i, c_i) + t(1, 0)\}$.

A generalization of Young's theorem is offered by Kaminski (2006). In his formulation, claimants are organized in types, and type space is a separable topological space. Types can be claims, as in the base model, or preference relations defined over $\mathbb{R}_+$, or utility functions defined over $\mathbb{R}_+$. The result is a characterization of the parametric rules on the basis of the same list of axioms as Young's. Here, the functions f of Young's definition are indexed not by claims but by types. Subsection 6.8 describes additional results along these lines.

A study with the same objective is by Stovall (2014a), who follows two approaches. One is similar to Kaminski's. The other approach involves an axiom of "internal consistency". If added to *consistency*, a characterization of generalized parametric rules is obtained.

3.11 Two notions of operators

Several notions of operators can be distinguished. An operator may be a mapping from the space of rules into itself. An operator may associate with an incompletely specified rule one that is defined everywhere. In a fixed-population framework, the rule may be defined only for some claims-endowment configuration (as is the case for Rabad's proposal). In a variable-population framework, it may be a mapping from the space of two-claimant rules to the space of rules defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$. In each case, a question of interest is whether the properties of the rule that is the starting point are inherited by the rule that results by applying the operator.

3.11.1 Operators as mappings from the space of rules into itself

A study of operators as mappings from the space of rules into itself is by Thomson and Yeh (2008). These authors derive several general properties of operators, but most of their results concern the following ones. Let S be a rule and $(c, E) \in \mathcal{C}^N$ be a problem. The **claims truncation operator** associates with S the rule defined by truncating claims at the endowment, obtaining $t(c, E) \equiv (\min\{c_i, E\})_{i \in N}$, and applying S to the problem $(t(c, E), E)$. The **attribution of minimal rights operator** associates with S the rule defined by adding to the vector of minimal rights, $m(c, E) \equiv (\max\{E - \sum_{N \setminus \{i\}} c_i, 0\})_{i \in N}$, the awards vector obtained by applying S to the problem $(c - m(c, E), E - \sum m_i(c, E))$. The **duality operator** associates with S the rule defined by subtracting from the claims vector the vector obtained by applying S to the problem with the same claims vector but in which the endowment is $\sum c_i - E$. Given $K \in \mathbb{N}$ and $w \equiv (w_k)_{k \in K} \in \Delta^{|N|-1}$, the **convexifying operator with weights $\boldsymbol{w}$** (studied in the original version of Thomson and Yeh, 2008) associates with an ordered list of $|K|$ rules $(S_k)_{k \in K}$, the w-weighted average awards vectors $\sum_{k \in K} w_k S_k$.

A number of algebraic relations between operators are uncovered by Thomson and Yeh who also identify which properties of rules they preserve and which they do not preserve.

Recall the manner in which Hougaard, Moreno-Ternero, and Østerdal (2012) generalize the claims truncation operator by using a baseline function instead of truncated claims, and similarly generalize the attribution of minimal rights operator by using a baseline function other than the minimal rights (Subsubsection 3.3.2). Virtually all of the results of Thomson and Yeh extend to this richer setting, as they show.

3.11.2 Operators as mappings from the space of two-claimant rules to the space of rules

In a variable-population framework, a different notion of an operator can be defined to provide extensions of two-claimant rules to the domain $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ of problems with arbitrary populations. Let us refer to this kind of operators as **extension operators**. In Subsubsection 3.10.1 we used *consistency* for that purpose. Our goal is the same here, except that we do not use a property of rules as the basis for the extension but an "aggregation-averaging" operation.

Aumann and Maschler (1985) had suggested the following way of deriving the awards vector of an n-claimant problem from the choice of concede-and-divide in the two-claimant case. Let $N \equiv \{1, \ldots, n\}$ and suppose that claimants are ordered by increasing claims: $c_1 \leq \cdots \leq c_n$. First, concede-and-divide is applied to the two-claimant problem in which claimant 1 faces a "composite" claimant whose claim is $c_2 + \cdots + c_n$. Claimant 1 leaves with his award unless a violation of *order preservation* (either of awards or of losses) occurs, in which case equal division among all members of N is carried out. Then, claimant 2 faces a composite claimant whose claim is $c_3 + \cdots + c_n$ and the endowment is what the first composite claimant received. Concede-and-divide is applied again and claimant 2 leaves with his award unless a violation of *order preservation* within the group $N \setminus \{1\}$ occurs, in which case equal division among this group is carried out, and so on. We refer to this process as the **AM process**. Aumann and Maschler show that, for each problem, it ends in the Talmud awards vector.

The AM process can be generalized in two ways. First, a rule other than concede-and-divide can be used to solve the successive two-claimant problems. Moreno-Ternero (2011a) considers instead rules in the TAL family. Recall that these rules are indexed by a point in the unit interval. For each $\theta \in [0, 1]$, redefining the AM process by using the two-claimant TAL rule T^θ produces the TAL rule T^θ for any number of claimants (Subsection 2.3). Other two-claimant rules could be used as input. Second, it is not clear why at each step the smallest claimant should face everyone else. Let us instead think of one claimant being chosen randomly among all claimants, have him face everyone else, and take an average of the resulting awards vectors.

A proposal to generalize the AM process in these two directions, made by Quant, Borm, and Maaten (2005), is investigated by Quant and Borm (2011), on which the next paragraphs are based.

We start from an arbitrary two-claimant rule S and associate with S a rule for all population sizes by the revised aggregation-averaging operation described above. Whereas, as we saw, not all two-claimant rules can be extended through *consistency*, here no such difficulty arises. Indeed, at each step, when a claimant is served, he receives a non-negative amount that is bounded above by his claim. These properties are preserved by averaging.

Let us call the operator just defined the **QB extension operator**.[17] Are the properties of S inherited by the rule that results when S is subjected to

[17] Quant and Borm refer to S^{QB} as the "random conjugate" of S.

this operator? The following theorem provides positive answers for three important ones.

Theorem 18 (Quant and Borm, 2011) *Claims truncation invariance, self-duality, and minimal rights first are preserved by the QB extension operator.*

3.11.3 Further generalizations of the Aumann-Maschler process

One may find it undesirable to always have a single claimant face everyone else and the following alternative formulation might be worth exploring. Again, let S be a two-claimant rule. Given a problem, we partition its claimant set into two subsets, add up the claims of all the members of each subset, and apply S to the resulting two-claimant problem. This determines an award for each of the subsets; we then partition each subset that has at least two claimants into two parts, and repeat the operation. We proceed until each claimant has been involved in a two-claimant problem as a singleton, and has been assigned an award. The successive partitioning of the claimant set can be described as a tree—the leaves are individual claimants—and to remove the arbitrariness that would be associated with the choice of a particular tree, we choose as awards vector the average of the awards vectors obtained for all possible trees. Again, an interesting question is what properties of S are inherited by this type of extension.

3.12 Ranking rules

An important characteristic of a rule is how evenly it distributes the endowment and we may be interested in comparing rules on the basis of the skewedness of the distributions of awards that they select. We can imagine several ways of doing so, but let us follow Hougaard and Tholund-Petersen (2001) and proceed by invoking the concept of Lorenz order, which is the central tool in comparing income distributions. It is defined as follows. Given two vectors x and $y \in \mathbb{R}^N_+$ with $\sum x_i = \sum y_i$, we say that ***x* Lorenz dominates *y***, which we write as $\boldsymbol{x \succ_L y}$, if, when the coordinates of these two vectors are rewritten in increasing order and denoting the results $\tilde{x}$ and $\tilde{y}$, the following inequalities hold: $\tilde{x}_1 \geq \tilde{y}_1$, $\tilde{x}_1 + \tilde{x}_2 \geq \tilde{y}_1 + \tilde{y}_2$, $\ldots$, $\tilde{x}_1 + \cdots + \tilde{x}_{n-1} \geq \tilde{y}_1 + \cdots + \tilde{x}_{n-1}$, with at least one strict inequality. If none of these inequalities holds strictly, we write $x \succeq_L y$. Given two rules S and S', we say that ***S* Lorenz dominates *S′***, which we write as $\boldsymbol{S \succ_L S'}$, if for each $(c, E) \in \mathcal{C}^N$, $S(c, E) \succeq_L S'(c, E)$ and for at least one problem $(c, E) \in \mathcal{C}^N$, $S(c, E) \succ_L S'(c, E)$.

The Lorenz order of vectors is incomplete and, *a fortiori*, so is the Lorenz order of rules. Thus, one should not expect to be able to order rules with great generality. Yet, a lesson to be drawn from this section is that the space of rules can be structured quite usefully by means of the Lorenz order.

3.12.1 Criteria for Lorenz ordering of rules

In the two-claimant case, Lorenz domination is easily checked. Let S and S' be two rules that satisfy *order preservation of awards*. Then, $S \succeq_L S'$ if and only if, for each claims vector, the path of S is everywhere at least as close to the 45° line as the path of S' is. Determining that two rules are Lorenz ordered for more than two claimants often requires complicated calculations. Nevertheless, most of the central rules can be Lorenz ordered. Part (d) involves **null-claims consistency**, the weak form of *consistency* obtained by imagining the departure of some agents whose claims are 0 (by definition of a rule, each such agent should be assigned nothing).

Theorem 19 (Hougaard and Thorlund-Petersen, 2001 for (a); Chun, Schummer, and Thomson, 2001, for (b); Bosmans and Lauwers, 2011, otherwise)

(a) The constrained equal awards rule Lorenz dominates each other rule.

(b) Among all rules satisfying endowment monotonicity *and the* midpoint property*, the constrained egalitarian rule is, uniquely, Lorenz-maximal.*

(c) Among all rules satisfying order preservation*, the* midpoint property*,* endowment monotonicity*, and* composition up from midpoint*, (the weaker version of* composition up *in which the smaller endowment is equal to the half-sum of the claims), Piniles' rule is, uniquely, Lorenz-maximal.*

(d) Among all rules satisfying order preservation*, the* reasonable lower bound on awards*,* order preservation under claims variations*, and* null-claims consistency*, the minimal overlap rule is, uniquely, Lorenz-minimal.*

(e) Among all rules satisfying order preservation of awards*, the* midpoint property*,* order preservation under endowment variations*, and* endowment monotonicity*, the minimal overlap rule is, uniquely, Lorenz-maximal.*

Bosmans and Lauwers also offer results on the domain of problems in which the endowment is at least as large as the half-sum of the claims, and on the domain of problems for which the opposite holds. On these domains, additional interesting comparisons can indeed be made.

The following result pertains to two parametric families introduced in Section 2.

Theorem 20 *(a) Given two TAL rules (Section 2) with parameters θ and θ', if $\theta > \theta'$, then T^{θ} Lorenz dominates $T^{\theta'}$ (Moreno-Ternero and Villar, 2006b).*

(b) Given two ICI rules S and S', associated with parameter pairs (F, G) and (F', G'), if $F \geq F'$ (equivalently, because of the ICI relations, $G \geq G'$) then S Lorenz dominates S' (Thomson, 2012a).[18]

3.12.2 Preservation by operators of Lorenz ordering of rules

A natural question about an operator is whether it preserves an order on the space of rules: formally, **operator p preserves order $\succeq$** if, for each pair of rules S and S', if $S \succeq S'$, then $S^p \succeq S'^p$ (denoting by S^p the image of S under operator p). One can add that if $S \succ S'$, then $S^p \succ S'^p$. The following theorem provides useful information about preservation, or reversal, of the Lorenz order by means of our four central operators:

Theorem 21 (Thomson, 2012a) *(a) The claims truncation operator preserves the Lorenz order on the space of rules.*

(b) The attribution of minimal rights operator preserves it for any two rules satisfying order preservation of awards*.*

(c) The duality operator reverses it for any two rules satisfying order preservation*.*

(d) The averaging operators preserve it for any two rules satisfying order preservation of awards*.*

3.12.3 Lifting by bilateral consistency of Lorenz ordering of rules

A property of rules is lifted by bilateral consistency if whenever a rule satisfies the property in the two-claimant case, and the rule is *bilaterally consistent*, it satisfies the property for any number of claimants (Hokari and Thomson, 2008). It is lifted with the **assistance of some other property**

[18]TAL rules are ICI rules, and two TAL rules necessarily satisfy the condition stated in (b). Thus, (a) can be obtained as a special case of (b).

if this implication holds for rules satisfying this other property (more than one property could provide the assistance) (Hokari and Thomson, 2008). The concepts can be applied to orders on the space of rules: **an order $\succ$ is lifted by bilateral consistency** (Thomson, 2012a) if whenever two rules S and S' are *bilateral consistent* and $S \succ S'$ in the two-claimant case, then $S \succ S'$ for any number of claimants; the **assisted lifting** of an order is defined in the obvious way. Our next theorem identifies two mild properties of rules providing enough assistance for the Lorenz order to be lifted. Theorem 20 can be derived as a corollary.

Theorem 22 (Thomson, 2012a) *Let S and S' be two rules satisfying* order preservation of awards *in the two-claimant case,* endowment monotonicity *in the two-claimant case, and* bilateral consistency*. If $S \succeq_L S'$ in the two-claimant case, then $S \succeq_L S'$ in general.*

3.12.4 Lorenz-inequality reducing rules

We inquire next about a possible relation between the distribution of claims in a problem and the distribution of awards that results when applying a rule. Consider an order $\succeq$ on $\mathbb{R}^N_+$ (here, we do not require the coordinates of the vectors that are compared to add up to the same number), and let us require of a rule that, for each problem, the difference between the claims vector and the awards vector it selects should dominate the claims vector. When applied to the Lorenz order, we add the prefix "Lorenz" to the name of the property. Rule S **reduces $\succeq$-inequality** if, for each $(c, E) \in \mathcal{C}^N$, $c - S(c, E) \succeq c$.

First are interesting logical relations. One of them involves **progressivity**, the requirement that, for each $(c, E) \in \mathcal{C}^N$ and each pair i and $j \in N$ with $c_i \leq c_j$, $\frac{S_j(c,E)}{S_i(c,E)} \leq \frac{c_j}{c_i}$.

Proposition 1 (Ju and Moreno-Ternero, 2008)

(a) Order preservation of losses *and* progressivity *imply* Lorenz-inequality reduction*.*

(b) Lorenz-inequality reduction *and* consistency *imply* progressivity*.*

(c) Lorenz-inequality reduction*,* endowment continuity*, and* consistency *imply* order preservation of losses*.*

Next is a characterization of a subfamily of the parametric family.

Theorem 23 (Ju and Moreno-Ternero, 2006) *A rule satisfies Lorenz inequality reduction, continuity, and consistency if and only if it is a parametric rule and, designating by* $f\colon \Lambda \times \mathbb{R}_+$ *a representation of it, (a) S is superhomogeneous in claims: for each* $\lambda \in \Lambda$*, each* $c_0 \in \mathbb{R}_+$*, and each* $\alpha > 1$*,* $f(\lambda, \alpha c_0,) \geq \alpha f(\lambda, \alpha c_0)$*, and (b) for each* $\lambda \in \Lambda$*, the function* $c_0 \in \mathbb{R}_+ \to c_0 - f(\lambda, c_0)$ *is nowhere decreasing.*

Next, we address the question of whether *Lorenz inequality reduction* is preserved by operators (Subsubsection 3.11.1).

Proposition 2 (Ju and Moreno-Ternero, 2006) *The averaging operators preserve Lorenz inequality reduction. So does the attribution of minimal rights operator.*

3.12.5 Lorenz rankings of claims vectors reflected in awards vectors

Another question can be asked about rankings of vectors in $\mathbb{R}^N_+$. Consider two claims vectors. We ask whether a ranking of these vectors is "reflected" in the awards and losses vectors chosen by a rule. We require that it should be so. Since claimants will not receive what they are entitled to, an impact is unavoidable; the requirement expresses the idea that this impact should be limited. Formally, a rule **preserves Lorenz rankings in awards** if, for each pair (c, E), $(c', E') \in \mathcal{C}^N$, if $c \succeq_L c'$ and $E = E'$, then $S(c, E) \succeq_L S(c', E')$; it **preserves Lorenz-ranking in losses** if under the same hypotheses, $(c - S(c, E)) \succeq_L (c' - S(c', E'))$. If it does both, it **preserves Lorenz rankings**. (These definitions are proposed by Hougaard and Thorlund-Petersen, 2001.)

Proposition 3 *The constrained equal awards rule preserves Lorenz rankings in awards. The constrained equal losses rule preserves Lorenz rankings in losses. The proportional rule preserves Lorenz rankings.*

The following is an additional characterization of the proportional rule.[19]

[19] Hougaard and Østerdal (2005) assert that if there are at least three claimants, only the proportional rule qualifies. This is true only if there are at least four claimants, as established by Kasajima and Velez (2011).

Theorem 24 (Hougaard and Østerdal, 2005; Kasajima and Velez, 2011) *For* $|N| \geq 4$*. The proportional rule is the only rule that preserves order and preserves Lorenz rankings.*

For $|N| = 2$, the axioms of Theorem 24 place almost no restriction on rules. For $|N| = 3$, they are satisfied by a large family of rules, but a characterization is possible (Kasajima and Velez, 2010). Interestingly, under the Lorenz order, the family has minimal and maximal elements (Kasajima and Velez, 2010).

4 Solving claims problems as games

One of the approaches developed by O'Neill (1982) to solve a claims problem is (i) to convert it into a game, either a cooperative game or a strategic game; (ii) to solve the game by applying some solution defined on the corresponding class of games; (iii) to choose the resulting payoff vector as the awards vector for the problem. We only give a short account of these developments in this area. Details, and a critical assessment, can be found in Thomson (2013a).

4.1 Mapping claims problems into bargaining games or coalitional games with transferable utility

• A claims problem can be mapped into a bargaining game (Dagan and Volij, 1993). For the two-claimant case, Ortells and Santos (2011) propose to apply the "equal area bargaining solution" to this game, thereby obtaining a new rule. Giménez-Gómez (2013a) proposes an alternative representation of claims problems as bargaining problems.

• A claims problem can be mapped into a transferable utility (TU) coalitional game, by setting the worth of each group of claimants to be what is left of the endowment after the complementary group has been given full satisfaction (or 0 is that is not possible) (O'Neill, 1982).[20] Alternatively, Alcalde, Marco, and Silva (2005, 2008) consider the TU game associated with a *pair* of problems that differ in their endowments. They specify how rules should (implicitly) deal with this game. They characterize the minimal

[20] Aumann and Maschler (1985) had proved that the nucleolus of this game is the Talmud awards vector. Fleiner and Sziklai (2012) give another proof.

overlap rule (Alcalde, Marco, and Silva, 2008) and the **recursive extension** (due to Bergantiños and Mendez-Naya, 2001) **of Rabad's proposal** (Alcalde, Marco, and Silva, 2005) .

• An invariance inspired by the notion of "self-consistency" proposed by Hart and Mas-Colell (1989) for TU games, plays the central role in a characterization of the random arrival rule (Albizuri, Leroux and Zarzuelo, 2010). The reduction involves a revision of claims.

• The proposal has also been made to define the worth of a group of claimants in a problem as the largest claim any of these claimants holds, truncated at the endowment (Aumann, 2010). The Shapley value, when applied to the examples presented in O'Neill, delivers most of the historical recommendations (Aumann, 2010), although it fails *claims boundedness*. Another pair of formulas of this type has been offered. By using one or the other, most of these historical recommendations can also be obtained, once again, if the Shapley value is applied (Guiasu, 2011). Finally is a proposal based on the constrained equal awards and constrained equal losses rules (Gadea-Blanco, Giménez-Gómez, and Marco-Gil, 2009). The rule it produces is the average of these two rules (an average studied by Thomson, 2007).

4.2 Strategic games

Another approach to solve claims problems is to turn them into strategic games.

• O'Neill (1982) proposed such a formulation, under the assumption that no claim is greater than the endowment. Imagine the endowment to consist of parts with distinct physical identities, which we call "units" (although we do not want to suggest integer amounts). Each claimant is given the opportunity to stake a subset of E of size equal to his claim; what he stakes is his strategy. Each unit is then divided equally among all claimants staking it. A claimant's payoff is defined to be the sum of the partial payments he receives from the various units that he staked. Dropping the assumption that claims are bounded by the endowment raises conceptual difficulties for the definition of the game and creates serious technical complications in its analysis. They are addressed by Atlamaz, Berden, Peters, and Vermeulen (2011), who give a complete description of all the equilibria. They also provide an asymptotic result, letting the number of claimants increase to infinity.

• Given a problem and an awards vector for it, is there a game form such that, at equilibrium, the awards vector is reached and nothing else? This question is addressed by García-Jurado, González-Díaz, and Villar (2006), who consider normal-form games—their equilibrium notions are Nash or strong Nash—and by Chang and Hu (2008) who define extensive-form games and examine their sub-game perfect equilibria. These game forms are allowed to depend on the data of the problem. The authors give positive answers to the above question under general conditions. In the game studied by Ashlagi, Karagözoğlu, and Klaus (2012), strategy spaces are unrelated to claims. They establish conditions under which equal division is the only equilibrium outcome.

• A game in which claimants propose rules and the outcome is obtained by recursively applying them and finding some compromise among the recommendations they make had been studied by Chun (1989). A version of such a game is formulated and studied by Giménez-Gómez (2013b).

• Another strategic opportunity that claimants may have is to merge their claims or to consolidate them, as already recognized by O'Neill (1982). More work has been done on this issue. **Merging-proofness** says that no group of claimants should benefit by merging their claims and appearing as a single claimant. **Multilateral merging-proofness** pertains to the following operation. Starting from some problem, we imagine a group of claimants redistributing their claims among the members of some second group and leaving: the requirement is that, in the problem that results, the second group should not be assigned in total more than the two groups were initially assigned in total. This is a significant strengthening of *merging-proofness*. Indeed, together with *consistency*, it implies **splitting-proofness**, the requirement that no claim should benefit by splitting his claim among several agents who then "represent him" (Ju and Moreno-Ternero, 2011). (It is of course not true that *merging-proofness* and *consistency* together imply *splitting-proofness*; the constrained equal awards rule satisfies the first two properties but not the last one).

Next is the requirement, **uniformity-preserving multilateral merging-proofness**, obtained from the previous one by adding the hypotheses that the claims of the members of the two groups involved in the merging are equal and that after this operation, so are the claims of the members of the group of claimants who stay. A parametric rule is *progressive* if and only if it is *merging-proof* and *uniformity-preserving multilaterally merging-proof.*

5 Experiments and surveys

In the last twenty years, a considerable literature has emerged devoted to the experimental testing of economic theories, but the theory concerning the adjudication of conflicting claims has only begun to be examined in the laboratory. Several types of questions can be asked.

1. A subject is confronted with numerical examples of claims problems and specific numerical resolutions for them, and is asked an opinion about which one he finds most appropriate.

2. A subject is asked to negotiate payoffs with a partner so as to resolve a particular problem. His payoff is what he extracts from these negotiations. This payoff is paid out to him (thus, a subject is not paid imaginary money).

3. A subject is presented with numerical examples of claims problems and with several rules and asked to indicate which rule provides the best resolution of the problems.

4. A subject is presented with properties of rules and asked to indicate which properties he or she endorses, the tradeoffs he or she perceives between properties.

One should expect answers to these questions to depend on a variety of factors:

(i) Context: the manner in which the problem is described, referred to as the "framing". For instance, whether the issue is distributing a good in short supply among households, or the liquidation value of a bankrupt firm among investors, or tax assessment (when the problem is phrased as one in public finance) probably matters.

(ii) Stakes: whether subjects are asked their opinion as disinterested third party, or invited to imagine being one of the parties and, if so, as holders of the larger claim or holders of the smaller claim.

(iii) Parameters of the problem: for instance, how unequal the coordinates of the claims vector are, and how far apart the endowment and the sum of the claims vector are.

(iv) Cultural differences: the type of society in which the testing is performed is probably relevant too, when negotiating payoffs, as described in (2), whether the test is performed in a society with a tradition of free enterprize, or a tradition of State intervention.

Considering (1), a study by Gächter and Riedl (2006) seems to provide more support for the proportional rule when claims are not too different from each other and relatively more support for the constrained equal awards rule when claims become more different. The study by Bosmans and Schokkaert (2009) throws light on the issue of context. They present subjects with two scenarios. One involves a firm going bankrupt, the issue being to divide its liquidation value among people having invested different amounts in it. The other involves retirees having been promised certain pensions, these pensions not being feasible anymore. They find that subjects are more likely to favor an outcome that is closer to the proportional outcome in the firm version and an outcome that is closer to the egalitarian outcome in the pensions version. Bosmans and Schokkaert (2009) are particularly concerned with understanding the relevance of (iii). Their finding is that the proportional outcome better approximates the respondents' choice.

As for (2), the constrained equal awards rule seems to provide a better fit for the experimental data reported by Gächter and Riedl (2006).

As for (3), the proportional rule emerges as a central rule in Gächter and Riedl (2006).

Herrero, Moreno-Ternero, and Ponti (2010), whose study—it was the first in this area—also addresses (1), confront subjects with the choice of one of the following three rules, the proportional, constrained equal awards, and constrained equal losses rules. Their conclusions are that the proportional rule acts as a coordination device in a game that is the choice of a large majority, although the extent to which it is perceived as superior to the others is shown to depend on the framing.

(4) We are not aware of any study addressing this question.

6 Variants and enrichments of the base model

One of the most interesting aspects of the literature under review have been various enrichments of the base model. In spite of its remarkable simplicity, this model is a rather faithful description of a number of situations encountered in the real world, but it is also true that additional data is needed for a more accurate modeling of others. We describe next the various ways in which authors have contributed to this enrichment.

6.1 Allowing for indivisibilities

In the base model, claims and endowment are infinitely divisible. They are non-negative real numbers and awards are also supposed to be non-negative real numbers. However, in a number of settings, the endowment comes in discrete amounts, say the natural numbers, and so do claims.

Discreteness creates significant difficulties. For example, a condition as fundamental as *equal treatment of equals* has no chance of being met by any rule: in a two-claimant problem in which the two claimants have equal claims but the endowment is an odd integer, nothing can be done to treat them symmetrically. However, in such situations it is natural to require then that the difference between their awards should be at most 1, a property we call **approximate equal treatment of equals**.

A (local) **deservingness relation**[21] is a strict binary relation $\prec$ on $\mathbb{N}\times\mathbb{N}$. This concept can be used in the following two ways. First, the **up rule associated with $\prec$** is defined by imagining that the endowment becomes available one unit at a time, and the relation specifies, for each newly available unit, who is *more* deserving of it: $(i,x) \prec (j,y)$ means that agent i with claim x is more deserving of it than agent j with claim y. More precisely, rank the agent-claim pairs; assign one unit to the agent in the pair that is ranked first; decrease his claim by one; place the pair consisting of that agent and his revised claim where it belongs in the order; assign one unit to the agent in the pair that is now first (it could be the same agent), and so on. The **down rule associated with $\prec$** is defined symmetrically but the starting point is the hypothetical situation in which all claimants have been fully compensated, and it involves subtracting from their assignments. This time, the relation specifies, for each unit, who is *less* deserving of it: $(i,x) \prec (j,y)$ means that agent i with claim x is less deserving than agent j with claim y. Rank the agent-claim pairs; delete one unit from the assignment of the agent in the pair that is ranked first; decrease his claim by one; place the pair consisting of that agent and his revised claim where it belongs in the order; delete one unit from the assignment of the agent in the pair that is now first, and so on.

Theorem 25 (Moulin and Stong, 2002) *The down rules are the only rules satisfying the dual of claims monotonicity, composition down, and consis-*

[21]The concept is introduced by Young (1994) who uses the expression "standard of comparison".

tency.

A deservingness relation is **monotonic** if for each pair (i, x), we have $(i, x+1) \prec (i, x)$. Down rules associated with monotonic deservingness relations satisfy *approximate equal treatment of equals*. In fact, the first two axioms of the next theorem imply *approximate equal treatment of equals*.

Theorem 26 (Herrero and Martínez, 2008) *The down rules associated with monotonic deservingness relations are the only rules satisfying conditional full compensation, composition down, and consistency.*

A deservingness relation is **claim-focused** if for each pair $i, j \in N$ and each pair $x, y \in \mathbb{N}$ with $x > y$, then $(i, x) \prec (j, y)$. It is **agent-focused** if for each pair $i, j \in N$ and each pair $x, y \in \mathbb{N}$, if $(i, x) \prec (j, x)$, then $(i, y) \prec (j, y)$.

Systematic favorability in awards says the following. Consider a problem in which some claimant i is assigned more than some claimant j. Then, if his claim increases, he should still receive more than claimant j. Suppose now that, in some problem, claimant i is fully compensated whereas agent j is not. Then, **systematic favorability with respect to full compensation** says that there should be no problem in which claimant j is fully compensated whereas claimant i is not. **Strong systematic favorability with respect to full compensation** involves comparing two problems with possibly different claimant sets. Given two claimants i and j belonging to both, suppose that, in the first problem, claimant i is fully compensated whereas claimant j is not. Then, in the second problem, claimant j should be fully compensated only if claimant i is too. The proofs of several statements in the following theorem involve the Elevator Lemma.

Theorem 27 (Chen, 2011)

(a) The agent-focused up rules associated with monotonic deservingness relations are the only rules satisfying systematic favorability in awards, composition up, and bilateral consistency.

(b) The claims-focused up rules associated with monotonic deservingness relations are the only rules satisfying order preservation of awards, composition up, and bilateral consistency.

(c) The sequential priority rules associated with monotonic deservingness relations are the only rules satisfying systematic favorability with respect to full compensation, composition down, and bilateral consistency.

(d) The sequential priority rules are the only rules satisfying strong systematic favorability with respect to full compensation and composition down.

When indivisibilities are present, it is quite natural to turn to rules that select probability distributions over awards vectors (Moulin, 2002; Moulin and Stong, 2002, 2003). The following theorem should be compared to its counterpart for the base model (Moulin, 2000; see also Chambers, 2006). The class of two-claimant rules satisfying these axioms is considerably larger.

Theorem 28 (Moulin, 2002) *The proportional probabilistic rule is the only probabilistic rule satisfying the probabilistic versions of equal treatment of equals, composition down, and composition up.*

It is the only rule satisfying the probabilistic versions of composition down and self-duality.

If *equal treatment of equals* is dropped and *consistency* added, a characterization of a large family of rules is obtained. It involves defining an ordered partition of the set of potential claimants and, for each problem, identifying the induced ordered partition: then, proceeding from class of class, fully satisfying the claims of the members of any class before awarding anything to any member of a lower-priority class. If a class is induced from a reference priority class with more than two claimants, the proportional rule has to be applied however.

Further results are obtained by Moulin and Stong (2002, 2003).

An adaptation of the recursive process studied by Giménez-Gómez and Marco (2012) (Subsubsection 3.3.3) to problems with indivisibilities is studied by Giménez-Gómez and Vilella (2013).

6.2 Relaxing requirements imposed on rules

An early study of situations when *non-negativity* and *claims boundedness* are not imposed is due to Chun (1998). His results were described in detail in T2003.

A study of *full additivity* and *endowment additivity* on four domains is due to Bergantiños and Vidal-Puga (2004). In addition to claims problems, they allow problems in which the sum of the claims is smaller than the endowment (surplus-sharing); they also consider rules that are not required to satisfy *claims boundedness*, and rules that are not required to satisfy *non-negativity*. Their main results are characterizations of generalizations of the proportional, constrained equal awards, and constrained equal losses rules. They also describe the implications of these properties on the counterparts of these domains when the data are integers, as in Subsection 6.1 (Bergantiños and Vidal-Puga, 2006).

Non-manipulability issues (**transfer-proofness** (which say that no group of agents should benefit by transferring claims among themselves), *merging-proofness*, and *splitting-proofness*) are the focus of Ju, Miyagawa, and Sakai (2007) in a model in which agents' characteristics are multi-dimensional, a possibility studied in Subsection 6.4.

In the game form formulated by Corchón and Herrero (2004), who consider a domain that includes surplus-sharing, a claimant is required not to claim more than his true claim. The main result is that a rule is implementable in dominant strategies and satisfies this requirement if and only if it is **strictly claims monotonic** (if an agent's claim increases, and the endowmentis positive, that claimant should get more).

6.3 Introducing uncertainty

In the base model, claims and endowment are known and fixed. However, in practice a claim is the result of a decision that an economic agent has made, on how much to lend to a business say, and it is natural to expect that this decision has been affected by the knowledge of what he will receive if the business fails. A first study of the possibility is by Hougaard and Tholund-Petersen (2001).

Understanding the incentives that rules give agents to invest requires that the model be enriched, and the literature is developing in that direction thanks to Karagözoğlu (2008, 2010) and A. Kıbrıs and Ö. Kıbrıs (2013). It is easy to imagine that there might not be a unique way of modeling this type of issue, a particularly important part of which is formalizing how uncertainty affects the endowment. For the former author, two firms compete for investors and each chooses a bankruptcy rule. Investors maximize expected payoffs, taking into account the possibility of bankruptcy. His main result is that, when firms have to choose among the proportional, constrained equal awards, and constrained equal losses rules, the proportional rule is a subgame perfect equilibrium, and under certain conditions, it is the only such equilibrium.

The study by A. Kıbrıs and Ö. Kıbrıs (2013) offers comparative statics results on equilibria of investment games concerning the total amount invested. It considers averages of the proportional and equal awards rules as well as averages of the proportional and equal losses rules. The results are that for the former, the greater the weight placed on the proportional rule, the greater the aggregate equilibrium investment. For the latter, the greater the weight placed on the proportional rule, the smaller the aggregate equilibrium investment. In the two-claimant case, this study also delivers welfare comparisons at equilibrium when the social welfare function is either egalitarian or utilitarian. Under interiority conditions, the proportional rule dominates both of the others according to the egalitarian criterion but these two cannot be ranked unambiguously. It dominates the constrained equal losses rule according to the utilitarian criterion, but the ranking of the proportional rule with the constrained equal awards rule, as well as the ranking of the constrained equal awards and constrained equal losses rules, are ambiguous.

The possibility that the endowment is not known is also addressed by Habis and Herings (2013). They associate with such a situation a transferable utility stochastic game and propose to solve this game by applying the notion of "sequential core" (Habis and Herings, 2011). None of the proportional, constrained equal losses, or Talmud rules select a point in it, but the constrained equal awards rule does.

A way of modeling situations in which there is uncertainty about claims is to represent them as intervals (Branzei, Dimitrov, Pickl, and Tijs, 2004). Rules can be defined for this variant of the model, inspired by standard rules. These situations can also be mapped into TU games (Branzei, Dimitrov, and Tijs, 2003). A counterpart of the Shapley value has been defined for them. The possibilities that the endowment be uncertain, or that both claims and endowment be uncertain, can be modeled in a similar way (Branzei and Dall'Aglio, 2009; Branzei and Alparslan Gök, 2008; Alparslan Gök, Branzei, and Tijs, 2010).

An experimental study of an enriched model in which agents strategically choose how much of a resource to invest in order to protect themselves against uncertainty in aggregate supply and possible shortages, and of the dependence of these decisions on the rule that is used to allocate shortages (the proportional, constrained equal awards, constrained equal losses rules are considered) is due to Lefebvre (2013).

6.4 Allowing claims to be multi-dimensional

In the base model, claims and endowment are one-dimensional. Some authors have considered the possibility that a claim may have several components, corresponding to different "issues". The endowment remains a single quantity, however. Two definitions of a solution have been proposed: each claimant may be assigned an award for each of the issues on which he has a claim; or each claimant may be assigned a single award.

Formally, a **multi-issue claims problem** is a list $(c, E) \in \mathbb{R}_+^{NK+1}$ such that $\sum_N \sum_K c_{ik} \geq E$. Let $\boldsymbol{\mathcal{D}^N}$ be the class of all such problems. A **multi-issue rule** is a function from $\mathcal{D}^N$ to $\mathbb{R}_+^{NK}$, which associates with each multi-issue problem $(c, E) \in \mathcal{D}^N$ a vector $x \in \mathbb{R}_+^{NK}$ such that $\sum_N \sum_K x_{ik} = E$. A **simple multi-issue rule** is a function from $\mathcal{D}^N$ to $\mathbb{R}_+^N$, which associates with each multi-issue problem $(c, E) \in \mathcal{D}^N$ a vector $x \in \mathbb{R}_+^N$ such that $\sum_N x_i = E$.

Characterizations of versions of the proportional, constrained equal awards, and constrained equal losses rules have been developed for this model. A composite rule based on the proportional and constrained equal awards rules has also come out of axiomatic work. Papers on the subject are by González-Alcón, Borm, and Hendrickx (2007), Calleja, Borm, and Hendrickx (2005), Borm, Carpente, Casas-Méndez, Hendrickx (2005), Moreno-Ternero (2009), and Bergantiños, Lorenzo, and Lorenzo-Freire (2010a, 2010b, 2011). Ju, Miyagawa and Sakai (2007) and Hinojosa, Mármol, and Sánchez (2012), also offer results that pertain to this model. In none of these contributions are preferences explicitly introduced, but in Ju and Moreno-Ternero (2012), they are added to the model, and they appear in some of the axioms.

6.5 Adding a network structure

A **flow problem** is given by a list of **nodes** and **directed arcs** between them. One node is the **source** and another is a **sink**. A resource has to flow from the source to the sink. To each arc is associated (i) a pair of numbers, a minimal amount of the resource that has to flow along the arc as well as a maximal amount, and possibly (ii) a cost of having the resource flow along the arc. The objective is to find the minimal total cost of transporting the resource from the source to the sink respecting all the constraints. This is the problem formulated by Branzei, Ferrari, Fragnelli, and Tijs (2008). They show how to map rules into profiles of cost functions so as to achieve it. They also apply this mapping to a number of rules that have been central in the literature.

Bjørndal and Jörnsten (2010) also propose to model a claims problem as a flow problem, but their formalism allows them to consider situations in which there are multiple estates, and estates can hold claims against one another. They define *consistency* directly in terms of the flow network and ask whether a two-claimant rule has a *consistent* extension. Similarly to what is the case for standard claims problems, concede-and-divide and the constrained equal awards rules have unique *consistent* extensions.

In the base model, the endowment takes only one form (it is homogeneous). Let us now imagine that it takes different forms and that, for each claimant, only some of these forms are acceptable. A network structure is added to the model describing who can get what. This model is proposed by Moulin and Sethuraman (2013), who characterize several families of rules in its context. Here too, *consistency* plays a key role.

6.6 Adding unions

Borm, Carpente, Casas-Méndez, and Hendrickx (2005) enrich the base model by imagining that players are grouped in unions. They adapt the constrained equal awards rule to this context and give two characterizations of this extension. One can be seen as an extension of the characterization of the constrained equal awards rule for the base model, on the basis of *equal treatment of equals*, *composition up*, and *claims truncation invariance* (Dagan, 1996). The other is a counterpart of the characterization of this rule on the basis of *conditional full compensation* and *composition down* (Herrero and Villar, 2002).

6.7 Allowing the endowment to be non-homogeneous

In the base model, the endowment is a homogeneous whole. A more general situation is when the dividend is a heterogeneous continuum. Land is an obvious example here. This situation is modeled by Pálvödgyi, Peters, and Vermeulen (2011), who mainly study it from the strategic viewpoint, extending the results of Atlamaz, Berden, Peters, and Vermeulen (2011) (Subsection 4.2). They establish the existence of limits of ϵ-equilibria. In the base model, every division is Pareto efficient, but at equilibrium, this is not necessarily the case here. However, no equilibrium Pareto dominates any other.

6.8 Introducing utility functions

In the base model, preferences are not explicitly indicated, but it is implicit that each claimant prefers more of the dividend to less. Also, no cardinal index of the satisfaction that a claimant derives from his award is specified. Representing the efficient boundary of the feasible set of a claims problem as a hyperplane normal to a vector of ones amounts to working in payment space, or to assuming that the utilities that claimants derive from their assignments are linear, or to ignoring utilities. If utilities are not linear and are not ignored, the feasible set takes a more general shape. NTU generalizations of a claims problem can be formulated to accommodate the possibility (Mariotti and Villar, 2005). Under a certain smoothness property of feasible sets, the core and a version of the prekernel based on a *consistency* property have a non-empty intersection (Orshan, Valenciano, and Zarzuelo, 2003). The model considered by Herrero and Villar (2010) also allows for a non-linear boundary of the feasible set and it includes surplus sharing.

Another enrichment of the model along these lines is due to Moreno-Ternero and Roemer (2006, 2012). In their 2006 paper, their main axioms are that no claimant should be assigned more of the endowment as well as a greater utility than some other claimant, and that as a result of a change in endowment and population, each of the claimants present before and after should receive the same amount, or that each of them should receive less, or that each of them should receive more. They characterize a family of rules that generalize the parametric family. In their 2012 paper, which also involves an axiom pertaining to joint changes in resources and population, they characterize a family of rules that generalize the constrained equal awards rule. They obtain further characterizations by adding *composition up*.

6.9 Adding a reference point

Parallel to a similar enrichment of Nash's (1950) classical model of bargaining (Chun and Thomson, 1992), a "reference point" in awards space can be added to the base model, a point to which claimants find it natural to compare proposed compromises in evaluating them (Pulido, Sánchez-Soriano, and Llorca, 2002; Pulido, Borm, Hendrickx, Llorca, and Sánchez Soriano 2008). The latter authors propose a rule for this class of problems and show that it can be obtained by invoking the concept of the theory of TU games known as the τ-value (Tijs, 1981).

In a study that extends Thomson and Yeh (2008), Hougaard, Moreno-Ternero, and Østerdal (2012b) study operators in a model enriched by the addition of "baselines" (Section 3.3). They derive a number of results pertaining to the composition of these operators with the duality operator.

6.10 Going further

As mentioned earlier, the theory concerning the adjudication of conflicting claims is remarkably well-developed. Progress in this area has been greatly facilitated by previous studies of other classes of allocation problems. Adapting the ideas and methodology that have been fruitful in its study to other classes of problems may also prove useful. An illustration is the model of contest incorporating claims developed by Ansink (2011).

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