Fully allocating a commodity among agents with single-peaked preferences

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Abstract

We survey the literature devoted to the study of the problem of allocating an infinitely divisible commodity among agents whose preferences are single-peaked. We formulate a number of normative and strategic requirements on rules, and study their implications when imposed in various combinations. A unique rule emerges as being the best-behaved from a variety of viewpoints: the uniform rule.

Key-words: single-peaked preferences; resource monotonicity; population monotonicity; consistency; Nash implementation; strategy-proofness; constrained equal-distance rule; constrained equal-preferred-sets rule; proportional rule; uniform rule.

JEL classification numbers: C72; D50; D63; D71

1 Introduction

An amount of an infinitely divisible and non-disposable commodity has to be fully allocated among a group of agents. Each agent has "single-peaked" preferences over what he consumes: up to some critical level, his "peak amount", an increase in his consumption raises his welfare; beyond that level, the opposite holds. Agents have equal rights (and equal responsibilities) over (for) the consumption of the commodity. What should be done? A "solution" recommends a set of allocations for each situation of this type, and a "rule" is a single-valued solution. We survey the literature concerned with the identification of the most desirable solutions and rules.

In a study that is the starting point of this literature, Sprumont (1991) characterized a rule known as the "uniform rule" as the only efficient and anonymous rule for which it is never in anybody's advantage to lie about his preferences. Multiple additional axiomatic characterizations of this rule followed, based on various requirements of fairness and robustness under strategic behavior: indeed, attacking the problem from a great variety of angles, normative or strategic, almost always leads to it, and our main conclusion will be that the uniform rule should probably be thought of as the most important rule to handle the class of allocation problems under consideration. In fact, and although it does not always satisfy the strongest forms of the relational fairness properties that one would have liked, no other class of problems seems to admit a rule that outperforms all others so systematically.

We first describe the model and give examples of situations it covers (Section 2).

We investigate efficiency and punctual distributional concepts that are central in the theory of fairness, no-envy and variants, as well as concepts based on comparisons to equal division. We present logical relations between these notions, and discuss the structure of the sets of allocations satisfying them. Taking advantage of the special features of the model, we also propose rules that do not always have counterparts in other contexts (Section 3).

Next, we study relational requirements of fairness. We consider in turn changes in the social endowment (Section 4), in the population of agents (Section 5), and in the preferences of some of the agents present (Section 6). Since receiving more is not always desirable, the monotonicity properties of rules that have been formulated to deal with such changes in the "classical" model of fair division are not applicable, but natural weakenings can be formulated that are still meaningful expressions of the idea of solidarity; we describe their implications when imposed in conjunction with efficiency and punctual fairness requirements. We then imagine simultaneous but related changes in the population of agents and in the social endowment (Sections 7 and 8), and formulate robustness requirements with respect to such changes.

We then turn to strategic issues. We look for strategy-proof rules (Section 9), discuss the manipulability of rules (Section 10), and investigate other implementability requirements (Section 11).

We conclude by exploring extensions of the model (Section 12). We sketch selected proofs in the appendix.

2 The base model

A **social endowment** $M \in \mathbb{R}_{++}$ of an infinitely divisible commodity has to be allocated among a group N of agents. Each agent $i \in N$ is equipped with a continuous and **single-peaked** preference relation R_i defined over the interval $[0, M]$: this means that there is a number in $[0, M]$, denoted $p(R_i)$, and called his **peak amount**, such that for each pair $x_i, x'_i \in [0, M]$, if $x'_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < x'_i$, then $x_i \; P_i \; x'_i$. For each $i \in N$, let u_i be a continuous numerical representation of R_i. Representations can be chosen to be linear to the left of the peak amount, as shown in the figures.

Let $\boldsymbol{\mathcal{R}}$ denote the class of all such preference relations. Single-peaked preferences are convex (although they do not exhaust the class of convex relations; preferences whose representations have a "plateau" or a "ledge", are convex too). The only standard assumption that they violate is monotonicity. Whenever the social endowment is kept fixed (variations in this parameter are mainly studied in Section 4), we simply refer to an **economy** as a list $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$. We use $p(R)$ to denote the profile of peak amounts of R, $(p(R_i))_{i \in N}$.

When it simplifies the treatment of an issue, we assume that preferences are defined over $\mathbb{R}_+$. Let $\boldsymbol{\mathcal{R}_\infty}$ be this class.

Agent i's preferences R_i can be described in terms of the function $r_i \colon [0, M] \to [0, M]$ that gives, for each $x_i \in [0, M]$, the amount on the other side of his peak amount that he finds indifferent to x_i if such an amount exists (Figure 1a); if not, it gives the endpoint of $[0, M]$ on the other side of his peak amount (Figure 1b). Formally, given $x_i \leq p(R_i)$, we have $r_i(x_i) \geq p(R_i)$ and $x_i \; I_i \; r_i(x_i)$ if such an amount exists, and $r_i(x_i) \equiv M$ otherwise; given

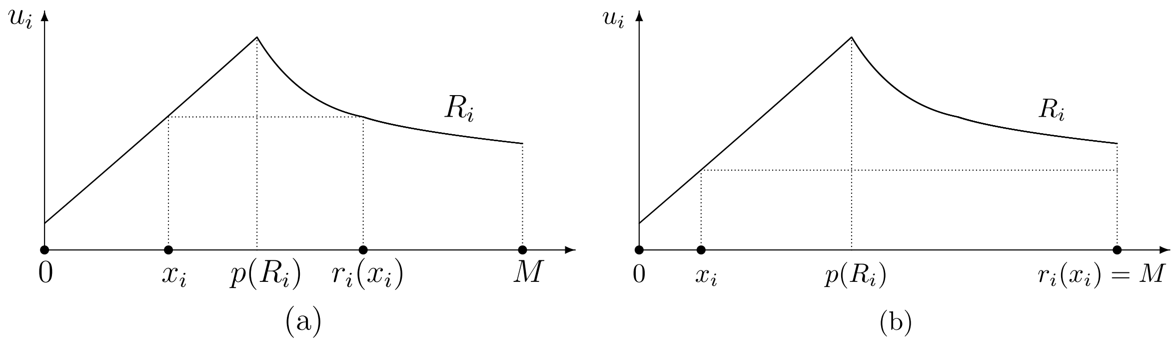

Figure 1: Definition of the function r_i in terms of which the preferences of agent i, R_i, can be described. (a) Here, there is an amount equivalent to x_i on the other side of the agent's peak amount: $r_i(x_i)\ I_i\ x_i$. (b) Here, there is none, so $r_i(x_i) \equiv M$.

$x_i \geq p(R_i)$, we have $r_i(x_i) \leq p(R_i)$, and $x_i\ I_i\ r_i(x_i)$ if such an amount exists, and $r_i(x_i) \equiv 0$ otherwise.

A (feasible) **allocation** is a list $x \equiv (x_i)_{i \in N} \in \mathbb{R}^N_+$ such that $\sum x_i = M$. Note that we do not assume that the commodity can be disposed of.[1] Let $\boldsymbol{X}$ denote the set of allocations.

When preferences are defined over $\mathbb{R}_+$ instead of $[0, M]$, requiring solutions to depend only on the profile of their restrictions to $[0, M]$ is equivalent to assuming that they are defined over $[0, M]$ (at least when M is fixed).

A **solution** is a correspondence that associates with each economy in our domain, or some subdomain of it, a non-empty subset of its allocations. A **rule** is a *single-valued* solution. To designate the intersection of two solutions, we juxtapose their names. For instance, we refer to the intersection of the no-envy solution with the efficiency solution (defined below) as the "no-envy and efficiency solution".

Here are three applications for the model. First, consider a two-good exchange economy in which resources are in principle allocated by operating the price mechanism, but suppose that prices have been thrown out of equilibrium by some exogenous shock, or that they are artificially kept from adjusting in order to achieve some social objective. Then, not all agents can be assigned the bundles they would prefer in their budget sets; rationing is needed. Now, note that if an agent's preference relation over his whole consumption space is strictly convex, then the restriction of his preference relation to the boundary of his budget set is single-peaked.

A task requiring so many hours of work has to be divided among a team

[1]If disposal were an option, we could, for each $i \in N$, replace agent i's preference relation R_i by one that is strictly monotonic up to some point and satiated above it. The possibility of disposal is discussed in Section 12, item (10).

of workers. Each worker is paid an hourly wage. If his disutility of labor is concave, then his induced preferences over the labor he supplies are single-peaked.

A social endowment of a good has to be allocated among a group of agents whose preferences exhibit the following form of altruism. Each agent cares about what he consumes but also about what the others consume in total. If he consumes little, he gives priority to himself however, and an increase in his consumption makes him better off in spite of the negative impact this has on what is left for the others. As his consumption increases, he turns his attention to them. At some point, his concern for them dominates, and increasing his consumption further makes him worse off.

3 An inventory of solutions and rules

We first address the issue of efficiency. We then adapt to the present model standard notions of fairness, for individuals first, and then for groups. Finally, we propose a number of rules that are based on the specific features of the model.[2]

• As always, an allocation is (Pareto) **efficient** if there is no other allocation that each agent finds at least as desirable and at least one agent prefers. Our first solution selects all of these allocations. We designate it by the letter P in reference to Pareto:

Efficiency solution, *P*: For each $R \in \mathcal{R}^N$, $x \in P(R)$ if $x \in X$ and there is no $x' \in X$ such that for each $i \in N$, $x'_i \; R_i \; x_i$, and for at least one $i \in N$, $x'_i \; P_i \; x_i$.

The efficiency of an allocation is easy to verify: all consumptions should be on the same side of the peak amounts.[3] Given $x \in X$, $x \in P(R)$ if and only if (i) when $\sum p(R_i) \geq M$—"there is too little" of the commodity then—for each $i \in N$, $x_i \leq p(R_i)$, and (ii) when $\sum p(R_i) \leq M$—now, "there is too much" of it—then for each $i \in N$, $x_i \geq p(R_i)$. The *efficiency* solution is convex-valued, a fact that will prove very useful.

[2]This section is mainly based on Thomson (1994b, 1994c).
[3]This is noted by Sprumont (1982).

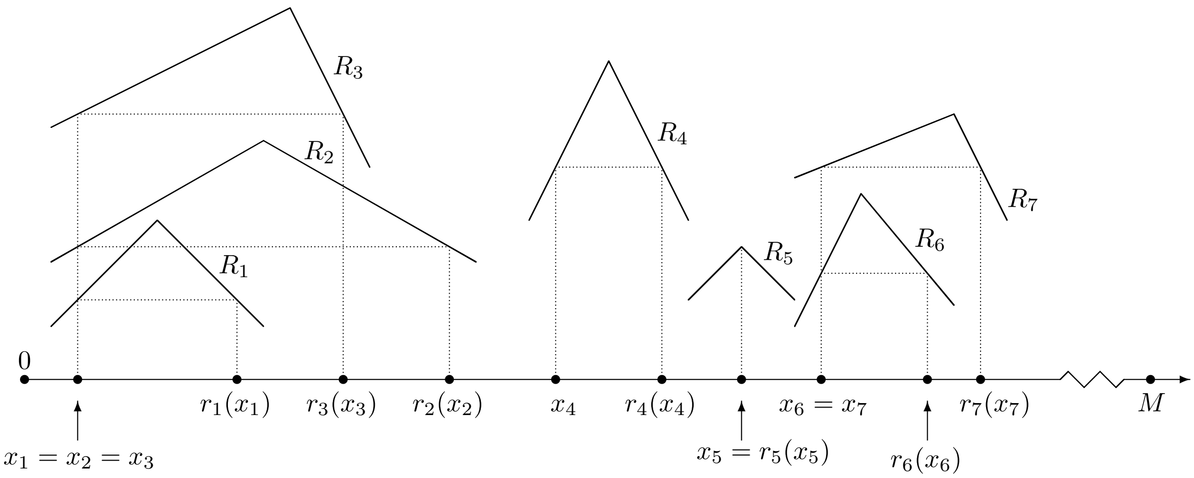

Figure 2: A typical envy-free and efficient allocation in a seven-agent example. In order not to clutter the Figure, for each agent, we draw the graph of a numerical representation of his preferences only in some subinterval of $[0, M]$ containing his peak amount.

• Our first fairness requirement is that each agent should find his assignment at least as desirable as anybody else's assignment (Foley, 1967):[4]

No-envy solution, *F*: For each $R \in \mathcal{R}^N$, $x \in F(R)$ if $x \in X$ and for each $\{i, j\} \subseteq N$, $x_i \; R_i \; x_j$.

Let us illustrate the geometry of the envy-free and efficient set with an example $R \in \mathcal{R}^N$ where $N \equiv \{1, \ldots, 7\}$, such that $p(R_1) < p(R_2) < \cdots < p(R_7)$, and $\sum p(R_i) \geq M$ (Figure 2). (The case $\sum p(R_i) \geq M$ can be handled in a symmetric manner.) By *efficiency*, as we just saw, each agent receives at most his peak amount. Let us assign some amount $x_1 \leq p(R_1)$ to agent 1. For him not to envy agent 2, we should not place x_2 in the interval $]x_1, r_1(x_1)[$ (agent 1's strict upper contour set at x_1). Thus, either $x_2 \leq x_1$ or $x_2 \geq r_1(x_1)$. If the former inequality holds, it cannot be strict; otherwise, agent 2 would envy agent 1. Also, since by *efficiency*, $x_2 \leq p(R_2)$, to be able to choose the latter, we need $r_1(x_1) \leq p(R_2)$, as is the case in the figure; altogether, x_2 should belong to the interval $[r_1(x_1), p(R_2)]$. Suppose that we choose $x_2 = x_1$ and let us turn to agent 3. We have two intervals, $]x_1, r_1(x_1)[$ and

[4]The allocation $x \in X$ is "egalitarian-equivalent for $R \in \mathcal{R}^N$" if there is a reference amount $x_0 \in \mathbb{R}_+$ such that for each $i \in N$, $x_i \; I_i \; x_0$. This concept (adapted from Pazner and Schmeidler, 1978) has been important in other contexts, but it will play no role here. The reason is that the set of egalitarian-equivalent and efficient allocations is typically empty. Suppose for instance that there is $x \in X$ such that for each $i \in N$, $x_i = p(R_i)$. Then, $P(R) = \{x\}$. Thus, if there is $\{i, j\} \subseteq N$ such that $p(R_i) \neq p(R_j)$, there is no egalitarian-equivalent and efficient allocation. Chun (2000) proposes a weakening of the definition guaranteeing non-emptiness.

$]x_2, r_2(x_2)[$, in which x_3 should not fall, otherwise either agent 1 or agent 2 or both, would envy him. So either $x_3 \leq x_1 (= x_2)$ or $x_3 \geq \max\{r_1(x_1), r_2(x_2)\}$ (which happens to be $r_2(x_2)$). Since $p(R_3) < r_2(x_2)$, the second choice is actually incompatible with *efficiency*. Thus, since as before, we cannot have $x_3 < x_1$ (otherwise agent 3 would envy both agents 1 and 2), we need $x_3 = x_1$. For agent 4, there is a third forbidden interval $]x_3, r_3(x_3)[$, and by the same reasoning, either $x_4 = x_1$ or $x_4 \geq \max\{r_1(x_1), r_2(x_2), r_3(x_3)\}$ (which is still $r_2(x_2)$). Once again, by *efficiency*, $x_4 \leq p(R_4)$. Since $r_3(x_3) \leq p(R_4)$, it is possible to choose $x_4 \in [r_2(x_2), p(R_4)]$, so let us do that. We proceed in this manner from agent to agent. When we reach agent n, what is left for him should be no greater than his peak amount, and such that he does not envy anyone and is not envied by anyone. This may or may not be the case. If it is not the case, we have to adjust the choices made along the way. At this point, it is not clear how to do so, but envy-free and efficient allocations do exist. We will prove this later by invoking the uniform allocation, an allocation that always exists and enjoys both properties. It is also easy to calculate.

Summarizing our discussion, at an envy-free allocation, agents are gathered in groups—let us denote these groups N^1, $N^2, \ldots, N^L$—and for each $\ell \in \{1, \ldots, L\}$, all members of N^ℓ consume the same amount a^ℓ, with $a^\ell < a^{\ell'}$ if $\ell < \ell'$. Let us number the groups so that $0 \leq a^1 < a^2 < \cdots < a^L$. Then, for each $\ell = 2, \ldots, L$ and each member of $N^{\ell-1}$, the consumption on the other side of his peak amount that he finds indifferent to $a^{\ell-1}$, his assignment, should be at most as large as a^ℓ. In the example represented in Figure 2, $L = 4$, $N^1 \equiv \{1, 2, 3\}$, $N^2 \equiv \{4\}$, $N^3 \equiv \{5\}$, and $N^4 \equiv \{6, 7\}$.

• Several fairness requirements can be defined by reference to equal division. The central one is that each agent should find his consumption at least as desirable as equal division:

Equal-division lower bound solution, $\boldsymbol{B_{ed}}$: For each $R \in \mathcal{R}^N$, $x \in B_{ed}(R)$ if $x \in X$ and for each $i \in N$, $x_i \; R_i \; \frac{M}{|N|}$.

For an allocation to pass this test, each agent's consumption should fall in his weak upper contour set at equal division, the interval $[\frac{M}{|N|}, r_i(\frac{M}{|N|})]$ if $\frac{M}{|N|} \leq p(R_i)$ and the interval $[r_i(\frac{M}{|N|}), \frac{M}{|N|}]$ otherwise. As in classical economies, if $|N| = 2$, an allocation meeting the equal-division lower bound is envy-free, but for $|N| > 2$, this implication fails. The implication in the other direction fails even for $|N| = 2$.

• Next, we formulate criteria to evaluate how fairly groups, as opposed to individuals, are treated. First, we require that no group of agents should be able to make each of its members at least as well off and at least one of them better off, by redistributing among themselves the resources assigned to each other group of the same size (Schmeidler and Vind, 1972):

Group no-envy solution, F^G: For each $R \in \mathcal{R}^N$, $x \in F^G(R)$ if $x \in X$ and for each pair $\{G, G'\}$ of subsets of N with $|G| = |G'|$, there is no $(y_i)_{i \in G}$ such that (i) $\sum_G y_i = \sum_{G'} x_i$ and (ii) for each $i \in G$, $y_i \; R_i \; x_i$, and for at least one $i \in G$, $y_i \; P_i \; x_i$.

If $x \in F^G(R)$, then $x \in P(R)$ (simply, take $G = G' = N$). Other definitions can be formulated according to which only distinct groups or only non-overlapping groups, are compared. Then, *efficiency* would not be implied any more.

The group envy-free allocations can be characterized as follows. Assume for the sake of illustration that $\sum p(R_i) > M$. Given $x \in F^G(R)$, and since $x \in P(R)$, then x passes the test if and only if for each pair $\{G, G'\}$ of subsets of N with $|G| = |G'|$, either $\sum_{G'} x_i \leq \sum_G x_i$ or $\sum_G r_i(x_i) \leq \sum_{G'} x_i$.

We could also evaluate the relative treatment of groups of different sizes by calculating resources per capita. Also, we could require of an objection by a group that it should make each of its members better off.

• Next, assuming that each agent is endowed with an equal share of the social endowment, we require that no group of agents should be able to make each of its members at least as well off, and at least one of them better off, by redistributing among themselves the resources it controls in total:

Equal-division core, C_{ed}: For each $R \in \mathcal{R}^N$, $x \in C_{ed}(R)$ if $x \in X$ and there is no $G \subseteq N$ and $(y_i)_{i \in G} \in \mathbb{R}^G_+$ such that (i) $\sum_G y_i = |S| \frac{M}{|N|}$ and (ii) for each $i \in G$, $y_i \; R_i \; x_i$, and for at least one $i \in G$, $y_i \; P_i \; x_i$.

We saw that if $|N| = 2$, an allocation meeting the equal-division lower bound is envy-free. If $|N| > 2$, this is not the case. In fact, an allocation in the *equal-division core* may not be envy-free. Non-emptiness of the *equal-division core* is actually not guaranteed then. However, a weaker definition—let us refer to it as the **weak equal-division core**—according to which an allocation is disqualified only if a group can make each of its members better off, is satisfied by the uniform allocation defined below.

• Our next rules can all be understood as attempts at equating some notion of "sacrifice" among agents.

The first one is the expression of the principle of proportionality. This principle is applicable in situations where each agent's characteristics can be meaningfully summarized into a single number. Here, it is of course most tempting to have the agent's peak amounts play this role. In fact, since obtaining *efficiency* already requires that the peak amounts be taken into account, this choice seems to be the most natural way to make the principle operational. We can interpret how far from unity the ratio of what an agent receives to his peak amount is, as a measure of the sacrifice he makes at an allocation. The idea then is to equate these sacrifices across agents.

Proportional rule, *Pro*: For each $R \in \mathcal{R}^N$, $x = Pro(R)$ if $x \in X$ and there is $\lambda \in \mathbb{R}_+$ such that $x = \lambda(p(R_i))_{i \in N}$. If not, $Pro(R) = (\frac{M}{|N|}, \ldots, \frac{M}{|N|})$.

The first case in the definition occurs if at least one peak amount is positive. When all peak amounts are zero, all preferences are the same, and it is certainly appealing to choose equal division.

The following variant of the proportional rule has the advantage over the previous definition to treat the case $\sum p(R_i) > M$ and the case $\sum p(R_i) < M$ symmetrically. Another advantage it has is continuity with respect to preferences (we omit a formal statement of this property), which the proportional rule fails (a discontinuity occurs when all peak amounts are zero).

Symmetrized proportional rule, *Pro*^{s}: For each $R \in \mathcal{R}^N$, $x = Pro^s(R)$ if $x \in X$ and (i) when $\sum p(R_i) \geq M$, there is $\lambda \in \mathbb{R}_+$ such that $x = \lambda(p(R_i))_{i \in N}$, and (ii) when $\sum p(R_i) \leq M$, there is $\lambda \in \mathbb{R}_+$ such that $(M, \ldots, M) - x = \lambda[(M, \ldots, M) - p(R)]$.

• Next, as a measure of the sacrifice made by an agent at an allocation, we use the distance between his assignment and his peak amount. Because assignments are non-negative numbers, equating these distances may not be feasible, so we make them as equal as possible subject to non-negativity.

Constrained equal-distance rule, *Dis*: For each $R \in \mathcal{R}^N$, $x = Dis(R)$ if $x \in X$ and (i) when $\sum p(R_i) \geq M$, there is $d \geq 0$ such that $x = (\max\{0, p(R_i) - d\})_{i \in N}$, and (ii) when $\sum p(R_i) \leq M$, there is $d \geq 0$ such that $x = p(R) + (d, \ldots, d)$.

• Our third measure of the sacrifice made by an agent at an allocation is the absolute value of the difference between what he receives and the amount on the other side of his peak amount that he finds indifferent to it if there is such an amount, and the endpoint of the interval $[0, M]$ that is on the other side of his peak amount otherwise. Here too, equating sacrifices may not be possible, so we make them as equal as possible subject to non-negativity.

Constrained equal-preferred-sets rule, *Eps*: For each $R \in \mathcal{R}^N$, $x = Eps(R)$ if $x \in X$ and (i) when $\sum p(R_i) \geq M$, there is $\sigma \geq 0$ such that for each $i \in N$, $r_i(x_i) - x_i \leq \sigma$, strict inequality holding only if $x_i = 0$, and (ii) when $\sum p(R_i) \leq M$, there is $\sigma \geq 0$ such that for each $i \in N$, $x_i - r_i(x_i) \leq \sigma$, strict inequality holding only if $x_i = M$.

The proportional, symmetrized proportional, constrained equal-distance, and constrained equal-preferred-sets allocations are efficient but they need not be envy-free and they need not meet the equal-division lower bound. However, their definitions can be adapted so as to recover these properties, if desired. For instance, to obtain *no-envy*, we could choose the envy-free allocation(s) whose associated vector of sacrifices, evaluated according to one or the other of these measures, is lexicographically minimal.[5] Similar selections from the *equal-division lower bound and efficiency* solution can be defined. In the process, we may lose *single-valuedness* however.

• Given an order on the set of agents, let them enter the scene in that order and assign to each of them his preferred amount subject to availability when his turn comes. We thereby obtain the **sequential priority rule associated with the order**. Such extreme asymmetric treatment of agents does not appear very desirable. Yet, each of the rules defined in this manner satisfies *efficiency* and many of the relational requirements formulated below. Besides, by choosing the orders with equal probabilities, and taking the average of the resulting allocations, we recover some measure of punctual fairness without losing *efficiency*—this property is preserved under convex operations, as we noted—nor the relational properties satisfied by the sequential priority rules, many of which are also preserved under convex operations. Let us refer to this average as the **random arrival rule**.

[5]Formally, let $\tilde{t}$ be obtained by rewriting the coordinates of $t \in \mathbb{R}^n$ in decreasing order. Given t and $t' \in \mathbb{R}^n$, we say that t is "lexicographically smaller than t'," written $t <_L t'$, if $[\tilde{t}_1 < \tilde{t}'_1]$, or $[\tilde{t}_1 = \tilde{t}'_1$ and $\tilde{t}_2 < \tilde{t}'_2]$, or $[\tilde{t}_1 = \tilde{t}'_1, \ldots, \tilde{t}_k = \tilde{t}'_k$ for each $k \leq \ell$ and $\tilde{t}_\ell < \tilde{t}'_\ell]$. An alternative order is obtained by focusing first on the agents whose sacrifices are the smallest and in successively making them as large as possible.

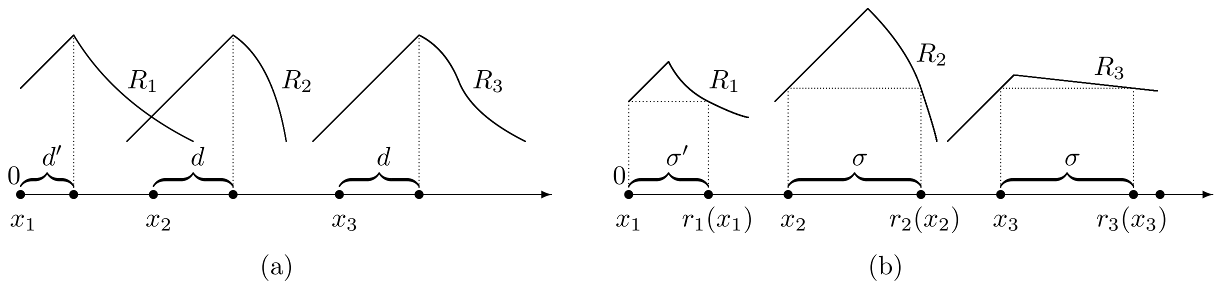

Figure 3: Two rules based on equating some notion of sacrifice made by agents. (a) Constrained equal-distance rule: if the consumptions of two agents are positive, they differ from the agents' peak amounts by the same amount. (Here, agent 1's consumption is closer to his peak amount than agents 2 and 3's consumptions are to theirs, but it is because he consumes 0.) (b) Constrained equal-preferred-sets rule: if the consumptions of two agents are positive, the upper contour sets of the two agents at these two points are of the same size.

• The following rule, introduced in the fixed-price literature[6], will play a central role throughout this essay. It is obtained by specifying a bound, the same for all agents—it is an upper bound if there is too little of the commodity and a lower bound if there is too much—and maximizing each agent's preferences subject to that bound; finally, adjusting the bound so that the list of maximizers yields a feasible allocation.

Uniform rule, *U*: For each $R \in \mathcal{R}^N$, $x = U(R)$ if $x \in X$ and there is $\lambda \in \mathbb{R}_+$ such that (i) when $\sum p(R_i) \geq M$, then $x = (\min\{p(R_i), \lambda\})_{i \in N}$, and (ii) when $\sum p(R_i) \leq M$, then $x = (\max\{p(R_i), \lambda\})_{i \in N}$.

Let us describe the allocation chosen by the uniform rule for a fixed preference profile as the social endowment grows from 0 to ∞. This description is not quite in agreement with the model as specified above, where the social endowment was given a fixed value M and preferences were defined on $[0, M]$, but it will be useful very soon when we turn to properties of rules involving variations in M. To simplify notation, we assume $p(R_1) < \cdots < p(R_n)$. As M starts increasing from 0, equal division prevails and it does so until each agent receives $p(R_1)$. At that point, agent 1 stops receiving anything for a while. Further increments in M are shared equally among the other agents until each of them receives $p(R_2)$. Then, agent 2 also stops receiving anything for a while ... This process continues until each agent receives his peak amount. The first increments beyond $\sum p(R_i)$ go entirely to agent 1 until he receives $p(R_2)$. Further increments are shared equally between agents 1

[6]It appears in Bénassy (1982).

and 2 until they receive $p(R_3)$ ... This goes on until agents 1 through $n-1$ receive $p(R_n)$. All further increments are shared equally.[7]

The uniform rule can be criticized on the grounds that it fully satisfies some agents (it gives them their peak amounts)—agents with the lowest peak amounts if $\sum p(R_i) \geq M$ and agents with the highest peak amounts if $\sum p(R_i) \leq M$—"at the expense" of the others, who receive equal amounts, any differences in the preferences of the members of the latter group being ignored. Nevertheless, the uniform allocation does satisfy *no-envy* and it meets the *equal-division lower bound*. We also have the following characterizations based on evaluating how differently a rule treats agents in terms of amounts received.

Theorem 1 (Schummer and Thomson, 1997) *(a) For each economy, the uniform allocation is the unique efficient allocation at which the difference between the greatest and smallest amounts any two agents receive is the smallest.*

(b) For each economy, the uniform allocation is the unique efficient allocation at which the variance of the amounts received by all the agents is the smallest.

The uniform allocation of an economy may not be *group envy-free* but it is always such that no group of agents, when given access to the resources assigned to a group of the same size, can make each of its members better off. Similarly, the uniform allocation may not be in the *equal-division core* (when this set is non-empty), but it satisfies the weaker property obtained, in defining blocking, by requiring that all members of a deviating coalition should be made better off by the deviation (the *weak equal-division core*).

A noteworthy feature of the uniform rule, as well as of the other rules that we have defined, is that they depend only on peak amounts. Let us write this property for a generic solution φ:

Peak-only: For each pair $\{R, R'\} \subset \mathcal{R}^N$, if $p(R) = p(R')$, then $\varphi(R) = \varphi(R')$.

[7] This similarity between the uniform rule and the rule advocated by Maimonides for the adjudication of conflicting claims (O'Neill, 1982; see Thomson, 2003, for a discussion) should be noted. Indeed, after replacing the vector of peak amounts by the vector of claims, the algorithm describing that rule is identical up to the point where each agent receives his peak amount.

Peak-only is not motivated by normative considerations. It should mainly be understood as a requirement of informational simplicity of solutions. It is also crucial in helping understand incentive issues. Interestingly, when imposed on subsolutions (non-necessarily *single-valued*) of the *no-envy and efficiency* solution, it leads to another characterization of the uniform rule:[8]

Theorem 2 (Thomson, 1994b) *The uniform rule is the only subsolution of the* no-envy and efficiency *solution to satisfy* peak-only.

• Solutions can be defined on the basis of the number of agents receiving their peak amounts. A **plurality rule** is defined by choosing the allocations x at which the number $pk(x,R) \equiv |\{i \in N : x_i = p(R_i)\}|$ is maximal, either when this maximization is performed within the efficient set, or when in addition punctual fairness requirements, such as *no-envy* or the *equal-division lower bound*, are imposed. None of the resulting solutions is *single-valued* however.

• We conclude with a family of rules whose definition is inspired by a concept (Young, 1987) that has played a central role in the literature on the adjudication of conflicting claims (O'Neill, 1982; see Thomson, 2003, for a survey of the literature).

Each member of the family is indexed by a function $f : \mathbb{R}_+ \times [\underline{\lambda}, \bar{\lambda}[\to \mathbb{R}_+$, where $[\underline{\lambda}, \bar{\lambda}[\subset \mathbb{R}$, that is continuous, nowhere decreasing in its second argument, and such that for each $R_0 \in \mathcal{R}$, $f(R_0, \underline{\lambda}) = 0$ and $f(R_0, \lambda) \to \infty$ as $\lambda \to \bar{\lambda}$. Let $\mathbf{\Phi}$ be the family of these functions.

Parametric rule of representation $\boldsymbol{f \in \Phi}$, $\boldsymbol{\varphi^f}$: For each $R \in \mathcal{R}^N$, $\varphi^f(R)$ is the allocation $x \in X$ such that there is $\lambda \in [\underline{\lambda}, \bar{\lambda}[$ for which, for each $i \in N$, $x_i = f(R_i, \lambda)$.

[8] *Single-valuedness* comes out of the axioms. Uniqueness is not maintained if the *equal-division lower bound* is imposed instead of *no-envy*. If *efficiency* is dropped and *single-valuedness* imposed, the class of selections from the *no-envy* solution that are *peak-only* constitute a lattice under the order on rules induced by Pareto domination, with the uniform rule dominating all of these selections and the equal-division rule being dominated by all of them (Sakai and Wakayma, 2012). Another notion of fairness is that each agent should find his consumption at least as desirable as any point in the convex hull of everyone's consumption (Kolm, 1973). It is studied by Chun (2000), Kesten (2006), and Sakai and Wakayma (2012). Kesten proves counterparts of Theorem 1 in which *efficiency* is replaced by this fairness notion.

The proportional, constrained equal-distance, constrained equal-preferred-sets, and uniform rules belong to the family. For the rule associated with a particular f to satisfy each given property, appropriate restrictions have to be imposed on f. To illustrate, for *efficiency*, there should be $\lambda^* \in [\underline{\lambda}, \bar{\lambda}[$ such that for each $R_0 \in \mathcal{R}$, $f(R_0, \lambda^*) = p(R_0)$.

4 Resource-monotonicity and a variant

In this section, we consider changes in the social endowment. Up to now, we have assumed preferences to be defined over the interval $[0, M]$, where M was fixed. This implied that even in situations where the natural domain of definition of preferences would be larger, only their restriction to the set of feasible consumptions of each agent was deemed relevant for the evaluation of an allocation. Since the property we now study pertains to variations in the social endowment, for its formulation to be in line with the above interpretation, the domain over which preferences are defined should be allowed to change. It will be a little simpler to assume instead that preferences are defined over $\mathbb{R}_+$. Thus, an economy is now a pair $e \equiv (R, M) \in \mathcal{R}_\infty^N \times \mathbb{R}_+$, with feasible set denoted $X(e)$. Let $\boldsymbol{\mathcal{E}_\infty^N}$ be the class of these pairs. Also, as we would like to be able to make unambiguous welfare comparisons between two economies that differ in their endowments, we consider *single-valued* mappings (rules) defined on $\mathcal{E}_\infty^N$, and not solutions, which may be multi-valued. The same comment applies to our study of two other solidarity properties in Sections 5 and 6, and to a property of robustness under strategic behavior in Section 9.

Starting from some economy and having applied to it a rule, suppose that the social endowment increases. If agents had monotone preferences, a natural solidarity requirement would be that they should all end up at least as well off as they were initially (Roemer, 1986; Chun and Thomson, 1988; Moulin and Thomson, 1988). Of course, when preferences are not monotone, this requirement does not make sense. What does instead is that when the social endowment increases—or more generally, changes, whether the change is an increase or a decrease—the welfares of all agents should be affected in the same direction: all agents should end up at least as well off as they were initially, or all should end up at most as well off:

Resource-monotonicity: For each $e \equiv (R, M) \in \mathcal{E}_\infty^N$, each $M' \in \mathbb{R}_+$, if

$x \equiv \varphi(e)$ and $x' \equiv \varphi(R, M')$, then either (i) for each $i \in N$, $x_i \; R_i \; x'_i$, or (ii) for each $i \in N$, $x'_i \; R_i \; x_i$.

We formulate the requirement in welfare terms and not in physical terms because welfare is what people care about. The same comment applies to the other requirements introduced below. However, for this model, requirements in physical terms are meaningful too and of some interest, and in the presence of *efficiency*, equivalence holds.

Resource monotonicity is quite strong. None of the uniform, proportional, constrained equal-distance, or constrained equal-preferred-sets rules satisfy it (although under a mild domain restriction, this last rule does). As for selections from the *no-envy* solution, or from the *equal-division lower bound* solution, general incompatibility results emerge (*efficiency* plays no role in precipitating them). One of them involves the basic requirement that agents with the same preferences should receive amounts that they find indifferent. Here too, for an efficient rule, a formulation in physical terms (Section 9) is equivalent. We state it for correspondences:

Equal treatment of equals in welfare: For each $e \equiv (R, M) \in \mathcal{E}^N_\infty$, each $x \in \varphi(e)$, and each pair $\{i, j\} \subseteq N$, if $R_i = R_j$, then $x_i \; I_i \; x_j$.

Proposition 1 (Thomson, 1994b) *(a) No selection from the* no-envy *solution is* resource-monotonic. *(b) No selection from the* equal-division lower bound *solution is* resource-monotonic. *(c) No rule satisfies* equal treatment of equals in welfare, peak-only, *and* resource-monotonicity.

Resource-monotonic selections from the *efficiency* solution do exist. Moreover, the class of all such solutions can easily be characterized, even though it is large:

Theorem 3 *Any selection from the efficiency solution that is* resource-monotonic *is defined as follows. For each $R \in \mathcal{R}^N$,*

(i) *There are $|N|$ non-decreasing functions $\varphi_i(R, \cdot)\colon [0, \sum p(R_i)] \to \mathbb{R}_+$ such that for each $M \in [0, \sum p(R_i)]$, we have $\sum \varphi_i(R, M) = M$, and for each $i \in N$, $\varphi_i(R, \sum p(R_i)) = p(R_i)$.*

(ii) *For each* $M' \in [\sum p(R_i), \sum r_i(0)[$*, let* $M \in [0, \sum p(R_i)]$ *and* $x' \in P(R, M')$ *be such that for each* $i \in N$, $x'_i \; I_i \; \varphi_i(R, M)$*. Then, let* $\varphi(R, M') \equiv x'$.[9] *This completes the construction if* $\sum r_i(0) = \infty$.

(iii) *If* $\sum r_i(0) < \infty$*, there are* $|N|$ *non-decreasing functions* $\varphi_i(R, \cdot) \colon [\sum r_i(0), \infty[\to \mathbb{R}_+$ *such that for each* $M \in [\sum r_i(0), \infty[$*, we have* $\sum \varphi_i(R, M) = M$*, and for each* $i \in N$, $\varphi_i(R, \sum r_i(0)) = r_i(0)$.[10]

An examination of the proof of the impossibilities of Proposition 1 shows that they arise when the change in the social endowment is such that it turns an economy from one in which there is too much to divide into one in which there is too little, or conversely. It is not surprising that this switch should play a role, and it suggests weakening *resource monotonicity* by limiting its applications to changes that are not so disruptive, in the sense that if initially, there is too little of the commodity initially, then there is still too little afterwards; and if initially, there is too much, there is still too much afterwards: if the social endowment changes but stays on the same side of the sum of the peak amounts, the welfare of all agents should be affected in the same direction. Let us call this variant **one-sided resource-monotonicity**.[11]

The uniform rule is *one-sided resource-monotonic* (to see this, it is most convenient to use the algorithmic description we gave of it in Subsection 3). The proportional and constrained equal-distance rules are too. Neither is a selection from the *no-envy* solution, but in fact, under the domain restriction that for each agent, there is a finite consumption indifferent to 0, no selection from the *no-envy and efficiency* solution other than the uniform rule is *one-sided resource-monotonic*. Let $\boldsymbol{\mathcal{R}_{\infty,fin}}$ be the subdomain of

[9]The number M' and the vector x' are defined uniquely. The equality follows from *resource-monotonicity*.

[10]The functions do not have to satisfy any other requirements. The reason for this freedom of choice in that interval is that, by giving to each agent more than $r_i(0)$, we ensure that each is worse off than he would be at any social endowment that would require that each receives less than his peak amount.

[11]We mentioned earlier that other monotonicity requirements have been considered. One says that as the social endowment increases, each agent should receive at least as much as he did initially (Otten, Peters, and Volij, 1996). Another requirement is that if the social endowment increases, but no reversal of the inequality between the sum of the peak amounts and the endowment occurs, then when the change is beneficial to society, each agent should be made at least as well off as he was initially (Sönmez, 1994). In conjunction with *efficiency*, it is clear that the three properties are equivalent. If consumption spaces are finite intervals, by itself, the latter implies *efficiency* (Ehlers 2002c). Then, a version of Theorem 4 can be obtained that does not involve *efficiency*.

$\mathcal{R}_\infty$ of preferences such that for each $i \in N$, $r_i(0)$ is finite, and let $\boldsymbol{\mathcal{E}^N_{\infty,fin}} \equiv \mathcal{R}^N_{\infty,fin} \times \mathbb{R}_+$.

Theorem 4 (Thomson, 1994b) *On the domain $\mathcal{E}^N_{\infty,fin}$, the uniform rule is the only selection from the no-envy and efficiency solution to be one-sided resource-monotonic.*

Without the domain restriction, rules other than the uniform rule become available. However, they all coincide with the uniform rule when there is too little of the commodity. A complement to Theorem 4 is Kesten (2006)'s observation that the uniform rule Pareto-dominates any selection from the *no-envy* solution to satisfy *one-sided resource-monotonicity* and the self-explanatory **resource continuity**. Also, in this theorem, *efficiency* can be replaced by *resource continuity* (Ehlers, 2011b).

If in Theorem 4, *no-envy* cannot be replaced by the *equal-division lower bound*, many selections from the *equal-division lower bound and efficiency* solution are *one-sided resource-monotonic.*[12]

The following variable-resource test on a rule is inspired by the literature on the adjudication of conflicting claims. Suppose that the social endowment is given in two installments, and consider the following two procedures to divide it: (ii) allocate the entire social endowment directly; (ii) divide the first installment first, x designating the division; for each $i \in N$, replace agent i's relation R_i by the relation R'_i defined by $a\ R'_i\ b$ if and only if $(a+x_i)\ R_i\ (b+x_i)$; then, divide the second installment in the economy that results. Thus, each agent receives his assignment in two parts. A solution satisfies **composition up** if the two procedures are equivalent. The constrained equal-distance rule satisfies *composition up* and a characterization of the rule is available in which the main axioms are *composition up* and a weak linearity requirement (Herrero and Villar, 2000).

No selection from the *equal division lower bound* solution (nor from the *no-envy* solution) satisfies the very weak version of *efficiency* according to which, if there is not enough of the commodity, each agent whose peak is zero is assigned zero, and *composition up*. The same impossibilities hold for a "dual" of *composition down*, pertaining to possible decreases in the social endowment. However, a hybrid of these composition properties, which reduces to *composition up* if there is not enough before and after the change in

[12]These solutions constitute a convex class.

the endowment, and to *composition down* if there is too much before and after, can be met. The uniform rule s the only selection from the *equal division lower bound* solution to be such that removing agents whose peak amount is zero should not affect the amounts assigned to the others and **hybrid composition**. A similar characterization holds if the *no-envy* requirement is imposed instead of the *equal-division lower bound* and *peak-only* is added. These results are due to Abizada and Chen.

5 Population-monotonicity and a variant

In this section and Section 7, we imagine changes in the population of agents and we study properties of solutions and rules pertaining to such changes. For that purpose, we generalize the model as follows. There is an infinite population of "potential agents", indexed by the natural numbers, $\mathbb{N}$. Let $\mathcal{N}$ denote the class of non-empty finite subsets of $\mathbb{N}$. An **economy** is defined by first specifying a set of agents $N \in \mathcal{N}$ and then a pair $(R, M) \in \mathcal{E}^N$. A **solution** is a correspondence defined on $\bigcup_{N\in\mathcal{N}} \mathcal{E}^N$ that associates with each $N \in \mathcal{N}$ and each $e \equiv (R, M) \in \mathcal{E}^N$ a non-empty subset of its feasible set, $X(e) \subset \mathbb{R}^N_+$. As before, a **rule** is a *single-valued* solution.

When preferences are monotone, the following is a natural expression of the solidarity idea, this time in response to population variations: starting from some economy and having applied a rule to it, imagine that some agents relinquish their rights on the resource, or that their claims are revealed to be invalid; we require of a rule that none of the agents who stay should be made worse off. (This requirement is formulated and studied by Thomson, 1983a,b, in the context of bargaining. For a survey of its applications, see Thomson, 1995b.) Here, if initially there is too little of the commodity, the departure of some of them is good news for the remaining agents, as it permits a Pareto improvement. Conversely, if initially there is too much of it, the departure of some of them is bad news for the remaining agents, and the natural requirement is that each of these agents should end up at most as well off as he was initially. To cover all cases, we require that upon the departure of some agents, the welfare of all remaining agents should be affected in the same direction.[13] Given $N, N' \in \mathcal{N}$ with $N' \subset N$, and $R \in \mathcal{R}^N$, the notation $R_{N'}$ designates the restriction of R to N', namely $(R_i)_{i\in N'}$.

[13]Chun (1986) studies it in the context of quasi-linear social choice.

Population-monotonicity: For each $N \in \mathcal{N}$, each $e \equiv (R, M) \in \mathcal{E}^N$, each $N' \subset N$, if $x \equiv \varphi(e)$ and $x' \equiv \varphi(R_{N'}, M)$, then either (i) for each $i \in N'$, $x_i \; R_i \; x'_i$, or (ii) for each $i \in N'$, $x'_i \; R_i \; x_i$.

None of the uniform, proportional, constrained equal-distance, and constrained equal-preferred-sets rules is *population-monotonic*. In fact, our next results are exact counterparts of the general incompatibilities of Proposition 1.

Proposition 2 (Thomson, 1995a) *(a) No selection from the* no-envy *solution is population-monotonic. (b) No selection from the* equal-division lower bound *solution is population-monotonic. (c) No solution satisfies* equal treatment of equals in welfare, peak-only, *and* population-monotonicity.

These results are disappointing, all the more so that they do not involve *efficiency*. On the other hand, if neither *no-envy* nor the *equal-division lower bound* is imposed, and only *efficiency* is insisted upon, *population-monotonicity* can be achieved by many rules, at least on a large subdomain of our base domain.

Note that as was the case when we considered variations in the social endowment, *population-monotonicity* allows for changes in the population that are quite disruptive, in the following sense: starting from an economy in which there is too much to divide, the arrival of additional agents may turn it into one in which there is too little. This is the underlying reason for the results of Proposition 2. It suggests weakening the requirement by applying it only when the change in the population is not so disruptive, that is, when the social endowment stays on the same side of the sum of the peak amounts. By analogy with the property that we proposed earlier concerning variations in the social endowment, we name this variant **one-sided population-monotonicity.**

It is easy to verify that the uniform, proportional, constrained equal-distance, and constrained equal-preferred-sets rules all satisfy this property. However, in conjunction with *no-envy*, *efficiency*, and a requirement that is quite mild since it is satisfied by most rules and solutions (we formulate it for solutions), only one rule remains acceptable, the uniform rule. The additional requirement is **replication invariance**: if an allocation is chosen for some economy, then for each $k \in \mathbb{N}$, its k-replica should be chosen for each k-replica of the economy (in a k-replica, each of the agents initially present has $k-1$ clones, and the social endowment is multiplied by k.)

Theorem 5 (Thomson, 1995a) *On the domain* $\bigcup_{N\in\mathcal{N}} \mathcal{E}^N_{fin}$*, the uniform rule is the only selection from the* no-envy *and* efficiency *solution to be* replication-invariant *and* one-sided population-monotonic*.*[14]

6 Welfare-dominance under preference-replacement and a variant

Next, we consider changes in the preferences of some agents and study the impact such changes may have on the others. Our conclusions exhibit some of the pattern of positive and negative results of the previous sections, but there are also significant differences.

Our first requirement is that when the preferences of some agents change, the welfare of all other agents should be affected in the same direction (Moulin, 1987, considers this property in the context of binary social choice; for a survey of the applications of the general idea, see Thomson, 1999). It is a fixed-population solidarity property, but since one of the auxiliary axioms in the main result pertaining to it is *replication-invariance*, we formulate it for the variable-population version of the model, using the notation of the previous section.

Welfare-dominance under preference-replacement: For each $N \in \mathcal{N}$, each $e \equiv (R, M) \in \mathcal{E}^N$, each $N' \subset N$, and each $R'_{N'} \in \mathcal{R}^{N'}$, if $x \equiv \varphi(e)$ and $x' \equiv \varphi(R'_{N'}, R_{N\setminus N'}, M)$, then either (i) for each $j \in N \setminus N'$, $x_j \; R_j \; x'_j$, or (ii) for each $j \in N \setminus N'$, $x'_j \; R_j \; x_j$.

For $|N| \leq 2$, the requirement has of course no force, and therefore, from here on, we assume $|N| \geq 3$. What are its implications? First, it is easy to see that the uniform rule, proportional, constrained equal-distance, and constrained equal-preferred-sets rules all violate it. However, violations extend much beyond these rules (compare to Propositions 1 and 2):

Proposition 3 (Thomson, 1997) *(a) No selection from the* no-envy *and* efficiency *solution satisfies* welfare-dominance under preference-replacement*. (b) No selection from the* efficiency *solution satisfies* equal treatment of equals in welfare*,* peak-only*, and* welfare-dominance under preference-replacement*.*

[14] The independence of *replication-invariance* from the other axioms in Theorem 5 is established by Klaus (2010).

Some selections from the *efficiency* solution do satisfy *welfare-dominance under preference-replacement* though, provided the domain is appropriately restricted. For instance, consider the domain of economies $(R, M) \in \mathcal{E}^N$ such that for each $i \in N$, $0 \; I_i \; M$. On this domain, the constrained equal-preferred-sets rule passes the test.

The equal-division rule, which is neither *resource-monotonic* nor *population-monotonic* (it does not even satisfy the one-sided versions of these properties), obviously satisfies *welfare-dominance under preference-replacement*. This rule reveals that *efficiency* cannot be dispensed with in the incompatibilities of Proposition 3. This is in contrast with the incompatibilities of either *resource-monotonicity* or *population-monotonicity* with *no-envy* (Propositions 1a and 2a), both of which hold even if *efficiency* is not imposed.

The *equal-division lower bound* is less restrictive than *no-envy* in the present context. Selections from the *equal-division lower bound and efficiency* solution satisfying *welfare-dominance under preference-replacement* can be defined on large subdomains of our basic domain.

In the examples used to establish Proposition 3, the change in preferences has, once again, the effect of turning an economy from one in which there is too much to one in which there is too little. Thus, here too, we propose to limit the range of application of our central axiom, *welfare-dominance under preference-replacement*, to situations where the social endowment stays on the same side of the sum of the peak amounts, thereby obtaining **one-sided welfare-dominance under preference-replacement**.

The uniform rule satisfies this property. In fact, many other rules do, including the proportional, constrained equal-distance, and constrained equal-preferred-sets rules. However, in the presence of *no-envy*, we have the following uniqueness result:

Theorem 6 (Thomson, 1997) *The uniform rule is the only selection from the* no-envy *and* efficiency *solution to satisfy* replication-invariance *and* one-sided welfare-dominance under preference-replacement.[15]

Are there selections from the *equal-division lower bound* (instead of *no-envy*) and *efficiency* solution other than the uniform rule satisfying *one-sided welfare-dominance under preference-replacement*? The answer is yes. Indeed, a large class of such rules exist.

[15]The independence of *replication invariance* from the other axioms in Theorem 6 is established by Klaus (2010).

7 Consistency and its converse

In this section, we pursue our study of the variable-population model.[16] We formulate two invariance properties that are meaningful for solutions, so we do not restrict attention to *single-valued* solutions. Let (R, Ω) be an economy with agent set $N \in \mathcal{N}$ and an allocation x that a solution chooses for it. We require that for each subgroup $N' \subset N$, and for the problem of dividing among them what they have collectively received at x, namely, $\sum_{N'} x_i$, the solution should choose the restriction of x to that subgroup. (For a survey of the applications of the general principle underlying the axiom, see Thomson, 2011.)

Consistency: For each $N \in \mathcal{N}$, each $e \equiv (R, M) \in \mathcal{E}^N$, each $x \in \varphi(e)$, and each $N' \subset N$, we have $x_{N'} \in \varphi(R_{N'}, \sum_{N'} x_i)$.

Bilateral consistency is the weakening of *consistency* obtained by applying it only to subgroups N' of two agents.

Many solutions are *consistent*. Examples are the *efficiency* solution and the *no-envy* solution (they have this property on arbitrary domains). So is their intersection (which is non-empty, as we know), *consistency* being preserved under arbitrary intersections. The uniform, proportional, constrained equal-distance, and constrained equal-preferred-sets rules are *consistent* too, and in fact, so are all parametric rules. However, neither the *equal-division lower bound and efficiency* solution nor the *equal-division core* (on its natural domain of definition) are.

The next requirement on a solution is the following. Consider an economy with agent set $N \in \mathcal{N}$ and a feasible allocation x. Suppose that for each two-agent subgroup N' of N, and for the problem of dividing between its members the sum of their components of x, the solution chooses the restriction of x to the subgroup. Then, the solution should choose x for the initial economy.[17]

[16] We could alternatively define an economy as a pair (R, M) in which preferences are defined on $[0, M]$. Now, given $N \in \mathcal{N}$ and such an economy (R, M), if we consider the problem of allocating some amount $M' < M$ among the members of $N' \subset N$, then for each $i \in N'$, we would replace R_i by $R_{i|[0,M']}$ and $R_{N'}$ by $(R_{i|[0,M']})_{i \in N'}$. We find it notationally easier to assume instead that preferences are defined over $\mathbb{R}_+$, and to permit solutions to depend on the whole preference relations. However, the results presented below hold for either specification.

[17] The weaker version of the property obtained by writing the hypotheses for each $N' \subset N$ is fact equivalent to our formulation.

Converse consistency: For each $N \in \mathcal{N}$, each $e \equiv (R, M) \in \mathcal{E}^N$, and each $x \in X(e)$, if for each $N' \subset N$ with $|N'| = 2$, we have $x_{N'} \in \varphi(R_{N'}, \sum_{N'} x_i)$, then $x \in \varphi(e)$.

The *efficiency* solution, the *no-envy* solution, their intersection (this property is also preserved under arbitrary intersections), and the uniform rule are *conversely consistent*. Here too, so are all parametric rules. On the other hand, the *equal-division core* is not. Neither is the *group no-envy* solution, which should not be a surprise since the hypotheses of *converse consistency* pertain only to two-agent groups whereas *group no-envy* is meaningful only when there are at least four agents. However, if *converse consistency* is weakened so as to make its hypotheses non-vacuous for four agents, the *group no-envy* solution is still disqualified.

Proposition 4 is key to our characterization, stated as Theorem 7, of the class of *consistent* subsolutions of the *no-envy and efficiency* solution satisfying in addition the mild following continuity requirement. Fix the population and their preferences. Consider a sequence of endowments, and for each element of the resulting sequence of economies, an allocation chosen by the solution. If the sequence of endowments and the sequence of allocations have limits, **social-endowment upper-semi-continuity** says that the limit allocation should be chosen by the solution for the limit economy.

Proposition 4 (Thomson, 1994c) *If a subsolution of the* no-envy and efficiency *solution satisfies* social-endowment upper-semi-continuity *and* consistency*, then it contains the uniform rule.*

Proposition 4 does not hold with *bilateral consistency* substituted for *consistency*. Indeed, any solution that coincides with the *no-envy and efficiency* solution on the subdomain of two-agent economies and is an arbitrary *social-endowment upper-semi-continuous* subsolution of the *no-envy and efficiency* solution for each greater cardinality, satisfies all the hypotheses of Proposition 4. Yet, the set of allocations it chooses for an economy with more than two agents need not contain its uniform allocation.

From Proposition 4 we derive complete characterizations of the classes of subsolutions of the *no-envy and efficiency* solution, or of the *equal-division lower bound and efficiency* solution, that satisfy both *social-endowment upper-semi-continuity* and *consistency*. For instance, we have:

Theorem 7 (Thomson, 1994c) *If a subsolution of the no-envy and efficiency solution, φ, satisfies social-endowment upper-semi-continuity and consistency, then it is given by the following recursive construction:*

1. *On the domain of two-agent economies, φ is a social-endowment upper-semi-continuous correspondence that contains the uniform rule.*
2. *Given $k \in \mathbb{N}$, assume that φ has been specified on the domain of k-agent economies, and let ψ be the correspondence defined on the domain of $(k+1)$-agent economies $e \equiv (R, M)$ by $\psi(e) \equiv \{x \in X(e)$: for each $N' \subset N$ with $|N'| = k$, $x_{N'} \in \varphi(R_{N'}, \sum_{N'} x_i)\}$. Then, on this domain, φ is any social-endowment upper-semi-continuous correspondence that contains the uniform rule and is contained in ψ.*

Here is another corollary of Proposition 4.

Theorem 8 (Thomson, 1994c) *The uniform rule is the only single-valued selection from the no-envy and efficiency solution, (or from the equal-division lower bound and efficiency solution), to be social-endowment upper-semi-continuous and bilaterally consistent.*

This characterization involves *social-endowment upper-semi-continuity* because we obtained it as a corollary of Proposition 4. However, in the presence of *single-valuedness*, a characterization of the uniform rule can be derived in which this continuity requirement does not appear (Dagan, 1996). It is based on Theorem 2 and the following lemma:

Lemma 1 (Dagan, 1996) *If a single-valued selection from the no-envy and efficiency solution is bilaterally consistent, then on the domain of two-agent economies, it is peak-only.*

Moreover, the uniform rule is the only *single-valued* selection from the *equal-division lower bound and efficiency* solution to be *replication-invariant* and *consistent*. Finally, it is the only *single-valued* selection from the *equal-division lower bound* solution to be *one-sided resource monotonic* and *bilaterally consistent*. This characterization, due to Ehlers (2002c)—Kesten (2006) imposes *no-envy* instead—generalizes one established by Sönmez (1994).

Because *consistency* is preserved under arbitrary intersections and the solution that associates with each economy its entire feasible set is *consistent*, it follows that for each solution, there is a smallest (in the sense of inclusion) *consistent* solution that contains it, (simply, the intersection of all the *consistent* solutions that contain it). It is its **minimal consistent enlargement** (Thomson, 1994a). Here is an application of the concept: the *minimal consistent enlargement* of the *equal-division lower bound and efficiency* solution is essentially (certain boundary allocations have to be excluded) the *efficiency* solution. Thus, in that case, the price of *consistency* is quite high: insisting on it forces us to give up our objective of fairness in distribution altogether.

The implications of *converse consistency* are striking if imposed on selections from the *equal-division lower bound and efficiency* solution and **anonymity** is imposed too. This says that the recommendation made by a rule should be covariant with renamings of agents:

Theorem 9 *The uniform rule is the only selection from the equal-division lower bound and efficiency solution to be single-valued, anonymous, and conversely consistent.*

Herrero and Villar (1998) characterize the constrained equal-distance rule as the only selection from the *efficiency* solution satisfying *equal treatment of equals in welfare*, *composition up* (Section 4), *consistency*, and a certain independence requirement.

8 Separability

Next, we formulate an invariance property of *single-valued* solutions pertaining to simultaneous changes in the preferences of the agents in some group and the social endowment. It says that if the total amount assigned to the complementary subgroup—the agents whose preferences have not changed—remains the same, then each of these agents' assignment should remain the same.

Separability: For each $N \in \mathcal{N}$, each pair $(R, M), (\bar{R}, \bar{M}) \in \mathcal{E}^N$, and each $N' \subset N$, if $R_{N'} = \bar{R}_{N'}$ and $\sum_{N'} \varphi_i(R, M) = \sum_{N'} \varphi_i(\bar{R}, \bar{M})$, then for each $i \in N'$, $\varphi_i(\bar{R}, \bar{M}) = \varphi_i(R, M)$.

The following characterizations of the uniform rule are available:

Theorem 10 (Chun, 2006) *(a) The uniform rule is the only selection from the no-envy and efficiency solution to be social-endowment upper semi-continuous and separable.*

(b) The uniform rule is the only selection from the equal-division lower bound and efficiency solution to be social-endowment upper semi-continuous, and separable.

In (a), *social-endowment continuity* can be replaced by *duplication invariance* (Klaus, 2006). Chun had stated (b) with the additional axiom of *replication invariance*, but this axiom is redundant (Klaus, 2006). *Social endowment continuity* can be dropped too if *duplication invariance* is added (Klaus, 2006).

Further characterizations of the uniform that do not involve *efficiency* but rely instead on several relational requirements are developed by Chun (2003).

9 Strategy-proofness

We turn to strategic issues. We are concerned about the fact that preferences are private information and that, in an attempt to get an assignment that he prefers, an agent may misrepresent his own. The set of agents is fixed here, so we return to our initial description of an economy as a list $R \in \mathcal{R}^N$ of preference relations. However, some of the results presented below rely on the possibility that the social endowment varies.

Given a rule φ, in the **direct revelation game form associated with $\boldsymbol{\varphi}$**, $\Gamma^{\varphi} \equiv (\mathcal{R}^N, \varphi)$, strategies are preferences in $\mathcal{R}$, and the outcome function is φ itself. Once preferences are specified, we have a game. We require that in this game, it should be a dominant strategy for each agent to announce his true preferences. Given $R \in \mathcal{R}^N$ and $i \in N$, we simplify the notation $(R_j)_{j \in N \setminus \{i\}}$ to R_{-i}.

Strategy-proofness: For each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R)\ R_i\ \varphi_i(R'_i, R_{-i})$.

Our main result here is the starting point of the literature under review:

Theorem 11 (Sprumont, 1991) *The uniform rule is the only selection from the efficiency solution to be anonymous and strategy-proof.*

This characterization has been much scrutinized. It has been shown that uniqueness still holds if *equal treatment of equals in welfare*, which in the presence of *efficiency* is weaker than *anonymity*, is substituted for *anonymity*. We also know that the following richness property of a subdomain of $\mathcal{R}$ guarantees that these axioms precipitate uniqueness. A domain of preferences defined on $\mathbb{R}_+$ satisfies **minimal richness** if each non-negative real is the peak amount of some admissible relation, and for each pair $\{x, y\} \subset \mathbb{R}_+$ with $x \neq y$, there is an admissible relation for which x is preferred to y and whose peak amount is between x and y. The following theorem collects the findings just described.

Theorem 12 (Ching, 1994a) *(a) The uniform rule is the only selection from the efficiency solution to satisfy equal treatment of equals in welfare and strategy-proofness.*

(b) (Mizobuchi and Serizawa, 2006) *On each minimally rich domain of preferences defined on* $\mathbb{R}_+$*, the uniform rule is the only rule satisfying these properties.*

Theorem 12a implies that on our base domain, the uniform rule is the only selection from the *no-envy and efficiency* solution to be *strategy-proof* (Sprumont, 1991; Ching, 1992). The uniform rule is still the only acceptable one if *strategy-proofness* is weakened by requiring that the peak amount of the announced relation be the peak amount of the true relation (Sakai and Wakayama, 2010b).

Suppose now that *efficiency* is dropped. **Non-bossiness** of a rule (Satterthwaite and Sonnenschein, 1981) says that if a change in an agent's preferences does not cause a change in his assignment, then it should not cause a change in any other agent's assignment. If *strategy-proofness* is weakened in the manner just mentioned and if *non-bossiness* is weakened in liked manner, the admissible rules form a lattice. Each rule in this lattice Pareto-dominates the equal division rule and is dominated by the uniform rule (Sakai and Wakayama, 2010a; these authors establish another structural result of this type when, in this list of axioms, the unrestricted version of *strategy-proofness* is imposed instead).

Consider now preferences that differ from single-peaked preferences only in that there is a non-degenerate interval of preferred consumptions: for each $i \in N$, there is an interval $[a_i, b_i] \subseteq \mathbb{R}_+$ such that for each $\{x_i, x'_i\} \subset [a_i, b_i]$, $x_i \; I_i \; x'_i$, and for each $\{x_i, x'_i\} \subset \mathbb{R}_+$, if $x'_i < x_i \leq a_i$ or $b_i \geq x_i > x'_i$, then

$x_i \; P_i \; x'_i$. A numerical representation of such a relation has a **plateau** (a peak is a degenerate plateau), so these preferences are called **single-plateaued**. The most natural way to adapt our rules to accommodate such preferences is to allow for multi-valuedness and to require that if two allocations are Pareto-indifferent and one of them is selected, then so should the other. This is the property of **Pareto-indifference**. (For *efficiency* to hold, this can only occur when all agents are simultaneously satiated.)

Let us call the **generalized uniform rule** the extension of the uniform rule obtained by the following minimal modification: if the sum of the left endpoints of the plateaus is smaller than the social endowment, apply the standard formula substituting in it the left endpoints instead of the peak amounts; similarly, if the sum of the right endpoints is greater than the social endowment, use the right endpoints in the formulas for that case; otherwise, select all the allocations at which all agents are satiated. Our earlier characterization of the uniform rule as the only selection from the *no-envy and efficiency* solution to be *strategy-proof* (the property needs to be restated to cover correspondences) extends to the single-plateaued domain with no difficulty, and it yields a characterization of the generalized uniform rule (Ching, 1992). In fact, uniqueness persists if *efficiency* is dropped (Ehlers, 2000). A family of rules emerges if no symmetry requirement is imposed either (Ehlers, 2002a). Returning to rules, when the distributional requirement is weakened to *equal treatment of equals in welfare*, and if the self-explanatory **agent-wise preference continuity** and the mild requirement that if there is not enough, any agent whose peak amount is zero is assigned nothing, are imposed, the uniform rule emerges once again (Ching, 2010).

When preferences are single-peaked but not necessarily continuous, a critical lemma in Sprumont's proof of Theorem 11a still holds, and his argument goes through (Weymark, 1999. Ching 1992, 1994a's proofs apply to these more general preferences with no modification.)

A sort of converse to the question answered by Theorem 11b is how large a domain can be, within some reference domain, for *strategy-proofness* to remain compatible with other prespecified properties. Say that **a domain of preferences is maximal within a reference domain for a particular list of properties** if it is contained in this reference domain, admits a rule satisfying these properties, but admits no such rule if a single preference relation from the reference domain is added.

The maximality question is most easily answered when preferences are defined on $\mathbb{R}_+$ and the social endowment is allowed to vary, so we consider the generalization of the model that allows such variations.

Theorem 13 *For the extension of the model obtained by allowing the social endowment to vary, the following hold:*

(a) (Ching and Serizawa, 1998) *There is a unique maximal domain containing the single-peaked domain and contained in the domain of continuous preferences, on which a selection from the efficiency solution exists that satisfies equal treatment of equals in welfare and strategy-proofness. It is the single-plateuaed domain.*

(b) (Mizobuchi and Serizawa, 2006) *In fact, for each minimally rich domain, there is a unique maximal domain containing it and contained in the domain of continuous preferences on which these properties are compatible. It is the single-plateaued domain.*

Any rule satisfying the properties of Theorem 13b is defined like the uniform rule when there is too little or when there is too much; if the social endowment lies between the sum of the left endpoints of the plateaus and the sum of the right endpoints, an example of a selection is obtained by giving to each agent the minimum of (i) his left endpoint augmented by some amount, this amount being the same for all agents, and (ii) his right endpoint.

A maximality result complementing Theorem 13 that does not involve *efficiency* is due to Ching and Serizawa (2009), and another pertaining to selections from the *no-envy* and *efficiency* solution satisfying *know-peak strategy-proofness* is due to Sakai and Wakayama (2012).

The next theorem also gives an answer to the maximality-of-domain question if the social endowment is kept fixed. **Equal treatment of equals in physical terms** is the requirement that if two agents have the same preferences, they should receive equal amounts. We do not state the formal definition of the **generalized single-plateaued** domain that appears in the next theorem, but only note that it is significantly broader than the single-plateaued domain: within a certain interval that depends on the social endowment, preferences are convex (thus intervals of local satiation—ledges—are allowed), and outside of this interval, there are no monotonicity requirement, only upper bounds on welfare. Part (b) involves a weak continuity requirement which we will not state explicitly, simply noting that it is only meaningful for rules required to satisfy *peak-only*, one of the axioms in the theorem. Let us refer to it as **continuity***.

Theorem 14 *For the version of the model in which the social endowment is kept fixed, the following hold:*

(a) (Massó and Neme, 2001) *There is a unique maximal domain containing the single-peaked domain and contained in the domain of continuous preferences on which efficiency, equal treatment of equals in physical terms, and strategy-proofness are compatible. It is the domain of generalized single-plateaued preferences.*

(b) (Massó and Neme, 2004) *There is a maximal domain (the authors identify such a domain) on which efficiency, continuity*, peak-only, and strategy-proofness are compatible.*

Strategy-proofness is a requirement of robustness to individual misrepresentation. However, a rule may be *strategy-proof* but such that a group of agents can coordinate their misrepresentation, each member of the group ending up at least as well off as he would have been otherwise, and at least one of them ending up better off. If a rule is not subject to this type of manipulation, it is **group strategy-proof**. A rule is **weakly group strategy-proof** if no group of agents can jointly misrepresent their preferences so that each of its members ends up better off. The uniform rule is *group strategy-proof.*

The definition allows any group to enter into such agreements. Obviously, the larger a group, the less likely will its members be able to coordinate their strategies, so let us consider the minimal form of this property (beyond *strategy-proofness*), one that pertains to manipulation by either one agent or two agents. In addition, let us require joint misrepresentations to be robust to double-crossing. The resulting property, **double-crossing–proof pairwise strategy-proofness**, together with **unanimity** (the requirement that if there is an allocation at which each agent receives his peak amount, then it should be chosen,) imply *group strategy-proofness* (Serizawa, 2006).

Misrepresenting their preferences does not exhaust the strategic opportunities that a group of agents have, however, because after doing so and receiving their assignments, they may be able to carry out transfers among themselves ("ex-post" transfers) so that each of them ends up at least as well off as he would have been otherwise, and at least one of them ends up better off. The uniform rule is not robust to such manipulations (the property is discussed in Thomson, 2014b).

A weaker form of it is the following: consider a group of agents and suppose that a subgroup misrepresent their preferences. There may be ex-post transfers among the members of the group that make all the members of the subgroup better off and the others at least as well off. If that is never the case, the rule is **bribe-proof** (Schummer, 2000). The next proposition reveals the strength of the stronger version obtained by letting groups of any size to engage in such manipulation. It involves a property of **weak replacement monotonicity**: if after the preferences of an agent change, he is assigned at least as much as initially, and either he did not receive his peak amount initially or does not receive it after the change, then each of the other agents should be assigned at most as much as initially.

Proposition 5 (Massó and Neme, 2007) *A rule is strongly bribe-proof if and only if it is a selection from the efficiency solution that satisfies weak replacement monotonicity and strategy-proofness.*

A maximal domain result for the existence of rules that satisfy *equal treatment of equals in welfare* and *bribe-proofness* is established by Wakayama (2013).

A large class of selections from the *efficiency* solution satisfy *one-sided welfare-dominance under preference-replacement* and *strategy-proofness* (Barberà, Jackson, and Neme, 1997) but a characterization is available. Each of these rules—let us refer to them as **BJN rules**—is defined by a sequential process. The process is parameterized by an initial allocation and a list of adjustment formulas for allocations. These formulas should satisfy certain monotonicity and independence conditions, which we omit.

Theorem 15 (Barberà, Jackson, and Neme, 1997) *A selection from the efficiency solution satisfies one-sided welfare-dominance under preference-replacement and strategy-proofness if and only if it is a BJN rule.*

The joint implications of several relational requirements have been investigated by several authors. First, characterizations of the uniform rule have been based on the *equal-division lower bound* together with any one of the following combinations: (i) *one-sided resource-monotonicity* and *consistency*; (ii) *one-sided resource-monotonicity* and *converse consistency*; (iii) *consistency*, *one-sided population-monotonicity*, and *replication invariance* (note that in none of these results is *efficiency* invoked) (Sönmez, 1994).

The next result concerns a minor variant of the model: consumption spaces are bounded above, as we have previously discussed, but the bounds may vary across agents. A **two-path–based rule** for population N is defined as follows: in $\mathbb{R}^N_+$, there are two continuous and monotone paths from the origin to the vector of the upper bounds for the members of N. Given any preference profile, as the social endowment increases from 0 to the sum of the peak amounts, the vector of assignments first follows one of these paths—let us call it the "path for excess demand"—until one agent receives his peak amount; this agent's assignment remains there for a while; the assignments to the others follow the projection of the path onto their own assignment space until a second agent reaches his peak amount; his assignment remains there for a while; the assignments to the others follow the projection of the path onto their own assignment space. The process continues in this manner until each agent has reached his peak amount. Symmetrically, as the social endowment decreases from the sum of the upper bounds (the maximal value it can take) to the sum of the peak amounts, the vector of assignments follows the other path—let us call it the "path for excess supply"—and its successive projections onto subspaces, each agent dropping out when he receives his peak amount. In the variable-population model, a pair of paths should be given for each $N \in \mathcal{N}$ (one for excess demand and one for excess supply), but the paths should be related: given two populations N and N' such that $N' \subset N$, the path for excess demand for N, when projected onto $\mathbb{R}^{N'}$, should be the path for excess demand for N', a similar projection requirement being imposed on the collection of paths for excess supply. We have the following characterization:

Theorem 16 (Moulin, 1999) *The two-path–based rules are the only selections from the efficiency solution to be one-sided resource monotonic, strategy-proof, and consistent*

A simplified proof is due to Ehlers (2002b). Ehlers (2002c) also pursues the analysis of the implications of these axioms when alternative forms of the monotonicity axiom are imposed but *efficiency* is dropped. (For selections from the *efficiency* solution, they are all equivalent.)

10 Manipulation

When a rule is found to be manipulable, a natural follow-up task is to investigate how manipulable it is. Given a rule φ, let $\boldsymbol{E(\Gamma^{\varphi}, R^0)}$ be the set of (pure strategy Nash) equilibria of the direct revelation game that results when the true preference profile is $R^0 \in \mathcal{R}^N$, and $\boldsymbol{E_X(\Gamma^{\varphi}, R^0)}$ be the corresponding set of equilibrium allocations.

The next proposition states that for several of the rules that we have discussed, the game has a unique equilibrium allocation. This allocation is none other than the uniform allocation for the true preferences.

Theorem 17 (Thomson, 1990) *Let Γ^{φ} be the direct revelation game form associated with a rule φ that may be one of the following: the symmetrized proportional rule,*[18] *the constrained equal-distance rule, or the constrained equal-preferred-sets rule. Let $R^0 \in \mathcal{R}^N$. Then, $E_X(\Gamma^{\varphi}, R^0) = \{U(R^0)\}$.*

A result analogous to Theorem 17 holds for any rule satisfying a certain **responsiveness** property. The large class of "equal-sacrifice rules" are *responsive*: such a rule is defined by first specifying, for each preference relation R_0 and each amount, a measure of the sacrifice imposed on an agent with preferences R_0 if he is assigned that amount. This sacrifice function should take the value 0 at the peak amount and it should be strictly monotonic as consumption moves away from the peak amount. Then, for each preference profile, the allocation at which sacrifices are as equal as possible is selected.

Next, we consider rules that satisfy *efficiency*, *peak only*, and the following axioms (which are meaningful because of the *peak-only* requirement). First are the self-explanatory **peak continuity**, **own-peak monotonicity** and its stronger version, **strong own-peak monotonicity**. Next is **others-oriented peak monotonicity**, which says that if an agent's peak increases, each of the other agents should be assigned at most as much as initially. A conclusion can also be reached about the set of strong Nash equilibria. Let $\boldsymbol{E^*_X(\Gamma^{\varphi}, R^0)}$ be the set defined as $E_X(\Gamma^{\varphi}, R^0)$ except that the requirement that no joint deviation be profitable to a group of agents is added.

Theorem 18 (Bochet and Sakai, 2007) *Let Γ^{φ} be the direct revelation game form associated with a selection φ from the efficiency solution that satisfies equal treatment of equals in welfare, peak only, peak continuity, and others-oriented peak monotonicity. Let $R^0 \in \mathcal{R}^N$. Then, $E_X(\Gamma^{\varphi}, R^0) \cap P(R^0) = E^*_X(\Gamma^{\varphi}, R^0) = \{U(R^0)\}$. Further, if φ is strictly own-peak monotonic, then $E_X(\Gamma^{\varphi}, R^0) = \{U(R^0)\}$.*

[18] However, the manipulation game associated with the proportional rule itself does not have that property.

11 Implementation

Next, we check the properties of solutions that are relevant for (Nash) *implementability* of a solution, starting with the following necessary property. Consider some profile of preferences and an allocation chosen for it by the solution. Now consider a second preference profile with the property that for each agent, the set of allocations that he now finds at most as desirable as that allocation (using his preferences over allocations induced from his preferences over consumptions in the natural way) contains the corresponding set for his initial preferences (the sets these allocations constitute for the relations R_i and R'_i in the formal definition below are denoted $L(R_i, x_i)$ and $L(R'_i, x_i)$). Then, the allocation should still be chosen by the solution for the new profile (Maskin, 1999).[19]

Invariance under monotonic transformations: For each $\{R, R'\} \subset \mathcal{R}^N$ and each $x \in \varphi(R)$, if for each $i \in N$, $L(R'_i, x) \supseteq L(R_i, x)$, then $x \in \varphi(R')$.

On any domain on which these solutions are well-defined, the *no-envy* solution is *invariant under monotonic transformations*, and so are the *equal-division lower bound* solution and the *group no-envy* solution. The following proposition tells us that essentially, any *invariant under monotonic transformations* subsolution of the no-envy solution contains the uniform rule. It involves the somewhat technical but very weak requirement of **closed-valuedness**: for each profile $R \in \mathcal{R}^N$, $\varphi(R)$ should be a closed set.

Proposition 6 (Thomson, 1990, 2010) *If a subsolution of the no-envy and efficiency solution is closed-valued and invariant under monotonic transformations, then it contains the uniform rule.*

The following characterization of the uniform rule is a direct consequence of Proposition 6. It does not involve *closed-valuedness*, since this property is implied by *single-valuedness*.

Theorem 19 (Thomson, 1990, 2010) *The uniform rule is the only single-valued selection from the no-envy and efficiency solution to be invariant under monotonic transformations.*

[19]The property is usually referred to as "Maskin monotonicity". A few paragraphs down, we also refer to as "invariance" properties two other properties that have been important to the understanding of implementability.

A solution satisfies **no veto power** if whenever an allocation is most preferred by all agents but possibly one, it is chosen. When there are at least three agents, a solution is *implementable* if and only if it is *invariant under monotonic transformations* and satisfies *no veto power* (Maskin, 1999). Unfortunately, on the domain under consideration, the *no-envy* solution violates *no veto power*: let $N \equiv \{1, \cdots, n\}$ and $R \in \mathcal{R}^N$ be such that $p(R_1) = \cdots = p(R_{n-2}) = 0$ and $p(R_{n-1}) = p(R_n) = M$. Let $x \equiv (0, 0, \ldots, M, 0)$. Then, agents 1 through $n-1$ prefer x to each other allocation. If the *no-envy* solution satisfied *no veto power*, x would be envy-free. But at x, agent n envies agent $n-1$. (The same example shows that the uniform rule also violates *no veto power*.)

Thus, *implementability* of the *no-envy* solution cannot be derived from invoking Maskin's theorem. However, the *no-envy* solution satisfies **strong invariance under monotonic transformations** (Danilov, 1992), a strengthening of *invariance under monotonic transformations* (we skip the statement) that is sufficient for *implementability*, as shown by Danilov (1992) for solutions defined on domains with a finite number of alternatives, and by Yamato (1992) on more general domains. The same is true of the *equal-division lower bound* solution.

The intersections of these solutions with the *efficiency* solution, which is also *strongly invariant under monotonic transformations*, are not *strongly invariant under monotonic transformations*—this property is not preserved under intersections—so their *implementability* cannot be decided by the abovementioned results. However, another property of this type (we omit the definition as well) is shown by Sjöström (1991) to be necessary and sufficient for *implementability*. The property is satisfied by the *no-envy and efficiency* solution, the *equal-division lower bound and efficiency* solution, as well as by the *equal-division core* and the *group no-envy* solution on their natural domains of definition. Summarizing, we have:

Theorem 20 (Thomson, 1990, 2010) *The* no-envy *and* efficiency *solution, the* equal-division lower bound *and* efficiency *solution, and on their natural domains of definition, the* equal-division core *and the* group no-envy *solution, are* Nash-implementable.

Further results on Nash implementation are due to Doghmi and Ziad (2008a,b), whose approach is to reformulate *no veto power*.

A rule is **securely implementable**, if there is a game form such that (i) for each preference profile, there is a profile of dominant strategies in the resulting game whose corresponding outcome if the allocation chosen by the rule for the profile, and (ii) this allocation is in fact obtained at each Nash equilibrium of this game (Saijo, Sjöström and Yamato, 2007). The following theorem reveals that this requirement is very strong. It involves the self-explanatory requirement on a rule that it should **make the same choice for the two profiles of extremists**, namely (i) the profile of preferences at which each agent's peak amount is 0 and (ii) the profile at which each agent's peak amount is the social endowment. This minimal requirement is met by most of the rules we have discussed.

Theorem 21 (Bochet and Sakai, 2010) *If a rule is securely implementable, and makes the same choice for the two profiles of extremists, then it is constant. Moreover, if a selection from the efficiency solution is one-sided resource monotonic, consistent, and securely implementable, then it is a sequential priority rule.*

These authors also consider other notions of implementability, such as implementability in coalition-proof Nash equilibrium (Bernheim, Whinston, and Peleg, 1987), and variants. Interestingly, for this model, *strategy-proofness* does not imply *invariance under monotonic transformations* (Klaus and Bochet, 2013).

12 Extensions of the model

(1) **Indivisibilities.** Here, the commodity is not infinitely divisible; instead, it comes in integer amounts, and consumptions also have to be integers. Discreteness creates conceptual and technical complications. For instance, properties such as *equal treatment of equals*, in physical terms or in welfare, cannot be met any more: in an economy with two agents whose preferences are the same and the dividend is an odd integer, it has to be violated. Allowing solution mappings to be correspondences is a way out, but as we have already pointed out, dealing with correspondences renders more delicate the formulation of relational solidarity requirements pertaining to changes in resources, populations, or preferences.

This model is studied by Moulin (1999), who extends to it his characterization of the family of two-path–based rules he had established for the continuous case (Theorem 16). Herrero and Martínez (2011) derive a characterization of a family of rules on the basis of *strategy-proofness* and *consistency*. Discreteness is also discussed in the context of probabilistic rules in (4) below.

(2) **Individual endowments.** Instead of having to divide a social endowment, each agent has his "personal" endowment, the issue being to redistribute these endowments. An **economy with individual endowments** is a pair $(R, m) \in \mathcal{R}^N \times \mathbb{R}_+^N$ where $R \in \mathcal{R}^N$ is, as before, a profile of single-peaked preferences, and $m \in \mathbb{R}_+^N$ is a profile of individual endowments. An **allocation for $(\boldsymbol{R}, \boldsymbol{m})$** is a list $x \in \mathbb{R}_+^N$ such that $\sum x_i = \sum m_i$.

This model is more complex. The solutions and rules that we have encountered earlier can be extended, but often in more that one way. However, properties pertaining to the individual endowments can be formulated, and they help in distinguishing between them. Characterizations of certain extensions of the uniform rule are available (Thomson, 1995c; Klaus, 1997a,b; 2001; Klaus, Peters and Storcken, 1997, 1998, Barberà, Jackson, and Neme, 1997; Moreno, 2002.).

(3) **Mixed ownership.** Now, not only does each agent have his own endowment, but in addition, there is a social endowment. A **mixed ownership economy** is a triple $(R, m, T) \in \mathcal{R}^N \times \mathbb{R}_+^N \times \mathbb{R}$ such that $\sum m_i + T \geq 0$, where (R, m) is as in (2), and T (unrestricted in sign) is interpreted as an amount of the commodity that has to be delivered to the outside world, if negative, or recovered from it, if positive. An **allocation for $(\boldsymbol{R}, \boldsymbol{m}, \boldsymbol{T})$** is a list $x \in \mathbb{R}_+^N$ such that $\sum x_i = \sum m_i + T$. (The inequality $\sum m_i + T \geq 0$ is imposed to guarantee the existence of feasible allocations.) For this extension of the model (Thomson, 1995c), a simple expression of the consistency principle, by contrast to what is the case in the model with only individual endowments.[20] Other properties of allocation rules in this context are studied by Thomson (1995c) and Herrero (2002).

(4) **Probabilistic rules.** So far, we have limited our attention to deterministic rules. Here, as in (1), the commodity is only available in integer amounts and assignments also have to take integer values, and allow a rule to choose a probability distribution over allocations. We will treat as one two rules that choose distributions with the same marginals; indeed, from the viewpoint of the agent's welfare, marginals are what matters.

[20] Thomson (1995c) uses the phrase "generalized economy".

There remains to specify how agents evaluate lotteries over assignments. One possible formulation is to assume that they are equipped with von-Neumann Morgenstern utility functions and compare assignments in terms of their expected values. This formulation is studied by Sasaki (1997). His main result is a characterization of the natural extension to the probabilistic framework of the uniform rule—let us call it the **probabilistic uniform rule**—as the only selection from the *ex ante efficiency* solution satisfying (a form of) *anonymity* and *strategy-proofness*. See also Kureishi (2001) who proves uniqueness with **equal treatment of equals in utilities**, which says that two agents with the same utility functions, up to a positive linear rescaling, should be assigned the same utilities; and Ehlers and Klaus (2003) who relate the different notions of *efficiency* for this model.

A probabilistic rule is **probabilistically same-sided** if, when the sum of the peak amounts is at least as large as the social endowment, then for each agent, the rule assigns weight 1 to the interval from 0 to his peak amount, and otherwise, for each agent, it assigns weight 1 to the interval from his peak amount to the social endowment. **Equal treatment of equals in marginals** says that if two agents have the same utility functions, the rule should assign to them the same marginals. The only probabilistic rule satisfying *probabilistic same-sidedness*, *equal treatment of equals in marginals*, and *weak group strategy-proofness* is the uniform probabilistic rule (Hatsumi and Serizawa, 2009).

To each preference relation over deterministic assignments can be associated an (incomplete) preference relation over lotteries by means of stochastic dominance comparisons: a lottery is preferred to another if for each amount, it places at least as much weight on the weak upper contour set at that amount as the second one does, and for at least one amount, it places a greater weight. When a property is rewritten in terms of these relations, let us add the prefix "sd" (for "stochastic dominance") to its name.[21] The only selection from the *sd-efficiency* (equivalently *ex-post efficiency*) and *sd-no-envy* correspondence satisfying *sd-strategy-proofness*, is the uniform probabilistic rule (Ehlers and Klaus, 2003). However, uniqueness is not preserved if *anonymity* is imposed instead of *sd-no-envy* (compare to Theorem 11).

[21]For instance, an allocation is "sd-envy-free" if for each pair $\{i, j\}$, and each $x_i \in [0, M]$ agent i's probabilistic assignment places on the set of amounts that he finds at least as desirable as x_i a weight that is at least as large as the weight that agent j's probabilistic assignment would place.

(5) **Game of migration.** The commodity is available at several locations and cannot be transferred between locations; agents are free to move however. At each location, the commodity is divided by applying a particular rule. In deciding whether to move, an agent takes into account the fact that the distribution at his new location will be adjusted to accommodate him, but ignores the inducement to move that others might then have. This is the model formulated by Gensemer, Hong, and Kelly (1996, 1998). Related stability questions are studied by Bergantiños, Massó, Moreno de Barreda, and Neme (2013) under somewhat different conditions.

A distribution of the population is in equilibrium if no agent would benefit from moving under the assumptions just listed. Allowing first the social endowment to differ between locations, these authors establish general theorems stating the non-existence of an equilibrium for each list of rules (one rule for each location) that are (i) selections from the *efficiency* solution satisfying *strategy-proofness* or (ii) selections from the *no-envy* solution satisfying *strategy-proofness*, or (iii) selections from the *no-envy and efficiency* solution (Gensemer, Hong, and Kelly, 1996). Suppose next that the social endowment is the same at all locations, and that the same rule is applied at all locations. Then, equilibria also fail to exist if the rule is any one of the equal division, proportional, sequential priority, and uniform rules (Gensemer, Hong, and Kelly, 1998).

Related questions are addressed by Kar and Kıbrıs (2008). Given a division rule to be operated at each location, they inquire about the possibility of assigning agents to locations in such a way that *efficiency* is obtained overall. If preferences are symmetric (when two consumptions are symmetric with respect to the peak amount, they are indifferent), such a matching rule exists as long as the division rule is a selection from the *efficiency* solution. Otherwise, and if the division rule is a selection from the *efficiency* solution that is *one-sided resource monotonic*, *strategy-proof*, and *consistent*, then no matching rule exists such that the pair delivers *efficiency* overall.

(6) **When the agent set is partitioned into demanders and suppliers.** Here, we add to the model an *a priori* partition of the agent set into demanders and suppliers, or buyers and sellers, of a good. Each of them has a single-peaked preference relation over $\mathbb{R}_+$. An economy is simply a list of preference relations for buyers and sellers. How much each agent receives can be decided in two steps: the volume of trade between the two sides; then, for each side, how much each agent on that side receives. The second part,

if treated separately, is formally identical to the problem formulated in the initial sections of this essay.

This problem is formalized and studied by Kıbrıs and Küçükçenel (2009), who propose and characterize a family of rules that can be seen as two-step extensions of the uniform rule. For each economy, the volume of trade is calculated using two functions, β and σ, whose domain of definition is the class of subsets of the agent set and whose range is $\bar{\mathbb{R}}_+$; each is such that if the subset contains only agents of one type, it takes the value 0. Then, if demand exceeds supply, the volume of trade is the median of $\beta(N)$, aggregate demand (the sum of the peaks of the buyers), and aggregate supply (the sum of the peaks of the buyers); if supply exceeds demand, the volume of trade is the median of $\sigma(N)$, aggregate demand, and aggregate supply. Then, for each side, the uniform rule is used among the agents on that side. The rules defined in this manner are the only selections from the **side-wise no-envy** (*no-envy* is applied to each side separately) and *efficiency* solution to be *strategy-proof* and such that the volume of trade only depends on aggregate demand and aggregate supply.

Richer network constraints are incorporated in the model formulated by Bochet, Ilkilic, and Moulin (2013). These authors characterize a rule that can be seen as an generalization of the uniform rule, and characterize it on the basis of *efficiency*, a form of *equal treatment of equals*, and *strategy-proofness*. Chandramouli and Sethuraman (2012) address the issue of its *group strategy-proofness*. Bochet, Ilkilic, Moulin, and Sethuraman is a study of a related model (2012). Szwagrzak (2011, 2012a,b,c) further generalizes and unifies them. He characterizes families of rules based on various relational principles of fairness and robustness under strategic behavior.

(7) **Relating the uniform rule to certain solutions to bargaining games.** Given $N \in \mathcal{N}$ and $(R, M) \in \mathcal{E}^N$, think of the comprehensive hull in $\mathbb{R}^N_+$ of the set of efficient allocations as a bargaining game. (Note that this mapping only takes into account the peak amounts of the preference relations.) We ask whether there are solutions to bargaining games that, when applied to this bargaining game, always yields the uniform allocation of (R, M). The answer is positive: the Nash and lexicographic egalitarian solutions are such solutions, as proved by de Frutos and Massó (1995) and Otten, Peters and Volij (1996). The authors exploit these connections to bargaining theory to develop characterizations of the uniform rule. Their strategy is to transcribe axiom systems on which characterizations of these

two bargaining solutions had been based to be applicable to the model under study here.

(8) **Several commodities.** The following is an ℓ-commodity extension of our base model. Say that agent i's preferences R_i over $\mathbb{R}^\ell_+$ are **commodity-wise single-peaked** if he has a most preferred bundle $p(R_i) \in \mathbb{R}^\ell_+$ and for each $x_i \in \mathbb{R}^\ell_+$, he prefers to x_i each bundle in the "box" in $\mathbb{R}^\ell_+$ whose sides are parallel to the axes and that has $p(R_i)$ and x_i as vertices. This is equivalent to saying that, for each commodity k, fixing his consumptions of all commodities but commodity k, his preferences over his consumption of commodity k are single-peaked in the sense we gave to this term in the one-commodity case; moreover, his peak amount for that commodity is independent of his consumptions of the others: it is $p_k(R_i)$. (Such preferences need not be convex.) His preferences may in addition be **separable**: two bundles that differ in how much they contain of a particular commodity, when complemented with the same bundle of the other $\ell - 1$ commodities, are ranked in the same way irrespective of this complementary bundle.[22]

Mainly strategic questions have been asked for this model. Having more than one commodity makes a considerable difference. In fact, the classical domain of general equilibrium theory is now included. It is well-known that on this subdomain, and even if convexity of preferences is imposed, no selection from the *efficiency* solution that satisfies standard distributional requirements is *strategy-proofness*. Let us then weaken *efficiency* to **commodity-wise efficiency**: for each commodity separately, either each agent receives at most his peak amount, or each agent receives at least his peak amount. As we have seen, if $\ell = 1$, this "same-sidedness" property is equivalent to *efficiency*. If $\ell > 1$, *commodity-wise efficiency* is not sufficient for *efficiency* but it remains necessary. The rule obtained by applying the uniform rule commodity by commodity—let us call it the **commodity-wise uniform rule**—satisfies a number of properties satisfied by the uniform rule in the one-commodity case. However, it is the only selection from the *no-envy* solution to be *commodity-wise efficiency* and *strategy-proof* (Amoros, 2002, proves this fact for two agents, and Adachi, 2010, for arbitrarily many agents). Also, for two agents, if *no-envy* is replaced by *equal treatment of equals in physical terms*, it is the only admissible rule. For $|N| > 2$, and if preferences are strictly convex and commodity-wise single-peaked and peak-separable, it

[22]Cho and Thomson (2013) review the various ways in which the single-peakedness property can be generalized to more than one commodity.

is the only rule to satisfy **unanimity** (here, this says that if the sum of the satiation bundles is equal to the endowment, each agent should get his satiation bundle), *equal treatment of equals in physical terms*, *strategy-proofness*, and *non-bossiness* (Morimoto, Serizawa, and Ching, 2013). This characterization remains true on the subdomain of continuous, strictly convex, and separable preferences. A maximality-of-domain result is also available (Cho and Thomson, 2013). A recent study, which allows for both divisible as well as indivisible resources, is by Erlanson and Szwagrzak (2013). They derive a family of rules defined by maximizing a separable concave function a polyhedral extension of the efficient set.

The following results (Anno and Sasaki, 2013) provide additional information on the structure of the set of *strategy-proof* rules. Given any such rule, there is a *strategy-proof* rule that is undominated (in welfare terms) in the space of all *strategy-proof* rules that dominates it. Thus, there is a *strategy-proof* rule that is undominated in the space of all *strategy-proof* rules that meet the *equal-division lower bound*. For $|N| = 2$, the commodity-wise uniform rule is *strategy-proof* and it is undominated among all *strategy-proof* rules satisfying the *equal-division lower bound* and the requirement that if an agent switches to a different preference relation but his peak amount remains the same, his assignment should not be affected.

Taking another step away from our base model, let us allow all convex but possibly satiated preferences. A "Walrasian allocation with equal slacks" is defined like a Walrasian allocation except that (i) prices are unrestricted in sign, and (ii) budgets are augmented by the same amount, an equilibrium requirement being imposed on this amount as well as on prices (Mas-Colell, 1992). Some agents may maximize their preferences in the interior of their augmented budget sets. The "slack" that interior maximization generates for these agents is distributed to the others. The "Walrasian solution with slacks" coincides with the standard Walrasian solution when preferences are monotone, and its equal-income version coincides with the uniform rule when specialized to the one-commodity case. Amoros (1999) applies it to economies with multi-dimensional single-peaked preferences and characterizes it along the lines of Theorem 1.

A production model with a linear technology is studied by Kıbrıs and Tapkı (2011), who establish a characterization of a multi-dimensional version of the uniform rule along the lines of Theorem 12a.

(9) **Introducing participation constraints.** In the variant of the model

we consider next, each agent has an **outside option** and is not forced to participate in the distribution, if when he participates, he is assigned an amount that he finds less desirable than his outside option. For each $i \in N$, let $\underline{x}_i$ be the amount of the resource to the left of his peak amount that agent i finds indifferent to his outside option (if there is such an amount) and $\bar{x}_i$ the amount to the right of his peak amount that is indifferent to it (if there is such an amount; neither amount needs exist of course). To model outside options, it suffices to replace agent i's preference relation by one in which he is indifferent between any two amounts to the left of $\underline{x}_i$, and any such amount is indifferent to any amount to the right of $\bar{x}_i$. The analysis of a variant of the model, a counterpart of one formulated for a related public good problem (Cantala, 2004), is carried out by Bergantiños, Massó, and Neme (2012b). Adding participation constraints complicates matters significantly. Nevertheless, these authors obtain characterizations of a version of the uniform rule analogous to results obtained for our base model. Bergantiños, Massó, and Neme (2012a) perform a similar analysis of a model in which each agent's consumption is bounded above, with the bounds possibly differing from agent to agent. This line of research is developed further in Bergantiños, Massó, and Neme (2012c).

(10) **When disposal is possible.** As in (9), let us assume that for each agent, there is an upper bound on how much he can consume, but this time let us allow free disposal. Such allocation problems are analogous to claims problems (O'Neill, 1982). They are formulated and studied by Kıbrıs (2003).

Another formulation is when for each agent, (i) there is a minimal amount below which the commodity is not useful to him, (ii) there is an upper bound beyond which it is not useful either, and (iii) preferences are monotone increasing between these bounds. It is proposed and studied by Manjunath (2012). Due to the lower bound, this model can be thought of as a hybrid between one in which resources are infinitely divisible and one in which indivisibilities are present.

Appendix: sketches of selected proofs.

Several proofs are based on understanding what it means for an allocation to be envy-free and efficient without being the uniform allocation. In sketching them below, we only discuss the case $\sum p(R_i) \geq M$ although the case $\sum p(R_i) \leq M$ is not always exactly symmetric (it is not symmetric when preferences are defined over $\mathbb{R}_+$ instead of over a finite interval).

Observation: Let $N \in \mathcal{N}$ and $e \equiv (R, M) \in \mathcal{E}^N$ be such that $\sum p(R_i) \geq M$. Let $x \in FP(e)$ be such that $x \neq U(e)$. At $U(e)$, an agent consumes less than his peak amount only if he consumes at least as much as anybody else. Thus, since $x \in P(R)$ and $x \neq U(e)$, **there are $\boldsymbol{i, j \in N}$ such that (i) $\boldsymbol{x_i < p(R_i)}$** and $x_j > x_i$, and since $x \in F(e)$, agent i does not envy agent j, so that in fact (ii) $\boldsymbol{x_j \geq r_i(x_i) > x_i}$.

In the proof of each of the theorems below, φ is a rule assumed to satisfy the hypotheses.

Proof of Theorem 2: Here, N is fixed. Suppose that there are $N \in \mathcal{N}$ and $e \equiv (R, M) \in \mathcal{E}^N$ with $\sum p(R_i) > M$ such that $x \equiv \varphi(e) \neq U(e)$. Let i and j be as in the Observation. Let $R'_i \in \mathcal{R}$ be such that $p(R'_i) = p(R_i)$ and $x_j \; P'_i \; x_i$. By *peak only*, $x \in \varphi(R'_i, R_{-i}, M)$. Yet, agent i now envies agent j.

Proof of Theorem 4: Here, N is fixed. A first step, whose proof we skip, is that φ is *resource-continuous*. Suppose that there are $N \in \mathcal{N}$ and $e \equiv (R, M) \in \mathcal{E}^N$ with $\sum p(R_i) \geq M$ such that $x \equiv \varphi(e) \neq U(e)$. Let i and j be as in the Observation. Let the social endowment decrease to 0. As this occurs, since $\varphi \in P$, and by *one-sided resource-monotonicity*, which applies, what each agent receives never increases, and it goes to 0. By *resource continuity*, it does so continuously. Thus, there is $M' < M$ such that $x'_j \in]x_i, r_i(x_i)[$, where $x' \equiv \varphi(R, M')$. Since $x'_i \leq x_i < p(R_i)$, $[x'_i, r_i(x'_i)]$ contains $[x_i, r_i(x_i)]$ so that $x'_j \in]x'_i, r_i(x'_i)[$. Thus, at x', agent i envies agent j, in violation of $\varphi \in F$.

Proof of Theorem 5: Suppose that there are $N \in \mathcal{N}$ and $e \equiv (R, M) \in \mathcal{E}^N$ with $\sum p(R_i) > M$ such that $x \equiv \varphi(e) \neq U(e)$. Let i and j be as in the Observation and $g \equiv r_i(x_i) - x_i$. Let $k \in \mathbb{N}$ be such that $\frac{x_j}{k} < g$. First, we consider the k-replica of e. By *replication invariance*, φ chooses this k-replica of x for the k-replica of e. We now introduce additional "clones" of agent j, one at a time. Let us consider the first step. Since $k \sum_N p(R_\ell) + p(R_j) > kM$, *one-sided population monotonicity* applies, and each of the $k|N|$ agents initially present should end up at most as well off as he was initially. Since $\varphi \subseteq P$, this means that each of these agents ends up with at most as much as at the k-replica of x. Thus, calling x' the allocation chosen for the new economy, $x'_i \leq x_i$, so that $[x'_i, r_i(x'_i)]$ contains $[x_i, r_i(x_i)]$. Since the new agent is a clone of agent j, by *no-envy* and *efficiency*, he and each of the other agents of type j receive the same amount, x'_j. Thus, if what the newcomer received came entirely from the k agents of type j initially present, each of these agents would contribute at most $\frac{x'_j}{k}$. Since $x'_j \leq x_j$ and $\frac{x_j}{k} < g$, this contribution would not be sufficient to bring x'_j below x'_i (to "jump over" the gap $[x'_i, r_i(x'_i)]$). Since $x'_i \leq x_i$, $x'_j \geq r(x'_i)$. This inequality holds a fortiori if agents other than the agents of type j also contributed to the amount received by the newcomer. We introduce a second additional clone of agent j, and repeat this reasoning to conclude that the common consumption of the agents of type j (there are $k+2$ of them now) remains to the right of the gap of the previous step. Since the common consumption of the agents of type i cannot increase, the gaps can only enlarge, and the total amount that goes to the agents of type j is unbounded above as $k \to \infty$. This contradicts the fact that, after the initial replication, the amount to divide is fixed.

Proof of Theorem 6: Suppose that there are $N \in \mathcal{N}$ and $e \equiv (R, M) \in \mathcal{E}^N$ with $\sum p(R_i) > M$ such that $x \equiv \varphi(e) \neq U(e)$. Let i and j be as in the Observation. Let e' be obtained by replicating e once. By *replication invariance*, $\varphi(e')$ is obtained by replicating x once. Let i' be the clone of agent i and let $\tilde{R}_{i'}$ be a new relation for him such that $p(\tilde{R}_{i'}) = p(R_i)$ and $x_j \;\; \tilde{P}_{i'} \;\; x_i$. Since the sum of the peak amounts remain the same, *one-sided welfare-dominance under preference-replacement* applies, and the welfares of all $2|N| - 1$ other agents should be affected in the same direction by the change. Since $\varphi \in P$, this means that either (i) each receives at least as much as he did in e', or (ii) each receives at most as much. Because agents i and i' have equal peak amounts, and $\varphi \subseteq FP$, they receive equal amounts. Thus, if (i) holds, *each of the* $2|N|$ *agents* receives at least as much as he did in e', and if (ii) holds, *each of them* receives at most as much. By feasibility, in either case, each receives the same amount as in e'. But then, in e', agent i' envies agent j.

Proof of Lemma 1: Let $N \in \mathcal{N}$ with $|N| = 2$—say $N \equiv \{1, 2\}$—and $(R, M), (R', M') \in \mathcal{E}^N$ be such that $p(R) = p(R')$ and $M = M'$. We introduce agents 3 and 4 with preferences $R_3, R_4 \in \mathcal{R}$ such that $p(R_3) = p(R_1)$ and $p(R_4) = p(R_2)$, and we double the social endowment. Let $x \equiv \varphi(R_1, R_2, R_3, R_4, 2M)$. Since $\varphi \in FP$, $x_1 = x_3$ and $x_2 = x_4$. Thus, $x_1 + x_2 = x_3 + x_4 = M$. By *consistency*, $(x_1, x_2) = \varphi(R_1, R_2, x_1 + x_2) = \varphi(R, M)$ and $(x_3, x_4) = \varphi(R_3, R_4, x_3 + x_4)$. Thus, $\varphi(R, M) = \varphi(R_3, R_4, M)$. The same argument applies to (R', M): $\varphi(R', M) = \varphi(R_3, R_4, M)$. Thus, $\varphi(R, M) = \varphi(R', M)$.

Theorem 2 and Lemma 1 together imply that any selection from FP satisfying *consistency* coincides with the uniform rule for two-agent economies. Now, let $N \in \mathcal{N}$ with $|N| > 2$, $e \equiv (R, M) \in \mathcal{E}^N$, and $x \equiv \varphi(e)$. Since φ is *bi-*

laterally consistent, for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \varphi(R_{N'}, \sum_{N'} x_i) = U(R_{N'}, \sum_{N'} x_i)$. Thus, x satisfies the hypotheses of *converse consistency* for U. Since U is *conversely consistent*, $x = U(e)$ (This is what is called the Elevator Lemma in Thomson, 2011).

Proof of Theorem 11: Here, N is fixed. Ching's proof is based on two lemmas describing how an agent's consumption is affected if his preferences change, the preferences of all other agents being held fixed. First is **own-peak monotonicity**: if an agent's preferences change, what he receives should not move in the opposite direction of his peak amount. Second is **uncompromisingness**: suppose that for some profile, an agent receives more than his peak amount. Then, if his preferences change but his new peak amount is at most as large as his initial assignment, he should still receive the same amount (a symmetric statement holding if initially, he receives less than his peak amount).

Suppose $\varphi \neq U$. Let $k \in N$ be an agent whose peak amount is the largest and $N' \subseteq N$ be the set of agents who have his preferences. At least one $i \in N \setminus N'$ receives an amount different from the amount the uniform rule would assign to him. We change agent i's preferences to R_k. Applying the two properties in turn, we deduce that for the new profile, φ and U still make different choices. We then identify one agent who receives an amount different from the amount the uniform rule would assign to him. We change his preferences to R_k too. We proceed in this manner until all agents have preferences R_k, and reach the conclusion that for that profile, the two rules still differ. However, they both satisfy *equal treatment of equals in welfare* and *efficiency*, and for this profile of identical preferences, they should both pick equal division.

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Index