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THE NON-SUBSTITUTION THEOREM: MULTIPLE PRIMARY FACTORS AND THE COST FUNCTION APPROACH

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ABSTRACT

The non-substitution theorem asserts that the choice of technique is independent of patterns of final demand when efficiency prevails as to the use of a single primary factor, say labor, while the asserted constancy of the input-output table no longer holds when more than one kind of primary factor is involved. No definite answer has yet been given as to whether the commodity price vector is determined independently of final demand patterns when there are multiple primary factors of production. This paper shows that the "unit cost = price" relation uniquely determines the commodity price relatively to a given factor price vector. The proof of such a commdity price is provided by the use of Tarski's fixed point theorem without recourse to topology.

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By Kiyoshi Kuga *

1 Introduction

The end and aim of the Non-Substitution Theorem lies in the justification of constancy of an input-output table, and it asserts

(i) that the choice of technique is independent of patterns of final demand when efficiency prevails as to the use of a single primary factor, say labor [1)], and

(ii) that the commodity price vector is uniquely determined by the production structure, independently of final demand patterns [2)].

These two findings have a solid foundation in the static context, but the dynamic non-substitution theorems allow various degrees of freedom (Mirrlees (1969), Burmeister and Kuga (1970), Stiglitz (1970), Samuelson (1991)) in that the input-output table including the capital input coefficients is stable in the long-run along the wage-profit frontier (Morishima(1964, Chapter 3)).

It is well known that the asserted constancy of the input-output table no longer holds when more than one kind of primary factor is involved (Koopmans (1951, p.153)). Yet, no question has been posed so far as to whether the commodity price vector is determined independently of final demand patterns when there are multiple primary factors of production. The purpose of this paper is to examine the question by exploiting the cost function framework.

The outline of this paper is as follows. Section 2 describes the framework of the model and proves the existence of a price that equals the unit cost. Section 3 deals with its uniqueness relative to a given factor price. Section 4 develops the discussion.

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[1)] See Arrow (1951), Georgescu-Roegen (1951), Koopmans (1951b), Samuelson (1951).

[2)] See Morishima (1964, Chapter 3), Mirrlees (1969), Burmeister and Kuga (1970). Stiglitz (1970, p.544) writes to the effect that the non-substitution theorem is equivalent to the statement that the set of equations $p = c(p)$ has at most one solution p^*.

2 The Model

Each industry indexed by $j \in \mathbb{N} \overset{\text{def}}{=} \{1, 2, \ldots, n\}$ has a no-joint, concave, constant returns to scale technology. The input consists of intermediate outputs from industries in $\mathbb{N}$ and primary factors indexed by $k \in \mathbb{M} \overset{\text{def}}{=} \{1, 2, \ldots, m\}$. We begin with the description of the unit cost functions. The domain of each cost function is $\mathbb{P} \times \mathbb{W}$, where $\mathbb{P} \overset{\text{def}}{=} \mathbb{R}^n_+$ consists of prices of n commodities $p = (p_1, \ldots, p_n)$, and $\mathbb{W} \overset{\text{def}}{=} \mathbb{R}^m_+ \setminus \{0\}$ consists of factor prices $w = (w_1, \ldots, w_m)$ of m primary factors. Our first concern is to examine the preliminary question of whether or not to the set of unit cost functions $c^j : \mathbb{P} \times \mathbb{W} \to \mathbb{R}^n_+$ of industry $j \in \mathbb{N}$, there exists a certain price vector $p = (p_1, p_2, \ldots, p_n) \in \mathbb{P}$ given $w = (w_1, w_2, \ldots, w_m) \in \mathbb{W}$ satisfying

$$p_j = c^j(p, w), \ j \in \mathbb{N}. \tag{1}$$

We may write (1) as $p = c(p, w)$ and call p that satisfies (1) **the unit cost equating price relative to** w. To ensure the existence of such a price $p \in \mathbb{P}$ to (1), we assume

(A.1) Given $w \in \mathbb{W}$, there is a certain $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n) \in \mathbb{P}$, depending upon w, satisfying
$\bar{p}_j \geq c^j(\bar{p}, w)$ for all $j \in \mathbb{N}$. [3)]

(A.2) Given $w \in \mathbb{W}$, c^j is non-decreasing, i.e., $c^j(p, w) \leq c^j(p', w)$ for $p \leq p'$ $(p, \ p' \in \mathbb{P})$ and $j \in \mathbb{N}$.

We are now ready to state

Theorem 1. *Suppose an economy satisfies (A.1) and (A.2). Then for each $w \in \mathbb{W}$, there is a certain $p \in \mathbb{P}$ satisfying $p_j = c^j(p, w)$, $j \in \mathbb{N}$.*

Proof of Theorem 1.

A subset $P \subset S \times S$ is a partial order in S if the following three conditions are met: (i) $(x, x) \in P$ for $x \in S$, (ii) $[(x, y) \in P \ \wedge \ (y, x) \in P] \Rightarrow x = y$, (iii) $[(x, y) \in P \ \wedge \ (y, z) \in P] \Rightarrow (x, z) \in P$. A function $f : P \to P$ is called non-decreasing if $u \leq v \Rightarrow f(u) \leq f(v)$. Exploiting these concepts, we use the well-known Tarski's fixed point theorem: [4)]

Tarski's Fixed Point Theorem (Tarski (1955)): Let $(P, \leq)$ be a conditionally complete partially ordered set and f a non-decreasing function from P into P. If P has two elements a and b such that $a \leq f(a) \leq f(b) \leq b$, then there exists an element $c \in P : c = f(c)$.

A partially ordered set in which every non-empty subset that is bounded above has a least upper bound is conditionally complete (Alexander (1965, p.178)). $\mathbb{P} = \mathbb{R}^n_+$ is conditionally complete with respect to the usual partial order $\leq$. We show that the required conditions are met in our case. Our (A.2) is the non-decreasingness in Tarski's Fixed Point Theorem. We use (A.1) for the existence of an upper bound $\bar{p}$ satisfying $c(\bar{p}, w) \leq \bar{p}$. The lower bound in our case may be taken to be zero. Then by (A.2), it follows that $0 \leq c(0, w) \leq c(\bar{p}, w)$. In sum, we have $0 \leq c(0, w) \leq c(\bar{p}, w) \leq \bar{p}$. □

[3)] This assumption may correspond to Assumption 3 in Morishima (1964, p.59).
[4)] For an exposition of the theorem, the reader is referred to Abian (1965, p.182).

Remark 1. *Fujimoto (1986) opened the way to applying Tarski's theorem to the solvability of an infinite dimensional non-linear Leontief system. In his case as well as in our Theorem 1, only the property of the partial order is involved, and no topology such as continuity is used. See Abian (1965, p.182) and Birkhoff (1967, p.115) for the explanation of Tarski's fixed point theorem.*

3 Uniqueness of the Unit Cost Equating Price Relative to a Factor Price

To proceed further we assume for each $j \in \mathbb{N}$

(A.3) c^j is homogeneous of degree one in $(p, w) \in \mathbb{P} \times \mathbb{W}$, i.e., $c^j(\mu p, \mu w) = \mu \cdot c^j(p, w)$ for $\mu \in \mathbb{R}_{++}$,

(A.4) c^j is concave and continuously differentiable in the interior of $\mathbb{P} \times \mathbb{W}$,

(A.5) The primary factors are indispensable, i.e., $c^j(p, w) > 0$ for $p \in \mathbb{P}$ and w in the interior of $\mathbb{W}$ for $j \in \mathbb{N}$, and furthermore $w \cdot A_w(p, w) > 0$ (component-wise strictly positive) for each (p, w) in the interior of $\mathbb{P} \times \mathbb{W}$.

$A_w(p, w)$ above is the factor input coefficient matrix that appears in the final part of the following exposition of notation:

$$p = (p_1, p_2, \ldots, p_n), \ q = (q_1, q_2, \ldots, q_n), \ w = (w_1, w_2, \ldots, w_m)$$

$$a_{ij}(p, w) \stackrel{\text{def}}{=} \frac{\partial c^j(p, w)}{\partial p_i}, \quad i, j \in \mathbb{N}$$

$$A(p, w) \stackrel{\text{def}}{=} \begin{bmatrix} a_{11}(p, w) & a_{12}(p, w) & \ldots & a_{1n}(p, w) \\ a_{21}(p, w) & a_{22}(p, w) & \ldots & a_{2n}(p, w) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(p, w) & a_{n2}(p, w) & \ldots & a_{nn}(p, w) \end{bmatrix}$$

$$\tilde{a}_{kj}(p, w) \stackrel{\text{def}}{=} \frac{\partial c^j(p, w)}{\partial w_k}, \ j \in \mathbb{N}, \ k \in \mathbb{M}$$

$$A_w(p, w) \stackrel{\text{def}}{=} \begin{bmatrix} \tilde{a}_{11}(p, w) & \tilde{a}_{12}(p, w) & \ldots & \tilde{a}_{1n}(p, w) \\ \tilde{a}_{21}(p, w) & \tilde{a}_{22}(p, w) & \ldots & \tilde{a}_{2n}(p, w) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{m1}(p, w) & \tilde{a}_{m2}(p, w) & \ldots & \tilde{a}_{mn}(p, w) \end{bmatrix}.$$

From the former part of (A.5), we may take that each component of the equality between unit cost and price is strictly positive. Let us now suppose, given w in the interior of $\mathbb{W}$, that there existed two different equilibrium price vectors, $p, \ q \in \mathbb{P}$ satisfying

$$\begin{aligned} p_j &= c^j(p, w) > 0, \ j \in \mathbb{N} \\ q_j &= c^j(q, w) > 0, \ j \in \mathbb{N}. \end{aligned}$$

Then we have for w in the interior of $\mathbb{W}$

$$\begin{aligned} p &= c(p,w) > 0 \text{ by (A.5)} \\ p &= p \cdot A(p,w) + w \cdot A_w(p,w) > 0 \text{ by (A.3) and (A.4)} \\ p(I - A(p,w)) &= w \cdot A_w(p,w) > 0 \text{ by (A.5).} \end{aligned}$$

Thus the matrix $I - A(p,w)$, with its off-diagonals being non-positive, has a non-zero nonnegative solution p for a positive vector $w \cdot A_w(p,w)$. Thus we have [5)]

$$(I - A(p,w))^{-1} \geq 0.$$

On the other hand, we also have

$$\begin{aligned} q &= c(q,w) = c(q - p + p, w) \\ &\downarrow \text{ by the concavity and differentiability (A.4)} \\ &\leq c(p,w) + (q-p) \cdot A(p,w) \\ &\downarrow \; p = c(p,w) \\ q - p &\leq (q-p) \cdot A(p,w) \\ &\downarrow \\ (q-p) \cdot (I - A(p,w)) &\leq 0 \\ &\downarrow \; (I - A(p,w))^{-1} \geq 0 \\ q - p &\leq 0. \end{aligned}$$

We have thus obtained $q \leq p$. By a symmetric argument, we also obtain $p \leq q$. We may therefore summarize these results in the following uniqueness theorem

Theorem 2. *Suppose an economy satisfies (A.1) - (A.5). Then for each w in the interior of $\mathbb{W}$, there exists a unique positive cost equating price $p \in \mathbb{P}$ satisfying $p = c(p,w)$.*

Remark 2. *Our Theorem 2 can be regarded as an extension of Theorem 1 of Stiglitz (1970, p.544). He assumes that each cost function c^i does not contain p_i as its arguments, and that there is only one kind of primary factor, labor. Our Theorem 2 is more general in that c^i does contain p_i as its arguments, in addition to the structure in which more than one kind of primary factor is involved.*

Remark 3. *Fujimoto(1980) deals with the non-substitution theorem targeting an extension of of Stiglitz's (1970) as well as of the uniqueness theorem of Nikaido (1968, p.170). However, his cost function approach does not cover the multiple primary factor case. See also Fujimoto (1987).*

Remark 4. *Samuelson (1991) deals with the case in which m primary factors are involved. He leaves open the values of an own rate of interest $1+r$, and ratios of factor prices $w_2/w_1, \ldots, w_m/w_1$,*

[5)] See Nikaido (1968), Chapter 2.

and asserts determinacy of the input coefficient matrix. This must be qualified by the fact that the other equilibrium conditions (tastes, etc.) then impinge on the cost and price relations. Our Theorem 2 shows that the unit cost equating price is uniquely determined relative to $(w_1, \ldots, w_m)$. But as will be discussed in the final part of our Section 4, the factor price vector depends upon the then existent combination of the primary factors.

Remark 5. *Kurz and Salvadori (1994) discusses the case in which wages are zero and prices are not uniquely determined. They conclude, however, that the direct and indirect use of labor leads finally to the uniqueness result. Their result may be compared with the non-substitution theorem of Morishima (1964).*

4 Discussion

The static non-substitution theorem ceases to be true when more than one scarce primary factor is required (Koopmans (1951, p.153)). Our theorem still asserts that a unique choice of activity prevails relatively to a given factor price w. When the number of the primary factors exceeds unity, the factor price w will be determined endogenously by the general equilibrium system. Even then there is an advantage in knowing that an input-output matrix is determined by the factor price vector w alone, and that the commodity price vector is a unique function of the factor price.

Morishima (1964, Chapter 3) developed a dynamic non-substitution theorem in which there is a trade-off between the real wage rate and the uniform rate of profit. His theorem asserts that the input-output and capital input coefficients table of a dynamic Leontief system is uniquely determined when either the real wage rate or the uniform rate of profit is given. Two comments are in order.

There is the question of whether or not our formulation is formally equivalent to the logical development of Morishima's dynamic non-substitution theorem. The answer is no. With the linear homogeneity of cost functions ((A.3)), we could have followed his line of argument by restricting all our variables (p, w) to the unit simplex, and could have obtained a uniqueness result almost along his line of argument (p.63). But such a uniqueness result holds only relatively to the simplex normalization, whereas our theorem asserts the uniqueness result relative to each absolute level of $w \in \mathbb{W}$.

The important question that remains is the economic interpretation. Morishima develops a world in which an equal rate of profit prevails in a dynamic economy, while our economy can be either static or dynamic. When the factor price combination in our economy is in a steady state, then our theorem can be embedded into his world. When a factor price vector yields different rates of return commoditywise,[6)] our theorem can be given a temporary equilibrium interpretation.

Mirrlees (1969)'s Theorem 1 (p.69) asserts that an efficiency price is unique up to scale in the timeless context of no-joint, constant-returns-to-scale economy. Mirrlees' Theorems 2-4 are attempts to embed dynamic economies into the timeless world of his Theorem 1. His method of examination can be extended to the case of many primary factors, but it turns out that the factor prices are

[6)] See also Burmeister and Kuga (1970) for the non-substitution theorem for the case in which rates of returns are sectorwise different.

allowed then to vary only along the efficiency price contour. By contrast, in our approach, the unit cost equating price is a function of factor prices that run freely over the whole interior of $\mathbb{W}$.

Mirrlees (1969, p.76) poses a question asking under what circumstances prices in one period are determined independently of consumer demand. In a sense, our Theorem 2 is a possible answer to his problem posing.

One word on the general equilibrium theory. Our cost function approach reduces commodity prices to a function of factor prices, while there are markets for commodities and rented factors. How do we interpret these markets which reach equilibrium only through changes in factor prices? In such a case, the received income consists of factor incomes, and it pays for expenditures. The demand supply relation of commodities is balanced by output adjustment through the input-output mechanism, and only the rented factor market remains to be balanced through the variation of factor prices. In this way, the Walras law for the excess demand functions for factors comes to hold. We may express this line of reasoning mathematically as

$$
\begin{aligned}
& w \cdot [A_w(p(w), w) \cdot (I - A(p(w), w))^{-1} \cdot f(p(w), w) - L] \\
= \; & p(w) \cdot f(p(w), w) - w \cdot L \\
= \; & 0
\end{aligned}
$$

where $p(w)$ denotes the unit cost equating price relative to w, $f(p(w), w)$ the final demand vector, and L is a supply vector of primary factors. In this way, we obtain a link between a large scale model and a small scale one, such as Uzawa (1961)'s two sector growth model. Data of the capital intensity in various times and places may provide a key to evaluating the workings of large scale as well as miniature models. The interested reader may refer to Kuga (1967) for such values.

When there are only two kinds of primary factors, say capital and labor, the input-output table including the static capital and labor coefficients remains stable as long as the wage rental ratio does. In the temporary equilibrium, the short-run wage-rental ratio is determined by the then existent capital labor ratio as in the economy of Uzawa (1961). When the capital labor ratio (and therefore the wage rental ratio) remains constant in the steady state, our input-output commodity table keeps a stable value as in the case of Morishima (1964) 's dynamic non-substitution theorem.

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