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NECESSARY AND SUFFICIENT MOMENT CONDITIONS FOR THE GARCH(r,s) AND ASYMMETRIC POWER GARCH(r,s) MODELS*

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Necessary and Sufficient Moment Conditions for the GARCH(r, s) and Asymmetric Power GARCH(r, s) Models*

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Abstract

Although econometricians have been using Bollerslev's (1986) GARCH (r, s) model for over a decade, the higher-order moment structure of the model remains unresolved. The sufficient condition for the existence of the higher-order moments of the GARCH (r, s) model was given by Ling (1999a). This paper shows that Ling's condition is also necessary. As an extension, the necessary and sufficient moment conditions are established for Ding, Granger and Engle's (1993) asymmetric power GARCH (r, s) model.

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1 Introduction

A process ε_t is said to follow Bollerslev's (1986) general autoregressive conditional heteroskedasticity (GARCH (r, s)) model if it satisfies the equations:

$$\varepsilon_t = \eta_t \sqrt{h_t}\,, \tag{1.1}$$

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{s} \beta_i h_{t-i}, \tag{1.2}$$

where $\alpha_0 > 0, \alpha_i \geq 0 \; (i = 1, \cdots, r)$ with at least one $\alpha_i > 0$, and $\beta_i \geq 0 \; (i = 1, \cdots, s)$. When $s = 0$, the GARCH(r,s) model (1.1)-(1.2) reduces to Engle's (1982) autoregressive conditional heteroskedasticity (ARCH (r)) model. Both the ARCH and GARCH models have been applied widely in the econometric and finance literature to model volatility (see Bollerslev, Chou and Kroner (1992), Bollerslev, Engle and Nelson (1994), and Li, Ling and McAleer (1999) for recent reviews).

Bollerslev (1986) showed that the necessary and sufficient condition for the second-order stationarity of model (1.1)-(1.2) is

$$\sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{s} \beta_i < 1. \tag{1.3}$$

Bougerol and Picard (1992) provided the necessary and sufficient condition for the strict stationarity and ergodicity of model (1.1)-(1.2). Ling and Li (1997) proved that, under (1.3), there exists a unique $\mathcal{F}_{t-1}$-measurable and second-order stationary solution to model (1.1)-(1.2), and that the solution is strictly stationary and ergodic, where $\mathcal{F}_t$ is a $\sigma-$field generated by $\{\eta_t, \eta_{t-1}, \cdots\}$. Thus, the second-order moment structure of model (1.1)-(1.2) is now complete.

However, the higher-order moment structure of model (1.1)-(1.2) remains unresolved. When $s = 0$, Milhϕj (1985) gave the necessary and sufficient condition for the existence of the $2m$-th moment of the ARCH model. Bollerslev (1986) provided the necessary and sufficient condition for the existence of the $2m$-th moment of the GARCH(1,1) model, and the necessary and sufficient condition for the fourth-order moments of the GARCH(1,2) and GARCH(2,1) models. Karanasos (1999) and He and Terasvirta (1999b) gave conditions for the existence of the fourth moment of model (1.1)-(1.2). He and Terasvirta (1999b) state that their condition is necessary and sufficient. From the proof in Karanasos (1999), it can be seen that his condition is necessary, but it is not clear whether the condition is also sufficient.

Based on Theorem 2.1 in Ling and Li (1997) and Theorem 2 in Tweedie (1988), Ling (1999a) provided a sufficient condition for the existence of the $2m$-th moment of model (1.1)-(1.2). Ling's result does not need to assume that the GARCH (r, s) process starts infinitely far in the past with finite $2m$-th moment, as is required in Bollerslev (1986) and He and Terasvirta (1999b), and has a far simpler form as compared with Milhϕj (1985), Karanasos (1999), and He and Terasvirta (1999b).

In this paper, it is shown that the sufficient condition in Ling (1999a) is also necessary. As an extension, the necessary and sufficient moment condition is established for the asymmetric power GARCH (r, s) model proposed by Ding, Granger and Engle (1993).

2 Main Results

Denote $A^{\otimes m} = A \otimes A \otimes \cdots \otimes A$ (m factors), where $\otimes$ is the Kronecker product. Our result for the GARCH (r,s) model is as follows.

Theorem 2.1. *The necessary and sufficient condition for* $E\varepsilon_t^{2m} < \infty$ *is* $\rho[E(A_t^{\otimes m})] < 1$, *where* $\rho(A) = \min\{|\text{eigenvalues of a matrix } A|\}$, *and*

$$A_t = \left(\begin{array}{cccc|ccc} \alpha_1 \eta_t^2 & \cdots & & \alpha_r \eta_t^2 & \beta_1 \eta_t^2 & \cdots & \beta_s \eta_t^2 \\ & I_{(r-1)\times(r-1)} & & O_{(r-1)\times 1} & & O_{(r-1)\times s} & \\ \hline \alpha_1 & \cdots & & \alpha_r & \beta_1 & \cdots & \beta_s \\ & O_{(s-1)\times r} & & & & I_{(s-1)\times(s-1)} & O_{(s-1)\times 1} \end{array} \right),$$

in which $I_{r\times r}$ *is the* $r \times r$ *identity matrix.*

The sufficiency comes from Theorem 6.1 of Ling (1999a). The proof of necessity is given in Section 4. Ling (1999a) showed that, when $r = s = 1$, the condition in Theorem 2.1 is equivalent to that in Bollerslev (1986).

Recently, He and Terasvirta (1999b) investigated the necessary and sufficient condition for the fourth moment of the GARCH (r, s) model. It is instructive to examine their condition carefully. By assuming that $Eh_t^2 < \infty$, He and Terasvirta (1999b) derive their conditions by the following equation:

$$Eh_t^2 = \alpha_0^2 + 2\alpha_0\gamma_1 Eh_t + \gamma_2 Eh_t^2 + 2\sum_{l<m} E(c_{l,t-l}c_{m,t-m}h_{t-l}h_{t-m}), \tag{2.1}$$

where γ_1 and γ_2 are some suitable constants and $c_{i,t-i} = \alpha_i\eta_{t-i}^2 + \beta_i$. Because of the assumption that $Eh_t^2 < \infty$, their condition is clearly necessary rather than sufficient[1]. Karanasos (1999) used an equation similar to (2.1), and hence his condition is also necessary rather than sufficient.

As in Engle (1982) and Bollerslev (1986), He and Terasvirta (1999b) assumed that the GARCH (r, s) process starts infinitely far in the past with finite $2m$-th moment. In fact, this assumption is unnecessary. It is interesting to clarify this point. For simplicity, we consider only the case $m = 2$ and the ARCH(1) model, i.e.

$$\varepsilon_t = \eta_t\sqrt{h_t} \ \text{ and } \ h_t = \alpha_0 + \alpha_1\varepsilon_{t-1}^2. \tag{2.2}$$

Ling and Li (1997) proved that, when $0 < \alpha_1 < 1$, there exists a unique second-order stationary solution to model (2.2), and the solution has the following expansion in mean square:

$$\varepsilon_t = \eta_t\sqrt{h_t} \ \text{ and } \ h_t = \alpha_0 + \alpha_0\sum_{j=1}^{\infty}\prod_{i=1}^{j}(\alpha_1\eta_{t-i}^2). \tag{2.3}$$

The structure of this solution actually is a transformation of a series of i.i.d. random variables η_t, and does not involve the initial value. The sufficient condition in Ling

[1]Moreover, the proof of necessity in He and Terasvirta (1999b) is incomplete as some steps in their proof, such as the one given below, do not hold in general, but require additional restrictions which were not established by the authors. Note that it cannot be claimed that $\sum_{i=1}^{\infty} A^i < \infty$ follows from $\sum_{i=1}^{\infty} A^i\xi < \infty$, where A and ξ are some suitable matrix and vector, respectively, except for some special A and ξ as in Ling (1999b) and in this paper. It is not clear whether He and Terasvirta's (1999b) necessary condition for the existence of the fourth moment, namely $\lambda(\Gamma) < 1$, holds generally because, for example, without other arguments, their (A.21) converging does not ensure that $\sum_{i=m-l+1}^{k-1}\Gamma^{i-(m-l+1)}$ converges and hence does not ensure that $\lambda(\Gamma) < 1$. A possible solution is to find a vector with all elements positive and to use the ideas established in this paper to prove that $\lambda(\Gamma) < 1$. However, in He and Terasvirta (1999b), e_{p-1} in (A.21) includes some zero elements, and Γ is quite complicated. Thus, such a vector with all elements positive would not be easy to establish, even if it were to exist.

(1999a), i.e. the sufficient condition in Theorem 2.1, is the moment condition for solution (2.3). Since the solution is unique, the ε_t from model (2.2) is almost surely the same as (2.3) if $E\varepsilon_t^2 < \infty$. The method of proof in Ling (1999a) used the drift criterion in Tweedie (1988). This method has been a common tool in nonlinear time series [see Chan and Tong (1985), Feigin and Tweedie (1985), and Ling (1999a)]. In the next section, we will use this method to analyze the asymmetric power GARCH (r,s) model proposed by Ding, Granger and Engle (1993).

3 Asymmetric GARCH (r,s) model

A special asymmetric power GARCH (r,s) model in Ding, Granger and Engle (1993) [see also He and Terasvirta (1999a)] is given by:

$$\varepsilon_t = \eta_t \sqrt{h_t}\,, \tag{3.1}$$

$$h_t^{\delta/2} = \alpha_0 + \sum_{i=1}^{r} \alpha_i (|\varepsilon_{t-i}| - \gamma \varepsilon_{t-i})^{\delta} + \sum_{i=1}^{s} \beta_i h_{t-i}^{\delta/2}, \tag{3.2}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ $(i = 1, \cdots, r)$ with at least one $\alpha_i > 0$, $\beta_i \geq 0$ $(i = 1, \cdots, s)$, $\delta \geq 0$ and $|\gamma| < 1$. The primary feature of the asymmetric power GARCH (r, s) model (3.1)-(3.2) is the presence of a Box-Cox power transformation of the conditional variances and the asymmetric absolute errors. It can be seen that model (3.1)-(3.2) is a general version of GARCH. Ding, Granger and Engle (1993) showed that the asymmetric power GARCH model includes as special cases seven other ARCH-type models, including ARCH, GARCH, Higgins and Bera's (1990) NARCH, Geweke's (1986) and Pantula's (1986) log-ARCH, and simple asymmetric and threshold GARCH models. In the following, Theorem 3.1 gives the sufficient condition for the stationarity, ergodicity and $\delta-$order stationarity of model (3.1)-(3.2). Theorem 3.2 provides the necessary and sufficient condition for the existence of the higher-order moments.

Theorem 3.1. *Suppose that $\alpha_0 > 0$, α_i, $\beta_i \geq 0$, $\delta \geq 0$, and $|\gamma| < 1$. Denote $Z_t = (|\eta_t| - \gamma\eta_t)^{\delta}$. Then $\sum_{i=1}^{r} \alpha_i E Z_t + \sum_{j=1}^{s} \beta_j < 1$ is a necessary and sufficient* *condition for the existence of a unique $\mathcal{F}_t$-measurable δ-order stationary solution $\{\varepsilon_t\}$ to model (3.1)-(3.2). The solution $\{\varepsilon_t\}$ has the following causal representation:*

$$\varepsilon_t = \eta_t h_t^{1/2} \text{ and } h_t = [\alpha_0 + \sum_{j=1}^{\infty} c' \, (\prod_{i=1}^{j} A_{\delta t-i}) \, \xi_{\delta t-j}]^{\frac{2}{\delta}} \text{ a.s.}, \tag{3.3}$$

where $\xi_{\delta t} = (\alpha_0 Z_t, 0, \cdots, 0, \alpha_0, 0, \cdots, 0)_{(r+s)\times 1}$, that is, the first component is $\alpha_0 Z_t$ and the $(r+1)$-th component is α_0, $c = (\alpha_1, \cdots, \alpha_r, \beta_1, \cdots, \beta_s)'$, and

$$A_{\delta t} = \left(\begin{array}{ccc|ccc} \alpha_1 Z_t & \cdots & \alpha_r Z_t & \beta_1 Z_t & \cdots & \beta_s Z_t \\ & I_{(r-1)\times(r-1)} & O_{(r-1)\times 1} & & O_{(r-1)\times s} & \\ \hline \alpha_1 & \cdots & \alpha_r & \beta_1 & \cdots & \beta_s \\ & O_{(s-1)\times r} & & & I_{(s-1)\times(s-1)} & O_{(s-1)\times 1} \end{array}\right). \tag{3.4}$$

Hence, $\{\varepsilon_t\}$ is strictly stationary and ergodic.

For a more general asymmetric power GARCH (r,s) model, Ding, Granger and Engle (1993) provided a condition for $E|\varepsilon_t|^\delta < \infty$, which is the same as that in Theorem 3.1 for model (3.1)-(3.3). When $r = s = 1$, $\gamma = 0$ and $\delta = 1$, the condition in Theorem 3.1 is also the same as the condition for the absolute-value GARCH model in Taylor (1986). The strict stationarity and ergodicity condition for the asymmetric power model is a new result. The uniqueness of second-order stationarity and the expansion in (3.3) are also useful results.

Theorem 3.2. *The necessary and sufficient condition for $E|\varepsilon_t|^{m\delta} < \infty$ in model (3.1)-(3.2) is $\rho[E(A_{\delta t}^{\otimes m})] < 1$, where $A_{\delta t}$ is defined by (3.4).*

To establish the sufficient moment condition, usually there are two drift criteria to be employed, the first being that in Tweedie (1983). The results derived from this criterion will ensure that there is a unique strictly stationary and (geometric) ergodic solution to the underlying model with finite moment, as in Chan and Tong (1985) and Feigin and Tweedie (1985). However, this criterion usually needs to assume that the density function of η_t is lower semi-continuous or positive in a neighbourhood of the original point to prove the irreducibility condition.

The second criterion is that in Tweedie (1988), but does not need any irreducibility condition for the moment condition. However, the results from this criterion ensure only that there exists a strictly stationary solution for the underlying model with finite moment. Such a result is not especially helpful in practice since, without uniqueness, one cannot guarantee that the process from the underlying model has finite moment. Our method here uses the criterion in Tweedie (1988) to avoid the assumption on the density of η_t, and then uses Theorem 3.1 to establish uniqueness. Thus, the result in Theorem 3.2 should be very useful in practice.

4 Proofs of the Main Results

Proof of Theorem 2.1. Multiplying (1.2) by η_t^2 yields

$$\varepsilon_t^2 = \alpha_0\eta_t^2 + \sum_{i=1}^{r}\alpha_i\varepsilon_{t-i}^2\eta_t^2 + \sum_{i=1}^{s}\beta_i h_{t-i}\eta_t^2. \tag{4.1}$$

Now rewrite (1.2) and (4.1) in vector form as

$$X_t = A_tX_{t-1} + \zeta_t, \tag{4.2}$$

where $X_t = (\varepsilon_t^2, \cdots, \varepsilon_{t-r}^2, h_t, \cdots, h_{t-s})'$ and $\zeta_t = \alpha_0(\eta_t^2, 0, \cdots, 0, 1, 0, \cdots, 0)'$, of which the first element is η_t^2 and the $(r+1)$-th element is 1. If $E(\varepsilon_t^{2m}) < \infty$, then $E(X_t^{\otimes m}) < \infty$, where "a vector $A < \infty$" means that all the elements of A are finite.

Note that all the elements of A_t, X_t and ξ_t are non-negative. We have

$$\begin{aligned} E(X_t^{\otimes m}) &\geq E(A_tX_{t-1})^{\otimes m} + E(\xi_t^{\otimes m}) \\ &= E(A_t^{\otimes m})E(X_{t-1}^{\otimes m}) + C_1R_1^{\otimes m} \\ &\geq C_1\sum_{i=0}^{k}[E(A_t^{\otimes m})]^iR_1^{\otimes m}, \end{aligned} \tag{4.3}$$

where $C_1 = \min\{\text{all the positive elements of } E(\xi_t^{\otimes m})\}$, $R_1 = (1, 0, \cdots, 0, 1, 0, \cdots, 0)'$, and "a vector $A >$ a vector B" means that each element of A exceeds the corresponding element of B. Similarly, define $A \geq B$. If k tends to infinity, from (4.3) we have

$$\sum_{i=0}^{\infty}[E(A_t^{\otimes m})]^iR_1^{\otimes m} < \infty. \tag{4.4}$$

As discussed in Section 2, we cannot claim directly that $\rho(E(A_t^{\otimes m})) < 1$ from (4.4). Our proof here makes full use of the advantage of the non-negativity of the elements of $E(A_t^{\otimes m})$ and $R_1^{\otimes m}$.

Denote $r^\star = \max\{r, s\}$. In the following, we first show that

$$[E(A_t^{\otimes m})]^{r^\star} R_1^{\otimes m} > 0. \tag{4.5}$$

We will prove (4.5) for the case $\alpha_1 > 0$. For other cases, (4.5) can be similarly proved. First, $E(A_t^{\otimes m})R_1^{\otimes m} = E(A_t R_1)^{\otimes m}$, where $A_t R_1 = (\alpha_1\eta_t^2 + \beta_1, 1, 0, \cdots, 0, \alpha_1 + \beta_1, 1, 0, \cdots, 0)'$. Let $C_2 = \min\{\text{all the positive elements of } E(A_t\ R_1)^{\otimes m}\}$, and $R_2 = (1, 1, \cdots, 0, 1, 1, 0, \cdots, 0)'$. It follows that

$$E(A_t^{\otimes m})R_1^{\otimes m} \geq C_2 R_2^{\otimes m}. \tag{4.6}$$

From (4.6), we have

$$[E(A_t^{\otimes m})]^2 R_1^{\otimes m} \geq C_2 E(A_t^{\otimes m}) R_2^{\otimes m} = C_2 E(A_t R_2)^{\otimes m}. \tag{4.7}$$

Now, $A_t R_2 = (\alpha_1\eta_t^2 + \beta_1 + \alpha_2\eta_t^2 + \beta_2, 1, 1, 0, \cdots, 0, \alpha_1 + \beta_1 + \alpha_2 + \beta_2, 1, 1, 0, \cdots, 0)$. Let $C_3 = \min\{\text{all the positive elements of } E(A_t R_2)^{\otimes m}\}$, and $R_3 = (1, 1, 1, 0, \cdots, 0, 1, 1, 1, 0, \cdots, 0)'$. From (4.7), we have

$$[E(A_t^{\otimes m})]^2 R_1^{\otimes m} \geq C_2 C_3 R_3^{\otimes m}. \tag{4.8}$$

Repeating the above procedure $r^\star$ times, we can show that

$$[E(A_t^{\otimes m})]^{r^\star} R_1^{\otimes m} \geq (\prod_{i=2}^{r^\star} C_i) R_{r^\star}^{\otimes m}, \tag{4.9}$$

where $C_i > 0$ and $R_{r^\star} = (1, 1, \cdots, 1)'_{(r+s)\times 1}$. Thus, (4.5) holds. From (4.4)-(4.5), we have

$$\sum_{i=0}^{\infty} [E(A_t^{\otimes m})]^i [E(A_t^{\otimes m})]^{r^\star} R_1^{\otimes m} < \infty. \tag{4.10}$$

Let $a_{kj}^{(i)}$ be the (k, j)-th element of $[E(A_t^{\otimes m})]^i$. From (4.10), we know that $\sum_{i=0}^{\infty} a_{kj}^{(i)} < \infty$ for all, $k, j = 1, \cdots, (r+s)^m$, that is,

$$\sum_{i=0}^{\infty} [E(A_t^{\otimes m})]^i < \infty,$$

and hence $\rho[E(A_t^{\otimes m})] < 1$. This completes the proof. $\square$

Proof of Theorem 3.1. Multiplying (3.2) by Z_t yields

$$(|\varepsilon_t| - \gamma\varepsilon_t)^\delta = \alpha_0 Z_t + \sum_{i=1}^{r} \alpha_i(|\varepsilon_{t-i}| - \gamma\varepsilon_{t-i})^\delta Z_t + \sum_{i=1}^{s} \beta_i h_{t-i}^{\delta/2} Z_t. \tag{4.11}$$

Rewrite (3.2) and (4.11) in vector form as

$$X_t = A_{\delta t} X_{t-1} + \zeta_{\delta t}, \tag{4.12}$$

where $X_t = [(|\varepsilon_t| - \gamma\varepsilon_t)^\delta, \cdots, (|\varepsilon_{t-r+1}| - \gamma\varepsilon_{t-r+1})^\delta, h_t^{\delta/2}, \cdots, h_{t-s}^{\delta/2}]'$ and $\zeta_{\delta t} = \alpha_0(Z_t, 0, \cdots, 0, 1, 0, \cdots, 0)$, of which the first element is Z_t and $(r+1)$-th element is 1. By (4.12), following exactly the steps of Theorem 2.1 in Ling and Li (1997), we can prove that there is a $\delta-$order solution to (3.1)-(3.2), and the solution has expansion (3.4), and hence it is strictly stationary and ergodic.

For necessity, since $E|\varepsilon_t|^\delta = Eh_t^{\delta/2} = a\ constant < \infty$, we have

$$Eh_t^{\delta/2} = \alpha_0 + \sum_{i=1}^{r} \alpha_i E Z_i E h_t^{\delta/2} + \sum_{i=1}^{s} \beta_i E h_t^{\delta/2},$$

that is,

$$(1 - \sum_{i=1}^{r} \alpha_i E Z_i - \sum_{i=1}^{s} \beta_i) E h_t^{\delta/2} = \alpha_0.$$

Since $0 < Eh_t^{\delta/2} < \infty$, we have $\sum_{i=1}^{r} \alpha_i E Z_i + \sum_{i=1}^{s} \beta_i < 1$. This completes the proof. $\square$

Proof of Theorem 3.2. Using (4.12), the drift criterion in Tweedie (1988), and following the steps of Theorem 4.2 in Ling (1999a), we can show that there exists a strictly stationary solution to model (3.1)-(3.2) with $E|\varepsilon_t|^{m\delta} < \infty$.

By Hölder's inequality, $E|\varepsilon_t|^\delta \le (E|\varepsilon_t|^{m\delta})^{1/m} < \infty$, i.e. $\{\varepsilon_t\}$ is $\delta-$order stationary. By Theorem 3.1, the solution is unique ergodic. This means that a process ε_t satisfying (3.1)-(3.2) has finite $m\delta-$th moment if $\rho(E(A_{\delta t}^{\otimes m})) < 1$.

In a similar manner to the proof of Theorem 2.1, it can be proved that the necessity of Theorem 3.2 holds. This completes the proof. $\square$

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