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ENDOGENOUS GROWTH AND CYCLES WITH A CONTINUUM OF TECHNOLOGIES

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Abstract

When a continuum of technologies is introduced to the model of Grossman and Helpman (1991), both continuous and discrete technological progress may occur as a result of technology choices by private firms. A good is created through R&D based on one of a continuum of technologies that differ in productivity, and the R&D cost is smaller when there is greater public knowledge about that technology, which accumulates through spillovers. When firms shift continuously to superior technologies, there is no incentive to retain existing technologies and the economy grows smoothly. By contrast, when many firms choose the same technology, accumulated knowledge makes this choice privately optimal for a certain time period, and the economy grows cyclically through a sequence of discrete progresses in technology. These two dynamics constitute multiple equilibria, and it depends on the size of the parameters which equilibrium is desirable for consumers.

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1 Introduction

Technology choice involves a trade-off between the initial cost of investment and the productivity and/or the quality of the product. It is one of the most important problems for firms and entrepreneurs. In addition, the pattern of technology choices by individual firms is a matter of great significance at the level of the macroeconomy, since adoption of superior technologies contributes not only to the total factor productivity of the economy, but also to the stock of public knowledge through various types of spillovers. Under certain conditions, the augmented public knowledge promotes the adoption of even more superior technologies at the level of individual firms, which in turn accumulates public knowledge of these new technologies. This micro-macro interaction brings about endogenous technological progress, which is one of the most important sources of sustained economic growth.

In the last decade, a considerable number of endogenous growth models have been presented that have investigated technological progress due to spillovers of knowledge. Most of these studies have focused on the dynamics where technology advances continuously.[1] However, as economic historians have pointed out, there has been not only continuous technological progress but also discrete advances in technology, such as the introduction of steam engines, electricity, and computers.[2] In these cases the pattern of growth was not necessarily smooth. Partly in response to these historical facts, some economic theorists have developed models which are capable of explaining this kind of discrete technological change and the resulting cyclical patterns in economic growth.[3] While most existing papers focus exclusively on either continuous or discrete technological progress, this paper shows that both patterns of technological progress may be realized as multiple equilibria when we explicitly model the technology choices by private firms from a continuum of technologies. We will clarify the way in which new technologies are chosen and when they are adopted, and derive a path of economic growth that is substantially different depending on the patterns of technological progress. In addition, our analysis makes it possible to compare the desirability of continuous and discrete

[1]Seminal works include Romer (1990) and Grossman and Helpman (1991).

[2]See Helpman (1998) for references.

[3]Representative works include Bental and Peled (1996), Helpman and Trajtenberg (1998) and Freeman, Hong and Peled (1999). See also Horii (2000) for cyclical growth arising from discrete changes in the industrial structure.

technological progress in terms of welfare.

This paper develops an endogenous growth model that generalizes the basic variety-expansion model of Grossman and Helpman (1991, hereafter GH) by allowing firms to choose from technologies that differ in productivity. Specifically, we extend the GH model in the following three respects.

1. There exists a continuum of technologies, and each firm can conduct R&D based on one technology to create a good. The marginal cost of production differs depending on the technology choice.

2. The labor cost of R&D on a certain technology is reduced by the accumulation of public knowledge on that technology. However, this effect gets weaker when the cost becomes sufficiently small.

3. R&D conducted by each firm contributes not only to the stock of public knowledge on the technology adopted by this firm, but also to the stock of knowledge on related technologies, according to the degree of association.

In this setting, firms confront a trade-off in choosing technologies. On the one hand, they have an incentive to choose technologies that have higher productivity, since this will enlarge the firm's profit stream. On the other hand, it is advantageous to choose technologies that are similar to those adopted by existing firms, since such technologies can be adopted with a small initial cost by virtue of the large stock of public knowledge. These two forces represent centrifugal force and agglomeration force in the technology space, respectively. Their relative magnitude determines the equilibrium pattern of technology choice in R&Ds. Subsequently, the trade-off itself is dynamically affected, through the process of knowledge accumulation. This creates an endogenous evolution in the pattern of technology choice and hence economic growth.

We will demonstrate that after a short time period of take-off, the economy enters a period of sustained growth. In this period, there are two patterns of equilibrium dynamics, which we call the 'continuous growth regime' and the 'cyclical growth regime,' respectively. In the continuous growth regime, technology choices by new firms at each date shift continuously in the direction of higher productivity. That is, the technology adopted by new firms today is slightly more advanced than that adopted yesterday, which is slightly more advanced than that adopted the day before yesterday, and so forth. Due to the R&D spillovers across technologies, the costs of R&D using superior technologies are gradually falling, and thus it is privately optimal to adopt a technology slightly more advanced than that adopted yesterday, given that all the other firms follow this pattern of technology adoption. The speed of technological progress cannot be infinite since the magnitude of R&D spillovers between quite different technologies is small. In fact, the equilibrium speed of technological progress is determined so that the marginal benefit of choosing a technology with higher productivity exactly equates to the marginal cost of adopting a technology with less public knowledge.

The dynamics in this regime differ from the original GH model in at least three respects. First, growth is sustained, not by exponential increases in the variety of products, but by steady improvements in the production technology. The economy will converge to a balanced growth path where both the cost of R&D and the number of new goods introduced at each instant become constant. Second, in this paper the rate of economic growth depends heavily on the magnitude of knowledge spillover across technologies. This factor is not considered in the GH model but is believed to have some significance in the real economy. Third, two economies that initially differ in their performances may or may not converge to the same state, depending on their initial structures. It is even possible that their relative performance will eventually be reversed.

While the above dynamics fully satisfy the equilibrium conditions, another quite different pattern of dynamics exists which still satisfies all of the conditions for a perfect foresight equilibrium. Suppose that the same technology is adopted by all entrepreneurs during a certain time period.

The cost of R&D declines throughout this period through spillovers of knowledge, but the rate of cost reduction is not uniform across technologies. In general, a non-negligible portion of the knowledge gained from experiences with a certain technology is specific to that technology, and cannot be applied to other technologies. This implies that the cost of R&D based on this technology is reduced more rapidly than R&D based on other technologies. Thus, provided that other entrepreneurs adopt the same technology, it is privately optimal to choose this technology.[4] Moreover, an additional adoption of this technology makes the cost of R&D even lower, resulting in a chain of adoption of the same technology by new entrepreneurs. As knowledge accumulates and the technology becomes mature, however, the R&D cost becomes less sensitive to marginal increases in knowledge, while it becomes possible to adopt increasingly advanced technologies at a given labor cost owing to spillovers across technologies. In fact, when the number

[4]This economy of scale is the primary cause of multiple equilibria.

of firms adopting the same technology reaches a certain threshold, it becomes more profitable for entrepreneurs to switch to a significantly advanced technology. With the adoption of the new technology, the rate of economic growth temporarily declines. It then gradually accelerates as knowledge on the new technology is accumulated through the R&D experiences, creating a cyclical pattern in the rate of economic growth.

This pattern of technology adoption and its resulting fluctuations in output is essentially similar to those in the models of *general purpose technologies* (GPTs),[5] among which Helpman and Trajtenberg (1998, hereafter HT) is the closest. Between our model and the HT model, however, there is a crucial difference in the process of the emergence of GPTs. The HT model assumes that GPTs are developed somewhere outside the model, and once a new GPT is introduced exogenously firms immediately switch to the new technology. By contrast, our model shows that GPTs are endogenously formed as a result of technology choices by private firms. Consequently, which technologies become GPTs depends on the equilibrium path. In addition, our model shows that even though a technology is available that has higher productivity than the current GPT, firms do not switch to it until the cost of adoption becomes sufficiently small through spillovers of knowledge between the current and superior technologies.

In terms of welfare, the formation of a GPT is beneficial in that it reduces the cost of R&D and promotes the introduction of a large variety of products, especially by the accumulation of technology-specific components of public knowledge, which are never utilized in the continuous growth regime. However, the formation of a GPT also has drawbacks. Once a GPT is formed, it creates a discrete difference between the R&D cost based on it and those based on other technologies. This makes entrepreneurs stick with the current GPT until a technology with sufficiently high productivity becomes available at a certain cost of adoption. Moreover, when there is a significant difference in productivity between the current GPT and the potential new technology, only a small portion of the experience with the current GPT can be applied to the new technology. This implies that the average speed of technological progress tends to be slower in the cyclical growth regime compared to the continuous growth regime. We will show that it depends on the size of the parameters whether or not the benefits flowing from the

[5]According to Lipsey, Bekar and Carlaw (1998), a technology is qualified as a GPT if it has scope for improvement, a wide variety of uses, a wide range of use, and strong technological complementarities with existing or potential new technologies. In fact, it will be shown that each technology adopted in this regime satisfies all these conditions.

formation of GPTs exceed the disadvantages.

The rest of the paper is organized as follows. Section 2 presents the model and derives the instantaneous equilibrium. Section 3 investigates the evolution of R&D activities and shows that the dynamics of economic growth can be divided into a transitory take-off period and a stage of long-term growth. Having analyzed the dynamics in the take-off period, sections 4 and 5, respectively, derive two regimes of dynamics in the long-term growth period, which are described above intensively. Section 6 investigates which regime is desirable by comparing welfare in the two regimes. Section 7 concludes the paper. The proofs of all claims are collated in the appendix.

2 The Model

2.1 Firms and Technologies

In the model, there are many firms competing monopolistically, and each firm manufactures a single variety of differentiated goods.[6] The number of firms is determined in equilibrium through the process of free entry. To set up a new firm, an entrepreneur needs to invest in R&D to develop a new differentiated product and its manufacturing process. At the time of entry, each firm chooses its own production technology from a continuum of available constant-returns-to-scale technologies.[7] Depending on technology choice, the quantity and/or quality of goods that can be produced from a given amount of labor differ. For ease of notation, we normalize the quantity of each good so that every good enters the utility function symmetrically, and thus all differences are expressed by productivity. Each technology is indexed by a positive number $z \in (0, \infty)$ such that when technology z is adopted this firm can produce amount $z^{(1-\alpha)/\alpha}$ of its product from a unit of labor, where $\alpha \in (0, 1)$ is a constant which will be defined in equation (4).[8]

Although in principle any technology $z \in (0, \infty)$ can be adopted, the labor cost of

[6]Like most variety expansion models, we assume that the potential variety of products is sufficiently large that it does not become exhausted on the path of economic growth. We also assume that imitating an existing product costs no less than creating a new variety of differentiated good, for both technical and legal reasons. Thus, entrepreneurs have no incentive to imitate existing products and thus each good is produced by exactly one firm.

[7]We assume that once a technology is adopted firms will not change their production technology since doing so is as costly as creating another new differentiated good.

[8]This method of indexing simplifies the mathematical expressions to come, but it is not essential.

R&D differs depending on the technology choice. The labor input required to develop a new product is small when a large stock of public knowledge already exists on the technology adopted. Specifically, we assume that to create a blueprint of a new good based on technology z, an entrepreneur must devote $c(K_t(z))$ units of labor to R&D.[9] Here $K_t(z)$ is the stock of public knowledge with respect to technology z, and $c(\cdot)$ is a cost function defined in $(0, \infty)$ which satisfies the following properties.

Assumption 1 $c(\cdot)$ *satisfies*

a. $c(k) > 0$ *and* $c'(k) < 0$ *for all* $k \in (0, \infty)$*,*

b. The elasticity of $c(k)$*,* $\epsilon_c(k) \equiv (k/c(k))c'(k)$*, is strictly decreasing in* k*,*

c. $\lim_{k \to +0} \epsilon_c(k) \geq 1$ *and* $\lim_{k \to \infty} \epsilon_c(k) = 0$.[10]

While our specification of the R&D cost is based on Romer (1990) and GH,[11] we incorporate two additional realistic features. First, rather than focusing on aggregate "general knowledge," we explicitly deal with the heterogeneous distribution of public knowledge across various technologies. Second, while GH assumed that the R&D cost is reduced at a constant rate when knowledge (or experience) accumulates, we incorporate the tendency that the R&D cost becomes less sensitive to marginal increases in knowledge as the technology becomes mature.

The public knowledge accumulates as a by-product of R&D activities. When a firm creates a blueprint based on technology z_0, it augments the stock of knowledge $K_t(z_0)$ by one unit due to the positive technological spillover 'within' the technology z_0. This condition is the same in the GH model. In our model, there are, in addition, positive spillovers 'across' technologies depending on the degree of association between them: for any technology $z > 0$, knowledge $K_t(z)$ is augmented by $s(z_0, z)$, which is assumed to satisfy the following properties.[12]

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[9] A more formal statement is that an entrepreneur who devotes l units of labor to R&D for a time interval of length dt can create blueprints of $lc(K_t(z))^{-1}dt$ new goods that are based on technology z.

[10] In fact, it is sufficient to assume a much weaker condition, $\lim_{k \to +0} \epsilon_c(k) > 1/\delta$ and $\lim_{k \to \infty} \epsilon_c(k) < 1/\delta$, where $\delta > 1$ is a constant that will be introduced in claim 1

[11] Note that when $c(k) = a/k$ and $K_t(z) = K_t$ for all z, our specification coincides with that of GH.

[12] For simplicity, we assume that the magnitude of spillover between the two technologies depends only on their productivities.

Assumption 2 *For any $z_0 > 0$ and $z > 0$, $s(\cdot, \cdot)$ satisfies*

a. $s(z_0, z) \in (0, 1)$ *and* $s(z_0, z_0) = 1$,

b. $s(z_0, z) = s(z, z_0)$,

c. $s(z_0, z) = s(\alpha z_0, \alpha z)$ *for all* $\alpha > 0$,

d. $s(z_0, z')s(z', z) = s(z_0, z'')s(z'', z)$ *for all* $z', z'' \in (z_0, z)$,

e. $\lim_{z' \to \infty}(z' - z_0)s(z_0, z') = 0$.

The first three items, a, b and c, specify respectively boundedness, symmetry and homogeneity of the spillover function $s(\cdot, \cdot)$. Property d assumes that the spillovers are consistent in that, for any given z_0 and z, 'the portion of experience in technology z_0 that can be applied to some intermediate technology' multiplied by 'the portion of experience in that technology that can be applied to z' is not changed by the choice of the intermediate technology. The final property e, means that spillover effects are not large between significantly different technologies.

Under these restrictions, the functional form of $s(\cdot, \cdot)$ can be expressed in terms of two parameters.

Claim 1 *There exist $\mu \in (0, 1]$ and $\delta > 1$ such that $s(z_0, z) = \mu \min\{z/z_0, z_0/z\}^\delta$ for all z_0 and $z \neq z_0$.*

In this expression, parameter δ measures the rate at which the magnitude of spillover gets smaller as the difference between the productivities of two technologies increases, and μ indicates the portion of the component of public knowledge that is not technology-specific. From claim 1 and assumption $s(z, z) = 1$, the whole process of knowledge accumulation is summarized by

$$\dot{K}_t(z) = \mu \int_0^\infty \min\left\{\frac{z'}{z}, \frac{z}{z'}\right\}^\delta dR_t(z') + (1 - \mu)(R_t(z) - R_t(z-)), \tag{1}$$

where $R_t(\cdot)$ is the cumulative distribution function of R&D at time t. That is, $R_t(z)$ represents the instantaneous flow of new products introduced at time t that are based on technologies whose productivity is lower than or equal to z, and $R_t(z) - R_t(z-)$ represents the flow of new products based exactly on technology z.[13] We assume that the public

[13] $R_t(z-)$ represents the limiting value of $R_t(z')$ when z' approaches z from below.

knowledge does not depreciate, in which case there is a one-to-one relationship between the distribution of knowledge and the distribution of firms.

$$K_t(z) = \mu \int_0^{\infty} \min\left\{\frac{z'}{z}, \frac{z}{z'}\right\}^{\delta} dF_t(z') + (1-\mu)(F_t(z) - F_t(z-)), \tag{2}$$

where $F_t(z)$ represents the cumulative distribution function of firms in the technology space $z \in (0, \infty)$.

2.2 Consumers and the Market Demand

The demand side of our model is essentially the same as the GH model. There are L identical consumers. Each consumer inelastically supplies one unit of labor service at the prevailing wage w_t at every date and maximizes utility over an infinite horizon,

$$U_t = \int_t^{\infty} e^{-\rho(\tau - t)} \ln D_\tau \, d\tau. \tag{3}$$

Here ρ is the subjective discount rate, and D_t represents an index of consumption at time t. Consumers have tastes for variety, expressed by Dixit and Stiglitz (1977) type preferences for bundles of differentiated goods,

$$D_t = \left[\int_0^{J_t} X_t(j)^{\alpha} \, dj\right]^{1/\alpha}, \tag{4}$$

where parameter $\alpha \in (0, 1)$ represents substitutability between goods, $J_t \equiv F_t(\infty)$ denotes the number of goods produced at time t, and $X_t(j)$ the consumption of good j.

For the same reason as in the GH model, growth may cease in economies that are too small (small L), too impatient (large ρ) and/or too elastic in terms of substitution between goods (large α), since R&Ds are not privately profitable in such economies. Because of space constraints, we focus exclusively on economies where growth can be sustained, though this is not essential to the results of the analysis.

Assumption 3 $(1-\alpha)L/\alpha\rho$ *is greater than* Γ*, where* Γ *is a finite constant depending only on functional forms of* $c(\cdot)$ *and* $s(\cdot,\cdot)$*. The definition of* Γ *is presented in the appendix.*

Now, we turn to the market demand for goods. Let $P_t(j)$ denote the price of good j. Preference (4) implies that given the spending E_t at time t the maximized value of D_t is $E_t/\bar{P}_t$, where $\bar{P}_t$ is a price index of differentiated goods,

$$\bar{P}_t = \left[\int_0^{J_t} P(j')^{-\frac{\alpha}{1-\alpha}} \, dj'\right]^{-\frac{1-\alpha}{\alpha}}.$$

Here, the consumer's problem reduces to maximizing

$$U_t = \int_t^\infty e^{-\rho(\tau - t)} \left(\ln E_\tau - \ln P_\tau \right) \, d\tau.$$

Since the above expression is separable in E_t and P_t, it is obvious that the optimal expending rule satisfies $\dot{E}_t/E_t = r_t - \rho$. We follow the GH model by normalizing prices so that aggregate consumption expenditure, LE_t, becomes unity at every instant. In this case, the nominal interest rate r_t equals the subjective discount rate ρ for all t, and the market demand for good j can be calculated as

$$LX_t(j) = P_t(j)^{-\frac{1}{1-\alpha}} \bar{P}_t^{\frac{\alpha}{1-\alpha}}. \tag{5}$$

2.3 Instantaneous Equilibrium

In the settings presented above, we can characterize the instantaneous equilibrium of this economy at a certain time t, given the current and future distribution of firms, $F_{t'}(\cdot)$ for all $t' \geq t$. At this point, we impose a simple assumption on the distribution of firms for all t.[14]

Assumption 4 *For any t, $\operatorname{supp} F_t(\cdot)$ is bounded and $\int_0^\infty z dF_t(z)$ is finite.*

Consider a firm j which has already developed a blueprint based on a certain technology z. Since this firm can produce amount $z^{(1-\alpha)/\alpha}$ of its good from a unit of labor, the marginal cost of production is $w_t/z^{(1-\alpha)/\alpha}$. Given the demand function (5), the firm maximizes operating profits

$$\left(P_t(j) - w_t/z^{(1-\alpha)/\alpha} \right) P_t(j)^{-\frac{1}{1-\alpha}} \left[\int_0^{J_t} P_t(j')^{-\frac{\alpha}{1-\alpha}} \, dj' \right]^{-1}$$

by charging a price $P_t(j) = w_t/(\alpha z^{(1-\alpha)/\alpha})$.

When all firms follow the above rule, each firm's quantity and operating profits are determined by its technology z, the current wage w_t and the current distribution of firm $F_t(\cdot)$.

$$x_t(z) = \frac{\alpha z^{1/\alpha}}{w_t \int_0^\infty z' dF_t(z')}, \tag{6}$$

$$\pi_t(z) = \frac{(1-\alpha) z}{\int_0^\infty z' dF_t(z')}. \tag{7}$$

[14] Here, $\operatorname{supp} F_t(\cdot)$ means the support of $F_t(\cdot)$, that is, the set of technologies that have been adopted by some firms by t. In fact, it is sufficient to assume that the initial distribution satisfies the property given in assumption 4, since the dynamic analysis to follow will show that when the economy starts with such a distribution it will never become inconsistent with assumption 4.

These equations characterize the equilibrium of the goods market.

Labor services supplied by consumers are applied to the production of differentiated goods and R&D. From (6), the labor demand for production workers is calculated as α/w_t. The labor market equilibrium requires

$$\frac{\alpha}{w_t} + L_t^R = L, \tag{8}$$

where the second term represents the total employment in R&D, defined as

$$L_t^R \equiv \int_0^\infty c(K_t(z))dR_t(z). \tag{9}$$

For simplicity, we rule out speculative bubbles so that the stock market value of a firm equals the present discount value of its profit stream,[15]

$$v_t(z) = \int_t^\infty e^{-\rho(\tau-t)}\pi_\tau(z)\,d\tau. \tag{10}$$

Under the assumption of free entry, the value of a firm or equivalently, the value of its blueprint, must not be higher than the cost of creating a new blueprint with the same technology.[16] On the other hand, firms invest in R&D based on technology z only when the value of the blueprint based on that technology is not less than the cost of creating it. The following *free entry condition* summarizes these requirements:

$$v_t(z) \leq w_t c(K_t(z)) \tag{11}$$

for all $z \in (0, \infty)$ *and with equality on the support of* $R_t(\cdot)$. Note that $K_t(\cdot)$ is a predetermined function, and $v_t(\cdot)$ is also given since we take the future distribution of firms as given.

To find the equilibrium wage level satisfying this condition, we introduce an index of the labor cost of R&D when a certain technology is chosen. Let Z_t denote the frontier technology, that is, the maximal element in the support of $F_t(\cdot)$, and let V_t denote the value of a current frontier firm, $v_t(Z_t)$. Then, the value of a new blueprint based on an arbitrary technology z can be expressed by $(z/Z_t)V_t$, since equation (7) and (10) imply that the profit stream of a firm and thus its discounted sum are both linear in its technology level z. Using this fact, we can construct the following index,

$$l_t(z) \equiv \frac{Z_t}{z} c(K_t(z)), \tag{12}$$

[15]Here we utilize the fact that $r_t = \rho$ holds for all t.

[16]Otherwise, an infinite number of firms would want to enter into R&D, leading to excess demand in the labor market, which in turn would lift the wage rate until the above condition is met.

which represents the amounts of R&D labor required to create a bundle of new blueprints based on technology z that has in total the same value as a current frontier firm. There is a bounded set of technologies with which the required labor input takes the minimum value.

Claim 2 $l_t^{\min} \equiv \min_{z\in(0,\infty)} l_t(z)$ *exists, and the set of minima,* $\mathcal{L}_t^{\min} \equiv \{z|\, l_t(z) = l_t^{\min}\}$*, is bounded.*

Using this index, condition (11) reduces to a simple form,

$$V_t \le w_t l_t(z). \tag{13}$$

The above expression shows that the demand curve for R&D labor is horizontal at the level of $V_t/l_t^{\min}$. Recall also that the demand curve for production labor, α/w_t, is downward sloping and crosses the inelastic supply curve at $w_t = \alpha/L$. Since the total labor demand is the sum of these two, the equilibrium wage is determined as

$$w_t = \max\{V_t/l_t^{\min}, \alpha/L\}. \tag{14}$$

Note that the demand for R&D labor is positive if and only if the equilibrium wage is larger than α/L, and that only the technologies that give the smallest $l_t(z)$ are adopted for R&D,

$$\text{supp } R_t(\cdot) \subset \mathcal{L}_t^{\min}. \tag{15}$$

The arguments so far have demonstrated that once the current and future expectations of the distribution of firms is given, instantaneous equilibrium conditions specify prices $p_t(\cdot)$, quantities $x_t(\cdot)$, profits $\pi_t(\cdot)$, and the prevailing wage w_t uniquely, and also specify the support of the R&D distribution $R_t(\cdot)$. For this instantaneous equilibrium to constitute a part of a dynamic equilibrium path, however, evolution of the distribution of firms must be consistent with actual R&Ds invested in at each instant,

$$\dot{F}_t(z) = R_t(z). \tag{16}$$

That is, a path of $F_t(\cdot)$ constitutes a perfect foresight equilibrium if and only if the R&D distribution $R_t(\cdot)$ implied by (16) satisfies the labor market clearing condition (8) and the free entry condition (15) at all t. Once an equilibrium path of $F_t(\cdot)$ is found, economic growth is measured by the rate of growth in consumption,

$$D_t = \frac{\alpha}{w_t L}\left(\int_0^{Z_t} z dF_t(z)\right)^{(1-\alpha)/\alpha}, \tag{17}$$

which is obtained by substituting (6) for (4). In the following sections, we investigate evolutions in the amount of R&D investments and the technology choice in the R&D, which determine the path of $F_t(\cdot)$ and hence economic growth.

3 Evolution of R&D Activities

At each date, entrepreneurs face two problems: they must decide whether or not to invest in R&D, and, if they invest, they must also choose the technology. Though this decision cannot be completed until the future expectation on the R&D activities is specified, this section shows that some essential properties of the evolution of R&D activities can be inferred from just the current and the past state of the economy.

First, we focus on the decision on whether or not to invest. Entrepreneurs invest in R&Ds only when the discounted sum of the future profits, V_t, is no smaller than the investment cost, $w_t l_t^{\min}$. Note that the profits become small when the number of competitors rises. This is expressed by $\pi_t(Z_t) = (1-\alpha)/N_t$, where N_t is a weighted sum of the number of firms, $N_t \equiv Z_t^{-1} \int_0^{Z_t} z \, dF_t(z)$.[17]

Since no firm exits the goods market, the amount of profits a firm can get never rises after entry. Thus, V_t must be no larger than $(1-\alpha)/\rho N_t$, and this maximum value is realized only if there is no subsequent entrance of competitors. On the other hand, the cost of R&D is no smaller than $\alpha l_t^{\min}/L$ since there is a lower bound on the wage rate at $w_t = \alpha/L$. Note also that this minimum R&D cost is realized only when there is no entry at time t. From these facts we can derive the exact condition under which a positive number of entrepreneurs invest in R&D.

Claim 3 *At each date in a perfect foresight equilibrium,*

a. If there is a positive amount of R&D, then $V_t \in (\alpha l_t^{\min}/L, (1-\alpha)/\rho N_t)$. *Otherwise,* $V_t = (1-\alpha)/\rho N_t$.

b. There is a positive amount of R&D if and only if $(1-\alpha)/\rho N_t$ *is larger than* $\alpha l_t^{\min}/L$.

Next, we turn to the issue of technology choice. When an entrepreneur chooses from a continuum of technologies, he or she will take into account two facts. On the one

[17] Note that when all firms use the same technology, such as in the GH economy, index N_t coincides with J_t. However, when some firms use a technology whose productivity is lower than the current frontier technology, the number of these firms is discounted according to the value of z relative to Z_t.

hand, as shown in (7), the profit stream to a firm is magnified proportionally to z when it adopts a technology with higher productivity. This provides an incentive to choose a technology significantly superior to past ones. On the other hand, it is advantageous to choose a technology with a large stock of public knowledge since such a technology can be adopted with a small initial cost. This creates an incentive to choose a technology that is similar to those chosen by past entrepreneurs, a dynamic agglomeration force. Note that as the stocks of knowledge on existing technologies become large, the agglomeration force gets weaker, since the R&D cost becomes less sensitive to a marginal difference in the amount of knowledge between technologies. The equilibrium pattern of technology choice is determined by the relative magnitudes of the centrifugal force and the agglomeration force.

Whenever a new technology is adopted, the centrifugal force and the agglomeration force must exactly balance at that technology level, otherwise adopting either more advanced or more conservative (less advanced) technologies will provide higher profitability. In fact, there is a certain level of knowledge under which these two forces cancel out.

Claim 4 *If there is a technology $z \in \operatorname{supp} R_t(\cdot)$ such that $Z_{t'} < z$ holds for all $t' < t$, then $K_t(z) = \bar{K} \equiv \epsilon_c^{-1}(1/\delta)$.*

Note that the above claim says only that the amount of knowledge must be $\bar{K}$ *when the technology frontier advances.* For example, suppose that the number of firms in the economy is quite small and that no technology has amount $\bar{K}$ of knowledge. Then, the implication of claim 4 is not that there is no R&D, but that the frontier does not advance unless knowledge accumulates up to $\bar{K}$.

Here, another question is whether the frontier advances whenever the amount of knowledge reaches $\bar{K}$?[18] The answer depends on both parameters and the current distribution of firms. Suppose that μ is less than unity and there is a positive mass of firms which have adopted the same technology. Then, equation (2) implies that there is a portion of accumulated knowledge that is specific to this technology. In this case, entrepreneurs may want to adopt this established technology even when knowledge exceeds $\bar{K}$, since adoption of even a slightly advanced technology will cause discrete increases in the R&D cost. While this creates a 'lock in effect', the following claim shows that the adoption of some advanced technologies eventually becomes more profitable when a sufficiently large amount of knowledge is accumulated. It also shows that the amount of

[18]This is equivalent to the question whether the reverse of claim 4 holds or not.

knowledge at the frontier never exceeds $\bar{K}$ when the frontier technology is not a *mass point*[19] or $\mu = 1$.

Claim 5

a. *Let* $\tilde{c}(k) \equiv c(\bar{K})(\bar{K}/\mu k)^{1/\delta}$*. Then, there exists a finite value* $K^{\max}$ *such that* $\tilde{c}(k) < c(k)$ *for* $k > K^{\max}$ *and* $\tilde{c}(k) > c(k)$ *for* $k \in [\bar{K}/\mu, K^{\max})$*.*

b. $\min$ *supp* $R_t(\cdot) > z$ *holds whenever* $K_t(z) \geq K^{\max}$*.*

c. *The minimum element of supp* $R_t(\cdot)$ *that is not a mass point in* $F_t(\cdot)$ *is greater than* z *whenever* $K_t(z) > \bar{K}$*.*

d. *If* μ *is unity,* $\min$ *supp* $R_t(\cdot) > z$ *holds whenever* $K_t(z) > \bar{K}$*.*

Though the exact dynamics cannot be specified by claim 4 and 5, they suggest that it is useful to break down the process of economic growth into two periods, a transitory take-off period and a period of sustained growth. Let us define the take-off period as the time period before the technology frontier, Z_t, starts to advance. During this period, economic growth is primarily led by reductions in R&D costs caused by accumulation of knowledge on existing technologies, since only a limited set of technologies are adopted.[20] This period lasts until the amount of knowledge on the initial frontier technology reaches some threshold value, which exists between $\bar{K}$ and $K^{\max}$ and depends on the initial distribution of firms. After the take-off, entrepreneurs start to adopt superior technologies. Throughout this period of sustained growth, the amount of knowledge on any technology on which new products are created is bounded between $\bar{K}$ and $K^{\max}$. Since this boundary is finite, there is no long-run trend in the amount of knowledge and hence in the cost of R&D. Thus, in the period of sustained growth, only technological progress explains long-term growth.

3.1 Dynamics in the Take-off Period

The rest of this section investigates the dynamics of the economy in the take-off period. We first derive the motion of macroeconomic variables in this period, assuming that the

[19]In what follows, we call a technology which is adopted by a positive mass of firms, *mass point* in distribution $F_t(\cdot)$. In mathematical terms, z is a mass point in $F_t(\cdot)$ if and only if $F_t(z) - F_t(z-) > 0$ holds.

[20]Note that, however, this process cannot sustain economic growth permanently because, as the number of firms grows, the returns to R&D will decline more rapidly than the cost of R&D declines.

economy starts from a quite simple distribution of firms.[21] Subsequently, the condition for the end of this period is derived.

Suppose that initially only one technology is adopted by all existing firms to produce their goods. Let us denote the number of initial firms by n_0 and, without losing generality, let the initial technology be unity.

$$F_0(z) = \begin{cases} 0 \text{ for } z < 1, \\ n_0 \text{ for } z \geq 1. \end{cases} \tag{18}$$

Given the initial distribution, subsequent evolution of $F_t(\cdot)$ is determined by R&D activities of entrepreneurs. In the following, we characterize these dynamics by focusing on two macroeconomic variables, $N_t \equiv Z_t^{-1} \int z dF_t(z)$ and $V_t \equiv v_t(Z_t)$.

From equation (2), the amount of knowledge available on the initial technology, $K_t(1)$, coincides with the number of firms, N_t, throughout the take-off period. Since all R&D investments are based on technology $z = 1$ within this period, $l_t^{\min} = l_t(1) = c(N_t)$ holds whenever $L_t^R > 0$. Applying this equation to (14), the equilibrium wage becomes $w_t = \max\{V_t/c(N_t), \alpha/L\}$. The flow of new goods developed at each instant is $\dot{N}_t$, and each R&D activity requires $c(N_t)$ units of labor. Thus, the total amount of R&D labor is $L_t^R = c(N_t)\dot{N}_t$. Substituting these expressions for w_t and L_t^R into the labor market equilibrium condition (8) yields a law of motion for N_t,

$$\dot{N}_t = \max\left\{\frac{L}{c(N_t)} - \frac{\alpha}{V_t}, 0\right\}. \tag{19}$$

On the other hand, derivative of equation (10) provides a standard arbitrage condition for the value of each firm, $\dot{v}_t(z) = \rho v_t(z) - \pi_t(z)$ for all z. When z is set to unity, this condition and equation (7) specify the dynamics of V_t in terms of N_t and V_t itself,

$$\dot{V}_t = \rho V_t - \frac{1-\alpha}{N_t}. \tag{20}$$

The above two equations characterize the dynamics of the economy in the take-off period, and enable us to draw a phase diagram in (N_t, V_t) space. Setting $\dot{N}_t = 0$ and $\dot{V}_t = 0$ in equations (19) and (20), respectively, we have two curves on which either N_t or V_t stays constant,

$$NN: \quad V_t = \frac{\alpha c(N_t)}{L}, \tag{21}$$

$$VV: \quad V_t = \frac{1-\alpha}{\rho N_t}. \tag{22}$$

[21] As will become clear in the following sections, the long-term dynamics are not affected by the specification of the initial distribution.

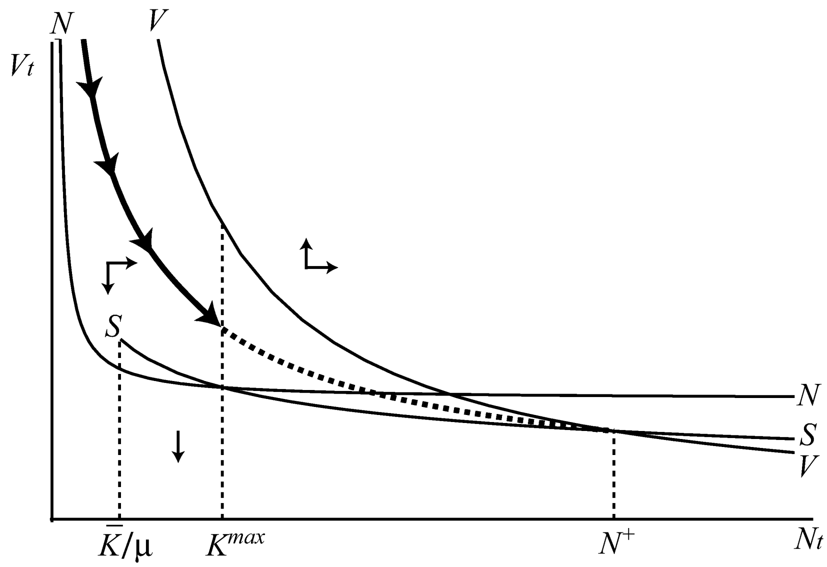

Figure 1: Phase diagram in the take-off period.

The take-off period ends when entrepreneurs adopt some technology z whose productivity is higher than that of the initial technology. Recall that claim 4 says $K_t(z)$ must be exactly $\bar{K}$ when the frontier advances. For this condition to be satisfied, N_t must be larger than $\bar{K}/\mu$ and the new technology must be $z = (\mu N_t/\bar{K})^{1/\delta}$. Thus, the exact condition of take-off can be found by comparing $l_t\left((\mu N_t/\bar{K})^{1/\delta}\right) = c(\bar{K})(\bar{K}/\mu N_t)^{1/\delta} = \tilde{c}(N_t)$ with $l_t(1) = c(N_t)$. In fact, claim 5 shows that $\tilde{c}(N_t)$ becomes smaller than $c(N_t)$ when N_t exceeds $K^{\max}$.

Figure 1 shows the locations of NN curve and VV curve, along with

$$SS: \quad V_t = \frac{\alpha c(\bar{K})}{L}\left(\frac{\bar{K}}{\mu N_t}\right)^{1/\delta} \quad \text{for } N_t > \bar{K}/\mu, \tag{23}$$

which represents the minimum cost of R&D by using technology $z = (\mu N_t/\bar{K})^{1/\delta}$. In the region of $0 < N_t < K^{\max}$, all entrepreneurs adopt the initial technology. The state variables follow (19) and (20), which we call the take-off period dynamics, until N_t, or $K_t(1)$, reaches the critical value $K^{\max}$. When N_t reaches $K^{\max}$, choosing either $z = 1$ or $z = (\mu N_t/\bar{K})^{1/\delta}$ yields the same rate of return, and then the economy immediately enters the period of sustained growth.[22] The dynamics in the period of sustained growth, which will be analyzed in the following sections, determine the value V_t at the end of

[22]Note that $K_t(1)$ is strictly increasing in time as long as growth does not cease, which is guaranteed

the take-off period.[23] Subsequently, the transitional dynamics in the take-off period are specified by solving (19) and (20) backwards from this point.

If N_0 is larger than $K^{\max}$, the return from investing in technology $(\mu N_t/\bar{K})^{1/\delta}$ is better than the returns from choosing the initial technology. Note also that, from claim 3, R&Ds are carried out if and only if VV curve locates above SS, which is equivalent to

$$N_0 < N^+ \equiv \left(\frac{(1-\alpha)L}{\alpha\rho c(\bar{K})}\right)^{\delta/(\delta-1)} \left(\frac{\mu}{\bar{K}}\right)^{1/(\delta-1)}. \tag{24}$$

Thus, if the number of initial firms is in an intermediate range of $(K^{\max}, N^+)$ the economy enters the period of sustained growth immediately, with V_0 settling somewhere between SS and VV. In an economy with too many existing competitors, however, profits are so small that no entrepreneur wants to invest in any R&D and thus no growth occurs. In the rest of this paper we assume (24).

4 Continuous Growth Regime

After the take-off period, the pattern of technology adoption and thus that of economic growth may not be unique. However, there is a special case in which the dynamics can be fully specified. Suppose that parameter μ is unity. Then, claim 4 and part d of claim 5 show that the amount of its knowledge $K_t(Z_t)$ is always kept at $\bar{K}$ and only the frontier technology is adopted by new entrepreneurs at each instant. This property implies that whenever there is a positive amount of R&D, the technology frontier, Z_t, must advance continuously without creating any mass point in $F_t(\cdot)$.[24]

In an economy where μ is smaller than unity, it may be possible that a mass of firms adopt the same technology, enabling them to utilize technology-specific components of knowledge. However, for the moment we suppose that no mass point is created in a certain time period. In such a situation, the second term in equation (1) vanishes and the pattern of knowledge accumulation becomes substantially equivalent to the case of $\mu = 1$. This implies that it is optimal for private entrepreneurs, who take knowledge distribution

under assumption 3. Then, the part b of claim 5 implies that no entrepreneur adopts technology $z = 1$ after $K_t(1)$ reaches $K^{\max}$.

[23]It follows from claim 3 that this value must settle somewhere between NN and VV. Under assumption 3, NN necessarily locates above VV when $N_t = K^{\max}$.

[24]Note that if a positive mass of firms adopted the same technology, then $K_t(Z_t)$ exceeds $\bar{K}$.

as given, to act similarly as in the economy with $\mu = 1$. That is, the entrepreneurs choose continuously advanced technologies without creating any mass point, consistent with the initial assumption. This consideration shows that, regardless of the value of μ, there is an equilibrium path which follows the continuous pattern of technology adoption. Below we derive the equilibrium dynamics in this 'continuous growth regime.'

Let us assume that any mass point does not emerge during the time interval in which the frontier locates in a certain interval $[Z^S, Z^E)$.[25] Combining claim 4 with part c of claim 5, we can confirm that the frontier continuously advances and only frontier technology is adopted in this regime, which corresponds to the $\mu = 1$ case. When there is no mass point, we can express $F_t(\cdot)$ in terms of a density function $f(\cdot)$,

$$F_t(z) = F_t(Z^S) + \int_{Z^S}^{\min\{z,Z_t\}} f(z')\,dz'. \quad \text{for } z \in [Z^S, Z^E). \tag{25}$$

Note that the value of $f(z)$ at each z does not depend on t, since it is only when $Z_t = z$ that new firms enter the market with technology z and thereafter the density at z does not change. For the amount of knowledge at the frontier to coincide with $\bar{K}$ at each date, density function $f(\cdot)$ must satisfy the following equation.

$$\left(\frac{Z_t}{Z^S}\right)^{-\delta} \bar{K} + \mu \int_{Z^S}^{Z_t} \left(\frac{Z_t}{z'}\right)^{-\delta} f(z')\,dz' = \bar{K} \quad \text{for } Z_t \in [Z^S, Z^E). \tag{26}$$

Differentiating both sides with respect to Z_t and conducting some transformation, equation (26) turns out to be equivalent to

$$f(z) = \frac{\delta \bar{K}}{\mu z} \quad \text{for } z \in [Z^S, Z^E). \tag{27}$$

Once the density of firms becomes clear, we can derive the speed at which the new technologies are adopted, using the labor market equilibrium conditions. Note that $l_t^{\min} = c(\bar{K})$ holds since only the frontier technology is adopted. Substituting it for (14), the equilibrium wage turns out to be $w_t = \max\left\{V_t/c(\bar{K}), \alpha/L\right\}$. The flow of the total labor input into R&D is calculated by substituting the above wage rate into the labor market clearing condition (8). On the other hand, from equation (27) we can calculate the cumulative input of R&D labor that is required for the technology frontier to move from Z_S to Z_t, as $\int_{Z^S}^{Z_t} c(\bar{K}) f(z')\,dz' = (\delta \bar{K} c(\bar{K})/\mu)\left(\ln Z_t - \ln Z^S\right)$. Differentiating this expression with respect to t and equating it to the derived flow of labor input, we have

[25] Here, Z^E may be infinite.

the growth rate of the technology frontier.

$$g_t \equiv \frac{\dot{Z}_t}{Z_t} = \max\left\{\frac{\mu}{\delta\bar{K}c(\bar{K})}\left(L - \frac{\alpha c(\bar{K})}{V_t}\right), 0\right\}. \tag{28}$$

This shows that the technology frontier advances if and only if V_t is greater than a certain value $\hat{V} \equiv \alpha c(\bar{K})/L$.

With equation (28) in hand, we can express the dynamics as an autonomous system of N_t and V_t in a similar fashion to our expression for the take-off period dynamics. When $F_t(\cdot)$ is expressed in the form of (25), where $f(\cdot)$ is given by (27), N_t becomes ${Z_t}^{-1}\left(\int_0^{Z^S} z\,dF_t(z) + (\delta\bar{K}/\mu)(Z_t - Z^S)\right)$. Differentiating this equation with respect to t, yields a simple expression $\dot{N}_t = (N^* - N)g_t$, where $N^* \equiv \delta\bar{K}/\mu$ is a stationary point of N_t. This equation can be expressed in terms of N_t and V_t by applying equation (28).

$$\dot{N}_t = \begin{cases} \dfrac{\mu}{\delta\bar{K}c(\bar{K})}\left(L - \dfrac{\alpha c(\bar{K})}{V_t}\right)(N^* - N_t) & \text{if } V_t > \hat{V}, \\ 0 \quad \text{if } V_t \leq \hat{V}. \end{cases} \tag{29}$$

As we have seen in the above, when V_t is less than or equal to $\hat{V}$, no firms enter the market and thus N_t does not change. When V_t is greater than V, new firms enter the market *and* the technology frontier advances. While the entry of new firms increases the total number of firms, the advance of the frontier decreases the relative profitability of existing firms to new frontier firms, which makes N_t small. Equation (29) shows that the overall effect on N_t is positive when N_t is less than N^*, while it is negative when $N_t > N^*$.

On the other hand, a fundamental rule of differentiation allows the time derivative of $V_t \equiv v_t(Z_t)$ to be deconstructed into two parts,

$$\dot{V}_t = \dot{v}_t(Z_t) + v_t'(Z_t)\dot{Z}_t, \tag{30}$$

where the first term represents the effect from variation in the value function $v_t(\cdot)$, while the second term is derived from the advance of the frontier.

The standard arbitrage condition gives the first term, $\dot{v}_t(Z_t) = \rho V_t - (1-\alpha)/N_t$. Since $v_t(z)$ is equivalent to $(z/Z_t)V_t$, its derivative is simply V_t/Z_t. Further, from the definition of g_t, $\dot{Z}_t = Z_t g_t$. Substituting these into formula (28), we have $\dot{V}_t = (\rho + g_t)\,V_t - (1 - \alpha)/N_t$. Then using equation (28), the dynamics can again be expressed in terms of N_t

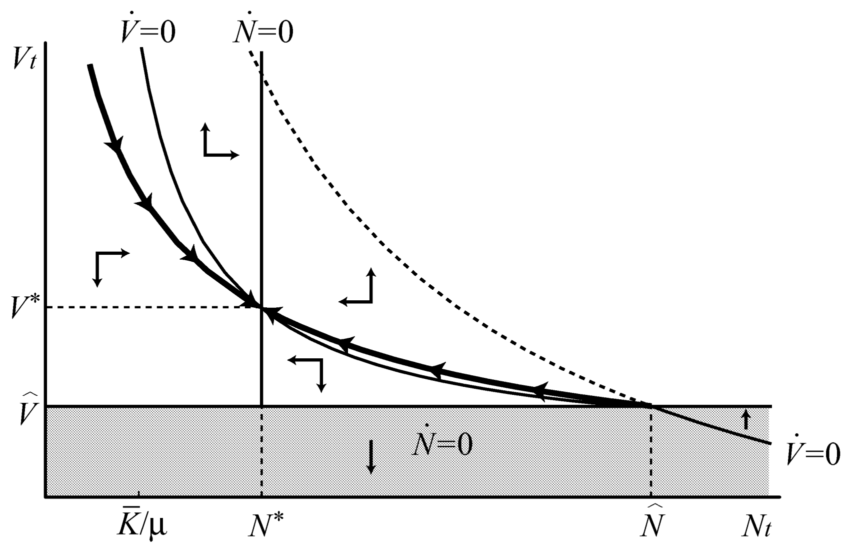

Figure 2: Phase diagram in the continuous growth regime.

and V_t ,

$$\dot{V}_t = \begin{cases} \left(\rho + \dfrac{\mu L}{\delta \bar{K} c(\bar{K})}\right) V_t - \dfrac{1-\alpha}{N_t} - \dfrac{\alpha\mu}{\delta\bar{K}} & \text{if } V_t > \hat{V}, \\ \rho V_t - \dfrac{1-\alpha}{N_t} & \text{if } V_t \leq \hat{V}. \end{cases} \tag{31}$$

By equating the above expression to zero, we also have a locus on which V_t does not change.

$$VV: \quad V_t = \begin{cases} \left(\rho + \dfrac{\mu L}{\delta \bar{K} c(\bar{K})}\right)^{-1} \left(\dfrac{1-\alpha}{N_t} + \dfrac{\alpha\mu}{\delta\bar{K}}\right) & \text{if } V_t > \hat{V}, \\ \dfrac{1-\alpha}{\rho N_t} & \text{if } V_t \leq \hat{V}. \end{cases} \tag{32}$$

If actual V_t is below this locus, V_t must fall to keep the arbitrage condition, while V_t rises when V_t is above the locus.

Equation (29) and (31) characterize the dynamics of (N_t, V_t) and enable us to draw a phase diagram. Figure 2 depicts the phase diagram, where the dotted curve represents the maximum possible value of a frontier firm, $(1-\alpha)/\rho N_t$, and the horizontal line at $V_t = \hat{V}$ expresses simultaneously the minimum possible cost of R&D, $\alpha l_t^{\max}/L$, and the $\dot{n} = 0$ locus. The configuration of these two curves shows that there is a positive amount of R&D

so long as N_t is smaller than the intersection of these two curves, $\hat{N} \equiv (1-\alpha)L/\alpha\rho c(\bar{K})$.[26] With the starting value of N_t and the final value of V_t specified, the dynamics of N_t and V_t within this regime is uniquely determined, and they also specify the dynamics of Z_t since (28) provides a one-to-one non-decreasing correspondence between V_t and the speed of technological progress. Specifically, when the economy stays in this regime forever, the initial value of V_t must be on the stable arm that converges to a saddle point (V^*, N^*), where

$$V^* \equiv \frac{\mu c(\bar{K})}{\delta\rho\bar{K}c(\bar{K}) + \mu L} \tag{33}$$

represents the value of a frontier firm on the balanced growth path.[27] On the balanced growth path, the technology frontier advances at a constant rate of

$$g^* = \frac{(1-\alpha)\mu L}{\delta\bar{K}c(\bar{K})} - \alpha\rho > 0. \tag{34}$$

Once the movements of N_t, V_t and Z_t are derived, it follows from equations (14) and (17) that consumption evolves according to

$$D_t = \frac{\alpha c(\bar{K})}{LV_t}(Z_t N_t)^{(1-\alpha)/\alpha}. \tag{35}$$

4.1 Comparison with the GH model

In this subsection, we compare the dynamics of our model in the continuous growth regime with that of the original GH model, where only one technology is adopted. This will make clear the difference caused by the introduction of multiple technologies, the modified R&D cost function, and spillovers across technologies.

First, let us focus on the workings of each model that make economic growth sustainable. In the GH economy, long-term economic growth is achieved through exponential increases in the flow of new goods. Such an explosion in variety is due to the assumption

[26] As we have seen in the previous section, the first technology adopted in the period of sustained growth is $(\mu N^E/\bar{K})^{1/\delta}$, where $N^E \in [K^{\max}, N^+)$ denotes the final value of N_t in the take-off period. Then, the starting value of the normalized number of firms is determined as $N_t = N^E/(\mu N^E/\bar{K})^{1/\delta} = (\bar{K}/\mu)^{1/\delta} N^{E(\delta-1)/\delta}$. Applying the fact that $K^{\max} \geq \bar{K}/\mu$ and the value of N^+ given by equation (24), the result is that N_t resides within a finite interval $[\bar{K}/\mu, \hat{N})$. In addition, it will become apparent that once this period starts with $N_t \in [\bar{K}/\mu, \hat{N})$, it will not get out of this interval whenever $K_t(Z_t) = \bar{K}$. Note also that N^* also resides in $[\bar{K}/\mu, \hat{N})$ under assumption 3.

[27] Otherwise the economy will enter the region below $\hat{V}$ or the one above the dotted curve, contradicting claim 3 either way.

that the labor cost of R&D is reduced at a constant rate as knowledge accumulates. By contrast, the economy in our model cannot grow through this mechanism because the cost becomes insensitive to a marginal increase in knowledge when it has become sufficiently small. Actually, the number of new goods introduced at each instant is constant on the balanced growth path, but the quantity (which can also be interpreted as quality) of each new product is steadily increasing because continuously more advanced technologies are adopted to produce the goods. Due to spillover effects, the cost of R&D based on every technology is declining, but at the same time entrepreneurs are willing to adopt increasingly advanced technologies which require large R&D labor inputs. In equilibrium, these two effects exactly cancel out and the cost becomes constant. Empirical observations show no exponential trend in the number of R&D projects or exponential decreases in the labor cost of R&D. Consequently, the latter story, expressed by our model, seems more plausible.

The second concern is the rate of economic growth, especially on the balanced growth path. The GH model specified the cost function by $c(k) = a/k$,[28] and derived the growth rate of consumption index D_t as

$$g^*_{D,GH} = \frac{(1-\alpha)^2 L}{a\alpha} - (1-\alpha)\rho.$$

To make the growth rate in our model comparable with the above equation, suppose the cost function of R&D takes a specific form $c(k) = a/k + b$, which is one of the simplest functional forms satisfying assumption 1.[29] The rate of economic growth on the balanced growth path is then calculated by applying (34) for equation (35),

$$g^*_D = \frac{1-\alpha}{\alpha} g^* = \frac{\mu}{\delta^2} \frac{(1-\alpha)^2 L}{a\alpha} - (1-\alpha)\rho. \tag{36}$$

It is observed that the only difference between the two expressions is the inclusion of μ/δ^2 in the first term of (36). That is, all the elements that enhance the growth rate in the GH model will also enhance the growth rate in our model. In addition, our model shows that the growth rate is also influenced by δ and μ, both of which concern the magnitude of

[28]Interestingly, if we weaken assumption 1 and let $c(k) = a/k$ in our model, the dynamics completely coincide with that of the GH model. Thus assumption 1 is crucial in the realization of technological progress.

[29]The small positive constant b can be interpreted as a fixed cost of R&D investment that cannot be eliminated by the accumulation of public knowledge. Note, however, that the assumption 1 does not necessarily require the existence of such a fixed cost.

R&D spillover effects across technologies. Specifically, when the larger portion of R&D experience can be applied to superior technologies, the term μ/δ becomes larger and hence the growth rate increases. This effect seems to have some relevance in the real process of economic growth.[30]

Third, we turn to the issue of convergence. In contrast to the neoclassical growth models that predict the convergence of economies, the GH model predicts that the proportional gap between economies will remain unchanged forever. Specifically, in the GH model, economies instantly jump to the balanced growth path and grow at the same constant speed. This absence of transitional dynamics is a by-product of the assumption of a one-to-one correspondence between the number of competitors (which determines the amount of profits) and the labor cost of R&D. Though such a correspondence is useful for many endogenous growth models to make growth sustainable, the dynamics in this section show it is not a necessary condition for the sustained growth: given the constant cost of R&D, $c(\bar{K})$, the number of competitors, N_t, can vary between $\bar{K}/\mu$ and $\tilde{N}$.

The phase diagram in figure 2 shows that N_t will converge to N^* in the long run, which seems somewhat like the dynamics in the neoclassical growth models. This does not mean, however, that convergence in the levels of GDP will occur, since GDP depends not only on the number of firms but also on their technologies. For example, suppose that there are two economies, both of which are in the continuous growth regime and differ only in the normalized number of firms, N_t. In transition, the amount of investments should be larger in the economy with small N_t since there are few competitors and thus expected profits are large.[31] Note that the difference in the amount of investments implies not only convergence in N_t, but also *divergence* in Z_t since the speed of technological progress, g_t, is proportional to investments in this regime. Thus, in the first half of the transition period, the difference in GDP between the two economies rapidly shrinks for both of the above reasons. However, before they settle into the balanced growth path, the initial gap in GDP is eventually reversed and, moreover, it is theoretically possible that the absolute value of the proportional gap becomes even larger than its initial value.

[30]Note that the growth rate does not depend on parameter b as long as it is positive. This implies that it is inappropriate to accept the GH model as an approximation when there is a positive lower bound in the labor cost of R&D, however small it is.

[31]This can be confirmed in the phase diagram where the stable arm is downward sloping.

Of course, such a reversal does not happen if the initial difference is not in N_t but in Z_t.[32] These examples suggest that the micro structure of the economy should be considered when the issue of convergence is analyzed.

Finally, there is yet another difference between the GH model and our model in the possibility of multiple equilibria. When μ is smaller than unity, there are significantly different equilibrium dynamics in which technology advances discretely rather than continuously. These will be investigated in the next section.

5 Cyclical Growth Regime

In the previous section, we concentrated on the situation where no mass point emerged in the distribution of firms, and demonstrated that there are equilibrium dynamics in which entrepreneurs choose continuously advanced technologies. When the economy follows such equilibrium dynamics, the technology-specific portion of knowledge is never accumulated. Without the opportunity to utilize technology-specific knowledge, there is no incentive for entrepreneurs to choose an existing technology at each instant. This explains exactly why no mass point emerges. In this sense, the behavior of each agent and macroeconomic movements are mutually consistent along this path. However, this consideration also suggests that there may be another equilibrium path in which these are mutually consistent but in a different way: when a positive mass of firms choose the same technology, this choice may become privately optimal for each firm, since they can utilize the technology-specific portion of accumulated knowledge to reduce R&D costs. Throughout this section, we focus on the economy where μ is less than unity and investigate these sorts of equilibrium dynamics by assuming that entrepreneurs create a sequence of discrete mass points in $F_t(\cdot)$ rather than choosing continuously advanced technologies.

Suppose that there is a small mass of firms that use the same technology z_0, which is the frontier technology at the time. When the number of firms adopting technology z_0 is n, this technology has amount $(1-\mu)n$ of its specific knowledge. This portion of knowledge makes technology z_0 relatively attractive for subsequent entrepreneurs because of low initial R&D costs. An additional adoption of this technology makes the cost of R&D even lower, resulting in a chain of adoption of the technology z_0 by new entrepreneurs.

[32] When the distribution of the two economies are given (18), where n_0 differs between them, reversal of the initial gap may or may not happen depending on the parameter values.

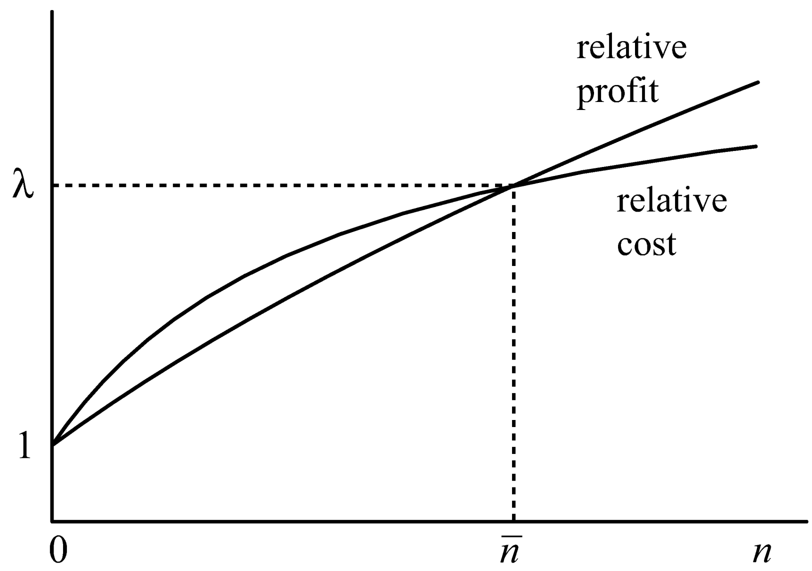

Figure 3: Determination of the size and time of technological improvement

However, this chain does not last forever, as shown in claim 5. Entrepreneurs eventually switch to a certain advanced technology, λz_0, when the number of firms adopting technology z_0 reaches a certain threshold, $\bar{n}$. Below, we show the ways in which the size of improvement and the threthould are determined.

As the number of firms with technology z_0 increases, the amount of knowledge on advanced technologies increases due to the R&D spillovers across technologies. Specifically, if an entrepreneur devotes $c(\bar{K})$ units of labor,[33] he or she can create a good by adopting a new technology $z = \left(\mu n/\bar{K} + 1\right)^{1/\delta} z_0$.[34] Since the amount of profit is proportional to the technology choice, the relative profit that can be gained by adopting the new technology, rather than z_0, is $\left(\mu n/\bar{K} + 1\right)^{1/\delta}$, which implies that the incentive to adopt a superior technology increases as n gets larger. On the other hand, the labor cost of R&D based on technology z_0 gradually falls due to the spillover effects within this technology. Specifically, the relative cost of adopting the new technology is $c(\bar{K})/c(\bar{K} + n)$, which means that the incentive to stick with technology z_0 is also increasing in n.

Figure 3 depicts the variations of relative profit and relative cost as the number of

[33]Recall that whenever an entrepreneur adopts a technology superior to the current frontier, the amount of knowledge on this technology must be $\bar{K}$ and thus the labor cost must be $c(\bar{K})$.

[34]This equation follows from $K_t(z) = \bar{K}$.

firms at the frontier increases. It shows that the relative cost dominates the relative profit for small values of n.[35] For these values of n, it is profitable for entrepreneurs to choose technology z_0 and thus the frontier does not advance. However, as the knowledge on technology z_0 grows, the cost of R&D becomes insensitive to a marginal increase in knowledge, which causes a slowdown in the increase of the relative cost. Eventually the magnitude of relative cost coincides with that of relative profit when n reaches a certain threshold value $\bar{n}$.[36] At this point, entrepreneurs are indifferent as to the choice between the current or new technology, but after that point the adoption of the new technology becomes more profitable.[37] Thus, immediately after the number of firms reaches the maximum value $\bar{n}$, the technology frontier jumps to the new technology, where the relative improvement of technology is the same as the relative profit at this time, $\lambda \equiv (\mu\bar{n}/\bar{K} + 1)^{1/\delta} = c(\bar{K})/c(\bar{K} + \bar{n})$. Entrepreneurs then continue to adopt technology λz_0 until an even more advanced technology $\lambda^2 z_0$ is adopted, when the number of firms with technology λz_0 reaches $\bar{n}$. This pattern of technological adoption creates cyclical growth dynamics, which we call the 'cyclical growth regime.'

To investigate the movements of macroeconomic variables, we first derive the dynamics within 'one cycle', that is, within the time interval throughout which all entrepreneurs adopt the same technology. Let $N^S \in [\bar{K}/\mu, \hat{N})$ denote the value of N_t at the beginning of the cycle.[38] Initially the amount of knowledge at the frontier is $\bar{K}$ and it increases with the number of firms at the frontier, which yields $K_t(Z_t) = N_t - N^S + \bar{K}$.[39] The motion of N_t within a cycle is then obtained in essentially the same way as we have derived (19).

$$\dot{N}_t = \max\left\{\frac{L}{c(N_t - N^S + \bar{K})} - \frac{\alpha}{V_t}, 0\right\}. \tag{37}$$

Note that the above expression includes $c(N_t - N^S + \bar{K})$ not $c(N_t)$, which is the only

[35]The derivative of the relative profit with respect to n at the point $n = 0$ is $\bar{K}/\delta$, while that of the relative cost is $\bar{K}/\delta\mu$. Note that the latter is larger as long as $\mu < 1$.

[36]It is straightforward to show that a unique solution exists to $\left(\mu n/\bar{K} + 1\right)^{1/\delta} = c(\bar{K})/c(\bar{K} + n)$ in the region of $n > 0$.

[37]Figure 3 shows that the adoption of the new technology becomes more profitable after n reaches $\bar{n}$ assuming that only the old technology is adopted. Actually, entrepreneurs start to adopt the new technology after that, which makes the profitability of adopting the new technology even higher relative to the old technology.

[38]See footnote 26.

[39]Note that the number of firms at the frontier is $N_t - N^S$.

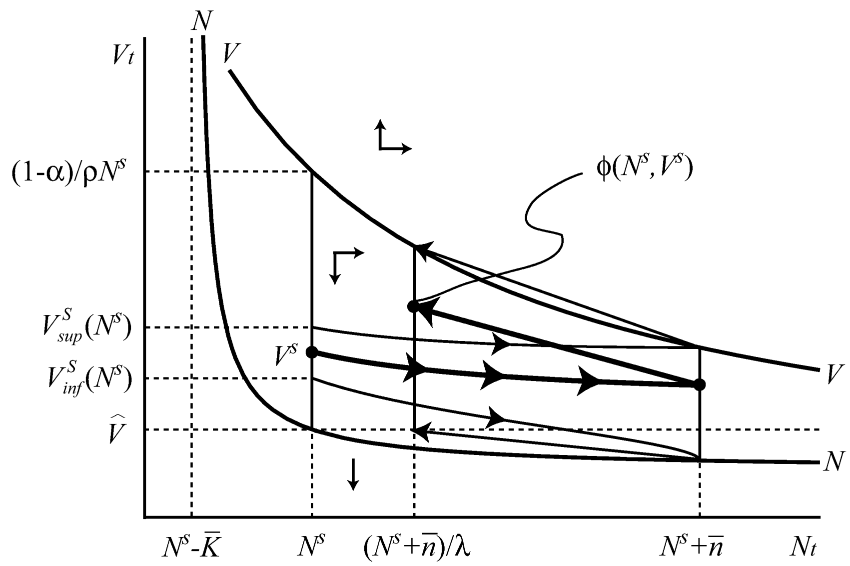

Figure 4: Dynamics within a cycle

difference between (19) and (37). Accordingly, the $\dot{N}=0$ locus becomes

$$NN: \quad V_t = \frac{\alpha c(N_t - N^S + \bar{K})}{L}. \tag{38}$$

From the arbitrage conditions we can also obtain the time derivative of V_t and the $\dot{V}=0$ locus, which are exactly the same as (20) and (22), respectively.

Figure 4 depicts the phase diagram of one cycle dynamics, defined by (37) and (20), along with a typical path. As shown in claim 3, the whole path must locate between the NN and VV curves. This condition is satisfied if and only if the initial value of V_t in this cycle, denoted by V^S, locates between $V^S_{\text{inf}}(N^S)$ and $V^S_{\text{sup}}(N^S)$, given in the figure.[40] When a cycle starts from a state (N^S, V^S) where V^S satisfies the latter condition, N_t reaches the critical point $N^S + \bar{n}$ in a finite time.[41] The technology frontier then jumps to a new technology which is λ times as productive as what it was. With the jump of Z_t, $N_t \equiv Z_t^{-1} \int z\, dF_t(z)$ is divided by λ, while $V_t \equiv v_t(Z_t)$ is multiplied by λ. This whole process constitutes a one-to-one mapping from the state of the economy

[40]Here, $V^S_{\text{inf}}(N^S)$ and $V^S_{\text{sup}}(N^S)$ are obtained by following the dynamics backward from the points $(N^S+\bar{n}, \hat{V}/\lambda)$ and $(N^S+\bar{n}, (1-\alpha)/\rho(N^S+\bar{n}))$, respectively. From this procedure, it is obvious that $\hat{V} < V^S_{\text{inf}}(N^S) < V^S_{\text{sup}}(N^S) < (1-\alpha)/\rho N^S$ holds.

[41]This is guaranteed under assumption 3.

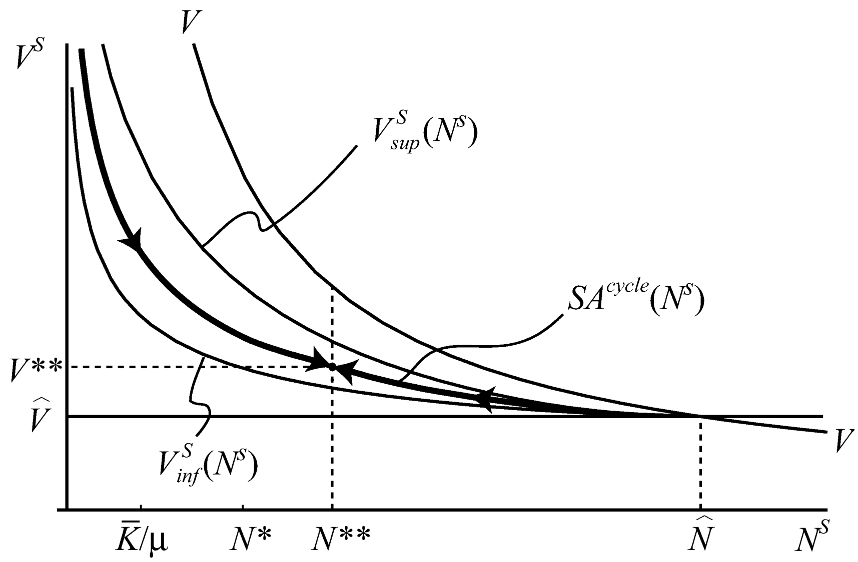

Figure 5: Phase diagram of $\Phi(\cdot,\cdot)$

at the beginning of a cycle, (N^S, V^S), to that immediately after the cycle, which we represent by a vector function $\Phi(N^S, V^S) \equiv ((N^S + \bar{n})/\lambda, \phi(N^S, V^S))$ defined in a space $\{(N, V)|\bar{K} < N < \hat{N}, V^S_{\text{inf}}(N) < V < V^S_{\text{sup}}(N)\}$.

The discrete dynamics of $\Phi(\cdot,\cdot)$ characterize the movements of cycles within the cyclical growth regime. The first element of the mapping, $(N^S + \bar{n})/\lambda$, shows that after many cycles N^S will converge to a stationary point $N^{**} = \bar{n}/(\lambda - 1)$.[42] By contrast, the following shows that the movement of V^S is unstable.

Claim 6 *The slope of $\phi(N^S, V^S)$ with respect to V^S is larger than λ for all $N^S \in (\bar{K}, \hat{N})$ and $V^S \in (V^S_{\text{inf}}(N_t), V^S_{\text{sup}}(N_t))$.*

Note also that, for any given N^S, $\phi(N^S, V^S)$ is a continuous function of V^S and the domain of this function, $(V^S_{\text{inf}}(N^S), V^S_{\text{sup}}(N^S))$, is contained in its region $(\hat{V}, (1-\alpha)/\rho N^S)$. These facts guarantee that there is a unique stationary point of $\Phi(\cdot,\cdot)$ and that this point has a saddle property.

Let us denote the saddle point by (N^{**}, V^{**}). Solving the discrete dynamics of $\Phi(\cdot,\cdot)$ backwards from (N^{**}, V^{**}), we can find the unique stable arm $V^S = SA^{cycle}(N^S)$ that

[42] N^{**} is greater than the stationary point in the continuous growth regime, N^*.

satisfies

$$\lim_{m\to\infty} \Phi^m(N^S, SA^{cycle}(N^S)) = (N^{**}, V^{**})$$

for all $N^S \in [\bar{K}/\mu, \hat{N})$. Figure 5 depicts the stable arm and the movements of N^S and V^S. If a cycle starts from a state off the stable arm, the deviation of V^S from the stable arm is multiplied by more than $\lambda > 1$ in each cycle and the economy will enter the infeasible region in finite cycles. Thus, provided that the economy stays in the cyclical growth regime permanently, the initial value of V^S is determined by $SA^{cycle}(N^S)$ and the economy eventually converges to the stationary cycle, that is, the cycle starting from (N^{**}, V^{**}). Combining the paths of N^S and V^S derived above with the dynamics of N_t and V_t within a cycle, and from the fact that the frontier technology is multiplied by λ at the beginning of each cycle, the movements of N_t, V_t and Z_t in the cyclical growth regime are uniquely determined. Then, from (14) and (17), the consumption of a representative consumer follows

$$D_t = \frac{\alpha c(N_t - N^S + \bar{K})}{LV_t}(Z_t N_t)^{(1-\alpha)/\alpha}. \tag{39}$$

5.1 Movements of Real Aggregate Variables

Let ω_t denote the real wage at time t, which also represents the real labor income of the representative consumer since each consumer inelastically supplies one unit of labor service. From the fact that the reciprocal of the price index coincides with the aggregate consumption, which is given by (17), the real wage rate can be calculated as

$$\omega_t \equiv w_t/\bar{P}_t = \alpha(Z_t N_t)^{(1-\alpha)/\alpha}. \tag{40}$$

Since $Z_t N_t$ represents $\int z dF_t(z)$, which increases smoothly,[43] real income always grows monotonically and smoothly.

Dividing both sides of equation (14) by the price index, the real value of a frontier firm is calculated as $V_t/\bar{P}_t = \omega_t l_t^{\min}$. This expression enables us to derive the movement of the real interest rate, denoted by R_t, within a cycle from the arbitrage condition in real terms.

$$R_t = \frac{\dot{\omega}_t}{\omega_t} + \frac{\dot{l}_t^{\min}}{l_t^{\min}} + \frac{1-\alpha}{N_t V_t}. \tag{41}$$

[43] Using equations (8), (9) and (14), the time derivative of $\int z dF_t(z)$ is calculated as $(Z_t/l_t^{\min})L - \alpha/v_t(1)$. Here, $Z_t/l_t^{\min} = \max_z z/c(K_t(z))$ varies continuously due to the envelope theorem. It is also clear from equation (10) that $v_t(1)$ is continuous in t. Thus, the time derivative of $\int z dF_t(z)$ is continuous in t, which means that $\int z dF_t(z)$ changes smoothly.

The above equation shows that the real interest rate is the sum of the growth rate of the real wage, that of R&D cost, and the reciprocal of the price-earning ratio, all of which vary continuously within each cycle.[44] However, as shown below, the real wage drops discontinuously when a new cycle starts. Note that the sensitivity of R&D cost to knowledge becomes discretely larger when the new technology is adopted, since there is a relatively small amount of knowledge about the new technology. This causes a sudden spur in the reduction of the labor cost of R&D, which means a discrete reduction in the second term of (41).[45] Thus, the real interest rate jumps downward when a new cycle starts.

Given the streams of real income and the real interest rate, consumption by a representative consumer with a log utility function is determined according to $\dot{D}_t/D_t = R_t - \rho$. The time path of consumption is smooth within each cycle, but not on the points between cycles since R_t drops discontinuously when new technologies are adopted. This implies that each cycle starts with a small, possibly negative, growth rate of consumption relative to the average growth rate within the whole cycle. As a result, the time path of consumption has a cyclical component.

Recall that consumption D_t also represents per capita production at each date. Using real wage ω_t, the total amount of production is expressed as $LD_t = \alpha^{-1}\omega_t(L - L_t^R)$, where $L - L_t^R = \alpha l_t^{\min}/V_t$ represents the number of production workers. This expression clarifies that the initial decline in the growth rate of D_t is associated with a sudden increase in the labor flow from production towards R&D. By adding the real value of new firms created through R&D, $\int (v_t(z)/\bar{P}_t)\, dR_t(z) = L_t^R \omega_t$, to the amount of production, real GDP is obtained,

$$\mathrm{GDP}_t = \left[\frac{1-\alpha}{\alpha}(L - L_t^R) + L\right]\omega_t. \tag{42}$$

Although the addition of R&D value reduces fluctuations in GDP relative to that in D_t, GDP is still affected by the distribution of the labor force between production and R&D. Note that while R&D activities are competitive, the goods market is monopolistic and

[44]Since the time path of the real wage is smooth, its growth rate is continuous. The denominator of the third term can be broken down into $N_t V_t = (N_t Z_t) v_t(1)$. As shown in footnote 43, both $N_t Z_t$ and $v_t(1)$ are continuous in t, and thus the third term is continuous. Finally, the growth rate of $l_t^{\min} = c(N_t - N^S + \bar{K})$ is also continuous in t *within* a cycle since $\dot{N}_t$ changes continuously according to (37) within a cycle.

[45]Let denote $K_t(Z_t)$ by k. Then, $\dot{l}_t^{\min}/l_t^{\min} = (c'(k)/c(k))(dN_t/dt) = (c'(k)/c(k)^2)L_t^R = -\epsilon_c(k)(c(k)k)^{-1}L_t^R$. Since both $\epsilon_c(k)$ and $(c(k)k)^{-1}$ are decreasing in k as long as $k \geq \bar{K}$ and L_t^R does not jump, the absolute value of $\dot{l}_t^{\min}/l_t^{\min}$ jumps upward when k drops.

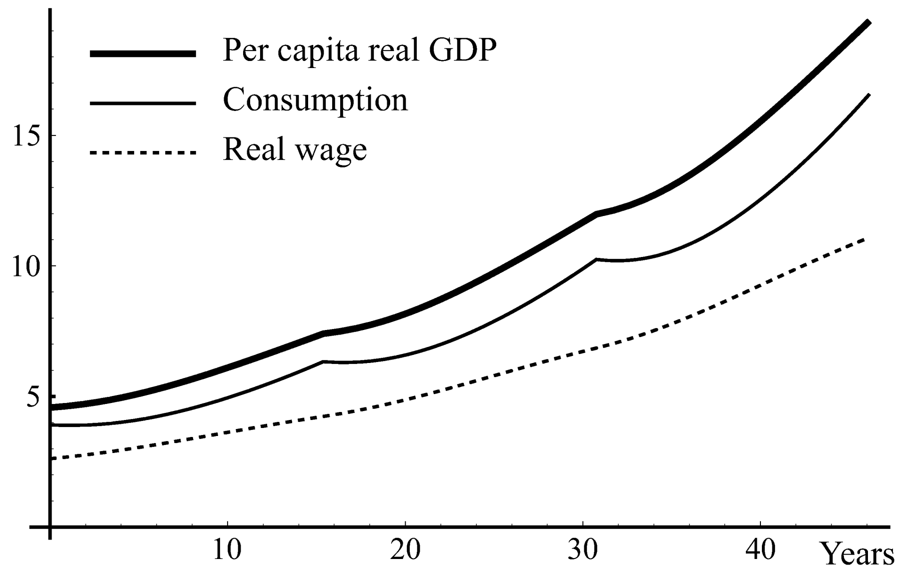

Figure 6: Movements of aggregate variables within three consecutive stationary cycles. $\delta = 2$, $\mu = 0.5$.

there is a positive markup. Thus, an increase in the labor flow from production toward R&D brings about a discontinuous fall in GDP growth, since it diverts resources away from the monopolistic sector to the competitive sector.

Figure 6 shows a representative time path of the real wage (ω_t), consumption (D_t), and per capita real GDP (GDP_t/L), within a few consecutive stationary cycles, calculated numerically.[46] All three variables have the same average growth rate ($(1-\alpha)/\alpha)\log\lambda/T^{**}$, where T^{**} represents the duration of each stationary cycle. In contrast to the steady increase in the real wage, consumption has a large fluctuation including negative growth at the beginning of each cycle and sustained growth only a few years after the adoption of the new technology, though exact dynamics depend on parameter values. The fluctuation in GDP is relatively small because of the inclusion of R&D value, but its pattern is essentially the same as that of consumption.[47]

[46] In all simulation results presented in this paper, we set parameter values to $\alpha = 0.5$, $\rho = 0.025$, and $L = 1$, which are the same as those adopted in the numerical simulation in the HT model. We use a simple R&D cost function $c(k) = 1/k + 1$ that satisfies assumption 1. Here, the two parameters of the spillover function are set to $\delta = 2$ and $\mu = 0.5$, though various values are examined later.

[47] This kind of downturn in economic performance immediately after the adoption of new technology has been referred to by some economists as *productivity slowdown puzzle*. See Greenwood and Yorukoglu

5.2 Relationships with the literature of GPTs

The dynamics investigated in this section are characterized by dramatic innovations in production technology and resulting discontinuities in the evolution of output and the pattern of resource allocation. These outcomes are consistent with a number of observations made by economic historians in which major improvements in technology had far-reaching and prolonged implications, such as the steam engine, electricity, and the computer. To stylize these dramatic historical innovations, the notion of *general purpose technologies* (GPTs) was recently introduced by Bresnahan and Trajtenberg (1995), and subsequently a number of theoretical research studies were carried out to investigate the relationship between GPTs and economic growth.

Although our model did not assume the existence of GPTs in its setup, it turns out that each technology, chosen endogenously in the cyclical growth regime, has four characteristics which are necessary and sufficient for any technology to qualify as a GPT.[48] First, a new technology has scope for improvement. Initially at the time of adoption the labor cost of R&D is a high value of $c(\bar{K})$, but it is continually reduced until $c(\bar{K})/\lambda$ at the end of cycle.[49] Second, each technology is used to produce a wide variety of products. Throughout each cycle, all entrepreneurs adopt the same technology to produce distinct products for different purposes. Third, the range of the new technology, defined as the proportion of productive activities in the economy using that technology, becomes considerably wider within the cycle. Specifically, the share of production based on the current frontier technology is calculated as $(N_t - N^S)/N_t$, which initially starts at zero and continuously increases until it reaches the maximum value $\bar{n}/(N^S + \bar{n})$ at the end of the cycle.[50] After that, the share of technology z gradually declines according to $z\bar{n}/Z_t N_t$.[51] Fourth, there are strong technological complementarities between R&D activities through spillovers of knowledge. Due to the spillover effect within the current

(1994) and Hornstein and Krusell (1996). Note also that our model predicts low real interest rates during the periods of slowdown.

[48]These criteria were presented by Lipsey, Bekar and Carlaw (1998).

[49]Our specification limits the improvement to the R&D cost because the number of firms is normalized so that their products enter the utility function symmetrically. Nonetheless, it is possible to reinterpret this cost reduction as an improvement in the quality of products by adopting another measure of firms.

[50]In a stationary cycle, this maximum value is $(\lambda - 1)/\lambda$, which is reasonably large.

[51]On this point, the workings of our model are more realistic than the dynamics of the HT model, where only one GPT is used in production at each date.

frontier technology, adoption of that technology by some firm makes it more attractive for subsequent entrepreneurs to adopt the same technology. Moreover, there exist spillovers across technologies, which enable the economy to shift to more advanced technology when the scope of cost reduction in the current technology has largely been exploited.

One major contribution of this paper is that it endogenizes the process by which GPTs emerge from a continuum of potentially available technologies. While existing models take the time of arrival of new GPTs and/or the productivity of GPTs as exogenous, our model explains the way in which both are determined. In addition, it shows that growth with GPTs is not the only possibility: as investigated in the previous section, there is another regime of economic growth where new technologies are continuously adopted and any single technology does not have a positive share (hence it is not a GPT). Thus, in our framework, it is possible to examine whether or not the formation of GPTs is beneficial for the economy. The next section deals with this issue.

6 Welfare

In the previous sections we have derived two patterns of equilibrium dynamics in the period of sustained growth, respectively called the continuous and cyclical growth regimes.[52] In a given economy, both patterns of dynamics may be realized and, moreover, it is possible that the economy switches from one regime to another at certain points in the equilibrium path.[53] This implies that there are many perfect foresight equilibria, each of which differs in the pattern of technological adoption. The pattern of technological adoption affects the evolution of consumption of a representative consumer, and consequently

[52]The dynamics in these two regimes are the only equilibrium dynamics that satisfy the condition that no entrepreneur adopts technologies that are behind the current frontier. Any equilibrium path not satisfying this condition is unstable because such an equilibrium path must contain a finite time period in which multiple technologies are adopted. Consequently, it has a 'knife-edge' property in that, once the distribution of R&D over these technologies is disturbed, these multiple technologies cannot have the same profitability over that time period and thus only one technology will be adopted. Unless all entrepreneurs are able to coordinate their pattern of technology adoption, such a path is unlikely to be realized.

[53]An economy can switch to the cyclical growth regime from any point in the continuous growth regime. By contrast, when an economy is in the cyclical growth regime, it can switch to the continuous growth regime only after the current cycle is finished (that is, only after the frontier jumps to a new technology). It is possible to show that once the pattern of switching between the two regimes is specified, a unique perfect foresight equilibrium exists that follows that pattern.

consumer welfare is also affected.

This section investigates whether discrete technological progress with GPTs is more desirable than continuous technological progress, in terms of the welfare of the representative consumer. First, we calculate the utility of the consumer in each regime, partly with the aid of a computer, and show that the desirable pattern of technology adoption differs depending on parameters. This result will then be interpreted by showing that there are both advantages and disadvantages in forming GPTs.

On the balanced growth path (BGP) of the continuous growth regime, N_t and V_t stay at their stationary values, N^* and V^*, respectively, and the technology, Z_t, progresses at a constant rate of g^*. Applying these facts for (35) and then substituting this into (3), we obtain the utility of the representative consumer on the BGP of the continuous growth regime.

$$U^*(Z_t) = \frac{1}{\rho}\left[\ln \frac{\alpha c(\bar{K})}{LV^*} + \frac{1-\alpha}{\alpha}\left(\ln N^* + \frac{g^*}{\rho}\right)\right] + \frac{1-\alpha}{\alpha\rho} Z_t.$$

On the other hand, state variables are cyclically evolving in the cyclical growth regime, even after both N^S and V^S settle to their stationary values. Let N_τ and V_τ ($0 \le \tau \le T^{**}$) denote the evolution of the state variables within the stationary cycle,[54] which is uniquely determined by (37) and (20) along with initial conditions N^{**} and V^{**}. Then, substituting these paths into (39) yields the movements of consumption within the stationary cycle, which can be denoted by $Z_t^{(1-\alpha)/\alpha} D_\tau^{**}$. While the technology, Z_t, is unchanged during one cycle, it is multiplied by λ when the economy enters the new cycle, implying that the whole path of consumption in the new cycle is $\lambda^{(1-\alpha)/\alpha}$ times larger than the last one. From these considerations, we can derive the utility level measured at the starting point of a cycle:

$$U^{**}(Z_t) = \frac{1}{1-e^{-\rho T^{**}}}\left[\int_0^{T^{**}} \ln D_\tau^{**} d\tau + e^{-\rho T^{**}} \frac{1-\alpha}{\alpha\rho} \ln \lambda\right] + \frac{1-\alpha}{\alpha\rho} Z_t.$$

The above two expressions give the utility of the representative consumer in the steady state of each regime, but they are not sufficient to judge which regime is better in a welfare sense since it is not possible to switch from the steady state of one regime to that of the other immediately. Suppose that the economy is currently on the BGP of the continuous growth regime and then switches permanently to the cyclical growth

[54]Here index τ is used instead of t to express the time passed after the current cycle started. Recall that T^{**} represents the time length of the stationary cycle.

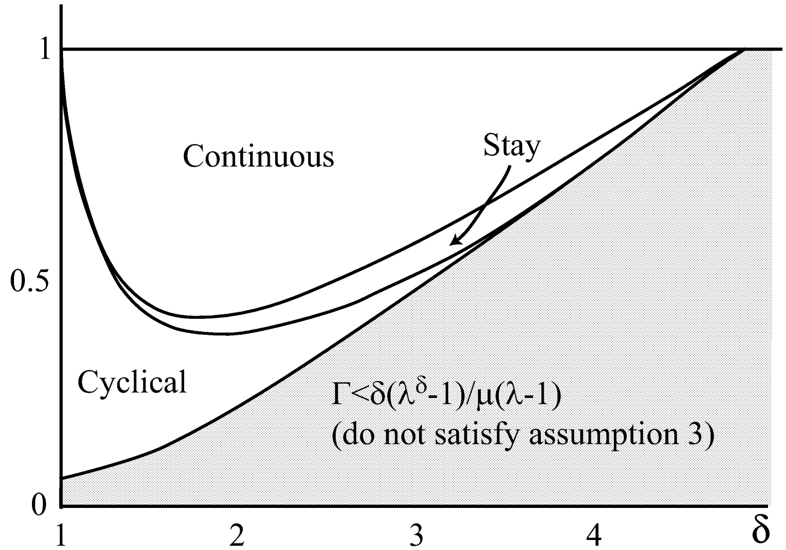

Figure 7: Comparison of welfare between the two regimes.

regime. In this case, the first cycle in the cyclical growth regime starts from $N^S = N^*$ and then, following the stable arm of $\Phi(\cdot, \cdot)$ shown in figure 5, the economy gradually converges to the stationary cycle, where $N^S = N^{**}$. Likewise, if the economy switches from the stationary cycle to the continuous growth regime permanently, the transitional dynamics start from $N_t = N^{**}$ and gradually converge to the BGP, where $N_t = N^*$, following the stable arm depicted in figure 2.

For various values for parameters,[55] we have numerically calculated the utility of the representative consumer when the economy switches from the steady state of one regime to the other regime, and compared them with $U^*(Z_t)$ and $U^{**}(Z_t)$. The results are summarized in figure 7, which shows that there are three regions in the relevant parameter space. When the parameters belong to the lower region labeled 'Cyclical' the cyclical growth regime is desirable regardless of the initial state of the economy.[56] Alternatively, when the parameters are in the upper region labeled 'Continuous' the

[55]In this simulation, we have focused on δ and μ, the parameters of the spillover function, since the most important difference between the two regimes is in the pattern of knowledge accumulation which depends on these parameters. For other parameters, see footnote 46.

[56]That is, the utility of remaining in the stationary cycle is larger than the utility gained from switching from it to the continuous growth regime, and the utility of staying at the BGP is smaller than the utility gained from switching from it to the cyclical growth regime.

continuous growth regime is desirable. In the intermediate region labeled 'Stay', it is better to stay in the current regime than to incur welfare losses during the transition.

This mixed result implies that there are both advantages and disadvantages in the formation of GPTs compared to shifting continuously to the advanced technologies. The magnitudes of these effects are influenced by the parameters of the spillover function, δ and μ. As we have seen in the previous section, when a GPT is formed the cost of R&D is efficiently reduced through the utilization of the technology-specific portion of knowledge. In fact, when μ is smaller than unity (i.e., there is a positive portion of knowledge that cannot be applied to other technologies), the growth rate of $Z_t N_t$ is larger in the cyclical growth regime, at least at the beginning of a cycle, than in the continuous growth regime for a given schedule of R&D labor input.[57] Thus, for some time after a cycle starts, the formation of a GPT promotes growth measured in terms of $N_t V_t$, which is beneficial for consumers since the per capita consumption is given by $(Z_t N_t)^{(1-\alpha)/\alpha}$ times the portion of production labor. This effect is stronger when μ is smaller, which is consistent with the configuration in figure 7.

However, in the long term, GPTs may harm growth. Once a GPT is formed, it creates a discrete difference between the cost of R&D based on the GPT and those based on other technologies. Consequently, entrepreneurs are not willing to switch to advanced technologies until a technology with a sufficiently high productivity becomes available at a certain cost of adoption. Thus, the productivity of the next technology must be significantly higher than that of the current GPT. However, this means the spillover of knowledge from the experiences in the current GPT to the next technology is small in magnitude, especially when δ is large.[58] In such a case, the adoption cost of new technology declines only slowly. This makes the economy stick to the current GPT even after the cost of R&D based on this technology becomes insensitive to the additional amount of knowledge. By contrast, in the continuous growth regime, spillovers between adjacent technologies promote adoption of advanced technologies and firms do not stick to any technology. In fact, unless δ is close to unity, the cumulative amount of R&D

[57]Let $A_t \equiv \int_0^t L_t^R dt$ denote the cumulative input of R&D labor. Then, at the beginning of a new cycle, $d(Z_t N_t)/dA = Z_t / c(\bar{K})$ and $d^2(Z_t N_t)/dA^2 = Z_t / \delta \bar{K} c(\bar{K})^2$. In the continuous growth regime, $d(Z_t N_t)/dA = Z_t / c(\bar{K})$ and $d^2(Z_t N_t)/dA^2 = \mu Z_t / \delta \bar{K} c(\bar{K})^2$. While the first derivatives are the same, the second derivative for the case of cyclical growth is larger than that for continuous growth, as long as $\mu < 1$.

[58]Note that a large δ means that the magnitude of spillover is highly sensitive to the difference in the productivity of technologies.

labor input that is required for the economy to switch to the next technology in the cyclical growth regime is larger than the amount required to reach the same technology in the continuous growth regime. This implies that, in the long run, the average rate of growth tends to be larger in the continuous growth regime than in the cyclical growth regime.

The relative gap in the average rate of growth between the regimes increases with δ, but this does not mean that the welfare disadvantages of the formation of a GPT monotonically increase with δ. As δ gets larger, the growth rate itself declines in both regimes. In addition, in the cyclical growth regime the length of one cycle becomes longer. When δ is very large, the delay in the adoption of the next technology caused by the formation of a GPT has only a minor effect on overall welfare, since consumption in such a distant future is heavily discounted. In fact, figure 7 shows that not only when δ is small but also when it is very large, the cyclical growth regime is more desirable than the continuous growth regime, while the disadvantages of a GPT outweighs its advantages when δ is an intermediate value (provided that μ is not small).

7 Conclusion

We have demonstrated that when there is a continuum of technologies differing in productivity, the pattern of growth is not unique. When the portion of technology-specific knowledge is large, it is desirable to form a sequence of GPTs. These GPTs produce cyclical evolutions in the macroeconomic variables, including a slowdown in GDP growth immediately after the economy switches to a new technology. On the other hand, in an economy where the portion of knowledge that can be applied to adjacent technologies is not quite so small, but it is reasonably difficult to apply knowledge to other technologies that differ substantially in productivity, shifting continuously to advanced technologies results in higher welfare than forming GPTs. In this case, all the macroeconomic variables are evolving smoothly, converging to the balanced growth path where GDP grows at a constant rate.

It is not guaranteed, however, that technologies are adopted in the desirable pattern since both patterns (and their possible hybrids) always constitute perfect foresight equilibria and it depends on the expectations of entrepreneurs which pattern is realized.[59] This suggest the possibility that some policies may improve the welfare of the

[59]In reality, it may even possible that agents have different expectations, though this viorate the

representative consumer by affecting the pattern of technological adoption. Specifically, with various measures to promote the adoption of specific technologies, such as direct subsidies or public research to aid the accumulation of knowledge, the government can manipulate the relative profitability of adopting various technologies, $l_t(\cdot)$, and lead the economy to the desirable regime.[60] In general, even when the economy is currently in the desirable regime, there remains room for government intervention since R&D has positive externalities. While this property is common to many endogenous growth models, our model suggests that policies must be implemented in a careful way: if the possibility of a regime change is not considered, some policies that are intended to improve welfare, assuming that the economy stays in the current regime, may worsen the situation by leading it to the Pareto inferior regime.

Appendix A: Proof of Claims

A.1 Proof of Claim 1

Let us define a function $g(x) \equiv \ln s(1, e^x)$. Note that $s(z_0, z) = s(1, z/z_0)$ holds since $s(\cdot, \cdot)$ is homogeneous of degree 0. Then, for any given $\zeta \equiv \ln(z/z_0)$, property d of assumption 2 implies that $g(\xi) + g(\zeta - \xi)$ does not depend on $\xi \equiv \ln(z'/z_0) \in (0, \zeta)$. From this, it is clear that the derivative of $g(\xi) + g(\zeta - \xi)$ with respect to ζ, $g'(\zeta - \xi)$, does not depend on $\xi \in (0, \zeta)$. Since ζ can be any positive number, the result is that $g'(x)$ is constant for all $x > 0$. Thus, for some constants c_0 and c_1, $g(x)$ can be expressed as $c_0 + c_1 x$. Then, we obtain $s(z_0, z) = e^{g(\ln(z/z_0))} = e^{c_0 + c_1 \ln(z/z_0)} = e^{c_0}(z/z_0)^{c_1}$ for any

conventional assumption of rational expectations. For example, suppose that there are two groups of entrepreneurs. One group expects that the economy will follow the continuous growth regime, while the other expects the cyclical growth regime to occur. Each group carries out R&Ds according to their belief. Depending on which belief is held, the discounted sum of profits that can be gained from investing in R&D, V_t, differs. In the labor market, the group of entrepreneurs who believe V_t is larger than the other group believes, hires labor for R&D until the wage level rises to satisfy the free entry condition under their belief. Since the other entrepreneurs believe that their investments will not pay at this wage level, only those entrepreneurs who expect large V_t invest in R&D and, as a result, their expectations are actually realized. The outcome is essentially the same as the perfect foresight equilibrium where all entrepreneurs expect the regime that realizes large V_t. However, since the high value of V_t means that the goods market in the future is less competitive, it tends not to be the desirable equilibrium in terms of welfare.

[60] To this end, the scale of intervention (such as the amounts of subsidies) can be arbitrarily small, which implies that a marginal intervention can improve welfare discretely.

$z > z_0$ and $s(z_0, z) = e^{c_0}(z_0/z)^{c_1}$ for $z < z_0$ from the assumption of symmetry. For the function $s(\cdot, \cdot)$ to be bounded in $(0, 1]$, e^{c_0} must be bounded in $(0, 1]$. From the property e, c_1 must be smaller than -1. This completes the proof.

A.2 Proof of Claim 2

We first show that $\mathcal{L}_t^{\min}$ is bounded. Let us introduce a function $\tilde{l}_t(\zeta) \equiv \ln c(K_t(Z_t)e^{-\delta\zeta}/\mu) - \zeta$, which is no larger than $\ln l_t(Z_t e^{\zeta}) = \ln c(K_t(Z_t e^{\zeta})) - \zeta$ since $K_t(Z_t)(z/Z_t)^{\delta}/\mu \geq K_t(z)$ holds for all z. Its derivative is $\tilde{l}_t'(\zeta) = \delta\epsilon_c(K_t(Z_t)e^{-\delta\zeta}/\mu) - 1$. From assumption 1, we have $\lim_{\zeta\to\infty} \tilde{l}_t'(\zeta) > 0$ and $\lim_{\zeta\to-\infty} \tilde{l}_t'(\zeta) < 0$. Then, it follows that $\lim_{z\to\infty} l_t(z) \geq \lim_{\zeta\to\infty} \exp(\tilde{l}_t(\zeta)) = \infty$ and $\lim_{z\to+0} l_t(z) \geq \lim_{\zeta\to-\infty} \exp(\tilde{l}_t(\zeta)) = \infty$. Thus, there exists a closed interval $[\underline{z}, \bar{z}]$ such that $l_t(z) > l_t(Z_t)$ for all $z \notin [\underline{z}, \bar{z}]$.

Next, we examine the existence of $l_t^{\min}$. If $\mu = 1$, the existence is obvious since $l_t(z)$ is a continuous function. In the following, we focus on the case $\mu < 1$. Let $\mathcal{M}_t \equiv \{z \in [\underline{z}, \bar{z}] | F_t(z) - F_t(z-) > 0\}$ denote the set of all mass points in $[\underline{z}, \bar{z}]$, and denote by $\bar{\mathcal{M}}_t$ the complementary set $[\underline{z}, \bar{z}] \backslash \mathcal{M}_t$. Since $l_t(z)$ is not continuous on $\mathcal{M}_t$, let us introduce a continuous variant of $l_t(z)$, $l_t^-(z) \equiv (Z_t/z)c(K_t^-(z))$, where

$$K_t^-(z) = \mu \int_0^{\infty} \min\left\{\frac{z'}{z}, \frac{z}{z'}\right\}^{\delta} dF_t(z'). \tag{A.2}$$

Note that $l_t^-(z)$ is larger than $l_t(z)$ on $\mathcal{M}_t$ and coincides with $l_t(z)$ on $\bar{\mathcal{M}}_t$. Since $l_t^-(z)$ is continuous and $[\underline{z}, \bar{z}]$ is compact, there exists $l_t^{-\min} \equiv \min_{z\in[\underline{z},\bar{z}]} l_t^-(z)$. If $\min_{z\in\mathcal{M}_t} l_t(z)$ exists, $l_t^{\min}$ is the smaller of this and $l_t^{-\min}$. If $\min_{z\in\mathcal{M}_t} l_t(z)$ does not exists but $\inf_{z\in\mathcal{M}_t} l_t(z)$ is no less than $l_t^{-\min}$, then $l_t^{\min} = l_t^{-\min}$. Finally, suppose that $\min_{z\in\mathcal{M}_t} l_t(z)$ is non-existent and that $\inf_{z\in\mathcal{M}_t} l_t(z)$ is smaller than $l_t^{-\min}$. Then, we can choose some $\xi > 1$ so that there are infinite elements in $\mathcal{M}_t^{\xi} \equiv \{z \in \mathcal{M}_t | \xi l_t(z) < l_t^{-\min}\}$. For each $z \in M_t^{\xi}$, $F_t(z) - F_t(z-) \geq \underline{k} \ln\xi/(1-\mu)\epsilon_c(\underline{k}) > 0$, where $\underline{k} \equiv \mu K_t(Z_t)\min\{\underline{z}/Z_t, Z_t/\bar{z}\}$. Thus $\int_0^{\infty} z dF_t(z) \geq \underline{z} \sum_{z\in\mathcal{M}_t^{\xi}} (F_t(z) - F_t(z-)) = \infty$, which contradicts assumption 4.

A.3 Proof of Claim 3

From equation (7) and (10) and the fact that $Z_t N_t \equiv \int_0^{\infty} z\, dF_t(z)$ is nondecreasing in time, it follows that $V_t = (1-\alpha)Z_t \int_t^{\infty} e^{-\rho(\tau-t)}(Z_\tau N_\tau)^{-1} d\tau \leq (1-\alpha)/\rho N_t$. Note that the equality in the latter applies if and only if $L_\tau = 0$ holds for all $\tau \geq t$, since otherwise $Z_\tau N_\tau$ increases at some date in the future. On the other hand, we have shown in the text that as long as there is a positive amount of R&D, $V_t = w_t l_t^{\min}$ and $w_t > \alpha/L$ hold, which implies that $V_t > \alpha l_t^{\min}/L$. This completes the proof of the part *a*.

It is clear from the above argument that when $\alpha l_t^{\min}/L$ is larger than or equal to $(1-\alpha)/\rho N_t$ there will be no R&D carried out in the economy. Conversely, consider the case where $(1-\alpha)/\rho N_t$ is larger than $\alpha l_t^{\min}/L$. Suppose that there is no R&D at time t. In this case, the free entry condition requires $V_t \leq \alpha l_t^{\min}/L$ since no R&D implies $w_t = \alpha/L$. If there is no R&D for all dates in the future, $V_t = (1-\alpha)/\rho N_t > \alpha l_t^{\min}/L$ holds, which contradicts the free entry condition. Thus, some entrepreneurs must carry out R&D in the future. Let t' be the first date after t with $L_{t'}^R > 0$. Then, the part *a* of the claim requires $V_{t'} > \alpha l_{t'}^{\min}/L$. Since there are no movements in the distribution of firms within the time interval between t and t', $l_{t'}^{\min} = l_t^{\min}$ and $Z_{t'} = Z_t$ hold. These facts and the non-decreasing nature of $Z_t N_t$ imply $\alpha l_t^{\min}/L = \alpha l_{t'}^{\min}/L < V_{t'} = (1-\alpha)Z_{t'} \int_{t'}^{\infty} e^{-(\tau-t')} Z_\tau N_\tau d\tau \leq (1-\alpha)Z_t \int_{t'}^{\infty} e^{-(\tau-t')} Z_{\tau-t'+t} N_{\tau-t'+t} d\tau = V_t$, which again contradicts the free entry condition at date t. Therefore, whenever $(1-\alpha)/\rho N_t > \alpha l_t^{\min}/L$, there must be a positive amount of R&D in the economy.

A.4 Proof of Claim 4

Let $K_t^-(z)$ be defined by equation (A.2). We first derive the slope of $K_t^-(z)$. Its left-hand derivative is calculated as

$$\begin{aligned}\lim_{\epsilon\to+0} \frac{K_t^-(z) - K_t^-(z-\epsilon)}{\epsilon} =& \mu \lim_{\epsilon\to+0} \frac{1}{\epsilon} \int_{(0,z-\epsilon]} \left\{ \left(\frac{z'}{z}\right)^\delta - \left(\frac{z'}{z-\epsilon}\right)^\delta \right\} dF_t(z') \\ &+ \mu \lim_{\epsilon\to+0} \frac{1}{\epsilon} \int_{(z-\epsilon,z)} \left\{ \left(\frac{z'}{z}\right)^\delta - \left(\frac{z-\epsilon}{z'}\right)^\delta \right\} dF_t(z') \\ &+ \mu \lim_{\epsilon\to+0} \frac{1}{\epsilon} \int_{[z,\infty)} \left\{ \left(\frac{z}{z'}\right)^\delta - \left(\frac{z-\epsilon}{z'}\right)^\delta \right\} dF_t(z')\end{aligned}$$

The absolute value of the second term is bounded by

$$\mu \lim_{\epsilon\to+0} \int_{(z-\epsilon,z)} \frac{1}{\epsilon} \left| \left(\frac{z'}{z}\right)^\delta - \left(\frac{z-\epsilon}{z'}\right)^\delta \right| dF_t(z') = \frac{2\mu\delta}{z} \lim_{\epsilon\to+0} \int_{(z-\epsilon,z)} dF_t(z'),$$

which is zero. Thus, the left-hand derivative of $K_t^-(z)$ is calculated as

$$\frac{\mu\delta}{z} \left[-\int_{(0,z)} \left(\frac{z'}{z}\right)^\delta dF_t(z') + \int_{[z,\infty)} \left(\frac{z}{z'}\right)^\delta dF_t(z') \right]. \tag{A.3}$$

In a similar way, we can derive the right-hand derivative of $K_t^-(z)$ as

$$\frac{\mu\delta}{z}\left[-\int_{(0,z]}\left(\frac{z'}{z}\right)^{\delta}dF_t(z')+\int_{(z,\infty)}\left(\frac{z}{z'}\right)^{\delta}dF_t(z')\right]. \tag{A.4}$$

Note that the two expressions, (A.3) and (A.4), coincide when $F_t(z)-F_t(z-)=0$, that is when z is not a mass point of $F_t(\cdot)$. In this case, $K_t^-(\cdot)$ is differentiable at z and its derivative is unambiguously expressed as

$$K_t^{-\prime}(z)=\frac{\delta\mu}{z}\left[-\int_0^z\left(\frac{z'}{z}\right)^{\delta}dF_t(z')+\int_z^\infty\left(\frac{z}{z'}\right)^{\delta}dF_t(z')\right]. \tag{A.5}$$

The assumption $Z'_t<z$ for all $t'<t$ means that z is not a mass point and that the second term in the square bracket in (A.5) is zero. In this case, $K_t^{-\prime}(z)$ reduces to $-(\delta/z)K_t(z)$. From this result, we can also calculate the derivative of $l_t^-(z)\equiv(Z_t/z)c(K_t^-(z))$ as $l_t^{-\prime}(z)=(l_t^-(z)/z)\,(\delta\epsilon_c(K_t(z))-1)$, which becomes zero if and only if $K_t(z)=\epsilon_c^{-1}(1/\delta)\equiv\bar{K}$. If $l_t^{-\prime}(z)$ were not zero, there would exist some technology z' such that $l_t^-(z')$ is smaller than $l_t^-(z)$. In this case, $l_t(z)=l_t^-(z)>l_t^-(z')\geq l_t(z')$ holds and thus z is not contained in $\mathcal{L}_t^{\min}$, contradicting (15).

A.5 Proof of Claim 5

a. Note that $d\ln c(k)/d\ln k=-\epsilon_c(k)>-1/\delta=d\ln\tilde{c}(k)/d\ln k$ holds for all $k>\bar{K}$. Since $c(\cdot)$ is a decreasing function, $\ln c(\bar{K}/\mu)$ is smaller than $\ln c(\bar{K})=\ln\tilde{c}(\bar{K}/\mu)$. Let $\tilde{K}\equiv(\bar{K}/\mu)\;(c(\bar{K})/c(\bar{K}/\mu))^{1/(1/\delta-\epsilon_c(\bar{K}/\mu))}>\bar{K}/\mu$. Then, $\ln c(\tilde{K})=\ln c((\bar{K}/\mu)\exp((\ln c(\bar{K})-\ln(\bar{K}/\mu))/(1/\delta-\epsilon_c(\bar{K}/\mu))))$ is larger than $\ln c(\bar{K}/\mu)-(1/\delta)(\ln c(\bar{K})-\ln(\bar{K}/\mu))/\;(1/\delta-\epsilon_c(\bar{K}/\mu))=\ln\tilde{c}(\tilde{K})$. It follows these facts that $\ln c(k)$ intersects with $\ln\tilde{c}(k)$ from below only once in the region $k>\bar{K}$ and the point of intersection $K^{\max}$ is located between $\bar{K}/\mu$ and $\tilde{K}$.

b. Suppose that a technology $z'\leq z$ exists such that $z'\in\mathcal{L}_t^{\min}$. Then $l_t(z')$ must be no greater than $l_t(z)$, which requires $K_t(z')\geq K_t(z)\geq K^{\max}$. Let us take another technology $z''=(\mu K_t(z')/\bar{K})^{1/\delta}z'$. Then, $K_t(z'')\geq\mu K_t(z')(z''/z')^{-\delta}=\bar{K}$. From these properties, it follows that $l_t(z'')\leq(Z_t/z'')c(\bar{K})=(Z_t/z')(\bar{K}/\mu K_t(z'))^{1/\delta}c(\bar{K})=(Z_t/z')\tilde{c}(K_t(z'))<(Z_t/z')c(K_t(z'))=l_t(z')$, which implies $z'\notin\mathcal{L}_t^{\min}$, a contradiction. Thus, $R_t(\cdot)\subseteq\mathcal{L}_t^{\min}\subset(z,\infty)$ follows.

c. Suppose that a technology $z'\leq z$ exists such that $z'\in\mathcal{L}_t^{\min}$ and $F_t(z')-F_t(z'-)=0$. Then $l_t(z')$ must be no greater than $l_t(z)$, which requires $K_t(z')\geq K_t(z)>\bar{K}$. Since

z' is not a mass point, $K_t(z')$ coincides with $K_t^-(z')$, which is defined by (A.2). The right-hand derivative of $K_t^-(\cdot)$ at z' is derived in a similar way to the way we have derived (A.4), which turns out to be no smaller than $-(\delta/z')K_t^-(z')$. From this, it follows that the right-hand derivative of $l_t^-(\cdot)$ at z' is well defined and no greater than $(l_t^-(z')/z')\left(\delta\epsilon_c(K_t^-(z'))-1\right)$, which is strictly negative because of $K_t^-(z')>\bar{K}$. This means that there exists some technology $z''>z'$ such that $l_t^-(z'')<l_t^-(z')$. Since $l_t(z')=l_t^-(z')>l_t^-(z'')\geq l_t(z'')$, z' is not contained in $\mathcal{L}_t^{\min}$, contradicting the initial assumption of $z'\in\mathcal{L}_t^{\min}$.

d. When $\mu=1$, the definition of $K_t^-(\cdot)$ coincides with that of $K_t(\cdot)$. Thus, the proof presented in part c applies regardless of whether z' is a mass point or not.

A.6 Proof of Claim 6

Consider a possible equilibrium path in one cycle that starts from (N^S,V^S), where $N^S\in(\bar{K},\hat{N})$ and $V^S\in(V^S_{\inf}(N_t),V^S_{\sup}(N_t))$. Then, at each point, the slope of the path in (N_t,V_t) space is $\dot{V}_t/\dot{N}_t$, where $\dot{N}_t$ and $\dot{V}_t$ are given by (37) and (20), respectively. Note that this slope is strictly negative because $\dot{N}_t>0$ and $\dot{V}_t<0$. Let V^E denote the value of V_t when N_t reaches $N^S+\bar{n}$.

Suppose that there is another possible path that starts from $(N^S,V^{S\prime})$, where $V^{S\prime}\in(V^S,V^S_{\sup}(N_t))$. Consequently, this path must be located above the previous one at each $N_t\in[N^S,N^S+\bar{n}]$. Moreover, the slope of this path at each N_t is larger than than that of the previous path since

$$\frac{d}{dV_t}\left(\frac{\dot{V}_t}{\dot{N}_t}\right)=\frac{1}{\dot{N}_t}\left(\rho+\frac{\alpha}{V_t^2}\frac{\dot{V}_t}{\dot{N}_t}\right)>0 \tag{A.6}$$

holds for all $N_t\in[N^S,N^S+\bar{n}]$. This means that the gap between the two paths gets wider as N_t increases, which implies $V^{E\prime}-V^E>V^{S\prime}-V^S$. After the frontier jumps, the gap widens: $\phi(N^S,V^{S\prime})-\phi(N^S,V^S)=\lambda V^{E\prime}-\lambda V^E>\lambda(V^{S\prime}-V^S)$.

Note that the opposite inequality holds when $V^{S\prime}<V^S$. Either way, we finally obtain

$$\frac{\phi(N^S,V^{S\prime})-\phi(N^S,V^S)}{V^{S\prime}-V^S}>\lambda \tag{A.7}$$

for all $V^{S\prime}\neq V^S$, which completes the proof.

Appendix B: Definition of Γ

Let $\bar{K}$ denote $\epsilon_c^{-1}(1/\delta)$, $K^{\max}$ the solution to $c(K^{\max}) = c(\bar{K})(\bar{K}/\mu K^{\max})^{1/\delta}$ in domain $(\bar{K}/\mu, \infty)$, and λ the solution to $c(\bar{K}(\lambda^\delta - 1 + \mu)/\mu) = c(\bar{K})/\lambda$ in domain $(1, \infty)$, the meanings of which are explained in claim 4, claim 5 and section 5, respectively. Then,

$$\Gamma = \max\left\{K^{\max}c(K^{\max}), \frac{\bar{K}c(\bar{K})(\lambda^\delta - 1)}{\mu(\lambda - 1)}\right\}.$$

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