Discussion Paper No. 566

EFFICIENT COLLUSION
IN REPEATED AUCTIONS
WITH COMMUNICATION

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May 2002

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May 24, 2002

Abstract

This paper studies collusion in repeated auctions when bidders communicate prior to each stage auction. The paper presents a folk theorem for independent and correlated private signals and general interdependent values. Specifically, it identifies conditions under which an equilibrium collusion scheme is fully efficient in the sense that the bidders' payoff is close to what they get when the object is allocated to the highest valuation bidder at the reserve price in every period.

Key words: collusion, auction, communication, folk theorem.

JEL classification numbers: C72, D82.

†Preliminary and incomplete.

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1. Introduction

Collusion is a wide-spread phenomenon in auctions as noted by many authors.[1] In particular, economic theory suggests that repeated auctions where the same set of bidders meet time and time again provide an ideal ground for collusion: In repeated auctions, not only is it easy to enforce the collusive agreement through the threat of reversion to competitive bidding in the event of a deviation, but it is also possible to transfer payoffs within a cartel without explicit exchange of money. For example, bidders in repeated auctions can employ a simple bid rotation collusion scheme which appoints the winning bidder of a stage auction in turn and hence transfers the continuation payoff from the current winner to other members of the cartel.

From the point of view of bidders, the optimal collusion scheme is one which is *fully efficient* in the sense that their equilibrium payoff is close to what they would get when the object is allocated at the reserve price to the highest valuation bidder in every stage auction. For example, the simple bid rotation collusion scheme as described above may be an improvement over the one-shot equilibrium, but is not fully efficient since the highest valuation bidder may not win just because it is not his turn. One important question then is if and when there exists a fully efficient equilibrium collusion scheme. This paper attempts to answer this question in a model of collusion with bidder communication.

Formally, the model of repeated auctions considered in this paper is a repeated game with private information in which the players' private signals are drawn identically and independently across periods. Because of the presence of private signals, it is known that standard folk theorems for repeated games do not apply except for some special cases as noted below. This paper shows, however, that with communication among bidders, an appropriate modification of the enforceability technique does yield an analytical framework for this class of games.

We formulate a collusion scheme in which bidders communicate their private signals with one another prior to every stage auction. At the beginning of each period, the bidders report their private signals to the *center,* which then return instructions to them based on the reported signal profile. An *instruction rule* is a functional relationship between the reported signal profile and the resulting instructions. For efficient collusion, of course, it is

[1] See, for example, Baldwin *et al.* (1997), Marshall and Meurer (1995), Pesendorfer (2000), Porter and Zona (1993), *etc.*

desirable to use the *efficient instruction rule,* which, based on the report profile, instructs the highest valuation bidder to bid the reserve price in the stage auction and all other bidders to stay out. It is easy to see, however, that this instruction rule is not incentive compatible as it gives the bidders an incentive to overstate their signals. Appropriate adjustment of continuation payoffs hence becomes important for the enforcement of such an instruction rule. We will show that this can be accomplished by transition to a collection of instruction rules that entail efficient allocation only within a subset of bidder(s).

In standard repeated games without private information, the enforceability conditions are expressed in terms of action profiles: They check whether taking a certain action is optimal given the discounted sum of today's stage payoff and the continuation payoff from tomorrow on. With private information, the enforceability conditions are instead expressed in terms of instruction rules. In other words, they check whether truth-telling is incentive compatible given the discounted sum of the current stage payoff implied by the instruction rule, and the continuation payoff.

By the standard argument, identifying the set of equilibrium payoff set reduces to finding a self-decomposable set of repeated game payoffs.[2] Following Fudenberg et al. (1994), the present paper solves the latter problem by finding a profile of transfer rules that satisfy weighted budget balance conditions. It describes when such a transfer rule profile exists in problems with two bidders and those with three or more bidders separately. It is shown that a fully efficient collusion scheme exists under fairly permissive conditions.

In repeated auctions with two bidders, we assume that the private signals are linearly ordered and affiliated across bidders. Under these conditions, we construct a *redistribution mechanism* in which the bidder who has reported the higher signal becomes the winner and his surplus is redistributed to the other bidder in the form of continuation payoffs. With only two bidders, one bidder's gain in the continuation payoff is necessarily the other bidder's loss. As will be seen, this trade-off creates a bound on the enforceable payoffs. In actual problems, it is easy to check if full efficiency can be achieved despite this bound.

When there are three or more bidders, we assume that signals are either (i) linearly ordered and independent across bidders, or (ii) correlated. In both cases, we show that there exists a desirable continuation payoff function profile that enforces any relevant

[2]See, for example, Abreu *et al.* (1990), and Fudenberg *et al.* (1994).

instruction rule. With three or more bidders, the key is to dissociate the inducement of truth-telling from the budget balance considerations. In other words, we can choose a continuation payoff function that ignores some bidder's report while letting him "absorb" the surplus or deficit caused by the inducement of truth-telling from another bidder.

In the analysis of repeated games with imperfect public monitoring, Fudenberg *et al.* (1994) discuss repeated adverse selection with communication. Specifically, they show that when players publicly announce their private signals, a folk theorem holds for an adverse selection model with *independent private values,* where private signals are independent across players and their values depend only on their own signals. Their theorem readily implies that under the independent private values assumption, fully efficient collusion is possible in repeated auctions when the bidders are sufficiently patient. On the other hand, it is not easy to see how we may generalize this construction to a more general environment without the independence of signals and/or the privateness of valuations. This paper takes an entirely different approach to enforceability and shows that the independent private values assumption is not crucial for a folk theorem.

The paper that is most closely related to the present one is Aoyagi (2002), which proves the existence of a collusion scheme in repeated auctions that improves on the one-shot Nash equilibrium of the stage auction and the simple bid rotation scheme as described above. It develops the idea of dynamic bid rotation whereby intertemporal transition between instruction rules takes place as a function of the reported signals. It does not, however, show the existence of a fully efficient collusion scheme. The key difference is that in Aoyagi (2002), the signal set is the unit interval $[0, 1]$, while it is *finite* in the present paper. The stronger conclusion in this paper benefits from the self-decomposability techniques available for finite-action games, and its extension to a continuous signal problem is not straightforward.

Skrypacz and Hopenhayn (2002) and Blume and Heidues (2002) both study tacit collusion in repeated auctions, where bidders do not communicate prior to each stage auction. They show that a certain degree of improvement over one-shot Nash equilibrium as well as simple bid rotation is possible in independent private values models. The difference in our modeling choice is based on the following considerations: First, bidder communication is often an integrated part of actual collusion practice, and it is important to understand its implications. Second, with communication or not, little is known about the full scope of collusion in repeated auctions. It is hence useful to present a simple framework in which full efficiency can be achieved.

Communication mediated by the center as assumed in this paper mimics the direct revelation mechanism applied to each stage and hence represents a natural mode of information transmission. It also allows for a clean presentation of the enforceability conditions through the use of an instruction rule. However, what is essential for the argument is the functional relationship between a report profile and bidding behavior in the stage auction as well as continuation play, and the folk theorem continues to hold if we instead assume public communication, where bidders publicly reveal their private signals.

Although the discussion in this paper is completely embedded in the repeated auctions framework, its analysis applies to other problems of repeated adverse selection including, for example, collusion in repeated Bertrand oligopoly with private cost signals as analyzed by Athey and Bagwell (2000) and Athey *et al.* (1998). Another interesting application that does not seem to have been studied elsewhere is a budget allocation problem in a dynamic setting. For example, consider the problem faced by a scientific foundation, which wants to allocate a unit of indivisible budget to the most promising research project each year. There is a fixed set of researchers, and each one of them comes up with a project every year whose true quality is random and privately observed. The foundation elicits proposals from the researchers to assess the quality of their projects, but cannot receive monetary transfer from them in return for awarding the budget. This problem has the same structure as our model once the foundation is reinterpreted as the center.

The paper is organized as follows: A model of repeated auctions is formulated in the next section. The enforceability of an instruction rule is defined in Section 3. Section 4 describes the feasible as well as self-decomposable payoff sets. Sections 5 and 6 study collusion by two bidders and by three or more bidders, respectively.

2. Model

The set I of I risk-neutral bidders participate in an infinite sequence of auctions, where a single indivisible object is sold in every period through a fixed auction format.[3] In each period, bidder i draws a private signal s_i from a *finite* set S_i. The signal profile

[3]Note that the symbol I represents both the set of bidders and its cardinality.

$s = (s_1, \dots, s_I)$ of I bidders has the joint distribution p in every period and is independent across periods.

Bidder i's valuation of the object sold in each period is a function of the signal profile $s = (s_1, \dots, s_I)$ in that period and denoted $v_i(s) \geq 0$. A *stage auction* is any transaction mechanism that determines the allocation of the good as well as monetary transfer based on a single sealed bid submitted by each bidder.[4] Participation in the stage auction is voluntary so that the set of each bidder's *generalized bids* is expressed as $B_1 = \cdots = B_I = \{N\} \cup \mathbf{R}_+$, where N represents "no participation." The rule of the auction is summarized by mappings ω_i and ξ_i $(i \in I)$ on the set $B = B_1 \times \cdots \times B_I$ of bid profiles $b = (b_1, \dots, b_I)$: $\omega_i(b)$ is the probability that bidder i is awarded the good, and $\xi_i(b)$ is his expected payment to the auctioneer. We assume that ω_i and ξ_i satisfy the following conditions.

(i) A bidder makes no payment when he does not participate: $\xi_i(b) = 0$ if $b_i = N$.
(ii) A bidder may win the object only if he submits a bid at or above the reserve price $R \in \big[0, \max_{s,i} v_i(s)\big)$: $\omega_i(b) = 0$ if $b_i \in \{N\} \cup [0, R)$.
(iii) If only one bidder participates and submits bid R, then he wins the object at price R: $\omega_i(b) = 1$ and $\xi_i(b) = R$ if $b_i = R$ and $b_j = N$ for all $j \neq i$.

Note that the above conditions hold for most standard auctions including the first- and second-price auctions. Consider the Bayesian game in which bidder i's (pure) strategy is a mapping $\eta_i : S_i \to B_i$ and his (*ex ante*) payoff function is

$$\sum_{s \in S} p(s) \big\{\omega_i(\eta(s)) v_i(s) - \xi_i(\eta(s))\big\}.$$

Let ΔB_i denote the probability distribution over B_i, and $\tilde{\eta}_i : S_i \to \Delta B_i$ denote bidder i's mixed strategy in this game. We assume that this game has a (mixed) Nash equilibrium $\tilde{\eta}^0 = (\tilde{\eta}_1^0, \dots, \tilde{\eta}_I^0)$, which describes the non-cooperative bidding behavior in the stage auction. Let g_i^0 be the corresponding (*ex ante*) Nash equilibrium payoff to bidder i.

Collusion in the repeated auction takes the following form: At the beginning of each period, all bidders report their private signals s_i to the center. Upon receiving the report profile $\hat{s} = (\hat{s}_1, \dots, \hat{s}_I) \in S$, the center chooses instruction to each bidder i on what (generalized) bid to submit in the stage auction.

[4] The restriction to a seal-bid auction is purely for simplicity.

In general, the bidders may report a false signal, and/or disobey the instruction. Bidder i's *reporting rule* $\lambda_i : S_i \to S_i$ chooses report $\hat{s}_i$ as a function of his true signal s_i, and his *bidding rule* $\mu_i : S_i^2 \times B \to B$ chooses bid b_i in the stage action as a function of his signal, report and instruction. The reporting rule is *honest* if it always reports the true signal, and the bidding rule is *obedient* if it always obeys the instruction. Denote by λ_i^* and μ_i^* bidder i's honest reporting rule and obedient action rule, respectively.

For simplicity, we assume that the (generalized) bids in the stage auction are observable to every party including the center.[5] It follows that any disobedience to the center's instruction is an observable deviation, while misreporting of one's signal is an unobservable deviation.

The *center* is simply a device that transforms the report profile into instructions. A (pure) *instruction rule* $d = (d_1, \ldots, d_I) : S \to B$ chooses an instruction to every bidder based on the report profile $\hat{s}$. Formally, it is possible to suppose that the one-shot Nash equilibrium $\tilde{\eta}^0$ of the stage auction specified above is played through the center if randomization over instructions is allowed. A *mixed instruction rule* is a mapping $d : S \to \Delta B$ with the interpretation that $d(\hat{s})$ is a probability distribution over instruction profiles when the report profile is $\hat{s}$.[6] Let d^0 be the (possibly) mixed instruction rule such that $d_i^0(\hat{s}) = \tilde{\eta}_i^0(\hat{s}_i)$.

Bidder i's *communication history* in period t in the repeated auction game is the sequence of his reports and instructions in periods $1, \ldots, t-1$. On the other hand, bidder i's *private history* in period t is the sequence of his private signals s_i in periods $1, \ldots, t-1$. Furthermore, the *public history* in period t is a sequence of instruction rules used by the center in periods $1, \ldots, t$ and (generalized) bid profiles in the stage auctions in periods $1, \ldots, t-1$.

Bidder i's (pure) *strategy* σ_i in the repeated auction chooses the pair (λ_i, μ_i) of reporting and bidding rules in each period t as a function of his communication and private histories in t, and the public history in t. Let σ_i^* be bidder i's *honest and obedient* strategy which plays the pair (λ_i^*, μ_i^*) of the honest reporting rule and obedient bidding rule for all

[5]If the center does not have monitoring capability, we can assume that the bidders report others' bids to the center so that the instructions in the next period will be conditioned on those reports.

[6]Note that the actual instruction to each player i is still an element of B_i.

histories.

The *collusion scheme* τ describes the center's choice of an instruction rule in every period as a function of communication and public histories. At the beginning of each period, it publicly informs the bidders which instruction rule is used in that period.

Our analysis will focus on the following class of "grim-trigger" collusion schemes with two phases: The game starts in the *collusion phase,* and reverts to the *punishment phase* forever if and only if there is disobedience by at least one bidder. In the punishment phase, the center chooses the one-shot Nash equilibrium instruction rule d^0 specified above.

Let $\delta < 1$ be the bidders' common discount factor, and $\Pi_i(\sigma, \tau, \delta)$ be bidder i's average discounted payoff (normalized by $(1-\delta)$) in the repeated game under the strategy profile (σ, τ). The collusion scheme τ is an *equilibrium* if the profile $\sigma^* = (\sigma_1^*, \ldots, \sigma_I^*)$ of honest and obedient strategies constitutes a Nash equilibrium of the repeated game: $\Pi_i(\sigma^*, \tau, \delta) \geq \Pi_i(\sigma_i', \sigma_{-i}^*, \tau, \delta)$ for any σ_i' and $i \in I$.

3. Enforceability of Instruction Rules

Denote by $q(s, \hat{s} \mid \lambda)$ the joint probability of the signal profile s and the report profile $\hat{s}$ when the bidders use the reporting rule profile $\lambda = (\lambda_1, \ldots, \lambda_I) \in \Lambda$:

$$q(s, \hat{s} \mid \lambda) = \begin{cases} p(s) & \text{if } \lambda(s) = \hat{s}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $q(\hat{s} \mid \lambda)$ be the corresponding (marginal) probability of $\hat{s}$:

$$q(\hat{s} \mid \lambda) = \sum_{s \in S} q(s, \hat{s} \mid \lambda).$$

Define

$$g_i^d(\lambda) = \sum_{s \in S} q(s, \hat{s} \mid \lambda) \left\{ \omega_i(\hat{s})\, v_i(s) - \xi_i(\hat{s}) \right\}.$$

to be bidder i's *ex ante* expected stage payoff under the instruction rule d when the bidders use the reporting rule profile $\lambda = (\lambda_1, \ldots, \lambda_I)$ and *obey* the center's instruction.

The construction below is an adaptation of Fudenberg *et al.* (1994) to the repeated adverse selection framework.

Let $W \subset \mathbf{R}^I$ be a set of payoff vectors. The instruction rule d is (*truthfully*) *enforceable with respect to* δ and W if there exists $y = (y_1, \ldots, y_I) : S \to W$, a profile of continuation payoff functions taking values in W, such that for every $i \in I$ and $\lambda_i \in \Lambda_i$,

$$(1-\delta)\, g_i^d(\lambda^*) + \delta \sum_{\hat{s}\in S} q(\hat{s} \mid \lambda^*)\, y_i(\hat{s}) \geq (1-\delta)\, g_i^d(\lambda_i, \lambda_{-i}^*) + \delta \sum_{\hat{s}\in S} q(\hat{s} \mid \lambda_i, \lambda_{-i}^*)\, y_i(\hat{s}).$$

In other words, truth-telling maximizes the (discounted) sum of today's stage payoff and the continuation payoff from tomorrow on among all possible reporting rules. Given a collection D of instruction rules, the set W is *locally self-decomposable with respect to* D if for each $w \in W$, there exist a discount factor $\delta < 1$ and an open neighborhood U of w such that for any $u = (u_1, \ldots, u_I) \in U$, there exists an instruction rule $d \in D$ such that d is enforceable with respect to δ and W through some continuation payoff function profile $y = (y_1, \ldots, y_I) : S \to W$, and

$$u_i = (1-\delta)\, g_i^d(\lambda^*) + \delta \sum_{\hat{s}\in S} q(\hat{s} \mid \lambda^*)\, y_i(\hat{s})$$

for every $i \in I$.

It readily follows from Lemma 4.2 of Fudenberg *et al.* (1994) that if W is compact, convex, and locally self-decomposable with respect to some D, then there exists a discount factor $\underline{\delta} < 1$ such that for any $\delta > \underline{\delta}$, any point $w \in W$ is sustained as a payoff vector of an equilibrium collusion scheme whose instruction rules are chosen from D.

Given vectors α, $\zeta \in \mathbf{R}^I$ such that $\alpha \neq 0$, let $H(\alpha, \zeta)$ denote the hyperplane in $\mathbf{R}^I$ through ζ with the normal vector α. Suppose that d is enforceable with respect to $\delta < 1$ and $W = H(\alpha, \zeta)$ through $y : S \to W$. Then for any $\delta' < 1$ and $\zeta' \in \mathbf{R}^I$, if we define $y' : S \to W' \equiv H(\alpha, \zeta')$ by

$$y'(\hat{s}) = \zeta' + \frac{\delta(1-\delta')}{\delta'(1-\delta)}\, \{y(\hat{s}) - \zeta\},$$

then d is enforceable with respect to δ' and W' through y'. For this reason, we may simply say that d is enforceable with respect to α when it is enforceable with respect to some δ and $W = H(\alpha, \zeta)$.

The set $W \subset \mathbf{R}^I$ is *smooth* if it is closed and convex, and if its interior is non-empty and its boundary is a C^2-manifold. As in Fudenberg *et al.* (1994), we will associate the local self-decomposability of a smooth set W with enforceability with respect to its supporting hyperplanes. Formally, a smooth set W is *decomposable on tangent hyperplanes* (given the set D of instruction rules) if for every point w on the boundary of W, there exists an instruction rule $d \in D$ such that (i) $g^d(\lambda^*)$ and W are separated by the supporting hyperplane H of W at w, and (ii) d is enforceable with respect to H.

Finally, by Theorem 4.1 of Fudenberg *et al.* (1994), if a smooth set W is decomposable on tangent hyperplanes given the set D of instruction rules, then W is locally self-decomposable with respect to D.

4. Feasible and Self-Decomposable Payoff Sets

For efficient collusion, the bidder with the highest valuation should be instructed to bid the reserve price in the stage auction while other bidders are instructed to stay out. We begin with the description of such an instruction rule.

Since the signal space S_i is finite, more than one bidder may share the same highest valuation with positive probability. This suggests the possible multiplicity of efficient allocations according to different tie-breaking rules. To capture this possibility, we introduce a *permutation* on I that describes each player's rank in tie-breaking. Let Φ_I be the set of permuations on I: each $\phi \in \Phi_I$ is a one-to-one mapping from I to itself. For any $\phi \in \Phi_I$, let $d^{\phi*}$ denote the *efficient* instruction rule defined as follows: Given the report profile $\hat{s} \in S$, $d^{\phi*}$ instructs the bidder with the highest valuation (based on $\hat{s}$) to bid R if his valuation is higher than the reserve price R. If there exist two or more bidders with the highest valuation, then bidder i becomes the winner if and only if his "rank" $\phi(i)$ according to ϕ is the smallest among all such bidders. Any other bidder is instructed to stay out. If we let $I^*(s) = \arg\max_{j \in I} v_j(s)$ be the set of bidders with the highest valuation under the signal profile s, then $d^{\phi*}$ can formally be described as:

$$d_i^{\phi*}(\hat{s}) = \begin{cases} R & \text{if } i \in I^*(\hat{s}),\ \phi(i) \le \phi(j) \text{ for every } j \in I^*(\hat{s}), \text{ and } v_i(\hat{s}) \ge R, \\ N & \text{otherwise.} \end{cases}$$

For each $i \in I$, denote by $g_i^{\phi*}$ bidder i's (ex ante) stage payoff $g_i^{d^{\phi*}}(\lambda^*)$ associated with $d^{\phi*}$. If we let

$$F = \mathrm{Co}\,\{g^{\phi*} : \phi \in \Phi_I\},$$

then F is the set of (first-best) efficient payoff vectors. Our analysis will assume that collusion is potentially profitable, *i.e.,* the one-shot Nash equilibrium is (strictly) Pareto dominated by any point on F: $g_i^0 < g_i^{\phi*}$ for every $i \in I$ and $\phi \in \Phi_I$.

We will next describe instruction rules that are used for the adjustment of continuation payoffs. For each $i \in I$, let d^i be an asymmetric instruction rule defined as follows: Given the report profile $\hat{s} \in S$, d^i instructs (i) bidder i to bid R if his valuation $v_j(\hat{s})$ exceeds R, and to stay out otherwise, and (ii) bidder j ($j \neq i$) to stay out:

$$d_i^i(\hat{s}) = \begin{cases} R & \text{if } v_i(\hat{s}) > R \\ N & \text{otherwise,} \end{cases} \quad \text{and} \quad d_j^i(\hat{s}) = N \text{ for any } \hat{s}.$$

In other words, bidder i is the only potential winner under d^i. Let $g_j^i = g_j^{d^i}(\lambda^*)$ be bidder j's (ex ante) stage payoff under d^i. We have $g_j^i = 0$ if $j \neq i$. For any i, we call d^i the *exclusion* rule.

LEMMA 4.1. *For any $\alpha \in \mathbf{R}^I \setminus \{0\}$, the instruction rules d^i ($i \in I$) and d^0 are truthfully enforceable with respect to α.*

PROOF: Let $y(\cdot) \equiv 0$. Whether $d = d^i$ or $d = d^0$, $g_i^d(\lambda^*) \geq g_i^d(\lambda_i, \lambda_{-i}^*)$ for every $i \in I$ so that d is enforceable with respect to α. //

Let $V \subset \mathbf{R}^I$ be defined by

$$V = \text{Co}\left\{g^0, (g^i)_{i \in I},\, (g^{\phi *})_{\phi \in \Phi_I}\right\} \cap \left\{u \in \mathbf{R}^I : u_i \geq g_i^0 \text{ for all } i \in I\right\}.$$

It can be seen that the efficiency frontier of V is given by F. Our analysis in what follows focuses on whether payoff vectors in V can be supported by equilibrium collusion schemes. To this end, let d be any instruction rule, and

$$A(d) = \left\{\alpha \in \mathbf{R}^I \setminus \{0\} : d \text{ is enforceable with respect to } \alpha\right\}.$$

Given the set D of instruction rules, define

$$V^*(D) = \left\{u \in V : \alpha \cdot u \leq \alpha \cdot g^d(\lambda^*) \text{ and } \alpha \in A(d) \text{ for some } d \in D\right\}.$$

The following lemma states that $V^*(D)$ is the set of equilibrium payoffs when the instruction rule in any period is chosen from D, provided that the bidders are sufficiently patient.

LEMMA 4.2. *Suppose that $V^*(D)$ has a non-empty interior for some set D of instruction rules. Then any smooth subset W of the interior of $V^*(D)$ is locally self-decomposable.*

Hence, for any $u \in V^*(D)$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D and yields the payoff vector $\Pi(\tau, \sigma^*, \delta)$ satisfying $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.

PROOF: Take any smooth set $W \subset \operatorname{int} V^*(D)$. Let w be a point on the boundary of W, and $\alpha \neq 0$ be the normal vector of the supporting hyperplane of W at w so that $\alpha \cdot u \leq \alpha \cdot w$ for any $u \in W$. If $\alpha \in A(d)$ and $\alpha \cdot w \leq \alpha \cdot g^d(\lambda^*)$ for some $d \in D$, then W is decomposable on the tangent hyperplane at w using d. Therefore, W is decomposable on tangent hyperplanes, and hence that it is locally self-decomposable with respect to D by Theorem 4.1 of Fudenberg *et al.* (1994). The desired conclusion then follows from their Lemma 4.2. //

In what follows, we will study the relationship between V and $V^*(D)$ for an approprioate choice of D.

5. Two Bidders: Redistribution Mechanism

In this section, we assume that the signals are linearly ordered in the sense below.

ASSUMPTION 1: For every $i \in I$, the signal set is such that $S_i = \{s^0, s^1, \ldots, s^K\} \subset \mathbf{R}_+$ for $s^0 < s^1 < \cdots < s^K$. Furthermore, the probability distribution p of signal profile s has full support over S, and satisfies the *monotone likelihood ratio property*:

$$(1) \qquad \frac{p_j(s'_j \mid s_i)}{p_j(s_j \mid s_i)} \leq \frac{p_j(s'_j \mid s'_i)}{p_j(s_j \mid s'_i)} \text{ if } s'_i \geq s_i \text{ and } s'_j \geq s_j \ (i = 1,\, 2,\, j \neq i).$$

With $I = 2$, the monotone likelihood ratio is equivalent to affiliation as specified by Milgrom and Weber (1982).

ASSUMPTION 2: The valuation functions $v_1, \ldots, v_I$ are *monotone* ($v_i(s') \geq v_i(s)$ if $s' \geq s$) and symmetric ($v_i(s) = v_j(s')$ if $s'_i = s_j$, $s'_j = s_i$ and $s'_{-i-j} = s_{-i-j}$).

Note that Assumption 2 holds in a private values model where $v_i(s) = s_i$ for every $s \in S$ and $i \in I$. The symmetry of the value functions is assumed mainly for simplicity. We also assume in this section that the reserve price R equals zero.

With two bidders, we consider a *redistribution mechanism* in which the winner's surplus in each stage auction is redistributed to the loser through an adjustment in continuation payoffs. Such a transfer has a natural interpretation, and is most likely at the heart of many actual collusion schemes. The theoretical analysis of such a mechanism is provided by McAfee and McMillan (1992) for the case where side transfer among bidders is possible. Aoyagi (2002) extends their analysis and shows that monetary transfer can partially be compensated by the adjustment in continuation payoffs for collusion without side transfer in repeated auctions. This section presents a further extension of this result when the signal space is finite.

For the permutation $\phi \in \Phi_I$ such that $\phi(i) = 1$ and $\phi(j) = 2$, we write d^{ij*} for $d^{\phi*}$.

For $i = 1, 2$ and $j \neq i$, let

$$\rho_j^{k,l} = \frac{p_j(s^l \mid s_i = s^k)}{\sum_{s_j \leq s^l} p_j(s_j \mid s_i = s^k)} \quad (k, l = 0, 1, \ldots, K).$$

By the monotone likelihood property (1), $\rho_j^{k,l} \leq \rho_j^{k',l}$ if $k \leq k'$. Define $\theta_{ij} \leq 1$ by

$$\theta_{ij} = \max_k \frac{\rho_j^{k-1,k}\, v_i(s^{k-1}, s^k)}{v_j(s^k, s^k) - \{1 - \rho_j^{k-1,k}\}\, v_i(s^{k-1}, s^k)}.$$

We consider the continuation payoff function $y = (y_i, y_j)$ that transfers payoff from the winner of the stage auction to the loser. Specifically, let $x_i : S_i \to \mathbf{R}_+$ and $x_j : S_j \to \mathbf{R}_+$ be non-negative functions of i's and j's reports, respectively, and let y be given by

$$(2) \qquad y_i(\hat{s}) = \begin{cases} -x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ \dfrac{\alpha_j}{\alpha_i}\, x_j(\hat{s}_j) & \text{otherwise,} \end{cases} \quad \text{and} \quad y_j(\hat{s}) = \begin{cases} \dfrac{\alpha_i}{\alpha_j}\, x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ -x_j(\hat{s}_j) & \text{otherwise.} \end{cases}$$

Clearly, $\alpha \cdot y(\hat{s}) = 0$ for any $\hat{s} \in S$. As seen, $x_i(\hat{s}_i)$ can interpreted as the compensation made by bidder i when he wins with report $\hat{s}_i$. Write $\alpha_j / \alpha_i = \theta$, and let x_i and x_j be defined recursively by

$$(3) \qquad \begin{aligned} & x_i(s^0) = x_j(s^0) = 0, \\ & x_i(s^k) = x_i(s^{k-1}) + \rho_j^{k-1,k} \Big[v_i(s^{k-1}, s^k) - t^{k-1} \\ & \qquad\qquad - \rho_i^{k-1,k-1} \left\{ \theta v_j(s^{k-1}, s^{k-1}) - t^{k-1} \right\} \Big], \\ & x_j(s^k) = x_j(s^{k-1}) + \rho_i^{k-1,k-1} \left[v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta}\, t^{k-1} \right] \\ & \qquad\qquad (k = 1, \ldots, K), \end{aligned}$$

where $t^k = x_i(s^k) + \theta x_j(s^k)$. Write $\Delta_i^k = x_i(s^k) - x_i(s^{k-1})$ and $\Delta_j^k = x_j(s^k) - x_j(s^{k-1})$ $(k = 1, \ldots, K)$.

LEMMA 5.1. *Suppose that Assumptions 1 and 2 hold. For* $\theta = \frac{\alpha_j}{\alpha_i} \in [\theta_{ij}, 1]$, x_i *and* x_j *are non-decreasing:* Δ_i^k, $\Delta_j^k \geq 0$ *for* $k = 1, \ldots, K$.

LEMMA 5.2. *Suppose that Assumptions 1 and 2 hold. If* $\alpha = (\alpha_1, \alpha_2)$ *is such that* α_1, $\alpha_2 > 0$ *and* $\alpha_j/\alpha_i \in [\theta_{ij}, 1]$, *then the efficient instruction rule* d^{ij*} *is enforceable with respect to* α *through* y *in (2) when* x_i *and* x_j *are defined by (3).*

Lemma 5.2 essentially states that the exchange rate for transfer between the two bidders should not be extreme. If, for example, money worth one dollar to bidder j is worth very little to bidder i, then any monetary transfer to/from bidder j designed to induce truth-telling from him will be insufficient to induce truth-telling from bidder i.

Let

$$V^0 = \{u \in V : u_i + \theta_{ij} u_j \leq g_i^i \text{ for } i = 1, 2, j \neq i\},$$

and

$$D^* = \{d^0, d^1, d^2, d^{12*}, d^{21*}\}.$$

We now show that $V^0 \subset V^*(D^*)$. Take any $u \in V^0$. Let $\alpha \neq 0$ be given. If $\alpha \cdot u \leq \alpha \cdot g^0$ or $\alpha \cdot u \leq \max_{i \in I} \alpha \cdot g^i$, then we are done since $\alpha \in A(d^0) = A(d^i)$ by Lemma 4.1. It can be readily verified that this is indeed the case if $\alpha_1 \leq 0$ and/or $\alpha_2 \leq 0$. Suppose then that $\alpha_i \geq \alpha_j > 0$ and that $\alpha \cdot u < \alpha \cdot g^{\phi *}$ for either $\phi = ij$ or $\phi = ji$. Since $g_i^{ij*} + g_j^{ij*} = g_i^{ji*} + g_j^{ji*}$ and $g_j^{ij*} \leq g_j^{ji*}$, we have

$$\alpha \cdot (g^{ij*} - g^{ji*}) = (\alpha_i - \alpha_j)(g_j^{ji*} - g_j^{ij*}) \geq 0.$$

Therefore, $\alpha \cdot u \leq \alpha \cdot g^{ij*}$ must hold. If $\alpha \in A(d^{ij*})$, then the proof is complete. Otherwise, $\alpha_j/\alpha_i \leq \theta_{ij}$ by Lemma 5.2 so that

$$\alpha \cdot u = \alpha_i \Big(u_i + \frac{\alpha_j}{\alpha_i} u_j \Big) \leq \alpha_i (u_i + \theta_{ij} u_j) \leq \alpha_i \, g_i^i = \alpha \cdot g^i.$$

We hence obtain the desired conclusion. This observation in conjunction with Lemma 5.2 yields the following theorem.

THEOREM 5.3. *Suppose that* $I = 2$ *and that Assumptions 1 and 2 hold. Then* $V^0 \subset V^*(D^*)$. *In other words, for any* $u \in V^0$ *and* $\epsilon > 0$, *there exists* $\underline{\delta} < 1$ *such that the following holds if* $\delta > \underline{\delta}$: *There exists an equilibrium collusion scheme* τ *which chooses* *instruction rules from* $D^* = \{d^0, d^1, d^2, d^{12*}, d^{21*}\}$ *and yields the payoff vector* $\Pi(\tau, \sigma^*, \delta)$ *satisfying* $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.

PROOF: See the Appendix.

In order to determine whether or not efficient collusion is possible, hence, we only need to check if $F \cap V^0 \neq \phi$. This is typically a straightforward task as demonstrated by the following example.

EXAMPLE 1: Suppose that the signals are independently drawn from the set $S_1 = S_2 = \{0, 1/K, \dots, 1 - 1/K, 1\}$ according to the uniform probability distribution $p_i(s_i) = 1/(K + 1)$ for any $s_i \in S_i$ ($i = 1, 2$). The value function is given by $v_i(s_i, s_j) = cs_i + (1 - c)s_j$ for $i = 1, 2$, and $j \neq i$, where $c \in [0, 1]$ is a constant. The efficient payoffs corresponding to d^{12*} can be calculated as:

$$g_1^{12*} = \sum_{k=0}^{K} \sum_{l=0}^{k} \frac{ck + (1-c)l}{K} \frac{1}{(K+1)^2} = \frac{1+c}{2K(K+1)^2} \sum_{k=0}^{K} k(1+k) = \frac{1+c}{2} \frac{K+2}{3(K+1)},$$

$$g_2^{12*} = \sum_{k=0}^{K} \sum_{l=0}^{k-1} \frac{ck + (1-c)l}{K} \frac{1}{(K+1)^2} = \frac{1+c}{2K(K+1)^2} \sum_{k=0}^{K} k^2 = \frac{1+c}{2} \frac{2K+1}{6(K+1)}.$$

The symmetric efficient payoff g_i^* is hence given by

$$g_i^* = \frac{1}{2} \sum_{s \in S} p(s) \max\{s_1, s_2\} = \frac{1}{2}(g_1^{12*} + g_2^{12*}) = \frac{1+c}{2} \frac{4K+5}{12(K+1)}.$$

On the other hand, bidder i's exclusion payoff equals $g_i^i = 1/2$, and the bounds on the exchange rate are given by

$$\theta_{12} = \theta_{21} = \frac{K-c}{K(1+c)}.$$

It can be verified that

$$g_1^* + \theta_{12}\, g_2^* = \frac{(4K+5)\{(2+c)K - c\}}{24K(K+1)} < \frac{1}{2}.$$

As depicted in Figure 1, hence, the symmetric efficient vector $g^* = (g_1^*, g_2^*) \in V^0$. By Theorem 5.3, therefore, there exists an equilibrium collusion scheme whose payoff vector approximates g^* provided that the discount factor is close to one.

6. Three or More Bidders

With three or more bidders, it is necessary to use a wider class of instruction rules to support efficient payoffs. Given the signal profile $s \in S$ and a subset $J \subset I$ of bidders, denote by $I^*(s, J) = \arg\max_{i \in J} v_i(s)$ the set of bidders with the highest valuation among those in set J. For each permutation $\phi \in \Phi_J$ on the set J, define $d(\cdot \mid \phi, J)$ to be the instruction rule such that

$$d_i(\hat{s} \mid \phi, J) = \begin{cases} R & \text{if } i \in I^*(\hat{s}, J) \text{ and } \phi(i) \leq \phi(j) \text{ for any } j \in I^*(\hat{s}, J), \\ N & \text{otherwise.} \end{cases}$$

In other words, this instruction rule allocates the good efficiently *within* the set J but excludes all other bidders. If $J = I$, then $d(\cdot \mid \phi, I)$ is equivalent to the efficient instruction rule $d^{\phi *}$, and if $J = \{i\}$, then it is equivalent to the exclusion rule d^i.

Given any α, let $J_\alpha = \{j : \alpha_j > 0\}$. Our objective is to identify conditions under which $d(\cdot \mid \phi, J_\alpha)$ is enforceable with respect to α.

Since $d(\cdot \mid \phi, \{i\}) = d^i$ is enforceable with respect to any $\alpha \neq 0$ by Lemma 4.1, suppose that $J_\alpha = \{1, \ldots, n\}$ for some $n \geq 2$, and let $\phi \in \Phi_{J_\alpha}$ be given. By the definition of $d(\cdot \mid \phi, J_\alpha)$, the enforceability conditions for bidders $n+1, \ldots, I$ are satisfied if we take $y_{n+1}(\cdot) = \cdots = y_I(\cdot) \equiv 0$. For bidders $1, \ldots, n$, we express their enforceability conditions in matrix form. For each $i = 1, \ldots, n$, let $\Lambda_i^0 = \Lambda_i \setminus \{\lambda_i^*\}$, $m_i = |\Lambda_i^0|$ and $m = \sum_{i=1}^n m_i$. For any reporting rule profile λ, let $q(\cdot \mid \lambda) = \big(q(\hat{s} \mid \lambda)\big)_{\hat{s} \in S}$ represent the $|S|$-dimensional vector of probability distributions of report profiles under λ. Denote by B_i the $m_i \times |S|$ matrix whose row equals

$$b_i(\lambda_i) = q(\cdot \mid \lambda^*) - q(\cdot \mid \lambda_i, \lambda_{-i}^*) \quad (\lambda_i \in \Lambda_i^0).$$

In other words, each row of B_i corresponds to the difference in probability distributions of report profiles between λ_i^* and λ_i ($\neq \lambda_i^*$). Write $d = d(\cdot \mid \phi, J_\alpha)$ for simplicity, and let $\hat{v}_i(\lambda_i) = g_i^d(\lambda_i, \lambda_{-i}^*) - g_i^d(\lambda^*)$, and $\hat{v}_i$ be the m_i-dimensional vector $\hat{v}_i = \big(\hat{v}_i(\lambda_i)\big)_{\lambda_i \in \Lambda_i^0}$. It follows that d is enforceable with respect to α if there exist $y_1, \ldots, y_n \in \mathbf{R}^S$ such that

$$B_i\, y_i \geq \hat{v}_i \text{ for } i = 1, \ldots, n, \text{ and } \sum_{i=1}^n \alpha_i y_i = 0. \tag{4}$$

Note that we have set $\delta = 1/2$ in the expression of enforceability above based on the remark in Section 3. Eliminating y_n using the second condition and writing $\beta_i = \alpha_i / \alpha_n > 0$ ($i = 1, \ldots, n-1$), we can further rewrite (4) as:

$$B\, \hat{y} \geq \hat{v}, \tag{5}$$

where

$$B = \begin{bmatrix} B_1 & & O \\ & \ddots & \\ O & & B_{n-1} \\ -\beta_1 B_n & \cdots & -\beta_{n-1} B_n \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}, \quad \text{and} \quad \hat{v} = \begin{bmatrix} \hat{v}_1 \\ \vdots \\ \hat{v}_n \end{bmatrix}.$$

Therefore, d is enforceable with respect to α if the inequality (5) has a solution $\hat{y}$. The following lemma is a simple application of the theorem of the alternatives.

Lemma 6.1. *Inequality (5) has a solution $\hat{y}$ if and only if for any $(\gamma_1, \ldots, \gamma_n)$ such that $\gamma_i \in \mathbf{R}_+^{m_i}$ $(i = 1, \ldots, n)$,*

$$\gamma_i\, B_i - \beta_i \gamma_n\, B_n = 0 \text{ for every } i = 1, \ldots, n-1 \quad \Rightarrow \quad \sum_{i=1}^{n} \gamma_i \cdot \hat{v}_i \leq 0.$$

Proof: See the Appendix.

The following lemma states the conditions under which the hyperplane requirement $\sum_i \alpha_i y_i = 0$ can be ignored.

Lemma 6.2. *For every $i \in J = \{1, \ldots, n\}$ and $j \neq i$, if there exists $z : S \to \mathbf{R}$ such that $B_i\, z \geq \hat{v}_i$ and $B_j\, z = 0$, then there exists $y = (y_1, \ldots, y_n) : S \to \mathbf{R}^n$ that satisfies (4).*

Proof: In view of Lemma 6.1, suppose that $\gamma_i\, B_i - \beta_i \gamma_n\, B_n = 0$ $(i = 1, \ldots, n-1)$ for some $(\gamma_1, \ldots, \gamma_n) \geq 0$. For any $i < n$, if we multiply from the right $z_i \in \mathbf{R}^S$ such that $B_i\, z_i \geq \hat{v}_i$ and $B_n\, z_i = 0$, then

$$0 = \gamma_i\, B_i\, z_i - \beta_i \gamma_n\, B_n\, z_i = \gamma_i\, B_i\, z_i \geq \gamma_i \cdot \hat{v}_i.$$

Likewise, if we multiply $z_n \in \mathbf{R}^S$ such that $B_i\, z_n = 0$ and $B_n\, z_n \geq \hat{v}_n$, then

$$0 = \gamma_i\, B_i\, z_n - \beta_i \gamma_n\, B_n\, z_n = -\beta_i \gamma_n\, B_n\, z_n \leq -\beta_i \gamma_n \cdot \hat{v}_n,$$

which implies $\gamma_n \cdot v_n \leq 0$. It hence follows that $\sum_{i=1}^{n} \gamma_i \cdot \hat{v}_i \leq 0$. //

The simplest way to have z such that $B_j\, z = 0$ for any $j \in I$ is to make z independent of j's report as shown in the lemma below. For $j \in I$, let $G_j \subset \mathbf{R}^S$ be the set of continuation payoff functions that do not depend on j's report:

$$G_j = \left\{ z \in \mathbf{R}^S : z(\hat{s}) = z(\hat{s}'_j) \text{ if } \hat{s}_{-j} = \hat{s}'_{-j} \right\}.$$

The following lemma is immediate.

LEMMA 6.3. *If $z \in G_j$, then $B_j z = 0$.*

PROOF: See the Appendix.

Refine D^* to be the set of the following instruction rules: the one-shot Nash equilibrium instruction rule d^0, exclusion rules $d^1, \ldots, d^I$, and $d(\cdot \mid \phi, J)$ for all $J \subset I$ such that $|J| \geq 2$ and $\phi \in \Phi_J$.

THEOREM 6.4. *Suppose that for any $\alpha \in \mathbf{R}^I$ such that $|J_\alpha| \geq 2$ and $\phi \in \Phi_{J_\alpha}$, $d(\cdot \mid \phi, J_\alpha)$ is enforceable with respect to α. Then $V \subset V^*(D^*)$. In other words, for any $u \in V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists a collusion scheme τ which chooses instruction rules from D^* and yields payoff $\Pi(\tau, \sigma^*, \delta)$ such that $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.*

PROOF: See the Appendix.

6.1. Independent Signals

In this section, we make the following assumption about the signal distribution.

ASSUMPTION 3: For every $i \in I$, the signal set $S_i = \{s^0, s^1, \ldots, s^K\} \subset \mathbf{R}_+$ for $s^0 < s^1 < \cdots < s^K$. Furthermore, the probability distribution p of signal profile s has full support over S, and is *independent*: $p(s) = \prod_i p_i(s_i)$.

We also maintain Assumption 2 of Section 5 and suppose that the valuation functions are symmetric and monotone. Fix any $J \subset I$ such that $|J| \geq 2$ and $\phi \in \Phi_J$ and consider the instruction rule $d = d(\cdot \mid \phi, J)$. For $i \in J$ and $l = 1, \ldots, K$, let $\bar{Z}_i(s^l \mid \phi, J) \subset S_{-i}$ be the set of signal profiles $s_{-i} = (s_j)_{j \neq i}$ of bidders other than i such that according to $d(\cdot \mid \phi, J)$, i is designated the winner and instructed to bid R under $(s_i = s^l, s_{-i})$:

$$\bar{Z}_i(s^l \mid \phi, J) = \Big\{ s_{-i} \in S_{-i} : d_i(s_i = s^l, s_{-i} \mid \phi, J) = R \Big\}.$$

Let $Z_i(s^l \mid \phi, J) = \bar{Z}_i(s^l \mid \phi, J) \setminus \bar{Z}_i(s^{l-1} \mid \phi, J)$. In other words, $Z_i(s^l \mid \phi, J)$ is the set of s_{-i}'s against which bidder i wins when reporting $\hat{s}_i = s^l$ but loses when reporting $\hat{s}_i = s^{l-1}$. For any $i \in J$, $j \neq i$, $k = 0, 1, \ldots, K$, and $l = 1, \ldots, K$, define

$$w_i^j(s^k, s^l \mid \phi, J) = \frac{\sum_{s_{-i} \in Z_i(s^l \mid \phi, J)} v_i(s_i = s^k, s_{-i})\, p_{-i}(s_{-i})}{p_j(s_j = s^l)}.$$

Since v_i is monotone increasing by Assumption 2, it can be immediately verified that for any k, $k' = 0, 1 \ldots, K$ and $l = 1, \ldots, K$,

$$w_i^j(s^k, s^l \mid \phi, J) \leq w_i^j(s^{k'}, s^l \mid \phi, J) \text{ if } k \leq k'. \tag{6}$$

For any $j \neq i$, let

$$\rho_j^l = \frac{p_j(s_j = s^l)}{\sum_{s_j \leq s^l} p_j(s_j)} \quad (l = 0, 1, \ldots, K),$$

and consider a continuation payoff function $z : S \to \mathbf{R}$ such that

$$z(\hat{s}) = \begin{cases} -x_i(\hat{s}_i) & \text{if } \hat{s}_i \geq \hat{s}_j \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $x_i : S_i \to \mathbf{R}_+$ is a non-negative function of i's report defined recursively by

$$\begin{aligned} & x_i(s^0) = 0, \\ & x_i(s^k) = \rho_j^k \, w_i^j(s^{k-1}, s^k \mid \phi, J) + (1 - \rho_j^k) \, x_i(s^{k-1}) \quad (k = 1, \ldots, K). \end{aligned} \tag{8}$$

Lemma 6.5. *Under Assumptions 2 and 3, $B_i \, z \geq \hat{v}_i$ for z given in (7).*

Proof: See the Appendix.

Since z is a function of the reports of only two bidders, there is at least one other bidder h for whom $z \in G_h$ and hence $B_h \, z = 0$ by Lemma 6.3. This along with Lemma 6.5 and Theorem 6.4 yields the following theorem.

Theorem 6.6. *Suppose that Assumptions 2 and 3 hold. For any $u \in V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D^* and yields the payoff vector $\Pi(\tau, \sigma^*, \delta)$ satisfying $\|\Pi(\tau, \sigma^*, \delta) - u\| < \epsilon$.*

6.2. Correlated Signals

When private signals are correlated, continuation payoffs can be determined using the functional relationship between a bidder's private signal and the probability distribution of other bidders' signal profiles. When there exist three or more bidders, use of such a mechanism yields an extremely powerful conclusion that does not depend on the detailed specification of the valuation functions or the signal distribution. The analysis in this subsection draws heavily on Aoyagi (1998).

For each $s_i \in S_i$, let $p_{-i}(\cdot \mid s_i)$ and $p_{-i-j}(\cdot \mid s_i)$ denote the following vectors of conditional probabilities:

$$p_{-i}(\cdot \mid s_i) = \big(p_{-i}(s_{-i} \mid s_i)\big)_{s_{-i} \in s_{-i}},$$
$$p_{-i-j}(\cdot \mid s_i) = \big(p_{-i-j}(s_{-i-j} \mid s_i)\big)_{s_{-i-j} \in S_{-i-j}}.$$

Assumption 4: For any $i \neq j$, $p_{-i-j}(\cdot \mid s_i) \neq p_{-i-j}(\cdot \mid s_i')$ for any $s_i \neq s_i'$.

When the set of probability distributions p of $s \in S$ is identified with the $(|S|-1)$-dimensional simplex $\Delta^{|S|-1}$, Assumption 4 holds generically in this set as long as $|S_i| \geq 2$ for each $i \in I$. In particular, it holds when the distribution satisfies affiliation with strict inequality. Fix any $J \subset I$ such that $|J| \geq 2$ and $\phi \in \Phi_J$ and consider the instruction rule $d = d(\cdot \mid \phi, J)$.

Lemma 6.7. *Suppose that Assumption 4 holds. For any i, $j \in I$ $(i \neq j)$, there exists a continuation payoff function $z \in G_j$ such that $B_i\, z \geq \hat{v}_i$.*

Proof: See the Appendix.

Combining Lemmas 6.2 and 6.7 and Theorem 6.4, we obtain the following theorem.

Theorem 6.8. *Suppose that $I \geq 3$ and that Assumption 4 holds. Then for any $u \in V$ and $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that the following holds if $\delta > \underline{\delta}$: There exists an equilibrium collusion scheme τ which chooses instruction rules from D^* and yields the payoff vector $\Pi(\tau, \sigma, \delta)$ satisfying $\|\Pi(\tau, \sigma, \delta) - u\| < \epsilon$.*

Appendix

Proof of Lemma 5.1: We first show by induction that

$$t^k \leq \theta v_j(s^k, s^k) \quad \text{and} \quad t^k \leq \frac{v_i(s^k, s^{k+1}) - \rho_i^{k,k}\theta v_j(s^k, s^k)}{1 - \rho_i^{k,k}}.$$

for $k = 0, 1, \ldots, K$. These clearly hold when $k = 0$. For $k \geq 1$, (3) implies that t^k satisfies

$$\begin{aligned} t^k &= \rho_j^{k-1,k}\, v_i(s^{k-1}, s^k) \\ &\quad + \rho_i^{k-1,k-1}\,\{1 - \rho_j^{k-1,k}\}\,\theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_i^{k-1,k-1}\}\{1 - \rho_j^{k-1,k}\}\, t^{k-1}. \end{aligned} \tag{b1}$$

Suppose that $t^{k-1} \le \theta v_j(s^{k-1}, s^{k-1})$. Then (b1) implies that

$$\begin{aligned} t^k &\le \rho_j^{k-1,k}\, v_i(s^{k-1}, s^k) + \rho_i^{k-1,k-1}\{1 - \rho_j^{k-1,k}\}\, \theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_i^{k-1,k-1}\}\{1 - \rho_j^{k-1,k}\}\, \theta v_j(s^{k-1}, s^{k-1}) \\ &= \rho_j^{k-1,k}\, v_i(s^{k-1}, s^k) + \{1 - \rho_j^{k-1,k}\}\, \theta v_j(s^{k-1}, s^{k-1}). \end{aligned}$$

Since $\theta \ge \theta_{ij}$, the RHS is $\le \theta v_j(s^k, s^k)$ as desired. Suppose next that

$$t^{k-1} \le \frac{v_i(s^{k-1}, s^k) - \rho_i^{k-1,k-1}\theta v_j(s^{k-1}, s^{k-1})}{1 - \rho_i^{k-1,k-1}}.$$

It then follows from (b1) that

$$\begin{aligned} t^k &\le \rho_j^{k-1,k}\, v_i(s^{k-1}, s^k) \\ &\quad + \rho_i^{k-1,k-1}\{1 - \rho_j^{k-1,k}\}\, \theta v_j(s^{k-1}, s^{k-1}) \\ &\quad + \{1 - \rho_j^{k-1,k}\}\left\{v_i(s^{k-1}, s^k) - \rho_i^{k-1,k-1}\theta v_j(s^{k-1}, s^{k-1})\right\} \\ &= v_i(s^{k-1}, s^k). \end{aligned}$$

Since $\theta \le 1$, we obtain the desired conclusion. Note next that $\Delta_i^k = x_i(s^k) - x_i(s^{k-1})$ and $\Delta_j^k = x_j(s^k) - x_j(s^{k-1})$ can be expressed in terms of t^{k-1} as

$$\begin{aligned} \Delta_i^k &= \rho_j^{k-1,k}\left[v_i(s^{k-1}, s^k) - t^{k-1} - \rho_i^{k-1,k-1}\left\{\theta v_j(s^{k-1}, s^{k-1}) - t^{k-1}\right\}\right], \\ \Delta_j^k &= \rho_i^{k-1,k-1}\left[v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta}\, t^{k-1}\right] \end{aligned}$$

for $k = 1, \ldots, K$. The above conclusions then imply that $\Delta_i^k,\ \Delta_j^k \ge 0$ for $k = 1, \ldots, K$. //

Proof of Lemma 5.2: Let $\pi_i(s_i, \hat{s}_i \mid d, y)$ denote bidder i's (*interim*) expected payoff under the instruction rule d and the continuation payoff function profile y when he has signal s_i and reports $\hat{s}_i$, and other bidders report their signals truthfully. The conclusion follows if

(b2) $$\pi_i(s_i, s_i \mid d^{ij*}, y) \ge \pi_i(s_i, \hat{s}_i \mid d^{ij*}, y)$$

for any s_i, $\hat{s}_i \in S_i$, and

(b3) $$\pi_j(s_j, s_j \mid d^{ij*}, y) \ge \pi_j(s_j, \hat{s}_j \mid d^{ij*}, y)$$

for any s_j, $\hat{s}_j \in S_j$. We will first show that x_i and x_j defined in (3) satisfy the inequalities (b4)-(b7) below for $k = 1, \ldots, K$. We will then show that these inequalities imply (b2) and (b3).

$$\text{(b4)} \qquad \frac{1}{\rho_j^{k-1,k}}\,\Delta_i^k + \theta\Delta_j^k \geq v_i(s^{k-1}, s^k) - t^{k-1},$$

$$\text{(b5)} \qquad \frac{1}{\rho_j^{k,k}}\,\Delta_i^k + \theta\Delta_j^k \leq v_i(s^k, s^k) - t^{k-1},$$

$$\text{(b6)} \qquad \frac{1}{\rho_i^{k-1,k-1}}\,\Delta_j^k \geq v_j(s^{k-1}, s^{k-1}) - \frac{1}{\theta}\,t^{k-1},$$

$$\text{(b7)} \qquad \frac{1}{\rho_i^{k,k-1}}\,\Delta_j^k \leq v_j(s^k, s^{k-1}) - \frac{1}{\theta}\,t^{k-1}.$$

It can be readily verified that x_i and x_j defined in (3) satisfy (b4) and (b6) with equality. Since Δ_i^k, $\Delta_j^k \geq 0$ by Lemma 1, (b5) holds since $\rho_j^{k,k} \geq \rho_j^{k-1,k}$ and $v_i(s^k, s^k) \geq v_i(s^{k-1}, s^k)$, and (b7) holds since $\rho_i^{k,k-1} \geq \rho_i^{k-1,k-1}$ and $v_j(s^k, s^{k-1}) \geq v_j(s^{k-1}, s^{k-1})$.

For y given in (2), $\pi_i(s_i, \hat{s}_i \mid d^{ij*}, y)$ can be written as

$$\begin{aligned}\pi_i(s_i, \hat{s}_i \mid d^{ij*}, y) = &\sum_{s_j \leq \hat{s}_i} v_i(s_i, s_j)\, p_j(s_j \mid s_i) \\ &- x_i(\hat{s}_i) \sum_{s_j \leq \hat{s}_i} p_j(s_j \mid s_i) + \sum_{s_j > \hat{s}_i} \theta x_j(s_j)\, p_j(s_j \mid s_i).\end{aligned}$$

Hence, (b2) for $s_i = s^{k-1}$ and $\hat{s}_i = s^k$ (*i.e.*, the "one-step upward" incentive compatibility condition) is equivalent to ($k = 1, \ldots, K$):

$$\text{(b8)} \qquad \begin{aligned}&\pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^k \mid d^{ij*}, y) \\ &= -v_i(s^{k-1}, s^k)\, p_j(s^k \mid s_i = s^{k-1}) + x_i(s^{k-1})\, p_j(s^k \mid s_i = s^{k-1}) \\ &+ \theta x_j(s^k)\, p_j(s^k \mid s_i = s^{k-1}) + \Delta_i^k \sum_{s_j \leq s^k} p_j(s_j \mid s_i = s^{k-1}) \geq 0,\end{aligned}$$

and (b2) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ (*i.e.*, the "one-step downward" incentive compatibility condition) is equivalent to:

$$\text{(b9)} \qquad \begin{aligned}&\pi_i(s^k, s^k \mid d^{ij*}, y) - \pi_i(s^k, s^{k-1} \mid d^{ij*}, y) \\ &= v_i(s^k, s^k)\, p_j(s^k \mid s_i = s^k) - x_i(s^{k-1})\, p_j(s^k \mid s_i = s^k) \\ &- \theta x_j(s^k)\, p_j(s^k \mid s_i = s^k) - \Delta_i^k \sum_{s_j \leq s^k} p_j(s_j \mid s_i = s^k) \geq 0.\end{aligned}$$

Rearranging, we see that (b8) and (b9) are equivalent to (b4) and (b5), respectively. For bidder j, $\pi_j(s_j, \hat{s}_j \mid d^{ij*}, y)$ can be written as

$$\begin{aligned}\pi_j(s_j, \hat{s}_j \mid d^{ij*}, y) = &\sum_{s_i < \hat{s}_j} v_j(s_j, s_i)\, p_i(s_i \mid s_j) \\ &- x_j(\hat{s}_j) \sum_{s_i < \hat{s}_j} p_i(s_i \mid s_j) + \sum_{s_i \geq \hat{s}_j} \theta^{-1} x_i(s_i)\, p_i(s_i \mid s_j).\end{aligned}$$

Hence, (b3) for $s_j = s^{k-1}$ and $\hat{s}_j = s^k$ is equivalent to ($k = 1, \ldots, K$):

$$\begin{aligned}&\pi_j(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_j(s^{k-1}, s^k \mid d^{ij*}, y) \\ &= -v_j(s^{k-1}, s^{k-1})\, p_i(s^{k-1} \mid s_j = s^{k-1}) + x_j(s^{k-1})\, p_i(s^{k-1} \mid s_j = s^{k-1}) \\ &+ \theta^{-1} x_i(s^{k-1})\, p_i(s^{k-1} \mid s_j = s^{k-1}) + \Delta_j^k \sum_{s_i \leq s^{k-1}} p_i(s_i \mid s_j = s^{k-1}) \geq 0,\end{aligned} \tag{b10}$$

and (b3) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ is equivalent to:

$$\begin{aligned}&\pi_j(s^k, s^k \mid d^{ij*}, y) - \pi_j(s^k, s^{k-1} \mid d^{ij*}, y) \\ &= v_j(s^k, s^{k-1})\, p_i(s^{k-1} \mid s_j = s^k) - x_j(s^{k-1})\, p_i(s^{k-1} \mid s_j = s^k) \\ &- \theta^{-1} x_i(s^{k-1})\, p_i(s^{k-1} \mid s_j = s^k) - \Delta_j^k \sum_{s_i \leq s^{k-1}} p_i(s_i \mid s_j = s^k) \geq 0.\end{aligned} \tag{b11}$$

Rearrangement shows that (b10) and (b11) are equivalent to (b6) and (b7), respectively.

As an induction hypothesis, suppose that (b2) holds for $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l-1}$ ($l = 1, \ldots, K-k$). When $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l}$, we have

$$\begin{aligned}&\pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^{k+l} \mid d^{ij*}, y) \\ &= \pi_i(s^{k-1}, s^{k-1} \mid d^{ij*}, y) - \pi_i(s^{k-1}, s^{k+l-1} \mid d^{ij*}, y) \\ &+ \left\{-v_i(s^{k-1}, s^{k+l}) + \theta x_j(s^{k+l}) + x_i(s^{k+l-1})\right\} p_j(s^{k+l} \mid s_i = s^{k-1}) \\ &+ \Delta_i^{k+l} \sum_{s_j \leq s^{k+l}} p_j(s_j \mid s_i = s^{k-1}).\end{aligned}$$

By the induction hypothesis, the RHS is ≥ 0 if

$$\begin{aligned}&- v_i(s^{k-1}, s^{k+l}) + \theta x_j(s^{k+l}) + x_i(s^{k+l-1}) + \Delta_i^{k+l} \frac{\sum_{s_j \leq s^{k+l}} p_j(s_j \mid s_i = s^{k-1})}{p_j(s^{k+l} \mid s_i = s^{k-1})} \\ &= -v_i(s^{k-1}, s^{k+l}) + \theta \Delta_j^{k+l} + t^{k+l-1} + \frac{1}{\rho_j^{k+l,k-1}} \Delta_i^{k+l} \geq 0,\end{aligned}$$

or equivalently,

$$\frac{1}{\rho_j^{k+l,k-1}}\,\Delta_i^{k+l}+\theta\Delta_j^{k+l}\geq v_i(s^{k-1},s^{k+l})-t^{k+l-1}.$$

Since $\rho_j^{k+l,k-1}\leq\rho_j^{k+l,k+l-1}$ by (1) and $v_i(s^{k-1},s^{k+l})<v_i(s^{k+l-1},s^{k+l})$, (a1) for $k+l$ implies the above inequality. Therefore, (b2) holds for $s=s^{k-1}$ and $\hat{s}=s^{k+l}$ ($k=1,\dots,K$, $l=0,\dots,K-k-1$). We can show by an analogous argument that (b3) holds for $s=s^k$ and $\hat{s}=s^{k-l}$ ($k=1,\dots,K$, $l=1,\dots,k$). The argument for j is similar and is omitted. //

Proof of Lemma 6.1: By the theorem of the alternatives (Rockafellar (1970, Theorem 22.1)), (5) has a solution if and only if for any $\gamma\in\mathbf{R}_+^s$,

$$\gamma\,B=0\Rightarrow\sum_{i=1}^{n}\gamma_i\cdot\hat{v}_i\leq 0.$$

Take any $\gamma\in\mathbf{R}_+^s$ and write $\gamma=(\gamma_1,\dots,\gamma_n)$, where γ_i is m_i-dimensional. Simple algebra shows that $\gamma\,B=0$ is equivalent to

$$\gamma_i\,B_i-\beta_i\gamma_n\,B_n=0\quad\text{for } i=1,\dots,n-1,$$

and that $\gamma\cdot\hat{v}\leq 0$ is equivalent to

$$\sum_{i=1}^{n}\gamma_i\cdot\hat{v}_i\leq 0.$$

Hence, the desired conclusion follows. //

Proof of Lemma 6.3: Fix any $\hat{s}_j\in S_j$. For any $\lambda_j\in\Lambda_j$,

$$\begin{aligned}
b_j(\lambda_j)\cdot z&=\sum_{\hat{s}\in S}\left\{p(\hat{s}\mid\lambda^*)-p(\hat{s}\mid\lambda_j,\lambda_{-j}^*)\right\}z(\hat{s})\\
&=\sum_{\hat{s}_{-j}\in S_{-j}}z(\hat{s}_j,\hat{s}_{-j})\sum_{\hat{s}_j\in S_j}\left\{p(\hat{s}_j,\hat{s}_{-j}\mid\lambda^*)-p(\hat{s}_j,\hat{s}_{-j}\mid\lambda_j,\lambda_{-j}^*)\right\}\\
&=\sum_{\hat{s}_{-j}\in S_{-j}}z(\hat{s}_j,\hat{s}_{-j})\left\{p(\hat{s}_{-j}\mid\lambda_{-j}^*)-p(\hat{s}_{-j}\mid\lambda_{-j}^*)\right\}=0,
\end{aligned}$$

where the second equality follows since $z(\hat{s}_j',\hat{s}_{-j})=z(\hat{s}_j,\hat{s}_{-j})$ for any $\hat{s}_j'\in S_j$.

PROOF OF THEOREM 6.4: Let $\alpha \neq 0$ be given. If $\alpha_i \leq 0$ for every $i \in I$, then $\alpha \cdot u < \alpha \cdot g^0$. Otherwise, set $J = J_\alpha$ and write $g_i(\phi, J)$ for $g_i^{d(\cdot|\phi,J)}(\lambda^*)$.

If $|J| = 1$, then $\alpha \cdot u < \alpha \cdot g_1^1$ and $\alpha \in A(d^1)$. Suppose now that $|J| \geq 2$. If $\alpha \cdot u < \alpha \cdot g^0$ or $\alpha \cdot u < \alpha \cdot g^i$ for some $i \in I$, then we are done since $\alpha \in \mathbf{R}^I \setminus \{0\} = A(d^0) = A(d^i)$. Otherwise, we must have $\alpha \cdot u < \alpha \cdot g^{\phi *}$ for some $\phi \in \Phi_I$. Let $\bar{\phi} \in \Phi_J$ be the *restriction* of ϕ to J: For any i, $j \in J$,

$$\bar{\phi}(i) \leq \bar{\phi}(j) \Leftrightarrow \phi(i) \leq \phi(j).$$

In other words, the relative ranking between any pair of bidders in J under $\bar{\phi}$ is the same as that under ϕ. Since $g_j^{\phi *} \leq g_j(\bar{\phi}, J)$ for each $j \in J$ and $g_j(\bar{\phi}, J) = 0$ for each $j \notin J$, we have

$$\alpha \cdot g^{\phi *} \leq \sum_{j=1}^{n} \alpha_j g_j^{\phi *} \leq \alpha \cdot g(\bar{\phi}, J).$$

It then follows that $\alpha \cdot u < \alpha \cdot g(\bar{\phi}, J)$. Since $\alpha \in A(d(\cdot \mid \bar{\phi}, J))$ by assumption, the proof is complete. //

PROOF OF LEMMA 6.5: As in the proof of Lemma 5.2, let $\pi_i(s_i, \hat{s}_i \mid d, y)$ denote bidder i's (*interim*) expected payoff under the instruction rule d and the continuation payoff function y when he has signal s_i and reports $\hat{s}_i$, and other bidders report their signals truthfully. Write $d = d(\cdot \mid \phi, J)$ for simplicity. The conclusion follows if

(b12) $$\pi_i(s_i, s_i \mid d, y) \geq \pi_i(s_i, \hat{s}_i \mid d, y)$$

for any $s_i \neq \hat{s}_i$. We will first show that x_i defined in (8) satisfies (b13) and (b14) for $k = 1, \ldots, K$. We will then show that x_i satisfies (b12).

(b13) $$x_i(s^k) \geq \rho_j^k \, w_i^j(s^{k-1}, s^k) + (1 - \rho_j^k) \, x_i(s^{k-1}),$$

(b14) $$x_i(s^k) \leq \rho_j^k \, w_i^j(s^k, s^k) + (1 - \rho_j^k) \, x_i(s^{k-1}),$$

As x_i satisfies (b13) with equality, it also satisfies (b14) since $w_i^j(s^{k-1}, s^k) \leq w_i^j(s^k, s^k)$ by (6).

Note now that if y is such that $y_i = z$ for z given in (7), $\pi_i(s_i, \hat{s}_i \mid d, y)$ can be written as

$$\pi_i(s_i, \hat{s}_i \mid d, y) = \sum_{s_{-i} \in \bar{Z}_i(\hat{s}_i)} v_i(s_i, s_{-i}) \, p_{-i}(s_{-i}) - x_i(\hat{s}_i) \sum_{s_j \leq \hat{s}_i} p_j(s_j).$$

Hence, (b12) for $s_i = s^{k-1}$ and $\hat{s}_i = s^k$ is equivalent to

$$\text{(b15)}\quad \pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^k \mid d, y) \\ = -w_i^j(s^{k-1}, s^k)\, p_j(s^k) + x_i(s^k) \sum_{s_j \le s^k} p_j(s_j) - x_i(s^{k-1}) \sum_{s_j \le s^{k-1}} p_j(s_j) \ge 0.$$

Since $\sum_{s_j \le s^{k-1}} p_j(s_j) = (1-\rho_j^k) \sum_{s_j \le s^k} p_j(s_j)$, (b15) simplifies to (b13). Likewise, (b12) for $s_i = s^k$ and $\hat{s}_i = s^{k-1}$ is equivalent to (b14).

As an induction hypothesis, suppose that (b12) holds for $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l-1}$ ($l = 1, \ldots, K-k$). When $s_i = s^{k-1}$ and $\hat{s}_i = s^{k+l}$, we have

$$\pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^{k+l} \mid d, y) \\ = \pi_i(s^{k-1}, s^{k-1} \mid d, y) - \pi_i(s^{k-1}, s^{k+l-1} \mid d, y) \\ - w_i^j(s^{k-1}, s^{k+l})\, p_j(s^{k+l}) + x_i(s^{k+l}) \sum_{s_j \le s^{k+l}} p_j(s_j) - x_i(s^{k+l-1}) \sum_{s_j \le s^{k+l-1}} p_j(s_j),$$

By the induction hypothesis, the RHS is ≥ 0 if

$$- w_i^j(s^{k-1}, s^{k+l})\, p_j(s^{k+l}) + x_i(s^{k+l}) \sum_{s_j \le s^{k+l}} p_j(s_j) - x_i(s^{k+l-1}) \sum_{s_j \le s^{k+l-1}} p_j(s_j) \\ = \Big\{ x_i(s^{k+l}) - \rho_j^{k+l}\, w_i^j(s^{k-1}, s^{k+l}) - (1-\rho_j^{k+l})\, x_i(s^{k+l-1}) \Big\} \sum_{s_j \le s^{k+l}} p_j(s_j) \ge 0.$$

Since $w_i^j(s^{k-1}, s^{k+l}) \le w_i^j(s^{k+l-1}, s^{k+l})$ by (6), (b13) for $k+l$ implies the above inequality. Therefore, (b12) holds for $s = s^{k-1}$ and $\hat{s} = s^{k+l}$ ($k = 1, \ldots, K$, $l = 0, \ldots, K-k-1$). An analogous argument shows that (b12) holds for $s_i = s^k$ and $\hat{s}_i = s^{k-l}$ ($k = 1, \ldots, K$, $l = 1, \ldots, k$). //

Proof of Lemma 6.7: Write $\| \cdot \|$ for the square norm, and let $z \in G_j$ be defined by $z(s_i, s_j, s_{-i-j}) = \bar{z}(s_i, s_{-i-j})$ for any $(s_i, s_j, s_{-i-j}) \in S$, where

$$\bar{z}(s_i, \cdot) = \frac{p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s_i)\|}.$$

Then for any s_i, $s_i' \in S_i$,

$$\bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) = \frac{p_{-i-j}(\cdot \mid s_i) \cdot p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s_i)\|} = \|p_{-i-j}(\cdot \mid s_i)\|, \quad \text{and}$$
$$\bar{z}(s_i', \cdot) \cdot p_{-i-j}(\cdot \mid s_i) = \frac{p_{-i-j}(\cdot \mid s_i') \cdot p_{-i-j}(\cdot \mid s_i)}{\|p_{-i-j}(\cdot \mid s_i')\|}.$$

If $s_i \neq s_i'$, then $p_{-i-j}(\cdot \mid s_i) \neq p_{-i-j}(\cdot \mid s_i')$ by Assumption 4 and hence by the Cauchy-Schwartz inequality,

$$p_{-i-j}(\cdot \mid s_i) \cdot p_{-i-j}(\cdot \mid s_i') < \|p_{-i-j}(\cdot \mid s_i)\| \, \|p_{-i-j}(\cdot \mid s_i')\|.$$

It follows that

$$\bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) > \bar{z}(s_i', \cdot) \cdot p_{-i-j}(\cdot \mid s_i).$$

This further implies that for any $\lambda_i \in \Lambda_i^0$, we have

$$b_i(\lambda_i) \cdot z = \sum_{s_i \in S_i} p(s_i) \left\{ \bar{z}(s_i, \cdot) \cdot p_{-i-j}(\cdot \mid s_i) - \bar{z}(\lambda_i(s_i), \cdot) \cdot p_{-i-j}(\cdot \mid s_i) \right\} > 0,$$

or equivalently, $B_i\, z > 0$. Therefore, we have $B_i\, z \geq \hat{v}_i$ if we redefine z to be kz for $k > 0$ sufficiently large. //

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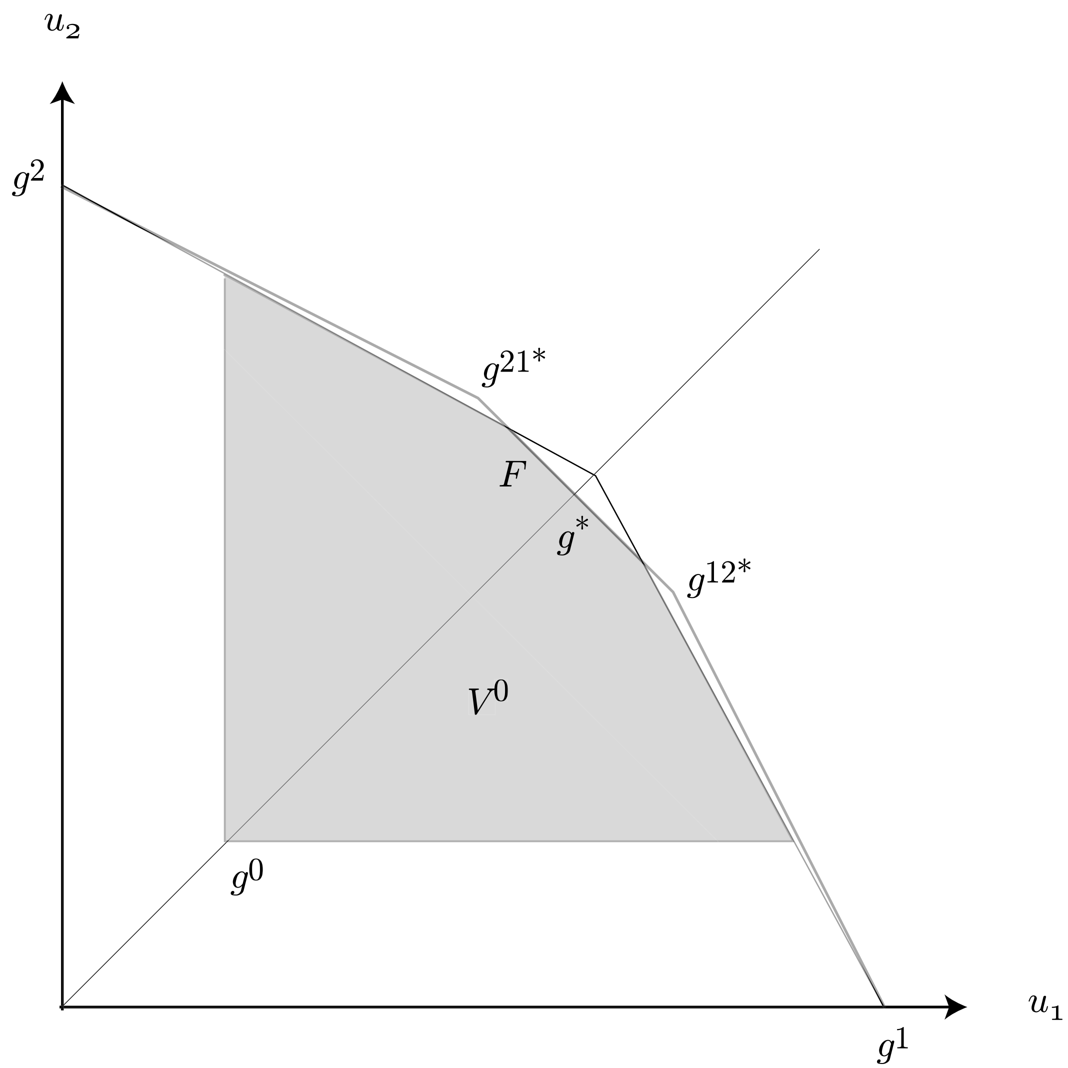

Figure 1