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ROBUST CYCLICAL GROWTH

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Abstract

The stability of cyclical growth within the context of a model in Matsuyama (1999) is examined. It is shown that but for an extreme situation, the two-cycles are unique and a range of parameter values which imply the stability of such cyclical growth is derived. The growth enhancing property of 2-cycles are shown to be retained by any cycle; the results of simulation exercises carried out are reported to show that for very wide range of parameter values, such cyclical growth paths are stable and thus robustness of the conclusions are established. Finally, the configuration of parameters for which the dynamics exhibits complicated (chaotic) behavior is also identified.

KEYWORDS : Cycles, stability of cycles, robustness, growth, innovation, chaos.

1 Introduction

Studies of growth processes usually consider two sources of growth: one is through process of accumulation and the other is through processes of technical change and innovation. Matsuyama (1999)(and (2001)) provides a neat model which combines the two sources. This strategy allows the feasibility of sustaining growth indefinitely through technical progress, since without such an avenue, accumulation can only take an economy some distance before diminishing returns destroys growth possibilities. With accumulation and innovation alternating, the studies referred to being in discrete time, growth possibilities are examined.

That neoclassical models of growth, in discrete time, may be capable of non-convergence and generating complicated behavior has been known for some time, now. Consequently, it is worthwhile to investigate whether the possibility of innovation affects long term behavior. In the studies mentioned above, it is claimed that there is a trade-off between growth and innovation; while innovation is necessary for sustaining growth, during the process of innovation, growth rate is shown to be low; on the other hand, when no innovation occurs, growth rate picks up. When the steady state is unstable, it is shown there are two-cycles, which alternate between these two phases and offers the possibility of growing at rates which are more than the one associated with the steady state.

While the existence of such cyclical possibilities are of interest, it is only when these cycles are stable that the conclusion has some relevance. We examine this issue closely and show that parameters of the process are such that with similar looking parameter values, two-cycles may either be stable or unstable. Consequently it is necessary to investigate the range of parameter values for which the results are valid. A range of parameter values which imply stability of the 2-period cycles is obtained. The range which imply stability for 2-cycles is shown to increase with an increase in a crucial parameter. However, if one is interested only in enhanced growth possibilities along cyclical paths, we show that any cyclical path will have this property and so we have to examine only if the cycle is stable. We report the results of some exercises to conclude that such cycles are stable for very wide range of parameter values. Consequently the conclusions are robust.

The paper proceeds as follows: we introduce the model, to keep the treatment self-contained[1], and consider the existence of two-cycles and their stability. It is shown that but for an extreme situation, the two-cycles are unique; we also derive a stability condition for such cycles. This allows us to obtain a range over which the 2-cycles are stable. We also investigate how this range changes when parameters change. Next, we focus attention on what happens when two-cycles lose stability. The growth enhancing property of 2-cycles are shown to be retained by any cycle and consequently it is of some interest to investigate the stability of such cycles; the results of simulation exercises carried out are reported to show that for very wide range of parameter values, we have stable cycles; finally, the configuration of parameter values for which there may be complicated (chaotic) behavior is also identified. The concluding section briefly discusses the significance of these results.

2 The Model

There is one final good which is produced under competitive conditions with the help of labor (L) supplied inelastically and several intermediate goods; the single final good is both consumed and invested; say that at the end of period $t-1$, K_{t-1} units of the final good is available for production in period t and is to play the role of capital.

Additionally in period t, there are several types of intermediate goods available denoted

[1] But we refrain from a discussion of related literature and refer the interested reader to Matsuyma (1999) and (2001).

by z, $z \in [0, N_t]$; prior to period t, types of intermediate goods up to N_{t-1} have been introduced ($N_0 > 0$); in period t, production of type z up to N_{t-1} (the old intermediate goods) require just a units of capital per each unit of any of the old intermediate goods. New intermediates in the range $[N_{t-1}, N_t]$ may also be introduced and sold exclusively by those who choose to innovate; for each of the new intermediate good, in addition to the a units of capital per unit, a fixed cost of F units of capital has to be incurred before production can be carried out. The old intermediate goods are sold competitively and hence, their price reflect marginal costs i.e.,

$$p_t(z) = ar_t, z \in [0, N_{t-1}]$$

where r_t is the price of a unit of capital in period t; the new intermediates are sold under conditions of monopoly by the investor who innovates and to compute sales we need to find out about the demand for the new intermediates.

Production of the final good in period t takes place according to the production function:

$$Y_t = AL^{1/\sigma}\{\int_0^{N_t} [x_t(z)]^{1-1/\sigma} dz\} \tag{1}$$

where $x_t(z), z \in [0, N_t]$ is the intermediate good of type z in period t; $\sigma \in [1, \infty]$ is the constant direct partial elasticity of substitution between any two pairs of intermediates; the following features should be noted:

•given N_t, production in period t satisfies constant returns to scale.

•the price elasticity of demand for each intermediate is σ.

•the share of labor is $1/\sigma$.

Given the above demand condition, the price of the new intermediates must satisfy

$$p_t^m(z) = \frac{\sigma a r_t}{\sigma - 1}, z \in (N_{t-1}, N_t]$$

Since all intermediate goods enter the production function symmetrically, we may take

$$x_t(z) = x_t^c \forall z \in [0, N_{t-1}]; x_t(z) = x_t^m \forall z \in (N_{t-1}, N_t]$$

and given the demand conditions specified above,

$$\frac{x_t^c}{x_t^m} = [\frac{p_t^c}{p_t^m}]^{-\sigma} = [1 - \frac{1}{\sigma}]^{-\sigma} \tag{2}$$

Note that the above has been obtained on the assumption that both the new and the old intermediate goods are produced. The new intermediates are produced by monopolists whose one period profit π_t^m is the sole incentive for production. Note that $\pi^m(t) = p_t^m.x_t^m - r_t(ax_t^m + F) = r_t(ax_t^m \frac{1}{\sigma-1} - F)$. Free entry guarantees that profits are non-positive so that we have $ax_t^m \leq (\sigma - 1)F$ with the proviso that: $N_t \geq N_{t-1}$ and

$$(N_t - N_{t-1})[a.x_t^m - (\sigma - 1)F] = 0 \tag{3}$$

Thus, when $N_t > N_{t-1}$, innovation occurs and new products are introduced; the innovator earns no extra profits and operates at its break even point; while if this is not possible, potential sales do not break even, then $N_t = N_{t-1}$ and no innovation takes place. In this situation, of course there is a constraint which needs to be looked into. Recall that available capital is K_{t-1} and the production of both types of intermediates requires the use of this resource; hence we must have

$$K_{t-1} = N_{t-1}a.x_t^c + (N_t - N_{t-1})(a.x_t^m + F)$$

Recall that if production takes place for both the sets of intermediate products (the new and the old), we have, $N_t > N_{t-1}$ and from (2), $a.x_t^c = a.x_t^m\{\frac{\sigma-1}{\sigma}\}^{-\sigma}$ and from (3), $ax_t^m = (\sigma - 1)F$, so that $ax_t^c = \theta F\sigma$ where $\theta = \{\frac{\sigma-1}{\sigma}\}^{1-\sigma}$. It should be pointed out that as $\sigma \in [1, \infty]$, $\theta \in [1, e]$. If on the other hand, $N_t = N_{t-1}$, $ax_t^c = \frac{K_{t-1}}{N_{t-1}}$. Thus, we have:

$$ax_t^c = \min[\frac{K_{t-1}}{N_{t-1}}, \theta\sigma F] \tag{4}$$

And further,

$$N_t = N_{t-1} + \max[0, \frac{K_{t-1}}{\sigma F} - \theta N_{t-1}] \tag{5}$$

Substituting the above into (1), we have:

$$Y(t) = \begin{cases} AL^{\frac{1}{\sigma}} N_{t-1}(\frac{K_{t-1}}{aN_{t-1}})^{1-\frac{1}{\sigma}} & \text{for } \sigma F\theta N_{t-1} \geq K_{t-1} \\ AL^{\frac{1}{\sigma}}[N_{t-1}(\frac{\theta\sigma F}{a})^{1-\frac{1}{\sigma}} + (\frac{K_{t-1}}{\sigma F} - \theta N_{t-1})(\frac{(\sigma-1)F}{a})^{1-\frac{1}{\sigma}}] & \text{otherwise} \end{cases} \tag{6}$$

And finally to close the model, we have that a constant fraction of the output is left unused so that it may be used as capital in the next period,[2] i.e.,

$$K_t = \mu Y_t \tag{7}$$

To simplify matters, we introduce the following notation:

$$\alpha = \frac{A(aL)^{\frac{1}{\sigma}}}{a(\theta\sigma F)^{\frac{1}{\sigma}}},\ k_t = \frac{K_t}{N_t\sigma F\theta}$$

The dynamics of the entire system is captured through the equations (5), (6) and (7). And we may combine them into a single equation, using the variable k_t introduced above:

$$k_t = \begin{cases} \mu\alpha k_{t-1}^{1-\frac{1}{\sigma}} & \text{if } k_{t-1} \leq 1 \\ \frac{\mu\alpha k_{t-1}}{1+\theta(k_{t-1}-1)} & \text{otherwise} \end{cases} \tag{8}$$

In short, we shall write

[2]It is this which provides the point of departure for the contribution of Matsuyama (2001), where capital accumulation is derived from intertemporal optimization of the infinitely lived agent. This does lead to a complication in the dynamics.

$$k_t = \phi(k_{t-1}) \tag{9}$$

and given any initial k^0, we shall study the iterates $\phi^t(k^0)$, for $t = 1, 2, ...$

It should be noted $\phi(1) = G = \mu\alpha$; further that $\phi'(k) = (1 - \frac{1}{\sigma})Gk^{-\frac{1}{\sigma}} > 0$ if $k < 1$ and $\phi'(k) = \dfrac{G(1-\theta)}{(1+\theta(k-1))^2} < 0$ if $k > 1$. Thus the map $\phi(.)$ is of the standard uni-modal variety with a maximum value at 1.

FIGURES 1a and 1b HERE

It may be easily seen by referring to (8) that a crucial parameter for the system is $G = \phi(1)$. Note also that as $k \to \infty$, $\phi(k) \to \dfrac{G}{\theta}$, which is finite, so that for all k large enough, $\phi(k) < k$; also since $\phi(k) > k$ for all k small enough (since $\phi(0) = 0$ and $\phi'(k) > 1$ for all k small enough), it follows that there exists $k > 0$ such that $k = \phi(k)$. Further such a positive k is unique too.

There are two possibilities depending on the magnitude of G; first of all notice that if $G < 1$, k^* is an equilibrium $\Rightarrow k^* < 1$; consequently $k^* \leq 1 \Rightarrow k^* = G^\sigma$; and if $G > 1$ then $k^* > 1 \Rightarrow k^* = \dfrac{G-1}{\theta} + 1$. Consider then these two cases one by one.

Consider first, $G \leq 1$; as indicated above, in this case, $k^* = G^\sigma$. In this equilibrium, notice that N remains fixed and hence no innovation occurs; since N is constant and given the definition of the variable k, K too remains fixed and hence Y, so that no growth takes place. Beginning from any arbitrary k^0, the iterates settle to the regime $k < 1$ (Matsuyama calls this the Solow regime) and converges to k^* since, $|\phi'(k^*)| = |(1 - \frac{1}{\sigma})| < 1$; thus k^* is locally asymptotically stable.

Consider next $G > 1$; $k^* = \dfrac{G-1}{\theta} + 1$ and $|\phi'(k^*)| = |\dfrac{\theta - 1}{G}|$; thus it is seen that local asymptotic stability requires that $\theta - 1 < G$. If on the other hand, $\theta - 1 > G$, the equilibrium is unstable. At this equilibrium notice that N is increasing continuously and keeping pace with it is K and as we may see Y too is increasing all at the same rate: the case of balanced growth. Along the balanced growth path, where $k_t = k^* > 1$, it follows from (5) that $\frac{N_{t+1}}{N_t} = G$ and consequently, $\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = G$ as well; thus $G - 1$ is the growth rate per period along the balanced growth path.

3 2-cycles and Their Stability

When $G > 1$ and $\theta - 1 > G$, the equilibrium is unstable as we have shown above; but there is a 2-cycle, i.e., $\exists k_1 > 1 > k_2$ such that $k_1 = \phi(k_2)$ and $k_2 = \phi(k_1)$.

Thus it is asserted that there exist $k_1 > 1 > k_2$ satisfying the following:

$$k_2 = \frac{Gk_1}{1 + \theta(k_1 - 1)}$$

and

$$k_1 = Gk_2^{1-\frac{1}{\sigma}}$$

Using the second relation to eliminate k_1, we have the following claim: there is $0 < k_2 < 1$ such that

$$k_2^{1-\frac{1}{\sigma}}[(1-\theta)k_2^{\frac{1}{\sigma}} + G\theta k_2 - G^2] = 0$$

Note that writing $f(k_2) = [(1-\theta)k_2^{\frac{1}{\sigma}} + G\theta k_2 - G^2]$, $f(0) = -G^2 < 0$ while $f(1) = (G-1)(\theta - 1 - G) > 0$ and hence the claim follows. Also note that $f'(0) < 0$ and $f'(k) = 0 \Rightarrow k = \overline{k} = \{\frac{\theta - 1}{\sigma\theta G}\}^{\frac{\sigma}{\sigma-1}}$ and $f(.)$ is decreasing for $k < \overline{k}$ and increasing for $k > \overline{k}$; consequently, there is a unique k_2 satisfying the above claim.

It is easy to see that there cannot be any two period cycle (k_1, k_2), $0 < k_1 < k_2 < 1$. For then $k_1 = Gk_2^{1-\frac{1}{\sigma}}$ and $k_2 = Gk_1^{1-\frac{1}{\sigma}}$; this implies that $k_1 = G^{\sigma} > 1$ which is a contradiction.

Nor can there be a cycle (k_1, k_2), $1 < k_2 < k_1$; for other wise $k_2 + k_2\theta(k_1 - 1) = Gk_1$ and $k_1 + k_1\theta(k_2 - 1) = Gk_2$ so that $(k_2 - k_1)(1 - \theta + G) = 0$ which too is a contradiction.

Thus:

1 *There is a unique 2-period cycle* (k_1, k_2) *with* $0 < k_2 < 1 < k_1$ *when* $\theta - 1 > G > 1$.

The stability of this cycle depends on whether the following is greater or less than unity:

$$|\phi'(k_1).\phi'(k_2)| = |\frac{G^2(1-\frac{1}{\sigma})(\theta-1)}{k_2^{\frac{1}{\sigma}}\{1+\theta(k_1-1)\}^2}| = [\frac{G^2(\theta-1)}{k_2^{\frac{1}{\sigma}}}].\frac{(1-\frac{1}{\sigma})}{\{1+\theta(k_1-1)\}^2}$$
$$= \frac{(\theta-1)(1-\frac{1}{\sigma}).G^2.k_1^2}{\{1+\theta(k_1-1)\}^2.k_1^2.k_2^{\frac{1}{\sigma}}}$$
$$= \frac{\theta-1}{G}.\frac{k_2^{\frac{1}{\sigma}}(1-\frac{1}{\sigma})}{G}$$
$$= A.B \text{ say}$$

Note that $A > 1$, while $B < 1$ so that the product could go either way. If the above happens to be less than unity then, the 2-period cycle is attracting while if the quantity is greater than unity the cycle is unstable. It should be clear that the smaller is the ratio A, the greater is the chance of the cycle being stable. As we shall see, this point is of crucial importance for locating a robust range of parameter values for which the cycles will be stable. There can be nothing more definite about the aspect of stability since as is clear from the above expression, both possibilities exist.

For instance, consider the following situations for $\sigma = 5$ and hence, $\theta = 2.44$; but for alternative similar looking values of G, we have:

Example 1 *Let* $G = 1.075 < \theta - 1 = 1.44$. *Then the two period cycle is given by* $k_1 = 1.06487 > 1 > k_2 = 0.988226$; $|\phi'(k_1).\phi'(k_2)| = 0.995503$ *hence the cycle is stable.*

Example 2 *Let* $G = 1.070 < \theta - 1 = 1.44$. *Then the two period cycle is given by* $k_1 = 1.06041 > 1 > k_2 = 0.988805$; $|\phi'(k_1).\phi'(k_2)| = 1.00491$ *hence the cycle is unstable.*

Since both kinds of cycles are possible, consider then the merit of the following proposition (Proposition 2, Matsuyama (1999)):

Let g_x *be the gross growth rate of the variable* x. *Along the period-2 cycles*

(a) $g_N = 1 < G < G(k_2)^{-\frac{1}{\sigma}} = g_K = g_Y$ *in the Solow Regime.*

(b) $g_N = 1 + \theta(k_1 - 1) > G = g_k = g_Y$ *in the Romer Regime.*

(c) $g_N = g_K = g_Y = \{1 + \theta(k_1 - 1\}^{\frac{1}{2}} = Gk_2^{\frac{1}{2\sigma}} > G$ *over the cycles.*

The claims made above were meant to show that output (Y) and investment (K) grow faster in the Solow Regime: i.e., (when $k < 1$ and no innovation takes place); in other words, even though innovation is essential to sustain growth, during the process of innovation (the Romer regime when $k > 1$) the economy registers lower growth.

As should be obvious, this conclusion is of some interest only if the cycle is stable, since otherwise the fact, that along the cycle a higher growth rate is possible, is of little interest. It is because of this that the calculations about the stability of the cycles provided above assume significance.

3.1 Are Robust Stable 2-Cycles Plausible?

What kind of parameter values are sure to provide us with stable cycles ? The Examples (1) and (2) show that with roughly similar looking values of G one may get different results. But there is some thing more that one may say in addition to the above.

Consider the situation when $G = \theta - 1$; it is immediate, by referring back to (8), that in this case, $\phi(1) = G$ and $\phi(G) = 1$; further $|\phi'(1).\phi'(G)| = (1 - \frac{1}{\sigma})/G < 1$ so that this cycle is stable (here $\phi'(1)$ is to be interpreted as the derivative on the left); this assertion follows by considering the expression $A.B$ derived earlier; now $A = 1$ and hence the assertion follows. But in this particular situation there is an embarrassment of cycles; since for any $k_1 > 1, k_1 < G$ there is $k_2 > 1$ such that $\phi(k_1) = k_2$ and $\phi(k_2) = k_1$. But these cycles disappear as soon as $G < \theta - 1$, as we have seen above; but, it should be clear, that for G close to but less than $\theta - 1$, the cycle close to $(1, G)$ remains and that cycle retains stability.

Thus if G is close but less than $\theta - 1$ then the resultant 2-cycle is stable. Note that for the situation described in Examples 1 and 2, $\theta - 1$ is about 1.44; thus some where between the values for G given by 1.07 and 1.075, the two-period cycles change their stability property and we have a point of bifurcation; for lower values of G there **may** be other points of bifurcation.

In other words, very complicated dynamics may be possible in this framework and consequently, the relevance of the Proposition such as the one mentioned above may be ascertained only if we are assured that parameters have certain values which guarantee the robust stability of the two-period cycles i.e., G is close to $\theta - 1$. This is robust because there would be an open set $O = (\overline{G}, \theta - 1)$ and any $G \in O$ would ensure stability of the two period cycles. The lower bound on this interval amounts to where the product $A.B$ becomes unity [3]. To obtain an estimate for $\overline{G}$, we may note that the expression in $A.B$ may be written as

$$\frac{(\theta - 1)(1 - \frac{1}{\sigma})}{G^2}.k_2^{\frac{1}{\sigma}} = C.D \text{ say}$$

where we know that $D < 1$ so now if $C < 1$ then we are sure that $A.B < 1$ and the cycle is stable. Thus a sufficient condition for the stability of the two-cycle is:

[3] See, Matsuyama (1999), p. 344.; it is clear from this, that the author is aware that there may be ranges over which the 2-cycles may be unstable; also over this range, the attracting cycles may have periods which are powers of 2. It appears that the author is only concerned about the empirical plausibility of the existence of 2-cycles, as his discussion on plausibility considers only the condition for the existence of 2-cycles, viz., $\theta - 1 > G > 1$, with G being 'close' to $\theta - 1$. The entire plausibility exercise considers $G = \theta - 1$.

2 *If* $G > \hat{G} = \sqrt{((\theta - 1)(1 - \frac{1}{\sigma}))}$ *then the 2-cycle is stable.*

The claim follows by noticing that whenever the condition is satisfied, the term $C < 1$. In fact for $\sigma = 5$, for example, we have, $\hat{G} = 1.07384$ so that, whenever $G > 1.07384$ the two-cycle is stable[4]. This allows us to obtain an estimate of the open interval O, viz., $(\hat{G}, \theta - 1)$ and for any G-value in this interval, the two-cycle is stable. Since we do not know what the magnitude of the parameter σ will be, let us take the range suggested in Matsuyama (1999), $[5, 22]$; we examine next, how the range of G-values which imply stability for the 2-cycle, $[\hat{G}, \theta - 1)$ behaves with a variation in the parameter σ.

3 *Let* H *be defined as the range of* G*-values which imply stability of the period 2-cycles, i.e.,* $H = \theta - 1 - \hat{G}$*. Then* H *is an increasing concave function of* σ *in the range* $[5, 22]$*.*

The proof of this claim is contained in the Figure 2 below[5]

FIGURE 2 HERE

Thus so far as two-cycles are concerned, the range of G-values which ensure stability increase with the value of σ. Be that as it may, we shall show below, that we do not need to confine attention to 2-cycles alone to establish the fact that cycles are growth enhancing. In fact, the range of values for the parameter G for which stable cycles are growth enhancing, as we indicate, cover a much wider range.

[4]It should be clear that $\hat{G} > \overline{G}$. Thus for $\sigma = 5$, $G \in [1.07384, 1.44)$ implies stability of the two period cycle. In Matsuyama (1999), as we indicated above, G is taken to be $\theta - 1 = 1.44$ while discussing plausibility. Our analysis thus reveals that the 2-cycles are a lot more "plausible".

[5]Given the nature of the functions involved, we have used Mathematica to generate the diagram which seems to be enough for the purpose at hand.

4 When 2-cycles fail to attract

When the 2-cycle is not attracting, we need to analyze further. In this section, we shall confine ourselves to the case when $1 < G < \theta - 1$. Given this restriction, we note that $\phi(G) < 1$. To proceed formally, note first of all,

4 $\phi : [0, G] \to [0, G]$.

Next, we note that[6]:

5 $[\phi(G), G]$ *is an absorbing state (i.e.,* $\phi(G) \leq k \leq G \Rightarrow \phi(G) \leq \phi(k) \leq G$*).*

In addition,

6 *For any initial point* $k^o \in [0, G]$*,* $k^o \neq 0, k^\star$*,* $\exists t$ *such that* $\phi^t(k^o) \in [\phi(G), G]$ *unless* $\phi^t(k^o) = 0$ *for some* t*.*

For, suppose to the contrary, there is some $k^o \in [0, G]$ such that for no t is $\phi^t(k^o) \in [\phi(G), G]$; then $\phi^t(k^o) \in (0, \phi(G)) \forall t$ i.e., $k_{t+1} = \phi^{(t+1)}(k^o) = \phi(k_t) = G k_t^{1-1/\sigma} > k_t$. Thus $\{k_t\}$ is a monotonically increasing sequence bounded above and hence must converge to some $\overline{k}$, say, $\overline{k} \in [0, \phi(G)]$. Note that since k_{t+1} and k_t both converge to $\overline{k} \in [0, \phi(G)]$, we have $\phi(\overline{k}) = \overline{k} \Rightarrow \overline{k} = 0$ which is a contradiction, since a monotonically increasing sequence of positive numbers cannot converge to zero.

Thus the structure of an arbitrary trajectory is that apart from hitting the unstable equilibrium 0 accidentally, it will enter the interval $[\phi(G), G]$ in finite time and remain inside, thereafter. Limit points for the trajectory will thus exist and will be located in the interval $[\phi(G), G]$. Our next task is to locate these limit points, if possible.

We note, however, that cyclical orbits, if these exist, have a special structure:

[6] See, for instance, Matsuyama (1999), p. 342.

7 *Consider any cycle of period n, $k_1, k_2, ..., k_n \in [0, G]$, say, where $\phi(k_n) = k_1$ with $k_1 = \min_{1 \le j \le n} k_j$ must have $k_1 < 1$ and $k_n > 1$, with $k_n = \max_j k_j$.*

First of all note that $k_n > k^\star$ since otherwise $k_1 = \phi(k_n) \ge k_n$: a contradiction. Thus $1 < k^\star < k_n \le G$. Next suppose that to the contrary, $k_r = \max_j k_j$, $r \ne n$; then since $\phi(k_r) = k_{r+1} < k_r$ given the maximal nature of k_r, it follows that $k_r > k_n > k^\star > 1$ and thus, $k_{r+1} < k_1$: a contradiction to the definition of k_1.

We have thus, $\phi(1) = G > k^\star > k_1 > \phi(G)$. Suppose that $k_1 \ge 1$ that is, we have: $1 \le k_1 < k^\star, k_n$. Then, $\phi(k_1) = k_2 > \phi(k^\star) = k^\star \Rightarrow k_3 = \phi(k_2) < k^\star$ but $k_3 > k_1$ since recall that k_1 was the minimum. Thus $k^\star > k_3 > k_1 \ge 1$ and $G > k_2 > k^\star$; proceeding in this manner, it may be concluded that n must be even i.e., $n = 2s$ say and we must have

$$k_1 < k_3 < k_5....k^\star < k_{2s=n} < k_{2(s-1)} < ...k_2 \le G$$

But now $k_1 \ne \phi(k_n)$: a contradiction. Hence $k_1 < 1$ as claimed. •

Thus **any** n-cycle must spend at least one period in the Solow-Regime. This fact allows us to note the following property of growth rates along any n-period cycle. Consider any such cycle $k_1, k_2, ..., k_n$; we shall write $k_{n+1} = k_1$ and we note the following: If $k_j < 1$ then $k_{j+1} = Gk_j^{1-1/\sigma}$; thus we have

$$\frac{k_{j+1}}{k_j} = Gk_j^{-1/\sigma}$$

which means that

$$g_K = \frac{K_{j+1}}{K_j} = Gk_j^{-1/\sigma} > G$$

since **no innovation occurs** and $N_{j+1} = N_j$. On the other hand, if $k_j > 1$, we have

$$\frac{k_{j+1}}{k_j} = \frac{G}{1 + \theta(k_j - 1)}$$

and further since **innovation takes place** , we have from equation (5)

$$g_N = \frac{N_{j+1}}{N_j} = 1 + \theta(k_j - 1)$$

consequently, we have:

$$\frac{K_{j+1}}{K_j} . \frac{N_j}{N_{j+1}} = \frac{g_K}{g_N} = \frac{G}{1 + \theta(k_j - 1)}$$

which means that $g_K = G$, given the expression for g_N obtained above. With these preliminaries, let us return to the n-cycle $k_1, k_2, ..., k_n$ with $k_{n+1} = \phi(k_n) = k_1$ which means that

$$\frac{K_{n+1}}{N_{n+1}} = \frac{K_1}{N_1} \Rightarrow \frac{K_{n+1}}{K_1} = \frac{N_{n+1}}{N_1}$$

in other words, we have:

$$\frac{K_{n+1}}{K_n} ... \frac{K_2}{K_1} = \frac{N_{n+1}}{N_n} ... \frac{N_2}{N_1}$$

on the left hand side, each term is either G if the corresponding $k_j \geq 1$ or is $G.k_j^{-1/\sigma}$ if the corresponding $k_j < 1$. Thus the product on the left hand side is greater than G^n, since we have shown that at least one of the k_j's namely k_1 is less than 1. Consequently, the **average gross growth rate** $g_K = g_Y = g_N$ **along the cycle must be greater than** G**.** We note this in the form of the following claim:

8 *Along any n-period cycle, the average gross growth rate $g_K = g_Y = g_N > G$.*

Let us return to the **Example 2** , where the 2-cycle had been shown to be unstable. It may be shown that in that context viz., $G = 1.07, \sigma = 5$: there is a stable 4-period cycle $k_1 < 1, k_2 > 1, k_3 < 1, k_4 > 1$, $\phi(k_i) = k_{i+1}, i = 1, 2, 3, 4$ with $k_5 = k_1$. $k_1 = 0.9888, k_2 = 1.060402, k_3 = 0.988814, k_4 = 1.0604139$; further it may be checked that $|\phi'(k_1)....\phi'(k_4)|$ $= 0.000583969 < 1$ which signifies that the 4 period cycle is stable. Figure 3 contains a demonstration of the existence and convergence to this cycle.

FIGURE 3 HERE

For the case $\sigma = 5$, it has not been possible to locate any more points of bifurcation; that is on the basis of simulation exercises, and the bifurcation diagram (Figure 4) it seems reasonable to conclude that there may be **at the most** a stable 4-period cycle.

FIGURE 4 HERE

The bifurcation diagram[7] plots for values of G between $\theta - 1 = 1.4$ and 1, the last 100 points in an iteration of 1000 points from an arbitrary initial point. For G between 1 and 1.4, beginning with the high G values, we note the first approach to a two period cycle; for lower values of G, the two period cycle disappears and a stable four period cycle appears. This happens for values of G close to 1 and it has not been possible to uncover other points of bifurcation[8]. Outside the range, we note convergence to a steady state, as the results indicate.

We consider next bifurcation diagrams for narrower ranges of the parameters but still for $\sigma = 5$: first consider the range of G values between 1.4 and 1.48.

FIGURE 5 HERE

Notice the shift from convergence to the fixed point to convergence to the two-period cycle as the value of G decreases. And consider next the range of G values between 1 and 1.08, still for $\sigma = 5$.

FIGURE 6 HERE

[7] All diagrams were prepared on Mathematica Version 4.1 .

[8] The exercises we have conducted seem to indicate that bifurcations leading to cycles with period 8 and period 16 and more are non-existent, at least when σ is small; see Matsuyama (1999), p.344. For such possibilities, σ has to exceed 22. See below.

Notice here the shift from convergence to a two period cycle to convergence to a four period cycle as G attains the value unity. Thus Figures 5 and 6 focus attention on the two extreme ranges of the parameter G contained in Figure 4.

We present next, an example of topological chaos when G is small and σ is large. For this purpose it would be helpful to note the following:

9 $\phi^2(1) = \phi(\phi(1)) < 1 \Leftrightarrow 1 < G < \theta - 1.$

Thus a necessary condition for topological chaos[9] is always satisfied, for every value of G whenever, the steady state is unstable. But to clinch matters and to show the presence of complicated dynamics, we need to show that $\phi^3(1) < k^\star$ [10]

Example 3 *Consider $\sigma = 22$ and $G = 1.001$; one may then check that $\phi^3(1) < k^\star$. We thus have topological chaos for this map. For values of $\sigma \geq 22$, low values of G imply the existence of topological chaos*[11].

The bifurcation diagram for $\sigma = 22$ is provided below.

FIGURE 7 HERE

The above set of exercises allows us to draw the following conclusions:

- When the steady state is unstable, two-cycles exist; further, any G value in the set $[\hat{G}, \theta - 1)$ implies that the two-cycle is stable. This range increases with σ over the range $[5, 22]$ of plausible σ-values.

[9] See, for instance, Mitra (2001), p. 140, (2.5).

[10] Mitra (2001), Proposition 2.3, p. 142.

[11] See Mitra (2001), p.142 where he looks at the case when $\sigma = 50, G = 1.01$ to apply his result.

- When σ is small (around 5), the steady state and the two period-cycle is unstable, there would be a stable cyclical orbit with period 4; cycles with higher periods do not appear possible.
- When σ is large (≥ 22), possibilities for chaos exist particularly if the value of G is small enough.
- Whenever the steady state is unstable and a stable cycle exists, the rate of growth along the cycle will be greater than that along the unstable steady state.

5 Concluding Remarks

We examined whether a growth cycle could be obtained as limiting behavior; it turned out this led us to a consideration of what could be the parameter values which would imply such behavior. We also noted the existence of bifurcation points around which the stability properties of these cycles change sharply and hence it is important to examine the robustness of the claims. For meaningful cyclical behavior of period 2, the parameters had to be in the right range so the cycles could be attractive. We computed a range of values which imply the stability of period 2 cycles. We also investigated how this range changes when the parameter σ varies. It turns out that large σ values enhances the chances of the stability of the 2-cycles.

With low G-values, outside the range, 2-cycles may lose stability but stable cycles with larger periods may exist. It was of some interest to note that the possibilities of higher rates of growth along period 2-cycles are retained by the cycles with larger periods. Thus the conclusion that growth paths, which are cyclical in nature, are capable of providing higher rates of growth turns out to be a quite robust conclusion.

Our analysis also indicates that the chances of encountering chaotic trajectories arise only when G is "small" and σ is "large". Recall that if σ is large, there would be a larger range of possible G-values which imply stability of the 2-cycles. So outside this range, we shall have higher period cycles and only for G values close to 1 there maybe chaotic trajectories. The simulation exercises seem to indicate that unless σ was greater or equal to 22 and G was "small"(close to 1), we would not encounter this phenomenon; how small G must be depends on the value of σ, of course. But so long as cycles were stable, the conclusion that cyclical paths lead to enhanced growth, remains valid.

It may be recalled that the parameter σ has two roles in the analysis: first of all in the determining the share of labor $(1/\sigma)$ and secondly, in fixing the demand elasticity of the the intermediate goods (σ) and thereby the monopoly margin. The other parameter of interest is G, the gross rate of growth along the steady state in the Romer regime. Thus increasing σ amounts to lowering the share of labor and at the same time, increasing the monopoly margin. This has been shown to lead to two effects: first, this makes the chances of a stable 2-cycle larger that is for a wider set of G values, the two cycles will be stable; secondly, if the 2-cycles were unstable that is, if G were to be low enough, we may also encounter chaotic trajectories. But so long as there are stable cycles, growing along them would lead to larger rates of growth[12].

[12]Even in the presence of topological chaos, there may be a stable cycle with period which is not a power of 2; this is the case when chaos may not be observable as in the case of the logistic map $f(x) = Kx(1-x)$; for $K \approx 3.83$, one may show that $f^3(1/2) = 1/2$ and so satisfies the conditions of topological chaos; but almost all trajectories converge to this 3-period cycle, see, for example Day and Pianigiani (1991) in this connection; our results show that even along such a cycle, the rate of growth would be larger than the one along the unstable steady state. The fundamental point, thus, is that there should be a stable cycle. The periodicity is of little concern in this connection.

Finally in any dynamic economic model, there are several aspects in the specification of the laws of change: one is the functional form; for example, whether processes are linear or non-linear; and if non-linear, what terms are present. Quite apart from these issues, there are, in addition parameters of various kinds. In economic theory, the problem has been that these parameters can never be specified with any degree of exactitude. It is because of this reason that the exercises carried out above assume significance.

References

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Figure 1A: The Map with G > 1

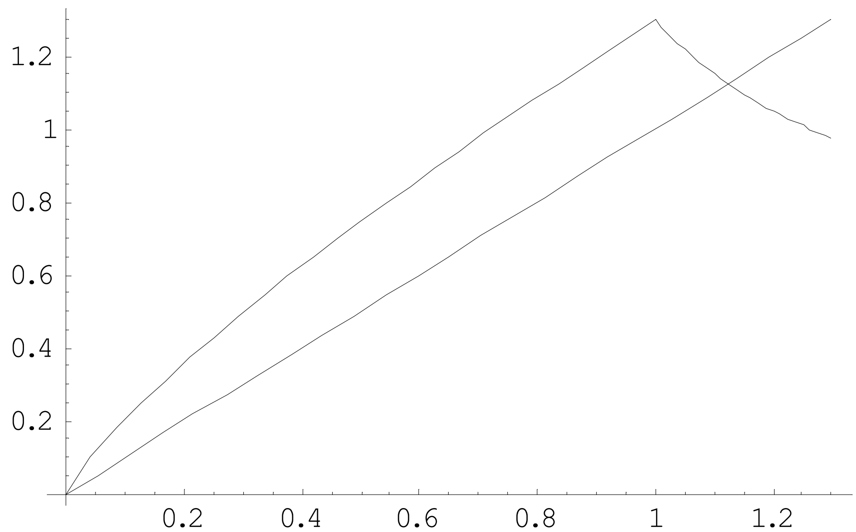

Figure 1B: The Map with G < 1

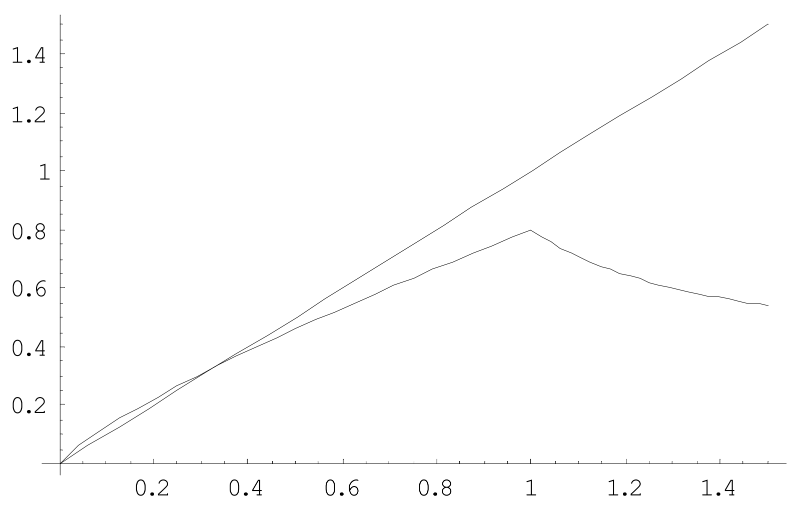

Figure 2: Range of G-values implying stability of 2-cycles

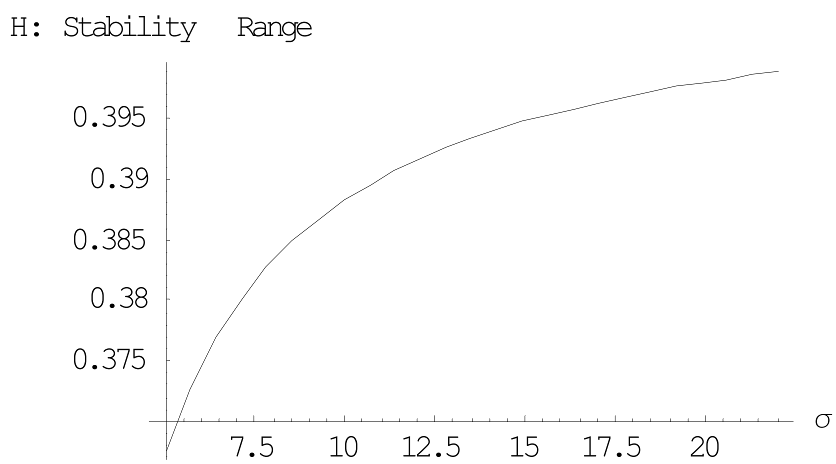

Figure 3: Convergence to a 4-cycle (σ = 5, G = 1.07)

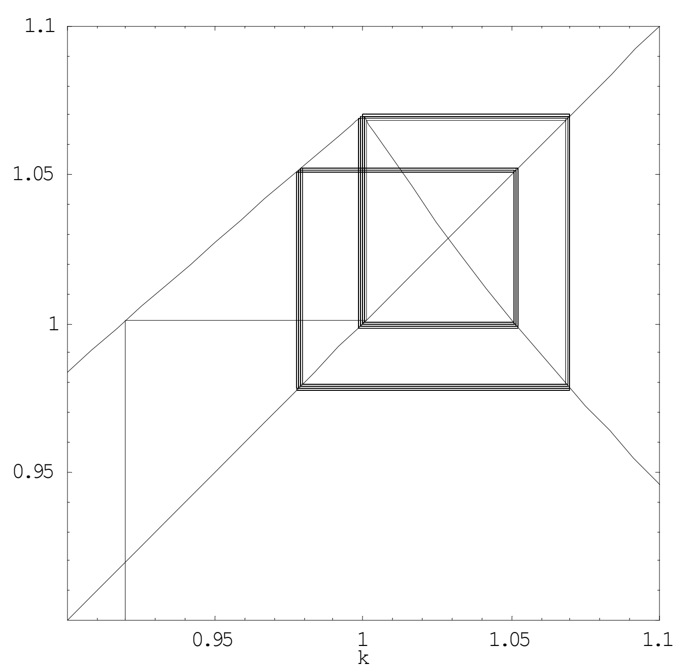

Figure 4: The Bifurcation Map
σ=5

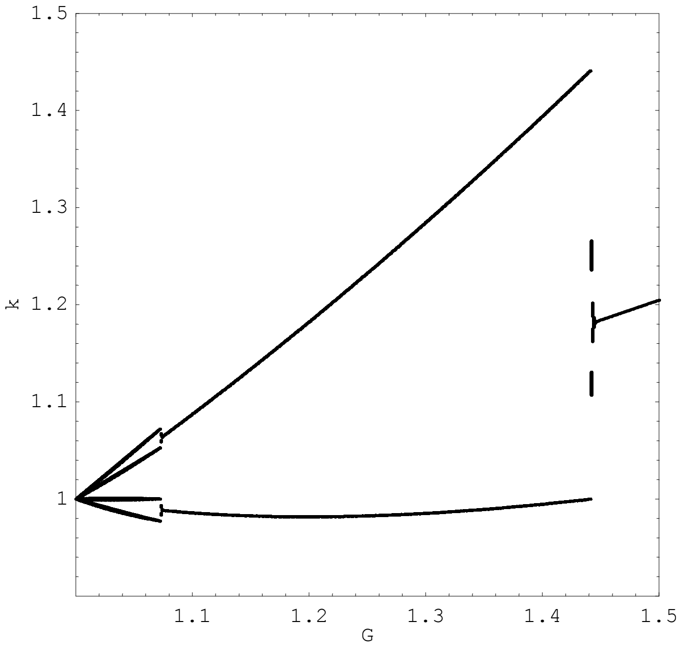

Figure 5: The Bifurcation Map
$\sigma = 5, 1.4 \le G \le 1.48$

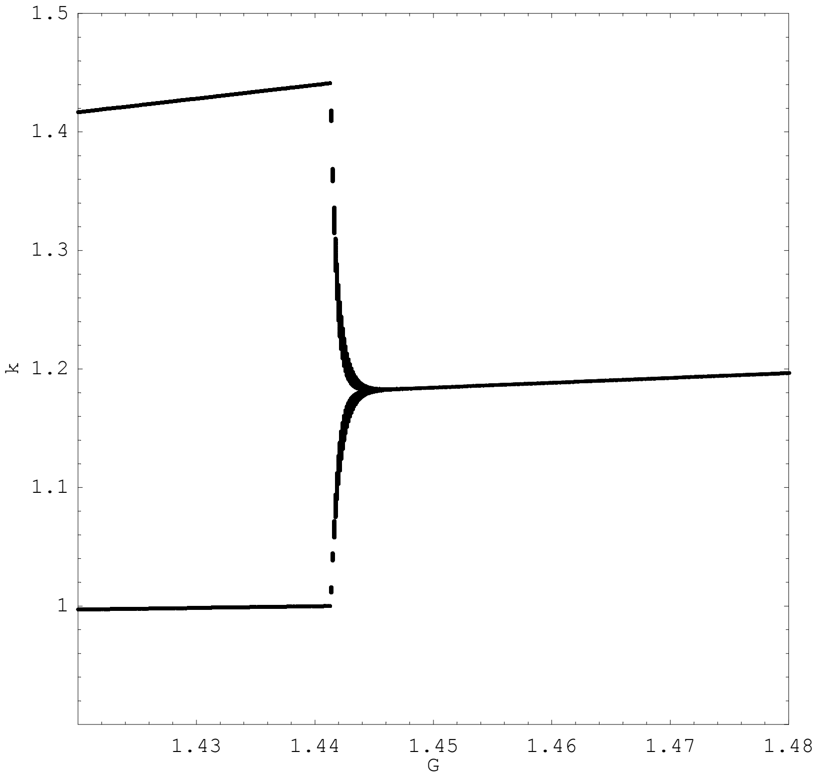

Figure 6: The Bifurcation Map
σ = 5, 1.0 ≤ G ≤ 1.08

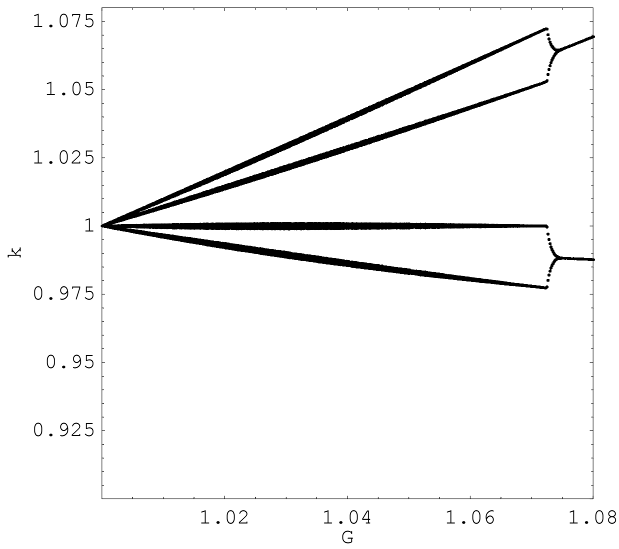

Figure 7: The Bifurcation Map σ = 22