3 Stability of Competitive Equilibrium

3.1 Introduction

The stability of competitive equilibrium is one area which attracted a lot of attention in the mid-fifties and the sixties. But thereafter, the interest some how shifted with the realization that unlike the question of the existence of competitive equilibrium, stability questions could not be resolved as satisfactorily. Thereafter, the stability question was never analyzed and neglected for reasons which are not really clear.

It may be of some interest to note what Malinvaud had to say in 1993 delivering the anniversary lecture on the eve of the Conference held to celebrate the 40th anniversary of the Arrow-Debreu-McKenzie contributions at the December 1952 Meetings of the Econometric Society: “Were there any failure of general nature? Thinking about the question, I am identifying two deficiencies, which would be considered as failures. They may be called imperfect competition and price stability” and then later, “The second deficiency follows from an ambiguity in the teachings of general equilibrium theory about the performance of the market system...... Here I am not referring to the formal problem of how equilibrium is reached, a problem about which, by the way, we shall perhaps be a little too silent in this conference”. More recently, McKenzie (1994), comments that “interest in this question (of stability of the competitive equilibrium) had been revived by the contribution of Arrow and Hurwicz (1958).... Although interest has lapsed in recent years, I do not regard this subject as completely obsolete”.

The views of these two eminent scholars provide the first motivation for re-examining
these issues more closely with the tools of non-linear dynamics. The views of McKenzie and Malinvaud notwithstanding, there is a more basic and and fundamental reason in carrying out such a reexamination in the current context. During the late 20th Century, the overwhelming favorite economic principle was based on a complete belief in the powers of the market mechanism. Clearly this logically implied that equilibria were stable. Since if these were not so, then the equilibrium would be attained by serendipity rather than design. Yet on the theoretical side of things, at the same time, there were no indication that there was any design which could guarantee attainment of equilibrium. This is what Malinvaud refers to in the paragraph noted above. One of the things that we wish to do in the present chapter is to reexamine the working of market mechanism.

The foundations of the subject had rested on the role of income effects. The belief, from the classical Samuelson (1947), Hicks (1946) contributions, had been that if the income effects could be overcome, the competitive equilibrium would at least be locally stable. One of the principal conditions for local stability has been that income effects cancel out at equilibrium. That some such restriction would be required became evident due to the contribution of Scarf (1960) and Gale (1963). Later contributions, which were interested in global stability primarily, introduced other conditions such as gross substitutes, dominant diagonals and the weak axiom of revealed preference. The really restrictive nature of these conditions were never fully appreciated till the contributions of Sonnenschein (1973) and

\footnote{This has been referred rather picturesquely to as being part of the consciousness of the profession. As we hope to show in the pages below, many such items in our professional consciousness need to be reformulated.}

\footnote{See Negishi (1962)or Hahn (1982) for surveys of these contributions.}

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Debreu (1974) established that excess demand functions could only be subjected to Walras Law and homogeneity of degree zero in the prices. With the Sonnenschein and Debreu contributions, and with the Scarf and Gale examples in the background, it was felt that anything could happen. Thus while the theorists among the profession felt that anything could happen, when they wrote in professional journals, the same theorists strangely remained quiet when the stability of equilibrium was routinely assumed and the virtues of competitive equilibria were extolled. Even as the present version is being prepared, the virtues of the competitive mechanism appear somewhat tarred but not on account of lack of stability. Accordingly the issues concerning stability of competitive equilibrium need a thorough reexamination.

The above provides, it is hoped, a satisfactory reason to be engaged in looking once more at question of stability of competitive equilibrium, we need to first set out the context or the model, which shall be the main vehicle of our discussion.

In the sections below, we hope to reexamine these issues to provide an unified view. This unification, as we shall show, serves to clarify issues and reveal connections which we believe are of fundamental interest. We begin with the nature of the price adjustment process and investigate how the standard form of the *tatonnement* may be derived from the Walrasian hypothesis of price behavior in disequilibrium. The question of what the numeraire should be, is also considered. Thus what has been called the market mechanism

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3This has more to do with the realization that the market mechanism may break down due to informational asymmetries. In other words, concerns were with the problems of existence of competitive equilibrium and not with its stability.
needs to be provided with some form which makes it amenable for analysis.

We hope to establish in the next stage, two things: first that it is the weak axiom of revealed preference which is the basic condition for stability of competitive equilibrium. We do this by first examining a necessary condition for local stability; it will be shown that this is related to the weak axiom; all the conditions mentioned above, the so-called sufficient conditions for local stability follow from the weak axiom. So far as global stability consideration are concerned, we try to pin down what the path of prices would be in the general situation. It turns out that although this path may be quite arbitrary, the weak axiom of revealed preference may again be used to provide some clues. The second thing that we hope to establish is that there are ways of determining stability conditions, in other words, place restriction on parameter choices, so that stability is obtained. Some well known models of instability are taken up for analysis in this context.

Apart from the above mentioned results being of interest on their own right, there is another aspect that we should point out. It is the recent contribution of Anderson et. al. (2002) in experimental economics. This is particularly important since one of the reasons for the neglect of this very fundamental area has been the feeling that the formalization of the market mechanism into the tatonnement has been inappropriate\textsuperscript{4}. The experiment conducted by Messrs Anderson et. al., however, seems to indicate that the predictions of results by the tatonnement process are accurate even in a non-tatonnement experimental double auction situation. Thus the predictions of a tatonnement process may be important after all.

\textsuperscript{4}Whether this was because the results were not quite satisfactory is not very clear.
Finally, it should be pointed out that on account of the theory of dynamical systems alone, we cannot expect very general conditions for convergence. This has been established relatively recently by Smale (1976). It may be worthwhile to consider what his result is. Let us denote by $\Delta^n$ the unit simplex in $\mathbb{R}^n$ spanned by the unit vectors $e_i = (\delta_{ik}), \delta_{ik} = 0, k \neq i, \delta_{ii} = 1$. The Smale result is as follows: Let $X$ be any $C^1$ vector field in $\Delta^{n-1}$; then there exists a $C^1$ vector field $F = (F_i)$ in $\mathbb{R}^n$ satisfying $F_i = x_i M_i(x), M_{ij}(x) < 0, j \neq i$ such that $F|\Delta^{n-1} = X$ and $\Delta^{n-1}$ is an attractor. In other words, for $n > 2$, “anything goes” on account of dynamical systems alone. Convergence therefore for any dynamical system involving more than 2 variables would require special conditions. This justifies the need for analyzing what these stability conditions are. And even more so, if the more than 2 variables are concerned. In the pages below, we shall identify “stability conditions” and also pay close attention to what happens on the plane.

3.2 Excess Demand Functions

The economy is considered to be made up of households $h, h = 1, 2, \cdots, H$, each with a consumption possibility set $X_h \subset \mathbb{R}^n$ where $X_h$ is convex and bounded below. Also each household $h$ has a strictly quasi-concave, strictly increasing and continuously differentiable utility function $U^h : X_h \to \mathbb{R}$; further each household $h$ has an endowment $w^h \in \text{Int}X_h$ where $\text{Int}X_h$ denotes the interior of the set $X_h$ and further $w^h \neq 0$. Firm $j, j = 1, 2, \cdots, J$ possesses a production possibility set $Y^j \subset \mathbb{R}^n$ which is assumed to be strictly convex and bounded above; also $Y^j \cap \mathbb{R}^n_+ = \{0\}; Y^j \cap -Y^j = \{0\}; y \in Y^j, z \leq y \Rightarrow z \in Y^j$ are assumed to hold for every $j$. Finally, $\theta_{hj} \geq 0$ is the share of household $h$ in
the share of firm $j$’s profit $\Pi_j$ with $\sum_h \theta_{hj} = 1$ for all $j$. Profits $\Pi_j$ are defined by the value of the following programme:

$$\text{Max } p.y$$

subject to $y \in Y^j$

Let $y^j(p)$ solve the above problem; note that, given our assumptions, this solution exists and is unique for all $p \in R^n_+$. $y^j(p)$ is the supply function and the profit function $\Pi^j(p)$ is defined by $\Pi^j(p) = p.y^j(p)$. Household $h$ solves the programme:

$$\text{Max } U^h(x)$$

subject to $p.x \leq p.w^h + \sum_j \theta_{hj} \Pi^j$, $x \in X_h$

Given our assumptions, a unique solution $x^h(p)$, the demand function exists to the above utility maximization exercise, for all $p \in R^n_+$. The excess demand function then is defined by:

$$Z(p) = \sum_h x^h(p) - \sum_j y^j(p) - \sum_h w^h = X(p) - Y(p) - W, \text{ say}$$

where $X(p), Y(p)$ and $W$ respectively stand for the aggregate demand, aggregate supply and the aggregate endowment. The excess demand function, so derived, will be taken to satisfy the following properties:

P1. $Z(p)$ is a continuously differentiable function of $p$ which is bounded below for all $p \in R^n_+$. 
P2. \( p.Z(p) = 0 \) for all \( p \in R_{++}^n \) (Walras Law)

P3. \( Z(\lambda p) = Z(p) \) for any \( \lambda > 0 \) and for all \( p \in R_{++}^n \).

P4. If \( p^s, s = 1, 2, \cdots, p^s \in R_{++}^n \), \( \|p^s\| \geq \delta > 0 \) for some \( \delta \) for all \( s \) and if \( p_k^s \to 0 \) as \( s \to \infty \) for some \( k \), then \( \sum_j Z_j(p^s) \to \infty \).

The above properties P1. - P4. are all standard properties\(^5\). The references provided below would also convince persons that the set \( E = \{ p \in R_{++}^n : Z(p) = 0 \} \neq \emptyset \). Let \( p^* = (p_1^*, \cdots, p_n^*, 1) \in E \). Unless otherwise stated, we shall choose all prices \( p = (p_1, \cdots, p_{n-1}, 1) \) i.e., with good \( n \) as the numeraire. Define the set \( K = \{ p \in R_{++}^n, p_n = 1 : p^*.Z(p) \leq 0 \} \).

It is relatively straightforward to see

**Claim 3.2.1** \( K \) is a nonempty and compact subset of \( R_{++}^n \) and has a positive distance from the boundary of \( R_{++}^n \).

Proof. Note that \( E \subset K \) and hence \( K \) is nonempty, since \( E \) is non-empty. The remaining part of the claim follows by virtue of the fact that if there is any sequence \( p^s \in K \) such that \( \|p^s\| \to \infty \) then clearly \( p_k^s \to \infty \) as \( s \to \infty \) for some \( k \). Define \( q^s = \frac{1}{p_k^s}p^s \). Note that by virtue of P3., \( Z(q^s) = Z(p^s)\forall s \); hence \( p^*.Z(q^s) \leq 0\forall s \). Note also that \( q_k^s = 1\forall s \) and further \( q_n^s \to 0 \) as \( s \to \infty \). Consequently P4. applies and given the bounded below nature of excess demand functions, one may conclude that \( p^*.Z(q^s) \to \infty \) as \( s \to \infty \); thus, \( p^*.Z(q^s) > 0\forall s \) large enough. But then \( \forall s \) large enough, \( p^s \notin K \): a contradiction; so no such sequence exists and \( K \) is bounded. The closure of \( K \) follows from the definition. Next note that the

distance of $K$ from the boundary of $R^n_+$ denoted by $B$, say is

$$d(K, B) = \inf_{x \in K, y \in B} d(x, y) = \alpha \text{ say}$$

where

$$d(x, y)^2 = \sum_i (x_i - y_i)^2$$

If $\alpha = 0$, then there is a sequence $p^s \in K \ \forall s$ such that $p^s_j \to 0$ as $s \to \infty$. Since $\|p^s\| \geq 1 \ \forall s$, P4 applies and $\sum_j Z_j(p^s) \to \infty$ as $s \to \infty$; thus exactly as argued above, $p^s \notin K \ \forall s$ large enough : a contradiction. Hence $\alpha > 0$ as claimed. ●

To need to apply our results on the existence of solutions to differential equations, we need to strengthen the property P1 to:

P1': For each $j$, $Z_j(p)$ is bounded below and continuously differentiable function of $p$ for all $p \in R^n_{++}$.

We shall say that **Weak Axiom of Revealed Preference (WARP) holds at $p$** if $p^* Z(p) > 0$; otherwise, we shall say that WARP is **violated at $p$**. If WARP holds for all $p \in R^n_{++}$, then **WARP holds**. For discussions of WARP, see Hildenbrand and Jerison (1989). Given the above, note that

$$K = E \cup \{p \notin E : WARP \text{ is violated at } p\}.$$

To investigate this notion further, let $p^* \in E$ and $p \neq \theta p^*$; then it should be noted that by virtue of Walras Law, using the notation introduced above:

$$p.(X(p) - W) = p.Y(p);$$
Further, by virtue of profit maximization, $p.Y(p) \geq p.Y(p^*)$; also by the definition of equilibrium, $Y(p^*) = X(p^*) - W$; consequently, putting all this together, we have $p.X(p) \geq p.X(p^*)$. Thus we have demonstrated the validity of:

**Claim 3.2.2** At any $p$, the aggregate demand $X(p)$ costs no less than the aggregate demand $X(p^*)$ at equilibrium.

**Remark 10** Had the aggregate demand $X(p)$ originated from the maximization of a single utility function, it would have been possible to argue that at $p^*$, $X(p)$ should be more expensive than $X(p^*)$, i.e., $p^*.X(p^*) < p^*.X(p)$ or retracing the steps taken above, and using the fact that from profit maximization, $p^*.Y(p) \leq p^*.Y(p)$, it would follow that $p^*.Z(p) > 0$: which, of course, is WARP$^6$.

### 3.3 Tatonnement Processes

The so-called tatonnement processes have two major properties: first of all, in dis-equilibrium situations price adjustment in each market occurs in the direction of excess demand in that market and secondly, trades occur only at equilibrium prices. These two assumptions about price adjustment and trades figure in Walras’ description of how the market figures out what the equilibrium prices are. The first treatment of the price adjustment equations as differential equations was provided by Samuelson (1946), where he wrote the price adjustment equations

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6Thus aggregation across individuals has deprived us of these nice properties. It is, in this connection, that the results of Sonnenschein (1972) and Debreu (1974) are crucial. The only restrictions on excess demand functions are that they satisfy homogeneity of degree zero in the prices and obey Walras Law.
equations as
\[ \dot{p}_j = F_j(p) \text{ for all } j \neq n \] (3.1)
where \( F_j(p) \) are required to satisfy the following restriction:
\[ \text{Sign of } F_j(p) = \text{Sign of } Z_j(p) \]

A special case of the above function \( F_j(p) \) has been mostly used where \( F_j(p) = k_j.Z_j(p) \)
with \( k_j > 0 \) are some constants. The adjustment equations thus become
\[ \dot{p}_j = k_j.Z_j(p) \text{ for all } j \neq n; k_j > 0 \forall j \] (3.2)

It has also been argued that in the above case, one may so define the units of measurement
of each commodity that \( k_j = 1 \) can be chosen without any loss of generality. Using this
strategy we shall use the following equation:
\[ \dot{p}_j = Z_j(p) \text{ for all } j \neq n; p_n = 1 \] (3.3)

We shall have a chance to study the solutions to both (3.1) and (3.3) in our analysis
below. The analysis of local stability of competitive equilibrium is restricted to situations
when the initial point of the solution to the (3.1) is a point \textit{close} to the equilibrium \( p^* \).

In such situations, the property of the functions \( F_j \), viz., that they are of the same sign as
\( Z_j \), allows us some advantages. To see these, we have first of all\(^7\).

\textbf{Claim 3.3.1} If \( f(x) \) and \( g(x) \) are real valued linear functions, \( f, g : \mathbb{R}^n \to \mathbb{R} \), such that
\( f(x) \neq 0 \) for some \( x \in \mathbb{R}^n \) and \( f(x) = 0 \iff g(x) = 0 \), then \( f(x) = \alpha.g(x) \), \( \forall x \in \mathbb{R}^n \), for
some \( \alpha \).

\(^7\)McKenzie (2002), p. 63.
Proof. Consider $\mathbf{x} \in \mathbb{R}^n$ such that $f(\mathbf{x}) \neq 0$. Consider $y \in \mathbb{R}^n$ and note that $f(y - \frac{f(y)}{f(\mathbf{x})} \mathbf{x}) = 0$; consequently $g(y - \frac{f(y)}{f(\mathbf{x})} \mathbf{x}) = 0$; hence $g(y) = \alpha f(y)$ where $\alpha = \frac{g(\mathbf{x})}{f(\mathbf{x})}$. \hfill \bullet

For any function $f(x)$, we write $\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_j} \right)$; by virtue of the above claim, if $\nabla F_j(p^*) \neq 0$ then it follows that $\nabla F_j(p^*) = \alpha_j \nabla Z_j(p^*)$ for some $\alpha_j > 0$. Consequently, the linear approximation to (3.1) may be taken to be

$$\dot{p} = D.A.p$$

(3.4)

where $A = \left( \frac{\partial Z_i(p^*)}{\partial p_j} \right)$ and $D$ is a diagonal matrix with positive $\alpha_j$ down the diagonal. Now for example, redefining units of measurement, we may choose the system

$$\dot{p} = A.p$$

(3.5)

Consequently, there are two routes to arriving at a system such as (3.5); one, from a system such as (3.1) and the other, from a system such as (3.3). It would be more convenient to adopt the convention that we have arrived from (3.3). If the matrix $A$ has all its characteristic roots with real parts negative, we shall say (3.3) is linear approximation stable. This is a stronger requirement than (3.3) being locally asymptotically stable. The former implies the latter; but one can have the latter condition being met without the former being true.\footnote{Consider, for example, $x \in \mathbb{R}, \dot{x} = -x^3$; and consider the linear approximation in a neighborhood of equilibrium ($x = 0$) given by $\dot{x} = 0$. The difference between the two, lies in the fact that even though linear approximation stability is violated, local asymptotic stability holds. This can happen only when the characteristic root of the matrix of partial derivatives evaluated at equilibrium (the Jacobian of $Z(p)$ without the numeraire row and column) has zero as a characteristic root.}

Now the full Jacobian of $Z(p)$ at the equilibrium $p^*$ has some interesting
properties which we note for future reference.

Claim 3.3.2 Consider the Jacobian $J(p^*)$ of the excess demand functions given by $\left(\frac{\partial Z_i(p^*)}{\partial p_j}\right)$ where $i, j$ run over all the goods, including the numeraire and $p^*$ is an equilibrium. Then

$$p^T.J(p^*) = 0$$

and further let $I = \{i : p^*_i > 0\}$; then $\forall i, j, r, s \in I$, $J_{ij}/p_i.p_j = J_{rs}/p_r.p_s$ and $J_{ij} = 0$ if either $i \notin I$ or $j \notin I$ where $J_{ij}$ denotes the cofactor of the $i$-th element in $J(p^*)$.

Proof. By virtue of Walras Law, we have $\sum_i p_i.Z_i(p) = 0$; hence differentiating, with respect to the $j$-th price $p_j$, we have $\sum_i p_i.Z_{ij}(p) + Z_j(p) = 0$ and evaluation at $p^*$ leads to $p^T.J(p^*) = 0$. Again noting the homogeneity of degree zero in the prices, we have from Euler’s Theorem that $J(p^*)p^* = 0$. Next consider the matrix $B = \text{adjoint } J(p^*) = (J_{ij})^T$. It is then known that $J(p^*).B = B.J(p^*) = \det J(p^*)I$. If $J_{ij} = 0 \forall i, j$, then the claim follows trivially; hence suppose $J_{ij} \neq 0$ for some $i, j$; then note that rank of $J(p^*)$ is $n - 1$ and hence the solutions to $x^T.J(p^*) = 0$ and $J(p^*).x = 0$ constitute vector spaces of rank 1. Thus rows of $B$ must be scalar multiples of $p^T$ and columns of $B$ must be scalar multiples of $p^*$; consequently, writing the $i$-th row of $B$ as $b_i$, $b_i = \theta_i.p^*$ for some scalar $\theta_i$ and the $j$-th column of $B$ as $b_j$, $b_j = \beta_j.p^*$ for some scalar $\beta_j$; it remains to note that $\theta_i = \beta_i$ and hence $J_{ij} = \theta_i.p^*_j = \theta_j.p^*_i$ and the claim follows. \hfill \bullet

Remark 11 It should be noted that the full Jacobian $J(p)$ of the excess demand function $Z(p)$, can be symmetric only at the equilibrium $p^*$. This follows from the proof of the above claim; note that $p^T.J(p) = -Z(p)^T$ and $J(p)*p = 0$ for any $p$. Consequently, $J(p)$ symmetric $\Rightarrow 0^T = p^T.J^T(p) = p^T.J(p) = -Z(p)^T$; hence $p$ must be an equilibrium.
We consider next, whether with conditions such as the above, we can ensure that the equilibrium is locally stable, at least.

### 3.4 Local Stability of Tatonnement Processes

We shall consider the local stability of the system (3.3), i.e., the process

\[ \dot{p}_j = Z_j(p) \text{ for all } j \neq n \]

and its linear approximation, around an equilibrium \( p^* \), (3.5), i.e.,

\[ \dot{p} = A.p \]

where \( A = (\frac{\partial Z_i(p^*)}{\partial p_j})(i, j \neq n) \). It may be recalled also that the stability of (3.5) implies local asymptotic stability of (3.3); but not conversely. We shall say that the equilibrium \( p^* \) is **linear approximation stable**, if the process (3.5) is stable i.e., if all the characteristic roots of \( A \) have negative real parts; we shall refer to \( A \) being **stable** if all the characteristic roots of \( A \) have their real parts negative. We shall say that the equilibrium \( p^* \) is **locally stable** if the process (3.3) is locally asymptotically stable. The conditions for the stability of the matrix \( A \) are contained in the Liapunov Theorem (see Section); to relate these conditions to excess demand functions, we have the following:

**Claim 3.4.1** \( A \text{ is stable} \Rightarrow \exists \text{ a positive definite matrix } B \text{ such that } (p^* - p)^T.B.Z(p) > 0 \)

\( \forall p \in N_\delta(p^*) = \{ p : p_n = 1, |p - p^*| < \delta \} \text{ for some } \delta > 0 \), where

\[
\overline{B} = \begin{pmatrix} B & 0 \\ 0^T & 1 \end{pmatrix}.
\]
Proof. Since $A$ is stable, by Liapunov’s Theorem, there is a positive definite matrix $B$ such that $B.A + A^T.B$ is negative definite. Let $\mathcal{B}$ be as defined above and let $f(p) = (p^* - p)^T.\mathcal{B}.Z(p)$. Note that $f(p^*) = 0$; further, writing the elements of the matrix $\mathcal{B}$ as $(\mathcal{b}_{ij})$, the partial derivative of the function $f(p)$ as $f_k(p) = \frac{\partial f(p)}{\partial p_k}$, $f_{kr}(p) = \frac{\partial^2 f(p)}{\partial p_k \partial p_r}$, we have the following:

$$f_k(p) = \sum_{i,j} (p_i^* - p_i) \mathcal{b}_{ij} . Z_{jk}(p) - \sum_j b_{kj} . Z_j(p)$$

and

$$f_{kr}(p) = \sum_{i,j} (p_i^* - p_i) \mathcal{b}_{ij} . Z_{jk}(p) - \sum_j b_{rj} . Z_{jk}(p) - \sum_j b_{kj} . Z_{jr}(p);$$

where $Z_{jk}(p) = \frac{\partial^2 Z_j(p)}{\partial p_k \partial p_r}$. Now consider $k \neq n; r \neq n$ and evaluate all the above partial derivatives at $p^*$; we have then:

$$f_k(p^*) = 0 \forall k$$

and further,

$$(f_{kr}(p^*)) = -(\sum_j b_{rj} . Z_{jk}(p^*) + \sum_j b_{kj} . Z_{jr}(p^*))$$

$$= -(\sum_j b_{rj} . Z_{jk}(p) + \sum_j b_{kj} . Z_{jr}(p))$$

using the symmetry of the matrix $\mathcal{B}$; hence, using the fact $k, r \neq n$, we have:

$$(f_{kr}(p^*)) = -(B.A + A^T.B)$$

which ensures the fact that the matrix of second order partial derivatives of the function $f(p)$ at $p^*$, i.e., the matrix $(f_{kr}(p^*))$ is positive definite. Consequently, the function $f(p)$ attains a regular minimum at $p = p^*$ and hence there is a neighborhood $N_\delta(p^*)$ such that $p \in N_\delta(p^*), p \neq p^* \rightarrow f(p) > f(p^*) = 0$. •
Remark 12 If the matrix $A$ is stable with the relevant $B = I$, then a local version of the Weak Axiom of Revealed preference holds; since then there is a neighborhood $N_\delta(p^*)$, such that $p \in N_\delta(p^*) \rightarrow (p^* - p)^T.\overline{B}.Z(p) = p^{*T}.Z(p) > 0$ when $B = I$.

The Claim made above has the following converse:

Claim 3.4.2 Suppose there is a positive definite matrix $\overline{B}$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$ for all $p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$ where

$$\overline{B} = \begin{pmatrix} B & b \\ b^T & 1 \end{pmatrix}$$

and the matrix $B.A + A^T.B$ has rank $(n - 1)$, then $A$ is stable.

Proof. Define $f(p) = (p^* - p)^T.\overline{B}.Z(p)$; note that by the conditions specified, $f(p)$ attains a local minimum at $p = p^*$; hence it follows that the matrix $(f_{kr}(p^*))$, $k, r \neq n$ must be positive semi-definite; as shown in the proof of the above claim, $(f_{kr}(p^*)) = -(B.A + A^T.B)$; hence it follows that $(B.A + A^T.B)$ must be negative semi-definite; given the rank condition, we may conclude that there is a positive definite matrix $B$ such that $(B.A + A^T.B)$ is negative definite; hence by Lyapunov’s Theorem, $A$ is stable. •

Without the rank condition, we have the following:

Claim 3.4.3 Suppose there is a positive definite matrix $\overline{B}$ such that $(p^* - p)^T.\overline{B}.Z(p) > 0$ for all $p \in N_\delta(p^*) = \{p : p_n = 1, |p - p^*| < \delta\}$ for some $\delta > 0$ where

$$\overline{B} = \begin{pmatrix} B & b \\ b^T & \alpha \end{pmatrix}$$

then $p^*$ is locally stable i.e., the process (3.3) is locally asymptotically stable.
Proof. Define $V(p) = (p^* - p)^T.\mathcal{B}.(p^* - p)$ and consider $\nu > 0$ such that $D_{\nu}(p^*) \subset N_\delta(p^*)$ where $D_{\nu}(p^*) = \{ p : p_n = 1, (p^* - p)^T.\mathcal{B}.(p^* - p) < \nu \}$. Now consider $p^o \in D_{\nu}(p^*)$ and the solution to (3.3) with $p^o$ as initial point, $p(t, p^o)$. Note that

$$\dot{V}(p(t, p^o)) = -2(p^* - p(t, p^o))^T.\mathcal{B}.Z(p) < 0, \forall p(t, p^o) \in D_{\nu}(p^*)$$

so long as $p(t, p^o) \neq p^*$; consequently $V(p(t, p^o)) \leq V(p^o) \forall t$; moreover, the function $V(p)$ has all the properties of a Liapunov function in the region $D_{\nu}(p^*)$ and the claim follows. •

**Remark 13** Suppose $A + A^T$ has rank $(n - 1)$. Then the following conditions are equivalent:

1. $p^*$ is an isolated point of the set $K$
2. $p^T.\mathcal{B}.Z(p) > 0, \forall p \neq p^*, p \in N_\delta(p^*)$
3. $x^T.A.x < 0$ for all $x \neq 0$.

It should be noted that the rank condition is only required for showing that Condition (ii) $\Rightarrow$ Condition (iii). Thus, the relationship between a local version of the Weak Axiom of Revealed Preference and stability of processes such as (3.3) or (3.5) are quite close.

Kihlstrom et. al. (1976), contain related results. To put our results in this section in a clear perspective, let us define **Local Generalised WARP** near an equilibrium $p^*$ by: $\exists$ a positive definite matrix $\mathcal{B}$ such that $(p^* - p)^T.\mathcal{B}.Z(p) > 0, \forall p \neq p^*, p \in N_\delta(p^*) = \{ p : p_n = 1, |p - p^*| < \delta \}$ for some $\delta > 0$.

Returning to the the Claim, one may note that linear approximation stability of the process (3.3) implies Local Generalised WARP near the equilibrium $p^*$ where the matrix
$\mathcal{B}$ is given by:

$$
\mathcal{B} = \begin{pmatrix}
  B & 0 \\
  0^T & 1
\end{pmatrix}
$$

and $B$, a positive definite matrix, satisfies the equation:

$$
A^T.B + B.A = -Q
$$

for any positive definite matrix $Q$. For every such matrix $B$, one may construct $\mathcal{B}$ and for each such $\mathcal{B}$, we shall have $(p^* - p)^T.\mathcal{B}.Z(p) > 0 \forall p \neq p^*, p \in N_\delta(p^*)$ for some $\delta > 0$.

Thus, Local Generalised WARP is necessary for linear approximation stability of the process (3.3) and is sufficient for the local asymptotic stability of the same process; only when a rank condition is met is the condition sufficient for linear approximation stability too. These results indicate what kinds of restrictions have to be in place to ensure local stability.

The major conditions on excess demand functions which have been used to ensure local stability are one of the following:

i. Gross Substitution i.e.,

$$
\mathcal{A} = (\frac{\partial Z_i(p^*)}{\partial p_j}), \forall i, j,
$$

has all its off-diagonal terms positive. Thus writing $Z_{ij} = \frac{\partial Z_i(p^*)}{\partial p_j}$, we must have $Z_{ij} > 0, i \neq j$.

ii. The matrix $A = (\frac{\partial Z_i(p^*)}{\partial p_j})$, $i, j \neq n$ has a dominant negative diagonal\(^{10}\) i.e., the diagonal terms are negative $\exists$ positive numbers $c_1, c_2, \cdots, c_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n - 9$.\(^9\)

\(^9\)In the literature, the weaker sign, i.e., $Z_{ij} \geq 0, i \neq j$, has been referred to as the weak gross substitute case. Usually the matrix $\mathcal{A}$ is required to be indecomposable, then. We shall consider this case later.

\(^{10}\)The fundamental property of dominant diagonal matrices is that they are non singular; if, in addition, the diagonal is negative, then all characteristic roots of the matrix have real parts negative: see, McKenzie (1959).
1, c_j|Z_{jj}| > \sum_{i \neq j} c_i|Z_{ij}|. An equivalent way of defining diagonal dominance is the following: \( \exists \) positive numbers \( d_1, d_2, \ldots, d_{n-1} \) such that \( \forall j, j = 1, 2, \ldots, n - 1, d_j|Z_{jj}| > \sum_{i \neq j} d_i|Z_{ji}| \). While the \( c \)'s ensure dominance of the diagonal terms over rows, the \( d \)'s ensure that the diagonal terms dominate across columns. Only in special cases (such as when the matrix is symmetric, for example), it is possible to conclude that the \( c \)'s and \( d \)'s match\(^{12}\).

However, if the \( c \)'s and \( d \)'s do match then one may show:

**Claim 3.4.4** If the matrix \( A \) defined above has a dominant negative diagonal with \( c_j = d_j \forall j \) then \( A \) is quasi-negative definite\(^ {13}\). Further WARP is satisfied locally i.e., \( \exists \) a neighborhood \( N(p^\ast) \) such that \( \forall p \in N(p^\ast), p \neq p^\ast, p^\ast.Z(p) > 0 \).\(^ {14}\)

Proof: Consider the matrix \( B = A + A^T \); let a typical element of \( B \) be written as \( b_{ij} \); writing elements of \( A \) as \( a_{ij}, b_{ij} = a_{ij} + a_{ji} \); we are given that \( \forall j, c_j|a_{jj}| > \sum_{i \neq j} c_i|a_{ij}| \) and \( c_j|a_{jj}| > \sum_{i \neq j} c_i|a_{ij}| \) for some positive numbers \( c_i \). It follows therefore that \( 2c_j|b_{jj}| > \sum_{i \neq j} 2c_i|b_{ij}|, \forall j \); consequently, the symmetric matrix \( B \) has a dominant negative diagonal and is thus negative definite; this, in turn, establishes the fact that \( A \) is quasi-negative definite. For the last part, we proceed as in the proof of Claim 3.4.1. Define \( f(p) = (p^\ast - p)^T.Z(p) \); note that \( f(p^\ast) = 0; \nabla f(p^\ast) = 0 \)\(^ {15}\) and that \( (\frac{\partial^2 f(p^\ast)}{\partial p_j \partial p_k})_{j,k=1,2,\ldots,n-1} = -(A + A^T) \). These facts imply that the function \( f(p) \) attain a local minimum at \( p = p^\ast \) and the claim follows. •

---

\(^{11}\)See, for example, Mukherji (1975).

\(^{12}\)For example, the matrix \( \begin{pmatrix} -1 & 0.5 \\ 1.5 & -1 \end{pmatrix} \) has a dominant negative diagonal; this may be checked by considering \( c_1 = 9, c_2 = 15 \) or by \( d_1 = 15, d_2 = 9 \); but choosing \( d_1 = c_1, d_2 = c_2 \), will not do.

\(^{13}\)\( A + A^T \) is negative definite.

\(^{14}\)A global version of this result may be found in Fujimoto and Ranade (1988).

\(^{15}\)Recall that \( \nabla f(p) \) stands for the vector of partial derivatives of the function \( f(p) \).

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It is easy to check that, given Walras Law and homogeneity of degree zero in prices of the excess demand functions:

**Claim 3.4.5** \( i. \Rightarrow ii. \) with \( c_i = d_i = p_i^* \).

Consequently, by virtue of Claim 3.4.4, it follows that a local version of WARP is satisfied for the Gross Substitute case, as well. Thus the importance of WARP should be apparent, even for local stability of equilibrium. It should therefore come as no surprise that we need additional requirements to ensure this property. We turn next, to examples of instability which may be set right by appropriate changes in parameter values without really affecting the substantive nature of the example.

### 3.5 The Gale Example: The Choice of the Numeraire

One of the first examples of an unstable equilibrium is due to Gale (1963). We take that up first to analyze the possible causes of instability in this set up. It is of interest to note that this analysis will also reveal another rather crucial element in the construction of the tatonnement process.

It may be recalled that the (3.1) involved choosing a numeraire; we chose good \( n \) to be the numeraire, the unit of account. Ideally of course, it should not matter which commodity is chosen; but before we can rest assured on this aspect, we need some further analysis on this question. One may argue, however, that in the real world, there is no choice of the numeraire available, since the medium of exchange is fixed and one should be interested in whether the economy is stable with respect to this choice of the numeraire; the fact that
some other choice of numeraire would have rendered the system stable or unstable should be of no interest. As Arrow and Hahn (1971)\textsuperscript{16}, indicate, such a remark should be objected to at two levels: first, the argument about the medium of exchange being fixed is what has been termed to be ‘casual empiricism’ by Arrow and Hahn, since our general equilibrium model has no adequate theory of money. Second in actual situations, there is a choice of numeraire: consider, for example, the recent worries on the international markets: whether dollar retains its position as the pre-eminent currency with all transactions being in dollars or whether this position is taken over by the new currency, the euro. The question that will be addressed is whether this difference matters so far as stability questions are considered.

We shall use the Gale example to demonstrate that, unfortunately, the choice of the numeraire does matter. Note first of all that in a two-good economy, with only one relative price to worry about, this kind of problem does not appear. The example due to Gale which we shall consider now, was introduced to exhibit the possibility of the tatonnement process being unstable. There are three goods labelled 0, 1 and 2 and there are two individuals, C and P. C possesses \( x^0 \) units of good 0 and wishes to exchange it for goods 1 and 2 and has no desire for good 0. P, on the other hand has stocks of goods 1 and 2 given by \( x^1, x^2 \) which he wants to exchange for good 0 with P having no desire for good 0. Gale chooses good 0 to be the numeraire and analyzes the situation when C maximizes a concave utility function \( U(x_1, x_2) \) subject to a budget constraint \( p_1.x_1 + p_2.x_2 = x^0 \), where \( p_i \) denotes the price of good \( i \) relative to good 0. The force of the paper was that if good 1 was a giffen

\textsuperscript{16}See p.305.
good, then one may so choose speeds of adjustment $r_i > 0$ such that the system

$$\dot{p}_i = r_i f_i(p) i = 1, 2 \quad (3.6)$$

is unstable; in the above, $f_i(p)$ denotes the excess demand for good $i$. In particular, Gale considers the linear approximation of the above system at equilibrium so that, as we have seen in the previous section, stability results depend entirely on the characteristic roots of the following matrix (where $f_{ij}$ denotes the partial derivative of $f_i$ with respect to the price of good $j$ evaluated at equilibrium):

$$ \begin{pmatrix} r_1 f_{11} & r_1 f_{12} \\ r_2 f_{21} & r_2 f_{22} \end{pmatrix} $$

By virtue of the giffen good assumption, $f_{11} > 0$; also, since $C$ consumes goods 1 and 2 only and no one else consumes goods 1 and 2, it follows that $f_{22} < 0$. However, $r_1$ can be so chosen that the trace of the above matrix $r_1 f_{11} + r_2 f_{22} > 0$, thus violating one of the necessary conditions for stability of (3.6). This was what Gale had shown.

Following Mukherji (1973), suppose we decide to choose good 1 to be the numeraire; clearly, this is possible, provided we assume that good 1 is not free. Note that no other giffen good can exist: we have already noted that good 2 cannot be giffen; and since $P$ consumes good 0 only and no one else consumes good 0, good 0 cannot be giffen. With the change in the numeraire, $C$ solves the problem

Maximize $U(x_1, x_2)$

subject to $x_1 + p_2 x_2 = p_0 x^0$
where $p_i$ for $i = 0, 2$ now represent the price of good $i$ relative to good 1. The demand functions are given by $x_i(p_0, p_2)$; consequently the excess demand functions are given by $f_i(p_0, p_2) = x_i(p_0, p_2) - x^i$ for $i = 1, 2$. So much for C. So far as P is concerned, P demands $x_0(p_0, p_2) = x^1 + p_2x^2$; consequently, the excess demand for good 0 is given by $f_0(p_0, p_2) = x_0(p_0, p_2) - x^o$. The first point to note is that, writing $f_{ij}$ as the partial derivatives of $f_i$ with respect to $p_j$, $1 = 1, 2, j = 0, 2$ evaluated at equilibrium:

**Claim 3.5.1**

\[
\det \begin{pmatrix} f_{10} & f_{12} \\ f_{20} & f_{22} \end{pmatrix} < 0
\]

provided that C’s utility maximization exercise is solved at an interior point and that second order conditions are met.

Proof. From the first order conditions at an interior maximum, we have the following:

\[
\begin{pmatrix}
U_{11} & U_{12} & -1 \\
U_{21} & U_{22} & -p_2 \\
-1 & -p_2 & 0
\end{pmatrix} \begin{pmatrix}
dx_1 \\
dx_2 \\
d\lambda
\end{pmatrix} = \begin{pmatrix}
0 \\
\lambda dp_2 \\
x_2 dp_2 - x^o dp_0
\end{pmatrix}
\]

(3.7)

Let us write the determinant of the matrix on the left hand side as $\det A$ which may be taken to be positive assuming that the second order condition is satisfied; note further that the determinant in the claim is given by

\[
f_{10}.f_{22} - f_{20}.f_{12} = x_{10}.x_{22} - x_{20}.x_{12}
\]

from the definition of the the excess demand functions. Substituting the values of $x_{ij}$ from (3.7), we have that

\[
\det \begin{pmatrix} f_{10} & f_{12} \\ f_{20} & f_{22} \end{pmatrix} = -\lambda.x^o/\det A < 0.
\]
Next, notice that with the change in the numeraire, the system that we ought to be considering is
\[ \dot{p}_i = r_i f_i(p_0, p_2), \ i = 0, 2 \]  
(3.8)

and consequently the matrix which will decide the (local) stability of the above is given by
\[
\begin{pmatrix}
  f_{00} & f_{02} \\
  f_{20} & f_{22}
\end{pmatrix}
\]

We already have that the trace of this matrix is negative (recall: no other giffen goods). To establish that the determinant is positive, we need to use a result we had proved earlier.

By virtue of the above and given the result of Claim 3.3.2, it follows that
\[
\det \begin{pmatrix}
  f_{00} & f_{02} \\
  f_{20} & f_{22}
\end{pmatrix} > 0
\]

since the Claim 3.3.2 establishes that a cofactor of order 2 is positive (the relevant cofactor is \(-1\) times the determinant in Claim 3.5.1). Consequently, the equilibrium is locally stable under the process (3.8).

Note that the example of instability provided by Gale has been converted to a stable case by changing the numeraire. There was only one good, in the Gale example, for which the price could move in the **wrong** direction; whereas Gale aggravated this movement, what has been done above is to eliminate the movement entirely. Note however, the two processes (3.6) and (3.8) are of course different. We can sum up the discussions here by means of the following:
**Proposition 3.1** In general, stability properties of equilibrium depend upon the choice of the numeraire.

We next provide a sufficient condition for stability of equilibrium to be insensitive to the choice of the numeraire. We shall focus entirely on the system (3.3):

\[ \dot{p}_i = Z_i(p) \ \forall i \neq n ; \]

Let the equilibrium \( p^* > 0 \). We shall say that the stability of equilibrium is **insensitive to the choice of the numeraire**, if the linear approximation to the above system is stable for all choice of \( n \). The following points should be obvious: choosing another good, say good 1, as the numeraire, amounts to considering a new equilibrium price vector \( q^* = \frac{1}{p^*_1} p^* \); note that the system (3.3) also changes to

\[ \dot{q}_i = Z_i(q) \ \forall i \neq 1 ; \quad (3.9) \]

The stability of the linear approximation to (3.3) depends on the matrix \( J(p^*) = (Z_{ij}(p^*); i, j = 1 \cdots n - 1) \); for the stability of the linear approximation to the system (3.9), on the other hand, we need to consider the matrix \( J(q^*) = (Z_{ij}(q^*); i, j = 2 \cdots n) \). We need to connect the stability properties of these two matrices.

It would be more convenient, at this stage to consider the non-normalized price vector \( P = (P_1, P_2 \cdots P_n) \) and the excess demand functions \( Z_i(P) \). We shall denote the Jacobian, including the numeraire row and column by \( A = (a_{ij}) = (\frac{\partial Z_i(P)}{\partial P_j}); i, j = 1, 2 \cdots n \); we write the cofactor of the \( i - j \)-th element in \( A \) by \( A_{ij} \). We have first of all,

**Claim 3.5.2** \( A_{11} = \frac{1}{P_n} J(p^*) \)
Proof. By virtue of homogeneity of degree zero:

\[ Z_i(P_1, P_2, \cdots, P_{n-1}, P_n) = Z_i(p_1, p_2, \cdots, p_{n-1}, 1) \]

where each \( p_i = P_i / P_n = g_i(P) \), say. Thus, for \( i, j \neq n \)

\[ \frac{\partial Z_i(P)}{\partial P_j} = \sum_k \frac{\partial Z_i(p)}{\partial p_k} \frac{\partial g_k}{\partial P_j} = \frac{1}{P_n} \cdot \frac{\partial Z_i(p)}{\partial p_j} \]

Evaluating at equilibrium \( p^* = 1/P_n^* \), the claim follows. •

The principal minors \( A_{ii}, A_{kk} \) of the matrix \( A \) defined above have the following relationship:

**Claim 3.5.3** Consider \( i, k \) such that \( P_i^*, P_k^* \neq 0 \). Then \( x^T A_{ii} x < 0 \) for all \( x \neq 0 \) \( \iff \) \( x^T A_{kk} x < 0 \) for all \( x \neq 0 \).

Proof. Let \( P_i^*, P_k^* > 0 \) and let \( x^T A_{ii} x < 0 \) for all \( x \neq 0 \). Writing \( B_{ii} = A_{ii} + A_{ii}^T \), note that then \( B_{ii} \) is negative definite. Consider \( B = A + A^T \), where \( A \) is as defined above; we first claim that \( A \) is negative semi-definite, i.e., \( x^T B x \leq 0 \) for all \( x \); for suppose to the contrary, there is some \( x^* \) such that \( x^T B x^* > 0 \); note that \( x^* \neq 0 \); in particular, \( x_i^* \neq 0 \); since otherwise, writing \( \overline{x}^* = (x_1^*, \cdots, x_{i-1}^*, x_{i+1}^*, \cdots, x_n^*), x^T B x^* = \overline{x}^T B_{ii} \overline{x}^* < 0 \): a contradiction. Without any loss of generality, we can take \( x_i^* > 0 \). Now note that \( P_i^T B P_i^* = 0 \); define \( t = x^*_{i} / P_i^* > 0 \) and \( v = x^* - t P_i^* \); then \( v^T B v = x^T B x^* > 0 \); on the other hand, since \( v_i = 0 \), writing \( \overline{v} = (v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_n) \), we have \( v^T B v = \overline{v}^T B_{ii} \overline{v} \leq 0 \): a contradiction; hence no such \( x^* \) can exist and \( x^T B x \leq 0 \) for all \( x \).
Consequently $B_{kk} = A_{kk} + A_{kk}^T$, being a principal minor of $B$ is also negative semidefinite i.e., $x^T B_{kk} x \leq 0$ for all $x$. Next we note that in case $x^T B_{kk} x = 0$ for some $x \neq 0$, then we must have $B_{kk} x = 0$; that is $B_{kk}$ must be singular; But since, $P^* x^T . B = 0$ and $B.P^* = 0$, the results of Claim are applicable, and we may conclude that $B_{kk}$ must be non-singular, since $B_{ii}$ is given to be so; i.e., we must have $x^T B_{kk} x < 0$ for all $x \neq 0$; i.e., $x^T A_{kk} x < 0$ for all $x \neq 0$. Interchanging the roles of $A_{ii}$ and $A_{kk}$, the converse follows.

By virtue of the above result, we may conclude as follows; suppose that we choose good 1 as numeraire; then we observe that the linear approximation to the process (3.3) is stable; i.e., the matrix $J(p^*)$ is stable. By virtue of Liapunov’s Theorem, this means that there is some positive definite matrix $C$ such that $C.J(p^*) + J(p^*)^T . C$ is negative definite. Suppose further that this $C = I$; then note that $x^T J(p^*) x < 0$ for all $x \neq 0$; further, by virtue of Claim, this means that $x^T A_{11} x < 0$ for all $x \neq 0$; consequently, by virtue of Claim, we have that $x^T A_{kk} x < 0$ for all $k$ which are permitted to be chosen as numeraire (i.e., they are not free); again using Claim, it follows then that $x^T J(q^*) x < 0$ for all $x \neq 0$ and this allows us to conclude that the linear approximation to the process (3.9) is stable. Thus, if the linear approximation to the tatonnement process for some choice of numeraire, is stable with the corresponding Jacobian to be quasi-negative definite\footnote{A matrix $M$ is quasi-negative definite if $x^T M x < 0$ for all $x \neq 0$; in case, $M$ is symmetric, then $M$ is negative definite.}, then stability of the linear approximation to the tatonnement with any other choice of the numeraire would follow.

The problem of sensitivity to numeraire choice appears therefore, because stability of
the linear approximation need not imply that the relevant Jacobian $J$ is quasi-negative definite; stability is equivalent to there being a positive definite $C$ such that $C.J + J^T.C$ is negative definite or that $C.J$ is quasi-negative definite for some positive definite $C$. Finally, it should be pointed out that for symmetric matrices, this gap is closed; i.e., a necessary and sufficient condition for all characteristic roots of a symmetric matrix to be negative (for symmetric matrices, all characteristic roots are real) is that the matrix be negative definite. Thus if the jacobian of the excess demand function happens to be symmetric, then stability of the linear approximation with one choice of numeraire implies that the linear approximation with any other choice of numeraire would also be stable.

### 3.6 The Scarf Example

We have already seen, in the last section, that the presence of giffen goods, for example, may destroy the stability properties of the tatonnement. In the present section we indicate that there may be other, seemingly more robust difficulties, for the stability of the tatonnement. We do this by considering an example due to Scarf (1960). Consider an exchange model where there are three individuals $h = 1, 2, 3$ and three goods $j = 1, 2, 3$. The utility functions and endowments are as under:

\[
U^1(q_1, q_2, q_3) = \min(q_1, q_2); \ w^1 = (1, 0, 0)
\]

\[
U^2(q_1, q_2, q_3) = \min(q_2, q_3); \ w^2 = (0, 1, 0)
\]

\[
U^3(q_1, q_2, q_3) = \min(q_1, q_3); \ w^3 = (0, 0, 1)
\]
Routine calculations lead to the following excess demand functions, where good 3 is treated as numeraire (i.e., \( p_3 = 1 \)):

\[
Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}
\]

\[
Z_2(p_1, p_2) = \frac{p_2(p_1 - 1)}{(1 + p_2)(p_1 + p_2)}
\]

\[
Z_3(p_1, p_2) = \frac{p_2 - p_1}{(1 + p_1)(1 + p_2)}
\]

and the tatonnement process, for this example is given by

\[
\dot{p}_i = Z_i(p_1, p_2) \quad i = 1, 2
\]  \( (3.10) \)

Notice that equilibrium for this exchange model (and for the process defined above) is given by \( p_1 = 1, p_2 = 1 \). It would be helpful to transform variables by setting \( x_i = p_i - 1 \) for \( i = 1, 2 \). With this change in variables, our process becomes

\[
\dot{x}_1 = -\frac{x_2(1 + x_1)}{(x_1 + 2)(x_1 + x_2 + 2)} \quad \dot{x}_2 = \frac{x_1(1 + x_2)}{(x_2 + 2)(x_1 + x_2 + 2)}
\]  \( (3.11) \)

In what follows, we shall analyse the answer to the following question: given an arbitrary \( x^o = (x_1^o, x_2^o) \), how does the solution \( x(t, x^o) \) to \( (3.11) \) behave as \( t \to \infty \)?

We introduce the function \( v : R \to R \) by

\[
v(x) = \frac{x^2}{2} + x - \ln(1 + x)
\]

which is continuously differentiable for all \( x \) such that \( 1 + x > 0 \). One may show that

**Claim 3.6.1** \( v(x) > 0 \) if \( x > -1, x \neq 0; v(0) = 0. \)
Proof. Note that $v(0) = 0$, and for $x > -1$ we have $v'(x) = \frac{x(x + 2)}{1 + x}$ and $v''(x) = 1 + \frac{1}{1 + x^2}$.
Thus for $x > -1$, $v(x)$ is strictly convex with $v'(0) = 0$; hence $x = 0$ yields a global minimum for $v(x)$ for all $x > -1$.

Next define $V(x_1, x_2) = v(x_1) + v(x_2)$. We have then:

**Claim 3.6.2** Along the solution $x(t, x^o)$ to (3.11), $\dot{V} = 0$ provided $x_i(t, x^o) > -1$ for $i = 1, 2$.

Proof. Note that

$$\dot{V} = v'(x_1) \dot{x}_1 + v'(x_2) \dot{x}_2 = 0$$


We may next claim

**Claim 3.6.3** Given $x^o = (x^o_1, x^o_2), x^o_i > -1, i = 1, 2$ the solution $x(t, x^o)$ to (3.11) is such that $\exists a_i, b_i$ such that $-1 < a_i < b_i$ and $x(t, x^o) \in [a_1, b_1] \times [a_2, b_2] \forall t > 0$.

Proof. Follows from the last two claims.

For local stability, it may be of some interest to note the following

**Claim 3.6.4** For $x$ small, $v(x) \approx x^2$.

Proof. This follows since for $x$ small one may use the following approximation:

$$\ln(1 + x) \approx x - \frac{x^2}{2}$$
The above may be used to classify the solution to (3.11) when the initial point $x^o$ is close to the equilibrium i.e., the origin. It is approximately a circle with the center origin and passing through $x^o$. In the general case, the nature of the orbit is provided by the Figure 5 below: the cyclical behavior of prices around equilibrium are revealed; however, since the figure cannot be taken as a demonstration, we provide such a demonstration, next.

**FIGURE 5: The Orbits of the Scarf Example**

First of all, note that since $V(t) = V(x_1(t), x_2(t)) = V(x^o_1, x^o_2)$ for all $t$, it follows that the solution or trajectory $x(t, x^o) = (x_1(t), x_2(t))$ is bounded and each $x_i(t)$ is bounded away from $-1$: since if either of these conditions is violated, $V(t)$ would tend to $+\infty$. Hence the $\omega$-limit set corresponding to $x^o$, $L_\omega(x^o)$, is non-empty and compact; also, $(0, 0) \notin L_\omega(x^o)$ if $x^0 \neq (0, 0)$ (remember, $(0, 0)$ is the equilibrium for the system) hence by the Poincaré-Bendixson theorem$^{18}$ $L_\omega(x^o)$ must be a closed orbit. This means that either we have a limit cycle or the trajectory $x(t, x^o)$ itself is a closed orbit.

If there is a limit cycle $L$, then by virtue of the Claim $??$, it follows that for any $y \in L, V(y) = V(x^o)$; further, in such circumstances, there would be a neighborhood $\mathcal{N}$ of $x^o$ such that for any solution $x(t, y)$ originating from any $y \in \mathcal{N}$, $x(t, y) \to L^{19}$. Consequently, we must have $V(y) = V(x^o) \forall y \in \mathcal{N}$: this of course, is not possible, since the function $V$ cannot be constant on an open set. Hence no such limit cycle exists. And the solution $x(t, x^o)$ must be a closed orbit. Thus we have shown the following to be true:

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Claim 3.6.5 For any initial configuration $x^o$, the solution to (3.11), $x(t,x^o)$ is a closed orbit around the equilibrium $(0,0)$.

We turn, next to the set $K$ introduced in the Section 3.2. For the particular case under consideration, the set $K$ is given by:

$$K = \{(p_1, p_2) : Z_1(p_1, p_2) + Z_2(p_1, p_2) + Z_3(p_1, p_2) \leq 0 \}.$$

Using the expressions for $Z_i(p_1, p_2)$, we have that

$$K = \{(p_1, p_2) : (p_1 - p_2).(1 - p_1).(1 - p_2) \leq 0 \}$$

FIGURE 6: The set $K$ for the Scarf Example

In Figure 6, the shaded portions constitute the set $K$; the arrows indicate the direction of price movements on the boundary of the set $K$. In the unshaded portions, i.e., on $K^c$, $\sum_{i=1}^3 Z_i(p) > 0$. Now consider $d(t) = \sum_{i=1}^2 (p_i(t, p^o) - 1)^2$ where $p(t, p^o)$ denotes the solution to (3.10). Hence along this solution, we have

$$\dot{d} = 2 \sum_{i=1}^2 (p_i(t, p^o) - 1) \dot{p}_i = 2 \sum_{i=1}^3 Z_i(p(t, p^o))$$

Hence, it follows that

Claim 3.6.6 $\dot{d} < 0$ in $K^c$ while $\dot{d} \geq 0$ in $K$.

Returning to the solution $p(t, p^o)$ of the system (3.10) through an arbitrary $p^o = (p^o_1, p^o_2) \neq (1,1)$, by virtue of the Claims made above, one may conclude the following:
Remark 14  

a. \( p_i(t, p^o) > 0 \) for all \( t \);

b. \( W(p(t, p^o)) = \sum_i \left\{ \frac{(p_i(t, p^o) - 1)^2}{2} + (p_i(t, p^o) - 1) - \ln p_i(t, p^o) \right\} \) is constant and = \( W(p^o) \) for all \( t \);

c. \( \dot{d} > 0 \) in the interior of \( K \), \( \dot{d} < 0 \) in \( K^c \) while \( \dot{d} = 0 \) on the boundary of \( K \) (See Claim 3.6.6, above);

d. The solution \( p(t, p^o) \) enters \( K \) and leaves \( K \) repeatedly.

For the Scarf example, the solution is a closed curve given by the above Remark (see (b)); consequently, the set of limit points \( L \) coincide with this curve; note that the Poincaré-Bendixson Theorem stated that so long as the set of limit points do not contain an equilibrium, again guaranteed by the point (b) noted above, a cycle is the only possible alternative.

As we hope to show below, there are some more interesting features of the Scarf Example.

3.6.1 Hopf Bifurcation for the Scarf Example

We introduce next, a parameter say \( b \), which stands for the amount of second good which individual 2 owns completely. Thus the value of \( b = 1 \) would revert back to the example considered above. We continue to treat good 3 as the numeraire and then compute excess demand functions for the non-numeraire commodities for the case at hand; it turns out that these are given, using the same notation as above, by the following expressions:

\[
Z_1(p_1, p_2) = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)}
\]
\[
Z_2(p_1, p_2) = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)}
\]
Consequently the system (3.10) now takes the form:

\[ \dot{p}_1 = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)} \quad \text{and} \quad \dot{p}_2 = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)} \]  

(3.12)

Once more standard computations ensure that the unique equilibrium is given by

\[ p_1^* = \frac{b}{2 - b} = \theta \text{ say, } p_2^* = 1 \]

Thus it may be noted that our choice of the parameter places a restriction on its magnitude

\[ 0 < b < 2; \]

and we shall take it that this is met. Notice also that when \( b = 1 \), \( \theta = 1 \) too, and we have the earlier situation. That there have been some changes to the stability property of equilibrium is contained in the next claim:

**Claim 3.6.7** For the process (3.12), \((\theta, 1)\) is a locally asymptotically stable equilibrium if and only if \( b < 1 \); for \( b > 1 \), the equilibrium is locally unstable.

Proof: The characteristic roots of the Jacobian of the system (3.10) evaluated at the equilibrium are given by:

\[ \frac{1}{8} (-b + b^2 \pm \sqrt{b\sqrt{(-32 + 49b - 26b^2 + 5b^3)}}) \]

and it should be noted that for \( 0 < b < 1.5 \) approximately, the characteristic roots are imaginary; moreover, the real part, viz., \(-b + b^2 < 0 \Leftrightarrow b < 1\) and the claim follows. •

We are now ready to show that:
Claim 3.6.8 For the system (3.12), the unique equilibrium (θ, 1) is globally stable whenever $b < 1$. When $b > 1$ any solution with an arbitrary non-equilibrium initial point is unbounded.

Proof. Consider the function:

$$W(p_1, p_2) = 2(1 - b)p_1 + (2 - b)p_1^2/2 - b \log p_1 + p_2^2/2 - \log p_2$$

Then consider the derivative of the function $W(.,.)$ along any solution to the system (3.12), we have:

$$\dot{W} = \{(2 - b)p_1 - b(1 + p_1)\} \frac{\dot{p}_1}{p_1} + (p_2^2 - 1) \frac{\dot{p}_2}{p_2}$$

$$= -(1 - p_2)^2 \frac{p_1(1 - b)}{p_2(p_1 + p_2)} < 0$$

whenever $b < 1$ and $p_2 \neq 1$. We may now conclude that the function $W(p_1, p_2)$ is a Liapunov function for the system and the first part of the claim follows. For the remaining part note that whenever $b > 1$, $\dot{W} \geq 0$ along any solution. The main point of interest about the function $W(p_1, p_2)$ is that it is a strictly convex function with an absolute minimum at $(p_1^*, p_2^*)$. Suppose then that some solution remains bounded and hence, limit points exist; consequently, along such a solution, $W(t)$ will be monotonically non-decreasing and bounded too and hence convergent and thus $\dot{W}$ must converge to zero; one may conclude that any limit point for the bounded solution must be the equilibrium and consequently, $W(t)$ is non-decreasing and converges to its minimum value: thus $W(t)$ must be constant and the only possibility for a bounded solution is that it must begin from the equilibrium, as claimed. •
Thus an easy stability condition for the Scarf Example is that $b < 1$; just as, for a meaningful equilibrium to exist, we need to have $b < 2$ a more stringent requirement has to be placed on the magnitude of $b$ to ensure global stability\(^{20}\). More importantly, it is clearly demonstrated that income effects need not necessarily be the villain of the piece. In the Scarf example, there are no substitution effects, yet it is possible to have global convergence.

### 3.7 Global Stability of Tatonnement Processes

Recall the equations (reftag:tg):

$$\dot{p}_j = F_j(p) \text{ for all } j \neq n$$

where $F_j(.)$ has the same sign as the excess demand functions $Z_j(.)$. For price adjustment, which is triggered off from some arbitrary initial $p \in R_{++}^n$, the above form seems to be the best suited as a candidate. We have already seen that unless the excess demand functions are restricted in some manner (i.e., beyond the properties P1-P4 mentioned in Section 3.2), the convergence to equilibrium can not be assured, even when the initial price is close to the equilibrium. When the initial price is not subjected to this restriction, it is not

\(^{20}\)In the context of the Scarf Example, several results of interest may be referred to: Hirota (1981) and (1985) and Anderson et. al. (2002). The first set of papers establishes the proposition that there are many distribution of the endowments which guarantee stability. Instead of the corner point chosen by Scarf (1960) and that we have followed, Hirota shows how redistribution of the totals will lead to global stability. In Anderson et. al., there is an example of an endowment distribution which ensures global stability; the endowment pattern is as follows: each individual possess the stock of the good that each is not interested in. A similar example in a two-good context is available in Gale(1963).
surprising that there have to be restrictions of some kind as well.

A principal condition under which convergence has been assured is that excess demand functions satisfy the assumption of gross substitution (GS):

\[ \frac{\partial Z_i(p)}{\partial p_j} > 0 \text{ for } i \neq j, \text{ for all } i, j, \text{ for all } p \in R^n_{++} \]

Note that the above definition is valid only over strictly positive prices. For, we have the following:

**Remark 15** The extension of the above definition to include the entire non-negative orthant, i.e., \( R^n_{+} \), conflicts with the homogeneity property of excess demand functions. For suppose, we are to insist that the above holds over the boundary of the non-negative orthant as well; consider \( p \) with \( p_j \geq 0 \) for all \( j \), \( p_1 = 0 \), \( p_k > 0 \) for some \( k \neq 1 \). Now, consider \( \lambda > 1 \) and \( Z_1(\lambda p) = Z_1(p) \), by homogeneity of excess demand functions; whereas by gross substitution, \( Z_1(\lambda p) > Z_1(p) \).

Sometimes gross substitution is defined with a weak inequality i.e.,

\[ \frac{\partial Z_i(p)}{\partial p_j} \geq 0 \text{ for } i \neq j, \text{ for all } i, j, \text{ for all } p \in R^n_{++} \]

we shall refer to this as the property of weak gross substitution (WGS) to distinguish it from the former. Let \( I = [1, 2, \cdots, n] \); the economy is said to be decomposable at \( p \), if there is a nonempty, proper subset \( J \subset I \) (i.e., \( J \neq I \)) such that

\[ \frac{\partial Z_i(p)}{\partial p_j} = 0 \text{ for } i \notin J, j \in J \]

If no such subset \( J \) exists, the economy is said to be indecomposable at \( p \). If the economy is indecomposable for every \( p \in R^n_{++} \), then the economy is indecomposable. Weak gross
substitution with indecomposability implies almost all the properties of gross substitution. First of all,

**Claim 3.7.1** GS implies that the equilibrium is unique, upto scalar multiples.

Proof: We note that by virtue of property P4, \( p \in R^{n}_{++} \). Suppose then, that \( p, q \) are equilibria with \( p \neq \delta q \) for any \( \delta > 0 \). Define \( \theta = \text{Max}_{i} p_i / q_i = p_k / q_k \), say. Note that \( \theta q_i \geq p_i \) for all \( i \); in particular note further that for \( i = k \), equality holds (i.e., \( \theta q_k = p_k \)) while there must be some \( i \) for which the strict inequality holds. By homogeneity property of excess demand functions, P3, it follows that

\[
Z_k(\theta.q) = Z_k(q) = 0;
\]
on the other hand, from GS and differentiability, we have:

\[
Z_k(\theta.q) = Z_k(p) + \sum_j Z_{kj}(p).(\theta q_j - p_j) > 0
\]

where \( p \) is some point lying on the line segment joining \( p \) to \( \theta q \); this establishes a contradiction and hence equilibrium must be unique upto scalar multiples. ●

**Remark 16** We shall show later that uniqueness follows if we have WGS together with indecomposability.

A second condition that has been used to generate global stability results is the condition of dominant negative diagonals, a condition which we saw also implied local stability in Section 3.4. We shall consider this restriction too, below. Finally, we shall consider motion on the plane and attempt to derive some global stability conditions.
3.7.1 Global Stability and WGS: The McKenzie Theorem

We first take up for consideration a result due to McKenzie (1960). The main interest lies in
the fact that this result uses the most general form of the price adjustment process. Recall
the equations (3.1):

\[ \dot{p}_j = F_j(p) \text{ for all } j \neq n \]

where \( F_j(.) \) has the same sign as the excess demand functions \( Z_j(.) \). We assume that
the excess demand functions satisfy the properties P1-P4; further, we assume that the
assumption of WGS holds and that the economy is indecomposable. And finally, the
functions \( F_j(p), j \neq n \) are assumed to be continuously differentiable for all
\( p \in R^n_{++} \). Under this assumption, for any \( p^o \in R^n_{++} \), the solution \( p(t, p^o) \) to the system (3.1) exists.

Let \( I = \{1,2,\cdots,n\} \): the set of all goods including the numeraire. We define the
following:

\[ P(p) = \{ i \in I : Z_i(p) \geq 0 \} \quad N(p) = \{ i \in I : i \notin P(p) \} \]

and

\[ P'(p) = \{ i \in I : \dot{p}_i \geq 0 \} \quad N'(p) = \{ i \in I : i \notin P'(p) \} \]

Thus \( P(p) \) denotes the set of goods at \( p \) which have a non-negative excess demand; while
\( P'(p) \) refers to the set of goods whose price adjustment is non-negative at \( p \). Notice that
\( n \in P'(p) \) since \( \dot{p}_n = 0 \) but \( n \) may or may not be an element of \( P(p) \). But \( P(p) \subset P'(p) \). This distinction allows us to define:

\[ V(p) = \sum_{i \in P(p)} p_i Z_i(p) \text{ and } V'(p) = \sum_{i \in P'(p)} p_i Z_i(p) \]
Further,

\[ V'(p) = \sum_{i \neq n} \max(p_i, Z_i(p), 0) + Z_n(p) \]

Thus

\[
V'(p) = \begin{cases} 
V(p) + Z_n(p) & \text{if } Z_n(p) < 0 \\
V(p) & \text{otherwise}
\end{cases}
\]

(3.14)

**Claim 3.7.2** \( Z(p) \neq 0 \Rightarrow V(p) > 0; p \in R^n_{++} \Rightarrow V'(p) \geq 0. \)

Proof. In case \( Z(p) \neq 0, \exists i \) such that \( Z_i(p) > 0 \) and hence \( V(p) > 0. \) For the other part of the claim, note that by virtue of Walras Law, we have for all \( p \in R^n_{++}: \)

\[
\sum_{i \in P'} p_i Z_i(p) + \sum_{i \in N'} p_i Z_i(p) = 0;
\]

Note that the first term is \( V'(p). \) In case \( N'(p) \) is empty, \( V'(p) = 0; \) in case \( N'(p) \) is non-empty, the second term above is negative; and hence the first term must be positive, i.e., \( V'(p) > 0. \) Since these are the only two possibilities, the claim follows. •

Let \( E, F \) be two disjoint nonempty subsets of \( I \) such that \( E \cup F = I. \) Since we have assumed that the economy is indecomposable, it follows directly that \( \forall p \in R^n_{++}, \exists i \in E, j \in F \) such that \( Z_{ij}(p) > 0. \) For our convergence argument, we require something weaker, which does not require indecomposability but follows from WGS. This was what McKenzie had used and for the sake of completeness we provide the following:

**Claim 3.7.3** Let \( p \in R^n_{++} \) and \( E, F \) be as defined above; further assume that \( \sum_{i \in E} p_i Z_i(p) \geq \epsilon > 0 \) for some \( \epsilon; \) WGS \( \Rightarrow \exists i \in E, j \in F \) such that \( Z_{ij}(p) > 0. \)
Proof. Differentiating the Walras Law expression \( \sum_i p_i Z_i(p) = 0 \) with respect to \( p_j \), we have:

\[
\sum_i p_i Z_{ij}(p) = -Z_j(p).
\]

Multiplying both sides by \( p_j \) and summing over for \( j \in E \), we have:

\[
\sum_i \sum_{j \in E} p_i Z_{ij}(p) p_j = -\sum_{j \in E} p_j Z_j(p) \leq -\epsilon \quad \text{from hypothesis}
\]

Thus \( \sum_{i \in E} \sum_{j \in E} p_i Z_{ij}(p) p_j + \sum_{i \in F} \sum_{j \in E} p_i Z_{ij}(p) p_j \leq -\epsilon \)

WGS implies that the second term is non-negative, given the fact that \( E, F \) are nonempty and disjoint. Hence we may conclude

\[
\sum_{i \in E} \sum_{j \in E} p_i Z_{ij}(p) p_j \leq -\epsilon \tag{3.15}
\]

Next from the property P3, that is the homogeneity property of excess demand functions, it follows that \( \forall \ p \in R^n_{++}, \sum_j Z_{ij}(p) p_j = 0 \). Hence \( \forall \ p \in R^n_{++} \)

\[
\sum_{i \in E} \sum_{j} p_i Z_{ij}(p) p_j = 0
\]

or \( \sum_{i \in E} \sum_{j \in E} p_i Z_{ij}(p) p_j + \sum_{i \in E} \sum_{j \in F} p_i Z_{ij}(p) p_j = 0 \)

or using ( 3.15 ), we have

\[
\sum_{i \in E} \sum_{j \in F} p_i Z_{ij}(p) p_j \geq \epsilon > 0
\]

and the claim follows. \( \bullet \)

 Returning to the functions, \( V(p), V'(p) \), note that these are both continuous functions of \( p \ \forall \ p \in R^n_{++} \). However, given the involvement of the function \( \text{Max} \), the functions may
lack derivatives at some $p$ such that $Z_i(p) = 0$ for some $i$; in such situations, the right hand and left derivatives will always exist; these however may not be equal. Keeping this in mind, we can proceed as follows:

**Claim 3.7.4** Let $V(t) = V(p(t, p'))$ and $V'(t) = V'(p(t, p'))$; whenever derivatives exist, $V'(t) \leq 0$; if $V'(t) > 0$, the inequality is strict. $V(t) \leq 0$ whenever derivatives exist.

Proof. Writing $p(t)$ for $p(t, p')$, we note that

$$V'(t) = \sum_{i \in P'(p(t))} p_i(t).Z_i(p(t));$$

if derivatives exist, (recall that $p_n(t) = 1 \forall t \Rightarrow \dot{p}_n = 0, \forall t$):

$$\dot{V}'(p(t)) = \sum_{i \in P'(p(t))} \{\dot{p}_i . Z_i(p(t)) + p_i(t). \sum_j Z_{ij}(p(t)). \dot{p}_j\}$$

$$= \sum_{i \in P'(p(t))} \{\dot{p}_i (- \sum_j p_j(t). Z_{ji}(p(t))) + p_i(t). \sum_j Z_{ij}(p(t)). \dot{p}_j\}$$

Splitting up the sum over $j$ into sum over $j \in P'(p(t))$ and $j \notin P'(p(t))$, and cancelling terms, we have:

$$\dot{V}'(p(t)) = - \sum_{i \in P'(p(t))} \sum_{j \notin P'(p(t))} \dot{p}_i . Z_{ji}(p(t)). p_j(t)$$

$$+ \sum_{i \in P'(p(t))} \sum_{j \notin P'(p(t))} p_i(t). Z_{ij}(p(t)). \dot{p}_j \leq 0$$

The last step follows because each term in the above is non-positive. Whenever, $p(t)$ is not an equilibrium price, $V'(p(t)) > 0$ and both $P'(p(t))$ and its complement are non-empty; consequently, either directly from indecomposability or from the Claim (3.7.3), it follows that the second term is negative. This demonstrates the validity of the claim regarding $\dot{V}'(t)$. The claim for $V'(t)$ follows exactly as above, replacing $P'(p(t))$ by $P(p(t))$. •
Next, we need to cover cases where derivatives of $V'(t)$ may not exist; say at $t = \bar{t}$; recalling the definition of $V'(p)$ from equation (3.14), it follows that at $t = \bar{t}$ the following must hold:

i. $\exists i \neq n$ such that $Z_i(p(\bar{t})) = 0$; and

ii. the right hand derivative, $\dot{V}^+(\bar{t})$ and the left hand derivative, $\dot{V}^-(\bar{t})$ exist but are unequal.

In view of the above, we define $Q(t) = \{ i \neq n : Z_i(p(t)) \geq 0 \}$; $Q_1(t) = \{ i \in Q(t) : Z_i(p(t)) > 0 \}$; $Q_2(t) = \{ i \in Q(t) : Z_i(p(t)) = 0 \}$. Thus $Q(t) = Q_1(t) \cup Q_2(t)$ and $P'(p(t)) = Q(t) \cup \{ n \}$. Note that $Q_1(t) \subset Q_1(t + h)$ for all $h > 0$ and small. The problems are created by virtue of the possibility that for $h > 0$ and small, there may be $i \in Q_1(t + h)$ but $i \notin Q_1(t)$. Clearly such an $i \in Q_2(t)$. We define $Q_3(t) = \{ i \in Q_2(t) : i \in Q_1(t + h) \}$ for all $h > 0$ and small. With these definitions, we have:

$$V'(\bar{t} + h) - V'(\bar{t}) = Z_n(p(\bar{t} + h)) - Z_n(p(\bar{t})) + \sum_{i \in Q_1(\bar{t} + h)} p_i(\bar{t} + h)Z_i(p(\bar{t} + h)) - \sum_{i \in Q_1(\bar{t})} p_i(\bar{t})Z_i(p(\bar{t}))$$

The right hand side of the above may now be broken up further as follows, using the fact that for $h > 0$ and small, $i \in Q_1(\bar{t} + h) \Rightarrow$ either $i \in Q_1(\bar{t})$ or $i \in Q_3(\bar{t})$:

$$Z_n(p(\bar{t} + h)) - Z_n(p(\bar{t})) + \sum_{i \in Q_1(\bar{t})} \{ p_i(\bar{t} + h)Z_i(p(\bar{t} + h)) - p_i(\bar{t})Z_i(p(\bar{t})) \}$$

$$+ \sum_{i \in Q_3(\bar{t})} \{ p_i(\bar{t} + h)Z_i(p(\bar{t} + h)) - p_i(\bar{t})Z_i(p(\bar{t})) \}$$

$$- \sum_{i \in Q_2(\bar{t}) - Q_3(\bar{t})} p_i(\bar{t})Z_i(p(\bar{t}))$$
Note that the last term is by definition zero. Let \( Q^* (\bar{t}) = Q_1 (\bar{t}) \cup Q_2 (\bar{t}) \cup \{ n \} \). Then we have:

\[
V' (\bar{t} + h) - V' (\bar{t}) = \sum_{i \in Q^* (\bar{t})} \{ p_i (\bar{t} + h) Z_i (p(\bar{t} + h)) - p_i (\bar{t}) Z_i (p(\bar{t})) \}
\]

Therefore \( \dot{V}^+ (\bar{t}) = \sum_{i \in Q^* (\bar{t})} \{ \dot{p}_i . Z_i (p(\bar{t})) + p_i (\bar{t}) . \sum_j Z_{ij} (p(\bar{t})). \dot{p}_j \} \)

We have already shown, by virtue of the Claim 3.7.4, that the right hand side is non-positive in general, while it is negative at dis-equilibrium price configurations. Consequently, we may now claim:

**Claim 3.7.5** \( V' (t + h) \leq V' (t) \) for all \( h > 0 \) and small.

Consequently, \( V'(t) \) is monotonically non-increasing and hence, \( 0 \leq V'(p(t, p^o)) = V'(t) \leq V'(p^o) \) for all \( t \). This allows us to make the following

**Claim 3.7.6** \( p(t, p^o) \) remains bounded for \( \forall \ t; \) further \( p_i (t, p^o) \geq \delta_i > 0 \ \forall \ t \) for some \( \delta_i \) for all \( i \).

Proof. Note that \( \| p(t, p^o) \| \rightarrow +\infty \Rightarrow \exists i \neq n \) such that \( p_i (t, p^o) \rightarrow +\infty \) since \( p_n (t, p^o) = 1 \ \forall \ t \). Define \( q(t) = p(t, p^o)/p_i (t, p^o); \) then \( q_i (t) = 1 \ \forall \ t \) while \( q_n (t) \rightarrow 0 \). By Properties P3 and P4, \( \sum_i Z_i (q(t)) = \sum_i Z_i (p(t, p^o)) \rightarrow +\infty \). Thus \( V'(t) \rightarrow +\infty \) which is a contradiction. Hence the solution remains bounded beginning from any initial point. Finally, a similar argument, using property P4 establishes a contradiction if any price is not bounded away from 0. This establishes the claim. ●

Considering the properties of the function \( V'(p) \) established above, it follows that it satisfies all the requirements of a Liapunov Function; consequently, by virtue of Claim 3,
we may conclude that every limit point of the solution $p(t, p^0)$ is an equilibrium of (3.1):
quasi global stability.

We shall show below, that under WGS and indecomposability, equilibrium is unique. Putting these conclusions together we have demonstrated the validity of the following result due to McKenzie (1960):

**Proposition 3.2** Given the properties P1-P4 of excess demand functions, WGS and indecomposability imply that the solution $p(t, p^0)$ to (3.1) from any initial point $p^0 \in \mathbb{R}^n_+$ converges to the unique equilibrium.

### 3.7.2 Global Stability and Dominant Diagonals

Let $Z : \mathbb{R}^n_+ \to \mathbb{R}^n$ be excess demand functions satisfying assumptions P1 - P3 introduced in 3.2; for this section, we strengthen P4, introduced there, to the following:

P4*. Consider a sequence $P^s \in \mathbb{R}^n_+$, $\forall s$, with $P^s_{i_0} = 1$, $\forall s$ for some fixed index $i_0$, such that $\|P^s\| \to +\infty$ as $s \to +\infty$; then $Z_{i_0}(P^s) \to +\infty$.

We shall consider the good $n$ as numeraire and consequently we represent prices by $(p_1, p_2, \cdots, p_{n-1}, 1)$ where we shall write $p = (p_1, p_2, \cdots, p_{n-1}) \in \mathbb{R}^{n-1}_+$, thus the price configuration will be written as $(p, 1)$. Thus excess demand functions will be written as $Z(p, 1)$. We shall use the symbol $p_{-i}$ to denote all the components of the vector $p$ except the $i$-th. Unless stated to the contrary, all prices will be considered to be positive. The main advantage in the strengthening of P4 to P4* lies in the following:

\[ \|x\| \text{ will be taken to be the Euclidean Norm, i.e., if } x \in \mathbb{R}^n, \text{ then } \|x\| = +\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \]
Claim 3.7.7 For each $i = 1, 2, \cdots, n - 1$, there exists $\varepsilon_i > 0$ such that $Z_i(p_{\sim i}, p_i, 1) > 0$ if $p_i \leq \varepsilon_i$ for any $p_{\sim i}$.

Proof: Suppose to the contrary, there is no such $\varepsilon_1$; i.e., for any sequence $\{p^*_s\}$, $p^*_i \to 0$ as $s \to +\infty$, one can find $p^*_{s-1} > 0$, such that $Z_1(p^*_s, p^*_{s-1}, 1) \leq 0$ for all $s$ large enough. Now consider the sequence $q^s = (1, 1/p^*_1, p^*_{s-1}, 1/p^*_1)$; notice that $\|q^s\| \to +\infty$ as $s \to +\infty$; hence by P4*, $Z_1(q^s) \to +\infty$ or by homogeneity of excess demand functions, (P3), $Z_1(p^*_1, p^*_{s-1}, 1) \to +\infty$; thus for all $s$ large enough, $Z_1(p^*_1, p^*_{s-1}, 1) > 0$: a contradiction. This establishes the claim.

We introduce, next the other main condition which has been imposed on excess demand functions to yield stability, is the condition of dominant negative diagonals, which we have encountered in Section 3.4. First of all, recall the definition of dominant diagonal condition. Consider the Jacobian of the excess demand functions $A(p, 1) = (Z_{ij}(p, 1)), i, j \neq n$.

For all $p \in \mathbb{R}^{n-1}_+$ we have:

DD i. the diagonal terms must be negative ($Z_{jj}(p, 1) < 0, j \neq n$) and

DD ii. there exist positive numbers $d_1, d_2, \cdots, d_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n-1, d_j|Z_{jj}(p, 1)| > \sum_{i\neq j} d_i|Z_{ji}(p, 1)|$.

An equivalent \(^{22}\) way of defining diagonal dominance is the following: $\exists$ positive numbers $c_1, c_2, \cdots, c_{n-1}$ such that $\forall j, j = 1, 2, \cdots, n-1, c_j|Z_{jj}(p, 1)| > \sum_{i\neq j} c_i|Z_{ij}(p, 1)|$; the numbers $d$'s in DD ii provide row dominant diagonals while the $c$’s provide column dom-

\(^{22}\)See, for example, Mukherji (1975).
inant diagonals. For the purpose at hand, we shall consider row dominant diagonals; also notice that the same constants d’s satisfy the condition \( DD \ ii \ \forall p \).

It is known that under the condition of dominant diagonals, equilibrium is unique\(^{23}\). We denote this equilibrium by \((p^*, 1)\). Recall the equations (3.1):

\[
\dot{p}_j = F_j(p) \quad \text{for all } j \neq n
\]

where \( F_j(p) \) has the same sign as the excess demand functions \( Z_j(p, 1) \). We shall assume, as before that the functions \( F_j(p) \) are continuously differentiable on \( \mathbb{R}^{n-1}_{++} \). And we shall investigate the behavior of the solution \( p(t, p^o) \) to (3.1) from any arbitrary initial point \( p^o \) in \( \mathbb{R}^{n-1}_{++} \). One immediate consequence of \( P4^* \) may now be noted:

**Claim 3.7.8** *For any adjustment process of the form (3.1), the solution \( p(t, p^o) \) remains within a bounded region and is bounded away from the axes.*

Proof: Suppose that \(||p(t, p^o)|| → +\infty\); then \( P4^* \) implies that \( Z_n(p(t, p^o), 1) → +\infty \); consequently, by Walras law, i.e., \( P2 \), we have, writing \( p(t, p^o) \) as \((p_1(t), p_2(t), \cdots, p_{n-1}(t))\):

\[
p_1(t)Z_1(p(t, p^o), 1) + \cdots + p_{n-1}(t)Z_{n-1}(p(t, p^o), 1) → -\infty
\]

Since excess demand functions are bounded below (P1), the above means that for some index \( j \neq n \) \( p_j(t)Z_j(p(t, p^o), 1) → -\infty \); note this possible only if \( p_j(t) → +\infty \) and \( Z_j(.) < 0 \); thus for all \( t \) large enough, say for \( t > T, p_j(t)Z_j(p(t, p^o), 1) < 0 \); which means that for all \( t > T, \dot{p}_j < 0 \); hence \( p_j(t) ≤ p_j(T) \forall t \): a contradiction. This establishes that \( p(t, p^o) \) remains within some bounded region. Using the result of Claim 3.7.7, it follows that there

\(^{23}\)See, for example, Arrow and Hahn (1971), p. 235.
is a rectangular region $R$ given by $0 < \varepsilon_i \leq p_i \leq M_i, i = 1, 2, \cdots, n - 1$, such that $p^o \in R \Rightarrow p(t, p^o) \in R \forall t$. 

Next, we shall deduce a consequence of assuming the dominant diagonal condition.

We begin by noting the following:

**Claim 3.7.9** Given $P1$-$P3$ and $P4^*$, $\exists \delta > 0$ such that for any $\lambda, 0 < \lambda < \delta$ and any $p \in \Re_{++}^{n-1}, p_i + \lambda Z_i(p,1) > 0, i = 1, 2, \cdots, n - 1$.

Proof: By virtue of $P1$, we know that for all $p \in \Re_{++}^{n-1}$, there is $b_i > 0, \text{ such that } Z_i(p,1) \geq -b_i, i = 1, 2, \cdots, n - 1$. Let $\bar{b} = \max_i b_i$.

Consider, next, the set $K_i = \{p \in \Re_{++}^{n-1}: Z_i(p,1) \leq 0\}, i = 1, 2, \cdots, n - 1$. The set $K_i$ is a closed subset of $\Re_{++}^{n-1}$ and we claim that $p \in K_i \Rightarrow p_i \geq \eta_i > 0$ for some $\eta_i$; if no such $\eta_i$ exists, then we can construct a sequence $p^s \in K_i \cap \Re_{++}^{n-1}, \forall s, p^s_i \to 0$ as $s \to +\infty$. Consider then the sequence $P^s = (p^s,1)$ and $q^s = P^s.1/p^s_i$; note that $q^s_i = 1 \forall s$ and $||q^s|| \to +\infty$; hence by $P4^*$, $Z_i(q^s) \to +\infty$; thus $Z_i(P^s) \to 0$ for all $s$ large enough and hence $p^s \notin K_i$ for all $s$ large enough: a contradiction. Hence there is $\eta_i$ with the claimed property. Let $\bar{\eta} = \min_i \eta_i$.

Define $\delta = \bar{\eta}/\bar{b}$; now choose any positive $\lambda < \delta$. If possible, suppose for some $i$ and for some $p \in \Re_{++}^{n-1}, p_i + \lambda Z_i(p,1) \leq 0 \Rightarrow p \in K_i \Rightarrow \lambda(-Z_i(p,1)) \geq p_i \geq \bar{\eta}$; hence $\lambda \geq \bar{\eta}/(-Z_i(p,1)) \geq \bar{\eta}/\bar{b} = \delta$: a contradiction. Hence no such $i$ and $p$ can exist and this proves the claim.

It would be appropriate to introduce in $\Re^{n-1}$ a norm which is somewhat different from the usual Euclidean norm which we have dealt with in most situations. We define the
following norm\textsuperscript{24}:

d-norm: For \( x \in \mathbb{R}^{n-1} \), \( ||x||_d = \max_{1 \leq i \leq n-1} |x_i/d_i| \) where the constants \( d_i \)'s are the same as the ones which appear in the definition of DD ii.

It is easy to check that this definition satisfies all the conditions required for being a norm. The reason for choosing such a norm will become clear with the next step:

Claim 3.7.10 \( ||x||_d \) is a norm on \( \mathbb{R}^{n-1} \); further if \( A = (a_{ij}), i, j = 1, 2, \cdots, n-1 \), \( a_{ij} \) real numbers for all \( i, j \), then \( ||A||_d = \max_{1 \leq i \leq n-1} \sum_{j=1}^{n-1} |a_{ij}|d_j/d_i \).

Proof: That \( ||x||_d \) satisfies all the conditions is straightforward. Now for any square matrix \( A \), \( ||A||_d \) is by definition \( \sup_{||x||_d = 1} ||Ax||_d \)\textsuperscript{25}. From this definition, we observe that:

\[
||Ax||_d = \max_i \left| \sum_j a_{ij} \cdot \frac{x_j}{d_i} \right| = \max_i \left| \frac{1}{d_i} \sum_j a_{ij} \cdot d_j \cdot \frac{x_j}{d_j} \right| 
\]

\[
\leq \max_i \left| \frac{1}{d_i} \sum_j |a_{ij}|d_j \cdot |x_j/d_j| \right| \leq \max_i \left| \frac{1}{d_i} \sum_j |a_{ij}|d_j \cdot ||x||_d \right|
\]

Now suppose that \( \max_i \sum_j |a_{ij}|d_j/d_i = \sum_j |a_{kj}|d_j/d_k \); then define:

\[
\bar{x}_j = \begin{cases} 
\frac{d_j a_{kj}}{|a_{kj}|} & \text{if } a_{kj} \neq 0 \\

\text{otherwise} 
\end{cases}
\]

(3.16)

Notice that \( ||\bar{x}||_d = 1 \); further for \( i = k \), \( \sum_j a_{kj} \bar{x}_j = \sum_j |a_{kj}|d_j \) and for any \( i \), \( |\sum_j a_{ij} \cdot \bar{x}_j| \leq \sum_j |a_{ij}|d_j \); hence note that for this particular definition of \( \bar{x} \), \( ||A\bar{x}||_d = \sum_j |a_{kj}|d_j/d_k \), the bound we had obtained above, is attained. This proves the claim. \( \bullet \)

We note next:

\textsuperscript{24}The analysis follows the elegant treatment in Fujimoto and Ranade (1988).

\textsuperscript{25}See, for example, Ortega and Rheinboldt (1970), p. 40-41.
Claim 3.7.11 Given any $\bar{p} \in \mathbb{R}^{n-1}_{++} \neq p^*$ there is some $\bar{\lambda} > 0$ such that $1 + \lambda Z_{ii}(p, 1) > 0 \forall \lambda, 0 < \lambda < \bar{\lambda}$ and for all $p \in [\bar{p}, p^*]$, where $[\bar{p}, p^*]$ consist of all points on the line segment connecting $\bar{p}$ to $p^*$.

Proof: By DD i, $Z_{ii}(p, 1) < 0 \forall p \in [\bar{p}, p^*]$; by virtue of P1,

$$\bar{\lambda} = \min_{p \in [\bar{p}, p^*]} \frac{1}{-Z_{ii}(p, 1)}$$

exists and is positive. It is straightforward to check that this definition of $\bar{\lambda}$ has the desired property. •

These preliminary steps allows us to prove the following crucial property of of dominant diagonal systems\footnote{This property was noted by Fujimoto and Ranade (1988).}

Proposition 3.3 Given DD i-ii, consider for any $p \in \mathbb{R}^{n-1}_{++} \neq p^*$, $\max_i \frac{|p_i - p^*_i|}{d_i} = \frac{|p_k - p^*_k|}{d_k}$, say. Then $p_k - p^*_k > 0 \Rightarrow Z_k(p, 1) < 0$; while $p_k - p^*_k < 0 \Rightarrow Z_k(p, 1) > 0$.

Proof: Consider any $p \in \mathbb{R}^{n-1}_{++} \neq p^*$; next choose $\lambda < \min(\delta, \bar{\lambda})$, where $\delta$ and $\bar{\lambda}$ are as in Claims 3.7.9 and 3.7.11 respectively; for such a fixed choice of $\lambda$, define $\psi(p) = (p_i + \lambda Z_{ii}(p))$; then by the Mean Value Theorem\footnote{See Ortega and Rheinboldt (1970), p. 69.}, we have the following:

$$||\psi(p) - \psi(p^*)||_d \leq \sup_{0 \leq t \leq 1} ||J_\psi(p^* + t(p - p^*))||_d ||p - p^*||_d$$
where the matrix $J_\psi$ is given by the $(n - 1) \times (n - 1)$ matrix:

$$
\begin{pmatrix}
1 + \lambda Z_{11} & \cdots & \cdots & \lambda Z_{1n-1} \\
\lambda Z_{21} & 1 + \lambda Z_{22} & \cdots & \lambda Z_{2n-1} \\
\cdots & \cdots & \cdots & \cdots \\
\lambda Z_{n-11} & \cdots & \cdots & 1 + \lambda Z_{n-1,n-1}
\end{pmatrix}
$$

Now by virtue of the Claim 3.7.10, we have:

$$
||J_\psi||_d = \max_i \left[ 1 + \lambda \sum_{j \neq i} |Z_{ij}| d_j \right]
$$

using the fact that by our choice of $\lambda$, $|1 + \lambda Z_{ii}| = 1 + \lambda Z_{ii}$; hence it follows from DD ii that $||J_\psi||_d < 1$ for all points on the line segment $[p, p^\star]$; consequently it follows that:

$$
||\psi(p) - p^\star||_d < ||p - p^\star||_d
$$

If the left hand side is $\frac{\psi_j(p) - p^\star_j}{d_j}$ and the right hand side is $\frac{p_k - p^\star_k}{d_k}$ then it follows that:

$$
\frac{\psi_k(p) - p^\star_k}{d_k} \leq \frac{\psi_j(p) - p^\star_j}{d_j} < \frac{p_k - p^\star_k}{d_k}
$$

which implies that $|p_k - p^\star_k + \lambda Z_k(p, 1)| < |p_k - p^\star_k|$; since $\lambda > 0$ this is possible only if $(p_k - p^\star_k)$ and $Z_k(p, 1)$ are of opposite signs. This establishes the claim. •

The above allows us to demonstrate the following:

**Proposition 3.4** Given DD i- ii, the unique equilibrium is globally asymptotically stable under the adjustment process (3.1).

Proof: Let $p(t, p^\circ)$ denote the solution to (3.1) from some arbitrary initial point $p^\circ \in \mathbb{R}_{++}^{n-1}$. Define $V(t) \equiv V(p(t, p^\circ)) = ||p(t, p^\circ) - p^\star||_d$. Note that $V(p(t, p^\circ))$ is continuous in $p$; in case
derivatives exist, \( \dot{V} = \text{Sgn}(p_k - p^*_k) \dot{p}_k \) where \( ||p(t, p') - p^*||_d = |p_k - p^*_k|/d_k \); hence by virtue of Proposition 3.3 and the properties of the system (3.1), it follows that when derivatives exist \( \dot{V}(t) < 0 \) if \( p(t, p') \neq p^* \). When derivatives do not exist, we need to investigate further.

Consider \( S(t) = \{ k : \frac{|p_k(t, p') - p^*_k|}{d_k} \geq \frac{|p_i(t, p') - p^*_i|}{d_i} \forall i \} \); thus \( ||p(t, p') - p^*||_d = \frac{|p_k - p^*_k|}{d_k} \) for \( k \in S(t) \). Notice that if at \( t \), \( S(t) \) is a singleton, \( \dot{V}(t) \) exists; when \( S(t) \) has more than one member, derivatives may fail to exist. In this case, note that if \( i \notin S(t) \) then \( i \notin S(t + h) \) for \( h \) positive and small. Thus \( S(t + h) \subseteq S(t) \) for all \( h \) positive and small enough. With these observations, consider some \( k \in S(t + h) \), \( h \) small enough; then \( k \in S(t) \) and we have:

\[
\lim_{h \to 0^+} \frac{V(t + h) - V(t)}{h} = \lim_{h \to 0^+} \frac{1}{h} \left( \frac{|p_k(t + h, p') - p^*_k|}{d_k} - \frac{|p_k(t, p') - p^*_k|}{d_k} \right)
\]

thus we may observe that:

\[
\lim_{h \to 0^+} \frac{V(t + h) - V(t)}{h} = \frac{1}{d_k} \frac{d|p_k(t, p') - p^*_k|}{dt} \text{ for some } k \in S(t) ;
\]

note that the right hand side exists and is negative whenever \( p(t, p') \neq p^* \).

Consequently, \( V(t + h) < V(t) \) for all \( h \) sufficiently small and positive, so long as the equilibrium is not encountered. Thus \( V(t) \) is a Liapunov function for the process (???); this also implies that \( V(p(t, p')) < V(p') \) which implies that the solution is bounded. So all the ingredients required for our claim are in place. •

The above result is crucially dependant on the fact that DD holds or that the same constants provide the dominance condition DD ii, for all \( p \). We present next an attempt to weaken this condition: but it will not be costless, since as we shall see the convergence
result will have to be in terms of a somewhat special version of the tatonnement. Given this trade-off, it has been thought best to present both of these results.

We shall maintain the assumptions P1-P3 and P4* on excess demand functions; in addition, we shall require a modified version of the conditions DD ii:

**DD ii***: There exist continuously differentiable functions $h_j: \mathbb{R}_+^{n-1} \to \mathbb{R}_+^{n-1}$, $j = 1, 2, \ldots, n-1$ satisfying $h_j(p)|Z_{jj}(p, 1)| > \sum_{i \neq j, n} |Z_{ji}(p, 1)| h_i(p)$ for all $j = 1, 2, \ldots, n - 1$.

We shall say that **DD** holds when we have **DD i** and **DD ii***.

For **DD**, the adjustment process (3.1) needs to be specialized to the following\(^{28}\):

$$\dot{p}_j = h_j(p).Z_j(p, 1) \text{ for all } j \neq n, p_n \equiv 1 \quad (3.17)$$

Notice that the weights which appeared in the definition of **DD ii*** also appear in the adjustment equations above; we shall return to an interpretation to these equations later.

We note that by virtue of our assumptions, and due to Claims 3.7.7 and 3.7.8, we have:

**Claim 3.7.12** Given any $p^o \in \mathbb{R}_+^{n-1}$, the solution to (3.17) through $p^o$, $p(t, p^o)$ exists and is continuous with respect to the initial point; further the solution remains within a bounded region and is bounded away from the axes.

We shall write the solution $p(t, p^o)$ in short as $p(t)$. Define $W(t) \equiv W(p(t)) = \max_{j \neq n} |Z_j(p(t), 1)| = |Z_k(p(t), 1)|$ say. Then, one may show:

**Claim 3.7.13** Given **DD***, $W(t)$ is strictly decreasing along the trajectory, if $p(t, p^o) \neq p^*$.

Proof: Define $S(t) = \{j \neq n : |Z_j(p(t), 1)| \geq |Z_i(p(t), 1)| \forall i \neq n\}$; notice that if at $t$, $S(t)$ is a singleton, $\{k\}$, say, and if $p(t) \neq p^*$, then $\dot{W}(t)$ exists and is given by $\dot{W}(t) = \ldots$

\(^{28}\)The treatment presented here is a generalization of Mukherji (1974 a).
\((\text{Sgn}Z_k(.)) \sum_{j \neq n} Z_{kj}(p(t), 1)h_j(p(t)).Z_j(p(t), 1)\). Suppose \(Z_k(p(t), 1) > 0\), we claim that the expression \(\sum_{j \neq n} Z_{kj}(p(t), 1)h_j(p(t)).Z_j(p(t), 1) < 0\); if to the contrary, this expression is non-negative, we have, using the fact that \(\text{DD} \; \text{i}\) holds:

\[
\left| Z_{kk}(.)h_k(.)Z_k(.) \right| \leq \sum_{j \neq k, n} |Z_{kj}(.)| |h_j(.)| Z_j(.) \leq |Z_k(.)| \sum_{j \neq k, n} |Z_{kj}(.)| h_j(.)
\]

where the first inequality follows from our contrary hypothesis and the fact that the diagonal term is negative; the second inequality follows on account of the absolute value of sum being less than or equal to the sum of absolute values and the last inequality follows because of the fact that \(k \in S(t)\). We note then that since \(Z_k(.)\) is not zero at dis-equilibrium, we have \( |Z_{kk}(.)| h_k(.) \leq \sum_{j \neq k, n} |Z_{kj}(.)| h_j(.) \) which violates condition \(\text{DD ii*} \) for \(j = k\). Hence \(k \in S(t)\) and \(Z_k(.) > 0\) implies that \(\sum_{j \neq n} Z_{kj}(p(t), 1)h_j(p(t)).Z_j(p(t), 1) < 0\). In a similar fashion, in case \(Z_k(.) < 0\), one may show that \(\sum_{j \neq n} Z_{kj}(p(t), 1)h_j(p(t)).Z_j(p(t), 1) > 0\).

Thus whenever, \(S(t)\) is a singleton, \(W(t) < 0\) provided, \(p(t) \neq p^*\). If \(S(t)\) is not a singleton, then derivatives may not exist. To cover such cases, notice that for \(h > 0\) and small, \(S(t + h) \subseteq S(t)\); so the situation is as in the proof of Proposition 3.4 and one may claim, exactly following those steps that \(W(t + h) < W(t)\) for all \(h\) positive and small, provided no equilibrium is encountered. Thus regardless of whether \(S(t)\) is a singleton or not, so long as no equilibrium is encountered, \(W(t)\) is strictly decreasing along the trajectory. \(

We may now claim, the following to be true:

**Proposition 3.5** Given \(\text{DD}^*\), conditions P1-P3 and P4* on excess demand functions, the unique equilibrium \((p^*, 1)\) is globally asymptotically stable under the process (3.17).

Proof: First of all, we note that the process (3.17) satisfies the requirement of being a
special case of the process (3.1) and hence, by virtue of Claims 3.7.7 and 3.7.8, any solution to (3.17) from an arbitrary \( p^0 \) will be trapped in some bounded region \( R \) which is also bounded away from the axes. Consequently limit points exist and since \( W(t) \) has been shown to satisfy the requirements for being a Liapunov function for the process (3.17), the claim is established. •

We have thus presented two results both of which assure global asymptotic stability for the unique equilibrium under appropriate adjustment processes. The first result, Proposition 3.4 considers the most general process but is more demanding in the restriction on excess demand functions, since the condition \( DD \ ii \) requires the same set of weights to provide row dominance for all prices. We used a condition derived by Fujimoto and Ranade (1988) to arrive at this conclusion; weakening the condition \( DD \ ii \) to \( DD \ ii^* \) and allowing these weights to vary over prices we obtained the second result, Proposition 3.5, where the adjustment process (3.1) had to be specialized to (3.17). As we had remarked earlier, the generalization was not costless. The results are of interest since the adjustment processes, for both of these cases are more general than the ones in the existing literature. It may be possible to provide some other results for special cases of (3.1) but such exercises have not been reported.

Finally, the dominant diagonal condition has an inherent interest because what it ensures is that in any market, the excess demand is “most” responsive to its own price; the various formalizations provided, namely \( DD \) and \( DD^* \) are alternative ways of capturing this simple idea. In contrast to the assumption of Gross Substitutes considered earlier, notice that
dominant diagonals offer a scope for the presence of complementarities in the system subject to the provision that own price effects dominate. We should also mention that the gross substitute case satisfies the Weak Axiom of Revealed Preference in the Aggregate (WARP); it is of interest that so does the Dominant Diagonal Condition DD provided the same weights also provide column dominant diagonals; this has been shown by Fujimoto and Ranade (1988). Some what loosely speaking, thus, the condition WARP lies at the common intersection of the conditions which guarantee the stability of equilibrium.

As we have seen above, there are two issues involved. One is the nature of excess demand functions and the other is the form of the tatonnement process. If we specialize one, we are able to consider general forms of the other. We consider, next, general forms for the excess demand functions and, as is to be expected, we specialize the form of the tatonnement process.

3.8 The Structure of Limit Sets for the General Case

In this section, we consider excess demand functions satisfying the properties P1-P3 and P4* and the adjustment on prices given by the process (3.3):

\[ \dot{p}_j = Z_j(p) \text{ for all } j \neq n; \quad p_n = 1 \quad (3.18) \]

Then for any \( p^\circ \in R_{++}^n \), the solution to (3.3), \( p(t,p^\circ) \) exists. In addition, recalling the results of Claims 3.7.7 and 3.7.8, which remain applicable since the process (3.3) is a special case of the process (3.1), we may conclude:

**Claim 3.8.1** There is a region \( R \) given by \( \{ (x_1, x_2, \cdots, x_{n-1}) : 0 < \varepsilon_i \leq x_i \leq M_i \} \) for some
Thus any solution to \((3.3)\) is bounded and bounded away from the axes. Let \(E = \{ p \in R^n_{++} : Z(p) = 0 \} \): the set of equilibrium prices. To provide this set with some minimal structure, we shall assume that the economy is \textbf{regular} i.e., P5. For every \( p \in E \), rank of

\[
J(p) = \left( \frac{\partial Z_i(p)}{\partial p_j} \right)
\]

is \((n - 1)\).

It is well known that P5 ensures that \(E\) is a finite set. Recall that for any \( p^* \in E \) we had defined the set \( K_{p^*} \) by \( K_{p^*} = \{ p \in R^n_{++} : p^*T . Z(p) \leq 0 \} \): the set of prices where the Weak Axiom is \textbf{violated}. We shall use P1-P3, P4* and P5 to analyze the behavior of \( p(t,p^o) \) as \( t \to \infty \). By virtue of the Claims made above, we may conclude that the set \( L = \{ p : \exists \text{ a subsequence } p(t_s, p^o) \text{ and } p(t_s, p^o) \to p \text{ as } s \to \infty \} \)

is \textbf{non-empty}. In addition, given the properties of the solution to the system \((3.3)\), one may also conclude that \( L \) is a \textbf{compact and connected} subset of \( R^n_{++} \). Further, we have:

\textbf{Claim 3.8.2} \( L \cap K_{p^*} \neq \emptyset \) for any \( p^* \in E \).

Proof. In case \( L \cap K_{p^*} = \emptyset \) for some \( p^* \in E \), then \( p(t, p^o) \notin K_{p^*} \) for all \( t > T \) for some \( T \) and consequently, \( d(p(t, p^o), p^*) \) is a Liapunov function for \( t > T \) and \( p(t, p^o) \to p^* \) as \( t \to \infty \), i.e., \( L = \{ p^* \} \subset K_{p^*} \): a contradiction. Hence the claim. \( \bullet \)

Next note that

\[
\epsilon_i, M_i, i = 1, 2, \cdots, n - 1 \text{ within which any solution to } p(t, p^o) \text{ lies provided } p^o \in R.
\]

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are well defined and further $d_*(p^*)$ and $d^*(p^*)$ are both attained in the set $L$. In particular, we have:

**Claim 3.8.3** $d_*(p^*)$ and $d^*(p^*)$ are attained in $L \cap K_{p*}$.

Proof. We shall write $K, d_*$ for $K(p^*), d_*(p^*)$ respectively. Suppose that $d_*$ is attained at $\bar{p} \notin K$; i.e., $\exists$ a subsequence $p(t_s, p^o) \rightarrow \bar{p}$ and $d(p, p^*) \leq d(p, \bar{p})$, $\forall p \in L$. Thus $p^* . Z(\bar{p}) = \delta > 0$ for some $\delta$. Also for some $\eta > 0$, $d(\bar{p}, K) > \eta$, where $d(\bar{p}, K) = \min_{p \in K} d(\bar{p}, p)$. Let $\epsilon > 0$ be such that $p \in N_\epsilon(\bar{p}) = \{p : d(p, \bar{p}) < \epsilon\} \Rightarrow d(p, K) > \eta$; further $\forall p \in N_\epsilon(\bar{p})$, $|p^* . Z(p) - p^* . Z(\bar{p})| < \delta/2$. Since $p(t_s, p^o) \rightarrow \bar{p}$, there is some number $S_\epsilon$ such that $\forall s > S_\epsilon$, $p(t_s, p^o) \in N_\epsilon(\bar{p})$ and there is a subsequence $p(t_s + \theta_s, p^o)$ such that $p(t_s + \theta_s, p^o) \notin N_\epsilon(\bar{p})$, while $p(t, p^o) \in N_\epsilon(\bar{p})$ for all $t$ such that $t_s \leq t < t_s + \theta_s$. Such a subsequence can always be constructed, since otherwise $p(t, p^o)$ has no other limit points and $L = \{\bar{p}\}$ and hence $L \cap K = \emptyset$: a contradiction. By definition, note that $d(p(t_s + \theta_s, p^o), K) \geq \eta$ and hence the subsequence $p(t_s + \theta_s, p^o)$ has all its limit points outside the set $K$. Now note that:

$$d(p(t_s + \theta_s, p^o), p^*) = d(p(t_s, p^o), p^*) + \theta_s \hat{d}(p(t_s + \lambda_s \theta_s, p^o), p^*)$$

for some $\lambda_s$, $0 \leq \lambda_s \leq 1$.

Recall that $\hat{d}(p(t, p^o), p^*) = -p^* . Z(p(t, p^o))$: thus, by hypothesis, $\forall s > S_\epsilon$, $\hat{d}(p(t_s + \lambda_s \theta_s, p^o), p^*) = -p^* . Z(p(t_s + \lambda_s \theta_s, p^o))$. Since $|p^* . Z(p(t_s + \lambda_s \theta_s, p^o)) - p^* . Z(\bar{p})| \leq \delta/2$ for all $s > S_\epsilon$, it follows that $\hat{d}(p(t_s + \lambda_s \theta_s, p^o), p^*) = -p^* . Z(p(t_s + \lambda_s \theta_s, p^o)) \leq \delta/2 - p^* . Z(\bar{p}) = -\delta/2$. Since $p(t_s + \theta_s, p^o) \notin N_\epsilon(\bar{p}) \forall s > S_\epsilon$, $\theta_s$ cannot have 0 as a limit point. Hence

$$d(p(t_s + \theta_s, p^o), p^*) \leq d(p(t_s, p^o) - \theta_s \delta/2$$

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\[ \leq d(p(t, p^\circ)) - \lim_{s \to \infty} \theta_s \cdot \delta / 2 \]

Hence all limit points \( \tilde{p} \) of \( p(t + \theta_s, p^\circ) \) satisfy

\[ d(\tilde{p}, p^\circ) \leq d_s - \lim_{s \to \infty} \theta_s \delta / 2 < d_s \]

which is a contradiction. Hence \( \overline{p} \in K \) and the claim follows. An analogous argument establishes that \( d^* \) is attained in \( K \) too. •

We now take into consideration the fact that \( E \) may have other equilibria; but given P5, \( E \) is a finite set. To take this aspect into account, we make the following definitions:

\[
K = \bigcup_{p^* \in E} K(p^*); d^* = \max_{p \in L} d(p, E); d_s = \min_{p \in L} d(p, E)
\]

Hence note that \( d_s = d(\hat{p}, p^\circ) \) for some \( \hat{p} \in L, p^\circ \in E \). By virtue of our last claim, \( \hat{p} \in K(p^*) \subset K \). In case this particular \( p^\circ \) happens to be an isolated point of \( K(p^*) \), we have the following:

**Claim 3.8.4** Since \( d_s = d(\hat{p}, p^\circ) \) for some \( \hat{p} \in L, p^\circ \in E, d_s = 0 \Rightarrow d^* = 0 \) and \( \lim_{t \to \infty} p(t, p^\circ) = p^\circ \) provided \( p^\circ \) is an isolated point of \( K(p^*) \).

Proof. Since \( d_s = 0 \), there is a subsequence \( p(t_s, p^\circ) \to p^\circ \) as \( s \to \infty \). Further since \( p^\circ \) is an isolated point of \( K(p^*) \), there is a neighborhood \( N(p^\circ) \) of \( p^\circ \) such that for all \( p \in N(p^\circ), p \neq p^\circ \Rightarrow p \cdot Z(p) > 0 \). Moreover, \( p \in N(p^\circ) \Rightarrow d(p, p^\circ) < 0 \). Let \( \epsilon > 0 \) be such that \( d(p, p^\circ) < \epsilon \Rightarrow p \in N(p^\circ) \). Since \( p(t_s, p^\circ) \to p^\circ, d(p(t_s, p^\circ), p^\circ) < \epsilon \) for all \( s > S_\epsilon \), say. Also by our choice of \( \epsilon, \epsilon, d(p(t, p^\circ), p^\circ) < \epsilon \Rightarrow d(p(t + h, p^\circ), p^\circ) < d(p(t, p^\circ), p^\circ) \) for all \( h > 0 \). Thus \( d(p(t, p^\circ), p^\circ) < \epsilon \) for some \( \bar{t} \) implies that \( d(p(t, p^\circ), p^\circ) < \epsilon \) for all \( t > \bar{t} \) and the claim follows. •
We are now ready to consider the general nature of the set of limit points $L$ of the solution $p(t,p^o)$. By virtue of what we have shown above $L \cap K \neq \emptyset$. The question is whether $L \cap E$ is empty or not. In case it is empty, $d_* > 0$ and there are the following logically feasible possibilities:

(i) there is $\overline{t}$ such that for all $t > \overline{t}$, $p(t,p^o) \notin K$;

(ii) there is $\overline{t}$ such that for all $t > \overline{t}$, $p(t,p^o) \in K$;

(iii) there are subsequences $p(t_s,p^o), p(t_k,p^o)$ such that $p(t_s,p^o) \in K$ and $p(t_k,p^o) \notin K$ for all $s,k$ respectively.

Note that in case (i), $d(p(t,p^o),p^*)$ is a Liapunov function for any $p^* \in E$ and consequently $p(t,p^o) \to p^*$ for any $p^* \in E$: a contradiction. For the case (ii), $\dot{d}(p(t,p^o)) \geq 0$ for any $p^* \in E$. Also since $p(t,p^o) \in K$ for all $t > \overline{t}$, which is a compact set, it follows that $d(p(t,p^o),p^*)$ converges for any $p^* \in E$ to some $D(p^*)$; thus all points of $L$ are at the same distance $D(p^*)$ from $p^*$ and this is true for every $p^* \in E$. In case (iii), the solution $p(t,p^o)$ fluctuates between a maximum distance $d^*$ and a minimum distance $d_*$ from the set of equilibria $E$, where the maximum and minimum are attained in the set $K$; recall this is what happened in the example due to Scarf considered above. When the set $L$ has this nature, we shall say that it has the Scarf property. We note these possibilities in the form of

**Proposition 3.6** The solution to the system (3.3) for any arbitrary initial point $p^o$ has a nonempty set of limit points $L$ which either contains an equilibrium or its points are equidistant from any equilibrium or $L$ has the Scarf property.
Without putting more structure on the the sets $K(p^*)$ or the set $K$, the set of limit points cannot be subjected to more restrictions. If the set of limit points $L$ does not contain an equilibrium, then it must be the case that either prices remain at some constant distance away from equilibria or fluctuate between a maximum and minimum distance from the set of equilibria. In some sense, therefore, if the type of phenomenon exhibited during our analysis of the Scarf example is not present, convergence to equilibrium may be ensured. We shall specifically consider these issues in smaller dimensions, next.

In particular, we shall consider motion on the plane.

### 3.8.1 The Scarf Example Once More

By virtue of the comments made above, we shall consider the example due to Scarf as the point of departure. The model and the our analysis has been presented in Section 3.6. Consequently we shall not repeat them here except we provide an alternative proof of Claim 3.6.8. This will be instructive since it will reveal exactly what is required for convergence and to eliminate the type of problems envisaged by Scarf. For this purpose recall the system (3.12):

\[
\dot{p}_1 = \frac{p_1(1 - p_2)}{(1 + p_1)(p_1 + p_2)} \quad \text{and} \quad \dot{p}_2 = \frac{p_2(p_1 - b) + (1 - b)p_1}{(1 + p_2)(p_1 + p_2)}
\]

Recall that the unique equilibrium is given by

\[
p_1^* = \frac{b}{2 - b} = \theta \quad \text{say}, \quad p_2^* = 1
\]

Also recall that the characteristic roots of the relevant matrix at equilibrium are given by:

\[
\frac{1}{8}(-b + b^2 \pm \sqrt{b\sqrt{-32 + 49b - 26b^2 + 5b^3}}).
\]
Consequently, for the process (3.12), \((\theta, 1)\) is a locally asymptotically stable equilibrium if and only if \(b < 1\); for \(b > 1\), the equilibrium is locally unstable.

We had made the much stronger assertion: *For the system (3.12), the unique equilibrium \((\theta, 1)\) is globally asymptotically stable whenever \(b < 1\); and any trajectory with \((p_1^0, p_2^0) > (0, 0)\) as initial point remains within the positive orthant. When \(b > 1\), any solution with an arbitrary non-equilibrium initial point is unbounded.* We offer below a somewhat different approach to the proving of this result, next.

We first note that for the system (3.12) there can be no closed orbit in \(\mathbb{R}^2_{++}\) so long as \(b\) is different from unity. For this purpose we shall use, Dulac’s Criterion\(^{29}\). Now consider the function:

\[
f(p_1, p_2) = \frac{(p_1 + p_2)(1 + p_1)(1 + p_2)}{p_1 p_2}
\]

on \(\mathbb{R}^2_{++}\). Notice that:

\[
\frac{\partial f(p_1, p_2) Z_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) Z_2(p_1, p_2)}{\partial p_2} = -(1 - b)/p_2^2
\]

Thus \(b \neq 1\) implies that Dulac’s Criterion is satisfied by this choice of \(f(p_1, p_2)\) and consequently there can be no closed orbits when \(b \neq 1\). Applying next, the Poincaré-\(^{29}\)See, Andronov et. al. (1966), p. 305. This criterion looks for a function \(f(p_1, p_2)\) which is continuously differentiable on some region \(R\) and for which

\[
\frac{\partial f(p_1, p_2) h_1(p_1, p_2)}{\partial p_1} + \frac{\partial f(p_1, p_2) h_2(p_1, p_2)}{\partial p_2}
\]

is of constant sign on \(R\) (not identically zero), then there is no closed orbit for the system \(\dot{p}_i = h_i(p_1, p_2), i = 1, 2\) on the region \(R\).
Bendixson Theorem, it follows that for any initial \( p^0 \in \mathbb{R}^2_{++} \), the unique equilibrium \( p^* = (\theta, 1) \in L_\omega(p^0) \) provided the \( \omega \)-limit set is non-empty.

Recall that for \( b > 1 \), the unique equilibrium is unstable; consequently no solution can enter a small enough neighborhood of \( p^* \); consequently, in this situation, \( L_\omega(p^0) \) must be empty, if \( p^0 \neq p^* \); thus the trajectories must be unbounded.

When \( b < 1 \), the unique equilibrium \( p^* \) is locally asymptotically stable; so if \( L_\omega(p^0) \neq \emptyset \), \( p^* \in L_\omega(p^0) \Rightarrow p^* = L_\omega(p^0) \); since once having entered a small enough neighborhood of the equilibrium, the trajectory cannot leave. Thus all that we need to guarantee convergence is that trajectories are bounded when \( b < 1 \).

This last step may be accomplished by considering the function

\[
W(p_1, p_2) = 2(1 - b)p_1 + (2 - b)p_1^2/2 - b \log p_1 + p_2^2/2 - \log p_2
\]

and noting that its time derivative, along any solution to the system (3.12):

\[
\dot{W} = \{(2 - b)p_1 - b)(1 + p_1)\} \frac{\dot{p}_1}{p_1} + (p_2^2 - 1) \frac{\dot{p}_2}{p_2}
\]

\[
= -(1 - p_2)^2 \frac{p_2(1 - \theta)}{p_2(p_1 + p_2)} \leq 0
\]

whenever \( b < 1 \). Thus for \( b < 1 \), \( W(p_1(t), p_2(t)) \leq W(p_1^0, p_2^0)(\forall t, \text{ where we write } (p_1(t), p_2(t)) \) as the solution to (3.12). Note that if \( p_i(t) \to +\infty \), for some \( i \), \( W(p_1(t), p_2(t)) \to +\infty \) and the boundedness and positivity of the solution are established\(^\text{30}\). This establishes the claim.

There are thus two things to be noted from the above demonstration: first that choosing a value of \( b \) different from unity negates the existence of a closed orbit; and a value of \( b\)

\(^\text{30}\)It may be recalled in our earlier approach, we had established that the function \( W(.) \) was a Liapunov function for the problem at hand.
less than unity is required to ensure that trajectories remain bounded. In a sense to be made precise below, these are the two aspects we need to account for if we are interested in identifying global stability conditions. This is why the above analysis is revealing.

3.8.2 General Global Stability Conditions

If there are three goods and one of them is the numeraire, then the price adjustment equations of the type used for the Scarf example introduces dynamics on the plane.

For motion on the plane, we shall use the results introduced in Section 1.5\textsuperscript{31}; we show next that it is possible to substantially weaken the conditions under which a global stability result may be deduced. This would allow us to conclude global stability for a competitive equilibrium as well as providing a general stability result which would be of some general interest, as well.

Consider the following systems of equations:

\[ \dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y) \quad (3.19) \]

where the functions \( f, g \) are assumed to be of class \( C^1 \) on the plane \( \mathbb{R}^2 \). For any pair of functions \( f(x, y), g(x, y) \) let \( J(f, g) \) or simply \( J \), if the context makes it clear, stand for the Jacobian\textsuperscript{32}:

\[
\begin{pmatrix}
  f_x & f_y \\
  g_x & g_y
\end{pmatrix}
\]

\textsuperscript{31}These results are Propositions 1.4, 1.5 and 1.6.

\textsuperscript{32}For any function \( f \) will refer to the partial derivative of \( f \) with respect to the variable \( x \).
We shall use the setting of the tatonnement to investigate motion on the plane and for this purpose we introduce the notion of the excess demand functions $Z_i(p_1, p_2, p_3): \mathbb{R}_+^3 \to \mathbb{R}, i = 1, 2, 3$ which are required to satisfy the following:

A Conditions P1- P3 and P4* hold.

To study the dynamics on the plane, we shall investigate the solutions to a system of equations of the following type:

$$\dot{p}_i = h_i(p), i = 1, 2 \text{ with } p_3 \equiv 1 \quad (3.20)$$

where the functions $h_i(p)$ are assumed to satisfy the following: (we write $p = (p_1, p_2) \in \mathbb{R}_+^2$)

B $h_i(p) = Z_i(p_1, p_2, 1), i = 1, 2.$

Thus the equation (3.20) defines motion on the positive quadrant of the plane and more importantly reduce to (3.3) for the plane.

A typical trajectory or solution to (3.20) from an initial $p^o \in \mathbb{R}_+^2$ will be denoted by $\phi_t(p^o)$; the price configuration will be $(\phi_t(p^o), 1)$ for each instant $t$; this is just to signify that the numeraire (the third good) price is always kept fixed at unity. Also we note that any equilibrium for the dynamical system (3.20), say $\bar{p}$ where $h_i(\bar{p}) = 0, i = 1, 2,$ implies that $(\bar{p}, 1)$ is an equilibrium for the economy, in the sense that $(\bar{p}, 1) \in E$ and conversely.

We shall denote the equilibrium for (3.20) by $E_R$.

We are interested in the structure of the $\omega$-limit set $L_\omega(p^o)$ i.e., the limit points of the trajectory $\phi_t(p^o)$ as $t \to +\infty^{33}$.

First, we recall from Claims 3.7.7 and 3.7.8 that there is a rectangular region $R =$

\[^{33}\text{On the plane, the structure of non-empty } \omega \text{-limit sets was discussed by Proposition 1.5 in Section 1.5.}\]
\{(p_1, p_2) : \varepsilon_i \leq p_i \leq M_i\} in the positive quadrant within which the solution gets trapped.

Incidentally, this fact together with Poincaré’s theory of indices for singular points\(^{34}\), implies that \(R\) contains equilibria; i.e., \(E_R \neq \emptyset\); recall that, by virtue of our assumptions on excess demands, \((p_1, p_2, 1) \in E \iff (p_1, p_2) \in E_R\).

We shall assume now the following:

\textbf{C i.} Trace of the Jacobian \(J(h_1, h_2)\) is not identically zero on \(R\) nor does it change sign on \(R\).

\textbf{C ii.} On the set \(E_R\), the Jacobian \(J(h_1, h_2)\) has a non-zero trace and a non-zero determinant.

Notice that while Olech (1963) demands an unique equilibrium, we do not. They demand a lot of other restrictions as well\(^{35}\). We have of course the properties of the excess demand function in A which have helped us to isolate a region such as \(R\); \textbf{C i} and \textbf{C ii} appear weaker than the requirements demanded in Olech (1963) and Ito (1978). \textbf{C ii} ensures that the equilibria in \(E_R\) have characteristic roots with real parts non-zero: this ensures that all equilibria or fixed points for the dynamic system (3.20) are hyperbolic or nondegenerate or simple. It follows that \(E_R\) contains a finite odd number of equilibria since the sum of the indices of all must add up to +1\(^{36}\).

\textbf{Proposition 3.7} Under A, B and C, for any \(p^0 \in R\), \(L_\omega(p^0) = p^* \in E_R\). Thus all solutions converge to an equilibrium.

\textbf{Proof:} Consider any \(p^0 \in R\) and the trajectory \(\phi_t(p^0)\): the solution to (3.20); by virtue

\(^{34}\)See, for instance, Section 1.5.

\(^{35}\)See, for example conditions listed as O1-O4, in Section 1.5 and Proposition 1.6.

\(^{36}\)See, for instance, Andronov et. al. (1966), p. 305.
of the Claim ??, the \( \omega \)-limit set \( L_\omega(p^0) \) is not empty. Again by the criterion of Bendixson
\(^{37}\) C i implies that there can be no closed orbits in the region \( R \). Thus there can be no
limit cycle and hence the Poincaré-Bendixson Theorem implies that \( L_\omega(p^0) \cap E_R \neq \emptyset \). It
follows therefore that \( p^* \in L_\omega(p^0) \) for some \( p^* \in E_R \); consequently there is a subsequence
\( \{t_s\}, t_s \to +\infty \) as \( s \to +\infty \) such that \( \phi_{t_s}(p^0) \to p^* \) as \( s \to +\infty \).

Since we know that the only types of equilibria are focii, nodes and saddle-points, the
characteristic roots of the Jacobian \( J(h_1, h_2) \) at \( p^* \) have real parts either both positive or
negative, or they are real and of opposite signs, given C ii.

In the first case, there would be an open neighborhood \( N(p^*) \) which no trajectory or
solution could enter; consequently since our trajectory \( \phi_{t}(p^0) \) does enter every neighborhood
of \( p^* \), it follows that at \( p^* \), the characteristic roots of the Jacobian, if complex, have real parts
negative; and if real, then at least one must be negative. Thus the fixed point \( p^* \) is either
a sink or at worst, saddle-point. If it is a sink, then any trajectory once having entered a
small neighborhood of the equilibrium, can never leave. Consequently, the trajectory \( \phi_{t}(p^0) \)
has no other limit point. Thus \( L_\omega(p^0) = p^* \). In the case of a saddle-point, there is only a
single trajectory which converges to the equilibrium; if \( p^0 \) happens to be on this trajectory,
\( L_\omega(p^0) = p^* \) but otherwise it is not possible for a trajectory to have a saddle-point as a limit
point. In any case therefore, the trajectory must converge to an equilibrium, as claimed.

We provide, next a set of remarks which highlight the implications of the above result.

\(^{37}\) See, for example, Remark 7.
Remark 17 The above result provides a set of conditions under which an adjustment on prices on disequilibrium, in the direction of excess demand, will always lead to an equilibrium. Notice also that these conditions guarantee that there will always be at least one sink i.e., an equilibrium at which the Jacobian has characteristic roots with real parts negative. To see this note that if no such equilibrium existed, then the only equilibria are saddle-points and sources. Also in aggregate they are finite in number and moreover, as argued above, no trajectory can come close to sources; so the only possibility for a limit is a saddle-point; but each saddle-point has only one trajectory leading to it and there are an infinite number of possible trajectories. Thus there must be a sink.

More importantly:

Proposition 3.8 Under \( A \), \( B \) and \( C \), if there is a unique equilibrium, it must be globally asymptotically stable.

Remark 18 As mentioned above, there must be at least one equilibrium where the characteristic roots have real parts negative. Hence the trace of the Jacobian at that equilibrium must be negative; further, since the trace at that equilibrium will be negative and the trace cannot change sign nor can it be zero at equilibria, it follows that the trace of the Jacobian at every equilibrium must be negative.

Consequently, we have:

Proposition 3.9 Under \( A \), \( B \) and \( C \), at every equilibrium, the sum of the characteristic roots of the Jacobian will be negative\(^{38}\).

\(^{38}\)Thus, if roots are complex, the real parts must be negative.
Remark 19 If we consider $h_i(p_1, p_2)$ to have the same sign as $Z_i(p_1, p_2, 1), i = 1, 2$ then the assumptions in C are restrictions placed on the functions $h_i$. Of course these become difficult to interpret. One may show that the Jacobian of $(h_1, h_2)$ at equilibria is related to the Jacobian of $(Z_1, Z_2)$, where the partial derivatives are with respect to $(p_1, p_2)$, again at equilibria by means of the following:

\[
\begin{pmatrix}
h_{11} & h_{12} \\
h_{21} & h_{22}
\end{pmatrix} = \begin{pmatrix}
Z_{11} & Z_{12} \\
Z_{21} & Z_{22}
\end{pmatrix} \cdot \begin{pmatrix}
d_1 & 0 \\
0 & d_2
\end{pmatrix}
\]

where all partial derivatives are evaluated at an equilibrium and $d_i > 0, i = 1, 2$ are some positive numbers. This would provide some link between the equilibria for the dynamic process and equilibria for the economy.

Let us reconsider the system (3.19); assume that the set of equilibria for this system $E = \{(x, y) : f(x, y) = 0, g(x, y) = 0\}$ is non-empty. The following general result follows from our analysis:

**Proposition 3.10** If

i. There is a rectangular region $R = \{(x, y) : 0 \leq x \leq M, 0 \leq y \leq N\}$ such that any trajectory of (3.19) on the boundary of $R$ is either inward pointing or coincides with the boundary;

ii. Trace of $J(f, g)$ is not identically zero and does not change sign in the positive quadrant;

iii. On the set $E$, the trace and determinant of the Jacobian $J(f, g)$ do not vanish;

then any trajectory $\phi_t(x^o, y^o)$ where $(x^o, y^o) > (0, 0)$ converges to a point of $E$.

A final remark considers the weakening of the assumption C i.
Remark 20  If we can find a function $\theta(p_1, p_2)$ which is continuously differentiable on the region $R$ and for which

$$\frac{\partial \theta(p_1, p_2)h_1(p_1, p_2)}{\partial p_1} + \frac{\partial \theta(p_1, p_2)h_2(p_1, p_2)}{\partial p_2}$$

is of constant sign on $R$, then there is no closed orbit for the system (3.20) on the region $R^{39}$.

In some situations, the above may provide a weakening of the condition $C_1$. It may be recalled that the sole purpose of $C_1$ was to rule out closed orbits in $R$. If, for example, $h_i(p_1, p_2) = Z_i(p_1, p_2, 1) = p_i g_i(p_1, p_2), i = 1, 2$, then we may replace $C_1$ by requiring that $p_1 g_1 (p_1, p_2) + p_2 g_2 (p_1, p_2)$ be of constant sign on $R$; note that we do not require the trace of $J (p_1 g_1, p_2 g_2)$ being constant on $R$. This follows by virtue of the fact that we may consider $\theta(p_1, p_2) = p_1^{-1} p_2^{-1}$ and then the condition in Remark 7 is satisfied for this choice of $\theta(p_1, p_2)$.

A more recent paper Anderson et. al. (2003), makes the important and significant contribution made that experiments conducted with agents with similar preferences and endowments, but engaging in double auctions would lead to price movements which are predicted by the tatonnement model. Thus the results provided by the tatonnement process, they argue, should be looked at with greater care because they seem to predict what price adjustments might actually occur.

As we showed in our analysis of the Scarf Example, the perturbation allowed us to get

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$^{39}$See Dulac's criterion, discussed in Section 1.5, Remark 7.

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rid of closed orbits; for convergence, we needed to show that the solution was bounded. One of the reasons for our being able to obtain such a different result was due to the fact that at the original equilibrium, the relevant matrix had purely complex characteristic roots, with zero real parts. It is not surprising that in such a situation, a perturbation changed the real parts of the characteristic root from zero to positive or negative.

Notice that $C_{ii}$ rules out the Jacobian of the excess demand functions from having characteristic roots with zero real part or from being singular; both serve to ensure that the properties we observe are robust; non-singularity of the Jacobian, some times called regularity preserves static properties of the equilibria of the economic system for small changes in parameters; the trace being non-zero at equilibrium, preserves the dynamic properties from small changes in parameters. While $C_{i}$ rules out the trace of the Jacobian from changing signs on the positive quadrant which eliminates cycles. Thus $C_{i}$ rules out periodic behavior and $C_{ii}$ ensures robustness. These two together imply that the process will always lead to an equilibrium, provided trajectories are bounded; the particular equilibrium approached will depend on the initial configuration of prices, of course. It is also important to note that if there is a unique equilibrium, then that has to be globally asymptotically stable. Thus the feature of the original Scarf example, of a unique equilibrium which cannot be attained, is removed. However, these conclusions are for motion on the plane. Their interest lie in the fact that in many applications in economics, only such motions are considered.

In Hicks (1946), there is an enquiry relating to the following questions: if a market is stable by itself, can it be rendered unstable from the price adjustment in other markets
? Alternatively, if a market is unstable when taken by itself, can it be rendered stable by the price adjustment in the other markets? To both an answer was provided in the negative. Notice that \( C_i \) essentially ensured (together with \( C_{ii} \)) that the trace of the relevant Jacobian remained negative; notice that this would be implied by assuming that \( Z_{ii} < 0 \) for each \( i \), that is when each market when taken in isolation, was stable. This in turn has been seen to imply that the markets together must also be globally stable. Under certain conditions, \( C_i \) may be weakened further; this involves the existence of a function \( \theta() \) satisfying Dulac’s criterion, as in the case of the perturbation of the Scarf example. But this is a matter of serendipity rather than design.

As we have seen, general results in this area are difficult to obtain. This is mainly due to two reasons: first of all, the excess demand functions are not expected, \textit{a priori} to satisfy any other property apart from Homogeneity of degree zero in the prices and Walras Law; secondly, dynamics in dimensions greater than 2 may be quite difficult to pin down. Even on the plane, a variety of dynamic motions are possible. It is not surprising that in higher dimensions matters become a lot more complicated and Walras Law and Homogeneity do not help too much. We have just seen what these considerations will lead us to. And consequently, we must impose additional restrictions which may be called \textbf{global stability conditions}; we have shown that an easy such condition for the Scarf example is \( b < 1 \). For the general case, on the plane, the conditions in \( C \) serve the same purpose.