Discussion Paper No. 628

MAXIMAL DOMAIN FOR STRATEGY-PROOF RULES IN ALLOTMENT ECONOMIES*

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February 2005

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*Forthcoming in *Social Choice and Welfare*

Maximal Domain for Strategy-Proof Rules in Allotment Economies[1]

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January 19, 2005

We consider the problem of allocating an amount of a perfectly divisible good among a group of n agents. We study how large a preference domain can be to allow for the existence of *strategy-proof, symmetric,* and *efficient* allocation rules when the amount of the good is a variable. This question is qualified by an additional requirement that a domain should include a *minimally rich domain.* We first characterize the *uniform rule (Bennasy, 1982)* as the unique *strategy-proof, symmetric,* and *efficient* rule on a *minimally rich domain* when the amount of the good is fixed. Then, exploiting this characterization, we establish the following: There is a unique *maximal domain* that includes a *minimally rich domain* and allows for the existence of *strategy-proof, symmetric*, and *efficient* rules when the amount of good is a variable. It is the *single-plateaued* domain.

[1] The authors thank Professors Miyake, M., and Thomson, W. and two anonymous referee for their helpful and detailed suggestions. They also thank Asano, T. and Suzuki, M. for their useful comments. The authors completed this article when Serizawa visited the Department of Economics at the University of British Columbia. He appreciates the good research environment of the department.

1. INTRODUCTION

We consider the problem of allocating an endowment of a perfectly divisible good among a group of n agents. An allotment economy is a pair of a preference profile and an amount of the good. A "rule" chooses a feasible allocation for each allotment economy; it is formally defined as a function from the class of allotment economies to the set of allocations. When the endowment is fixed, an allotment economy is described by a preference profile alone. A class of preference profiles are called a "preference domain." Since preferences are only privately known, the choices of rules must be based on preferences agents announce. Thus, agents might strategically misrepresent their preferences to obtain outcomes they prefer. As a result, the decision that the rule should make for agents' true preferences may not be realized. Thus, the condition called "strategy-proofness" is often imposed on rules to guarantee that agents have the incentive to reveal their true preferences. It says that no agent is better off by misrepresenting his preferences, no matter what his true preference is and no matter what preferences others announce. "Symmetry" and Pareto efficiency are also often imposed on rules. "Symmetry" is the distributional requirement that two agents with the same preference should be given indifferent amounts of the good.

If a rule satisfies strategy-proofness on a preference domain, it also satisfies strategy-proofness on any subdomain. Therefore, the larger the domain on which rules are required to satisfy strategy-proofness, the stronger the requirement is. For instance, in the model of public alternatives, it is well-known that strategy-proofness is so strong on the universal domain (the class of all preferences on the set of the public alternatives) that only dictatorships can satisfy strategy-proofness.[2] Similarly, the requirements of efficiency and symmetry are stronger on larger domains. Thus, the smaller a domain is, the more rules satisfying the three requirements potentially exist. On the other hand, if the domain is too large, the three requirements may become so strong that there exist no rules satisfying them. In this paper, we identify (i) minimal domains on which the three requirements imply uniqueness of the rule, and (ii) maximal domains on which rules satisfying the three requirements exist.

Many authors have analyzed strategy-proof rules on the class of "single-peaked" preferences. A preferences is "single-peaked" if more is preferred to less up to some level, called the "peak", and less is preferred to more beyond the peak. The single-peaked domain is the set of all single-peaked preferences. By fixing the endowment, Sprumont (1991) and Ching (1994) establish that there exists a unique rule satisfying strategy-proofness, symmetry, and efficiency on the single-peaked domain, the "uniform rule".[3] The "uniform rule" allocates the total endowment to agents as follows. When the sum of agents' peaks is greater than or equal to the total endowment, an agent gets his peak if that level is less than the common upper bound; otherwise he receives the common bound; and the common bound is chosen so as to satisfy the feasibility. When the sum of agents' peaks is less than the total endowment, the opposite principle is applied, that is, an agent gets his peak if that level is more than the common lower bound; and so on. The results by Sprumont (1991) and Ching (1994) motivate us to investigate the following two questions.

[2] This is the so-called Gibbard-Satterthwaite Theorem. (Gibbard, 1973, and Satterthwaite, 1975).

[3] First, Sprumont (1991) establishes that the uniform rule is the unique rule satisfying strategy-proofness, "anonymity", and (Pareto) efficiency on the single-peaked domain. "Anonymity" is the distributional requirement that the names of agents should not matter. Later, Ching (1994) strengthens Sprumont's (1991) characterization by replacing anonymity with the weaker requirement of symmetry.

The first question is how much we can shrink the domain of single-peaked preferences while preserving the uniform rule as the unique rule satisfying strategy-proofness, symmetry, and efficiency. A "minimally rich domain" is a small subset of the single-peaked domain satisfying the following two conditions: (i) for each consumption level, there exists only one preference whose peak coincides with the consumption level; (ii) given two distinct consumption levels, say x and y, there exists at least one preference whose peak is between x and y such that x is preferred to y. The "symmetric"[4] domain is an example of the minimally rich domain. Since any minimally rich domain is much smaller than the single-peaked domain, strategy-proofness on a minimally rich domain is weaker than strategy-proofness on the single-peaked domain, and so are efficiency and symmetry. Therefore, potentially there exist more rules satisfying the weaker requirements. However, we show that the uniform rule is still the unique rule satisfying the three properties on a minimally rich domain. We establish that *a rule on a minimally rich domain satisfies strategy-proofness, symmetry, and efficiency if and only if it is the uniform rule.* Following Sprumont (1991) and Ching (1994), we first fix the endowment and establish this characterization.

The second question is how much we can enlarge the preference domain while allowing for the existence of strategy-proof, symmetric, and efficient rules. Ching and Serizawa (1998) also study the same question. Ching and Serizawa (1998) show that the "single-plateaued" domain is the unique maximal preference domain that includes the single-peaked domain and allows for the existence of strategy-proof, symmetric, and efficient rules. "Single-plateaued" preferences are variants of single-peaked preferences for which the sets of most preferred consumption levels are intervals.[5] The single-plateaued domain is the set of all single-plateaued preferences. The setting of Ching and Serizawa (1998) is different from Sprumont (1991) and Ching (1994). Ching and Serizawa (1998) consider the situation in which rules have the amount of the good to be allocated as a variable and each economy is represented by a pair of one preference profile and the endowment of the good. We adapt the same setting as Ching and Serizawa (1998) in studying the second question, and establish a similar result. However, our result is stronger than theirs for the following reason. Ching and Serizawa (1998) require that the maximal domain should include the single-peaked domain. Here, we only require the maximal domain to include a minimally rich domain. Since any minimally rich domain is much smaller than the single-peaked domain, our requirement is weaker than Ching and Serizawa's (1998). Therefore, potentially there may exist a domain which includes a minimally rich domain and allows for the existence of strategy-proof, symmetric, and efficient rules, but is larger than or different from the single-plateaued domain. However, we establish that *the single-plateaued domain is still the unique maximal domain including a minimally rich domain for strategy-proofness, symmetry, and efficiency.* Our result implies Ching and Serizawa's (1998) result as a corollary.

Recently, Massó and Neme (2001) succeed in obtaining a maximal domain result when the amount of the good to be allocated is fixed. Since they adopt "strong symmetry"[6] as

[4]A single-peaked preference is "symmetric" if its utility representation is symmetric around the peak. Symmetric preferences are also called "quadratic" preferences in Border and Jordan (1983) since they are represented by quadratic utility functions.

[5]It is studied by Moulin (1984) and Berga (1998).

[6]Strong symmetry means that agents with the same preferences are given the same amount of the good.

a property of rules in place of symmetry in our result, and since they require the maximal domain to include the single-peaked domain, our result (Theorem 2) is independent of theirs.

Besides Ching and Serizawa (1998) and Massó and Neme (2001), there are several articles that study the maximal domain for strategy-proofness. Massó and Neme (2004) and Ching and Serizawa (2003) study the maximal domain for strategy-proof rules in allotment economies by assuming the continuity of rules and establish different maximal domain results.[7] Barberà, Sonnenschein, and Zhou (1991) and Serizawa (1995) study models of public alternatives and give the maximal domain where the class of rules called "the generalized median voter schemes" satisfies strategy-proofness. While they exclude rules other than the generalized median voter schemes, we do not restrict the rules a priori, but obtain the maximal domain by just imposing properties on rules. Berga and Serizawa (2000) also investigate the two questions that are parallel to ours in a model of public alternatives. Recently, Ehlers (2002) studies maximal domains for coalitional strategy-proofness in an indivisible goods model.

This paper is organized as follows. Section 2 explains the model and the main results. Section 3 is devoted to the proof of the results of Section 2. Section 4 raises open questions.

2. THE MODEL AND THE RESULTS

We consider the problem of allocating one perfectly divisible private good among a finite number of agents. Let $N = \{1, \cdots, n\}$ be the set of agents, where $n \geq 2$. Let $M \in \mathbb{R}_{++}$ be the amount of the good. Each agent $i \in N$ is equipped with a preference relation R_i on $\mathbb{R}_+ \cup \{\infty\}$. Let P_i be the strict preference relation associated with R_i, and I_i the indifference preference relation. We assume that preferences are continuous, that is, for all $x \in \mathbb{R}_+ \cup \{\infty\}$, the sets $\{y \in \mathbb{R}_+ \cup \{\infty\} : yR_ix\}$ and $\{y \in \mathbb{R}_+ \cup \{\infty\} : xR_iy\}$ are closed. Let $\mathcal{R}_C$ be the class of all continuous preferences. Given $R_i \in \mathcal{R}_C$, let $p(R_i) = \{x \in \mathbb{R}_+ \cup \{\infty\} | \forall y \in \mathbb{R}_+, xR_iy\}$ be the set of preferred consumptions for R_i. Let $\underline{p}(R_i) = \inf p(R_i)$ and $\overline{p}(R_i) = \sup p(R_i)$. When $p(R_i)$ is a singleton, we slightly abuse notation and use $p(R_i)$ to denote its single element. A preference profile is a list $R = (R_1, \cdots, R_n) \in \mathcal{R}_C^n$. When we emphasize the role of agent $i \in N$, we write $R = (R_i, R_{-i})$ where $R_{-i} = (R_j)_{j \in N \setminus \{i\}}$. An *economy* is a pair of $(R, M) \in \mathcal{R}_C^n \times \mathbb{R}_{++}$.

When the amount of the good is M, an allocation is a vector $z = (z_1, \cdots, z_n) \in \mathbb{R}_+^n$ such that $\sum_{i \in N} z_i = M$. Let $Z(M)$ be the set of all allocations for the economy (R, M). A *rule* associates to each economy (R, M) an allocation $z \in Z(M)$. It can be regarded as a recommendation for each economy. A *domain* is a subset $\mathcal{R}$ of $\mathcal{R}_C$. A *rule* is a function $\varphi : \mathcal{R}^n \times \mathbb{R}_{++} \to \mathbb{R}_+^n$ such that for all $(R, M) \in \mathcal{R}^n \times \mathbb{R}_{++}, \varphi(R, M) \in Z(M)$. For all $i \in N$, $\varphi_i(R, M)$ represents the amount of the good allocated to agent i. When we want to emphasize the domain of a rule, we call it a rule on $\mathcal{R}^n \times \mathbb{R}_{++}$

Agents are assumed to be equipped with the following preferences.

Definition 1. A preference $R_0 \in \mathcal{R}_C$ is *single-peaked* if $p(R_0)$ is a singleton, and for all $x, y \in \mathbb{R}_+ \cup \{\infty\}$, we have xP_0y whenever $y < x \leq p(R_0)$ or $p(R_0) \leq x < y$.

Let $\mathcal{R}_S \subseteq \mathcal{R}_C$ be the domain of all single-peaked preferences. We call it the single-peaked domain.

[7] Massó and Neme (2004) are only concerned with tops-only rules and Ching and Serizawa (2003) use unanimity instead of efficiency.

DEFINITION 2. A single peaked preference $R_0 \in \mathcal{R}_C$ is *symmetric* if for all $x, y \in \mathbb{R}_+$, xP_0y if and only if $|x - p(R_0)| < |y - p(R_0)|$.

Let $\mathcal{R}_M \subseteq \mathcal{R}_C$ be the domain of all symmetric preferences. We call it the symmetric domain.

DEFINITION 3. A preference $R_0 \in \mathcal{R}_C$ is *single-plateaued* if $p(R_0)$ is an interval $[\underline{p}(R_0), \overline{p}(R_0)]$; and for all $x, y \in \mathbb{R}_+ \cup \{\infty\}$ with $[x < y \leq \underline{p}(R_0)]$ or $[\overline{p}(R_0) \leq x < y]$, we have xP_0y.

Let $\mathcal{R}_P \subseteq \mathcal{R}_C$ be the domain of all single-plateaued preferences. We call it the single-plateaued domain.

Rules are required to satisfy the following three properties. The first one is the strongest incentive compatibility. No one can ever benefit by misrepresenting his preferences.

DEFINITION 4. A rule φ on $\mathcal{R}^n \times \mathbb{R}_{++}$ is *strategy-proof* if for all $(R, M) \in \mathcal{R}^n \times \mathbb{R}_{++}$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R, M)R_i\varphi_i(R'_i, R_{-i}, M)$.

The second property is a distributional requirement. If two agents have the same preference, their allocations are indifferent to each other.

DEFINITION 5. A rule φ on $\mathcal{R}^n \times \mathbb{R}_{++}$ is *symmetric* if for all $(R, M) \in \mathcal{R}^n \times \mathbb{R}_{++}$ and all $i, j \in N$ such that $R_i = R_j$, $\varphi_i(R, M)I_i\varphi_j(R, M)$.

The third property is the standard efficiency requirement.

DEFINITION 6. A rule φ on $\mathcal{R}^n \times \mathbb{R}_{++}$ is *efficient* if for all $(R, M) \in \mathcal{R}^n \times \mathbb{R}_{++}$, there is no $z \in Z(M)$ such that for all $i \in N$, $z_iR_i\varphi_i(R, M)$ and for some $j \in N$, $z_jP_j\varphi_j(R, M)$.

The fact below is derived directly from the definition of efficiency (Definition 6). It is useful in the proofs of Lemmas 1, 4, and Theorem 1.

FACT. Let $\mathcal{R} \subseteq \mathcal{R}_S \subseteq \mathcal{R}_C$. If a *rule* φ on $\mathcal{R}^n \times \mathbb{R}_{++}$ is *efficient*, then for all $(R, M) \in \mathcal{R}^n \times \mathbb{R}_{++}$, the following properties hold.

(i) If $\sum_{i\in N} p(R_i) \leq M$, then for all $j \in N$, $p(R_j) \leq \varphi_j(R)$.
If $M \leq \sum_{i\in N} p(R_i)$, then for all $j \in N$, $\varphi_j(R) \leq p(R_j)$.

(ii) If there exists an agent $i \in I$ such that $p(R_i) < \varphi_i(R)$, then $\sum_{j\in N} p(R_j) < M$.
If there exists an agent $i \in I$ such that $\varphi_i(R) < p(R_i)$, then $M < \sum_{j\in N} p(R_j)$.

(iii) If there exists an agent $i \in I$ such that $\varphi_i(R) < p(R_i)$, then for all agent $j \in I$, $\varphi_j(R) \leq p(R_j)$.
If there exists an agent $i \in I$ such that $p(R_i) < \varphi_i(R)$, then for all agent $j \in I$, $p(R_j) \leq \varphi_j(R)$.

2. 1. *Characterization*

Sprumont (1991) and Ching (1994) show that the following rule, the uniform rule, is the only one satisfying all three properties of strategy-proofness, symmetry, and efficiency on the single-peaked domain, $\mathcal{R}^n_S$.

DEFINITION 7. The *uniform rule* $U = (U_1, \cdots, U_n)$ is defined as follows. For all $(R, M) \in \mathcal{R}^n_S \times \mathbb{R}_{++}$ and all $i \in N$,

$$U_i(R) = \begin{cases} \max\{p(R_i), \lambda(R, M)\} & \text{if } \sum p(R_j) \leq M \\ \min\{p(R_i), \lambda(R, M)\} & \text{otherwise,} \end{cases}$$

where $\lambda(R, M)$ solves $\sum_{j \in N} U_j(R, M) = M$.[8]

How much can we weaken the assumption of strategy-proofness, symmetry, and efficiency on the single-peaked domain, while preserving the uniform rule as the unique rule satisfying strategy-proofness, symmetry, and efficiency? By the definition of strategy-proofness, if a rule satisfies strategy-proofness on a domain, it also satisfies strategy-proofness on any subdomain. Therefore the smaller domain (in the sense of inclusion) on which we assume a rule satisfies strategy-proofness, the less demanding we are. Similarly, the conditions of efficiency and symmetry are weaker on the smaller domains. We characterize the uniform rule on the following minimally rich domains.[9] Since any minimally rich domain is much smaller than the single-peaked domain, our characterization is stronger than Sprumont (1991) and Ching (1994).

DEFINITION 8. A domain $\mathcal{R} \subseteq \mathcal{R}_C$ is *minimally rich* if
(1) $\mathcal{R} \subseteq \mathcal{R}_S$
(2) for all $x \in \mathbb{R}_+$, there exists a unique preference $R_0 \in \mathcal{R}$ such that $p(R_0) = x$, and
(3) for all $x, y \in \mathbb{R}_+$ such that $x \neq y$, there exists $R_0 \in \mathcal{R}$ such that xP_0y and $p(R_0) \in (\min\{x, y\}, \max\{x, y\})$.

We denote a generic minimally rich domain by $\mathcal{R}_R$. The symmetric domain is an example of minimally rich domain. The domain $\mathcal{R}_0$ defined below is one of many other minimally rich domains.

$\mathcal{R}_0 = \{R_0 \in \mathcal{R}_S$: for all $x, y \in [0, M]$ such that $x < p(R_0) < y$, xI_0y if and only if $|x - p(R_0)| = 2|y - p(R_0)|\}$.

Now we establish a characterization of the uniform rule on minimally rich domains. Sprumont (1991) and Ching (1994) characterize the uniform rule when M is fixed. Their results also imply the characterization when M is not fixed. Therefore, we first show our characterization when M is fixed. The characterization when M is not fixed follows.

Given $M \in \mathbb{R}_{++}$, we define $\mathcal{R}_S(M)$, $\mathcal{R}_M(M)$, and $\mathcal{R}_R(M)$ as the set of preferences obtained by restricting on $[0, M]$ all preferences in $\mathcal{R}_S$, $\mathcal{R}_M$, and $\mathcal{R}_R$ respectively. We denote a generic element of $\mathcal{R}_S(M)$ by $R_i(M)$. When M is fixed, we consider that agent i's preference is defined only on $[0, M]$, an economy is represented by a list $R(M) = (R_1(M), \cdots, R_n(M)) \in \mathcal{R}(M)^n$, and that when a domain is a subset $\mathcal{R}(M)$ of $\mathcal{R}_S(M)$, a rule is a function $\varphi(\cdot, M) : \mathcal{R}(M)^n \to Z(M)$. When we want to emphasize the domain of a rule, we call it a rule on $\mathcal{R}(M)^n$. Strategy-proofness, symmetry, and efficiency are similarly defined in this setting. Thus, we omit their definitions.

<u>THEOREM 1.</u> *Let $M \in \mathbb{R}_{++}$. A rule $\varphi(\cdot, M)$ on a minimally rich domain $\mathcal{R}_R(M)^n$ is strategy-proof, symmetric, and efficient if and only if $\varphi(R(M), M) = U(R(M), M)$ for all $R(M) \in \mathcal{R}_R(M)^n$.*

Theorem 1 says that when M is fixed, the uniform rule is the unique rule satisfying strategy-proofness, symmetry, and efficiency on a minimally rich domain. The characterization when M is not fixed is a corollary of Theorem 1.

<u>COROLLARY 1.</u> *A rule φ on a minimally rich domain $\mathcal{R}_R^n \times \mathbb{R}_{++}$ is strategy-proof, symmetric, and efficient if and only if $\varphi(R, M) = U(R, M)$ for all $(R, M) \in \mathcal{R}_R^n \times \mathbb{R}_{++}$.*

[8] The uniform rule can be regarded as a system of equations. Therefore, $\lambda(R, M)$ is endogenously determined and depends on a profile of agents' preferences and the total amount of the good to be allocated. Sönmez (1994) establishes the algorithm to find $\lambda(R, M)$.

[9] Minimally rich domains are firstly studied by Berga and Serizawa (2000).

Since the symmetric domain is a minimally rich domain, the next corollary is also obtained.

COROLLARY 2. *A rule φ on the symmetric domain $\mathcal{R}_M^n \times \mathbb{R}_{++}$ is strategy-proof, symmetric, and efficient if and only if $\varphi(R, M) = U(R, M)$ for all $(R, M) \in \mathcal{R}_M^n \times \mathbb{R}_{++}$.*

2.2. *Maximal Domain*

We proceed to the next question; how much larger can a preference domain be while allowing for the existence of strategy-proof, symmetric, and efficient rules? To answer this question precisely, we need the following notion.

DEFINITION 9. A domain $\mathcal{R}_m$ is a *maximal domain* for a list of properties if (i) $\mathcal{R}_m \subseteq \mathcal{R}_C$, (ii) there is a *rule* on $\mathcal{R}_m^n \times \mathbb{R}_{++}$ satisfying the properties; and (iii) there is no rule satisfying the same properties on any $\mathcal{R}_A^n \times \mathbb{R}_{++}$ such that $\mathcal{R}_m^n \subsetneq \mathcal{R}_A^n \subseteq \mathcal{R}_C^n$.

Note that a maximal domain for a list of properties may not be unique. However, Ching and Serizawa (1998) show that the single-plateaued domain is the unique maximal domain including the single-peaked domain for strategy-proofness, symmetry, and efficiency. Any domain that includes the single-peaked domain also includes a minimally rich domain. Thus the set of domains that includes the single-peaked domain is contained by the set of domains including a minimally rich domain. Theorem 2 below says that even among the more candidate, the single-plateaued domain still remains the unique maximal domain. Theorem 2 strengthens Ching and Serizawa's result (1998).

THEOREM 2. *The single-plateaued domain is the unique maximal domain that includes a minimally rich domain for strategy-proofness, symmetry, and efficiency.*

Note that we obtain Ching and Serizawa's result (1998) as a corollary of Theorem 2.

COROLLARY 3 (Ching and Serizawa 1998). *The single-plateaued domain is the unique maximal domain including the single-peaked domain for strategy-proofness, symmetry, and efficiency.*

Since the symmetric domain is a minimally rich domain, we also have the following corollary.

COROLLARY 4. *The single-plateaued domain is the unique maximal domain including the symmetric domain for strategy-proofness, symmetry, and efficiency.*

3. PROOFS

3. 1. *Proofs for Theorem 1*

For the proofs, we introduce the notion of "the option set." In the model of public alternatives, Berga and Serizawa (2000) employed the notion in their proof of characterization. Similarly to them, we prove that option sets are convex in allotment economies. Our proof of Theorem 1 is similar to Ching's proof (1994) in structure. However, because a minimally rich domain is smaller than the single-peaked domain, his procedure cannot be applied throughout our proof. The convexity of option sets plays an important role in overcoming that difficulty, relevant in Lemma 5. Although our proofs of Lemmas 1, 2, 3, and Theorem 1 are similar to the proofs of Sprumont (1991) and Ching (1994), we state all proofs for completeness.

Before stating all the previous results, let us simplify notation. In this subsection, M is fixed. Thus each economy is represented by a profile of preferences. To simplify notation, we denote $\mathcal{R}_S(M)$, $\mathcal{R}_R(M)$, and $\mathcal{R}_M(M)$ by $\mathcal{R}_S$, $\mathcal{R}_R$, and $\mathcal{R}_M$ respectively throughout this subsection. We denote a generic element of $\mathcal{R}(M)$ by R instead of $R(M)$. For the same reason, $\varphi(R, M)$ and $U(R, M)$ are replaced by $\varphi(R)$ and $U(R)$ respectively.

Ching (1994) shows that strategy-proof rules on the single-peaked domain satisfy the property of "own peak monotonicity." It says that if the peak of an agent decreases, his share does not increase. Similarly to him, we can show that the uniform rule is own peak monotonic on any minimally rich domain.

LEMMA 1 ("Own Peak Monotonicity"). *If a rule φ on $\mathcal{R}_R^n$ is efficient and strategy-proof, then for all $R \in \mathcal{R}_R^n$, all $i \in N$, and all $R'_i \in \mathcal{R}_R$ such that $p(R'_i) \leq p(R_i)$, $\varphi_i(R'_i, R_{-i}) \leq \varphi_i(R)$.*

Proof. Let $R \in \mathcal{R}_R^n$, $i \in N$ and $R'_i \in \mathcal{R}_R$ be such that $p(R'_i) \leq p(R_i)$. By contradiction, suppose that $\varphi_i(R) < \varphi_i(R'_i, R_{-i})$.

First suppose that $\sum_{j \in N} p(R_j) \leq M$. By Fact (i), for all $j \in N$, $p(R_j) \leq \varphi_j(R)$. Since $p(R'_i) \leq p(R_i)$, $p(R'_i) \leq p(R_i) \leq \varphi_i(R) < \varphi_i(R'_i, R_{-i})$. Thus $\varphi_i(R) P'_i \varphi_i(R'_i, R_{-i})$. So agent i manipulates φ at (R'_i, R_{-i}) via R_i, contradicting strategy-proofness.

Next, suppose that $\sum_{j \in N} p(R_j) > M$. There are two subcases.

SUBCASE 1. $M \leq p(R'_i) + \sum_{j \neq i} p(R_j)$.

By Fact (i), $\varphi_i(R'_i, R_{-i}) \leq p(R'_i)$. Since $\varphi_i(R) < \varphi_i(R'_i, R_{-i})$ and $p(R'_i) \leq p(R_i)$, $\varphi_i(R) < \varphi_i(R'_i, R_{-i}) \leq p(R'_i) \leq p(R_i)$. Thus $\varphi_i(R'_i, R_{-i}) P_i \varphi_i(R)$. Therefore agent i manipulates φ at R via R'_i, contradicting strategy-proofness.

SUBCASE 2. $M > p(R'_i) + \sum_{j \neq i} p(R_j)$.

By Fact (i), for all $j \in N \backslash \{i\}$, $\varphi_j(R) \leq p(R_j)$. Thus $\sum_{j \neq i} \varphi_j(R) \leq \sum_{j \neq i} p(R_j)$. Since $p(R'_i) + \sum_{j \neq i} p(R_j) < M$, $p(R'_i) < M - \sum_{j \neq i} \varphi_j(R) = \varphi_i(R)$. Since $\varphi_i(R) < \varphi_i(R'_i, R_{-i})$, $p(R'_i) < \varphi_i(R) < \varphi_i(R'_i, R_{-i})$. Thus $\varphi_i(R) P'_i \varphi_i(R'_i, R_{-i})$. Therefore agent i manipulates φ at (R'_i, R_{-i}) via R_i, contradicting strategy-proofness.

We obtain contradictions in all cases. Therefore, $\varphi_i(R'_i, R_{-i}) \leq \varphi_i(R)$.

Q. E. D.

We introduce the notion of the option set.

DEFINITION 11. Given $i \in N$, $R_{-i} \in \mathcal{R}_R^{n-1}$, and a rule φ on $\mathcal{R}_R^n$, agent i's *option set* at R_{-i} is the set $\sigma_i(R_{-i}) = \{x \in [0, M] : \exists R_i \in \mathcal{R}_R \text{ such that } \varphi_i(R_i, R_{-i}) = x\}$.

The following lemma is well-known and used for the proof of Lemma 3.

LEMMA 2. *Let a rule φ on $\mathcal{R}_R^n$ be strategy-proof. Let $R \in \mathcal{R}_R^n$ and $x \in \sigma_i(R_{-i})$ be such that for all $x' \in \sigma_i(R_{-i}) \backslash \{x\}$, $x P_i x'$. Then $\varphi_i(R) = x$.*

Proof. By contradiction, suppose that $\varphi_i(R) \neq x$. By definition, there exists $R'_i \in \mathcal{R}_R$ such that $\varphi_i(R'_i, R_{-i}) = x$. By assumption, $\varphi_i(R'_i, R_{-i}) P_i \varphi_i(R)$. Therefore agent i manipulates φ_i at R via R'_i, contradicting strategy-proofness. Therefore, $\varphi_i(R) = x$.

Q. E. D.

In allotment economies, Sprumont (1991) shows that option sets are closed on the single-peaked domain. We can prove the same property on a minimally rich domain as he does.

LEMMA 3 ("Closedness of Option Set"). *Let a rule φ on $\mathcal{R}_R^n$ be strategy-proof. Let $i \in N$, and $R_{-i} \in \mathcal{R}_R^{n-1}$. Then $\sigma_i(R_{-i})$ is closed.*

Proof. By contradiction, suppose that $\sigma_i(R_{-i})$ is not closed. Then, there exists a sequence $\{x_k\}_{k\in\mathbb{N}}$ in $\sigma_i(R_{-i})$ such that x_k converges to $x \in \mathbb{R}$ as k goes to infinity and $x \notin \sigma_i(R_{-i})$. Since $[0, M]$ is closed, $x \in [0, M]$. By the definition of minimally rich domain, there is $R_i' \in \mathcal{R}_R$ such that $p(R_i') = x$. Let $\varphi_i(R_i', R_{-i}) = x'$. Since $x \notin \sigma_i(R_{-i})$ and $x \neq x'$, $xP_i'x'$. Moreover, since $x = \lim_{k\to\infty} x_k$, and R_i' is continuous, there exists $k \in \mathbb{N}$ such that $x_kP_i'x'$. Since $x_k \in \sigma_i(R_{-i})$, there exists $R_i'' \in \mathcal{R}_R$ such that $\varphi_i(R_i'', R_{-i}) = x_k$. Thus agent i manipulates φ at (R_i', R_{-i}) via R_i'', contradicting strategy-proofness. Therefore, $\sigma_i(R_{-i})$ is closed.

Q. E. D.

In the model of public alternatives, Berga and Serizawa (2000) show that option sets are convex on any minimally rich domain. We prove the convexity of the option set in allotment economies.

LEMMA 4 ("Convexity of Option Set"). *Let φ be a strategy-proof and efficient rule φ on $\mathcal{R}_R^n$. Let $i \in N$, and $R_{-i} \in \mathcal{R}_R^{n-1}$. Then the set $\sigma_i(R_{-i})$ is convex.*

Proof. Suppose, by contradiction, that $\sigma_i(R_{-i})$ is not convex. Then there exist $x < z < y$ such that $x, y \in \sigma_i(R_{-i})$ and $z \notin \sigma_i(R_{-i})$. But the convexity of $[0, M]$ guarantees $[x, y] \subseteq [0, M]$. Let $x' = \sup\{x'' : x'' \in \sigma_i(R_{-i}) \text{ and } x'' < z\}$ and $y' = \inf\{y'' : y'' \in \sigma_i(R_{-i}) \text{ and } y'' > z\}$. By the closedness of $\sigma_i(R_{-i})$ (Lemma 3), $x', y' \in \sigma_i(R_{-i})$. By the definition of a minimally rich domain, there exist $R_i^*, R_i^{**} \in \mathcal{R}_R$ such that $x'P_i^*y'$, $p(R_i^*) \in (x', y')$, $y'P_i^{**}x'$ and $p(R_i^{**}) \in (x', y')$. Since $x'P_i^*x''$ for all $x'' \in \sigma_i(R_{-i})\backslash\{x'\}$ and $y'P_i^{**}y''$ for all $y'' \in \sigma_i(R_{-i})\backslash\{y'\}$, it follows from Lemma 2 that $\varphi_i(R_i^*, R_{-i}) = x'$ and $\varphi_i(R_i^{**}, R_{-i}) = y'$.

Since $\varphi_i(R_i^*, R_{-i}) < p(R_i^*)$, Fact (ii) guarantees $M < p(R_i^*) + \sum_{j\neq i} p(R_j)$. Since $p(R_i^{**}) < \varphi_i(R_i^{**}, R_{-i})$, Fact (ii) also guarantees $p(R_i^{**}) + \sum_{j\neq i} p(R_j) < M$. From these two inequalities, we obtain $p(R_i^{**}) < p(R_i^*)$. Then, by own peak monotonicity (Lemma 1), $\varphi_i(R_i^{**}, R_{-i}) \leq \varphi_i(R_i^*, R_{-i})$. However, since $x' < y'$, this is a contradiction. Therefore, $\sigma_i(R_{-i})$ is convex.

Q. E. D.

Ching (1994) establishes a property called "uncompromisingness" on the single-peaked domain. It says that if an agent's original share is too much (less) to him, he cannot change it by revealing any other preferences for which it is too much (less). Lemma 5 below says that the same conclusion holds on a minimally rich domain. The structure of our proof of Lemma 5 is also similar to Ching (1994). However, Ching's proof does not work here since preferences employed in his proof may not be in a minimally rich domain. We exploit the convexity of option sets (Lemma 4) to overcome this difficulty. Note that the uniform rule is uncompromising on any minimally rich domain.

LEMMA 5 ("Uncompromisingness"). *If a rule φ on $\mathcal{R}_R^n$ is efficient and strategy-proof, the following property holds. For all $R \in \mathcal{R}_R^n$, all $i \in N$, and all $R_i' \in \mathcal{R}_R$,*

if $p(R_i) < \varphi_i(R)$ *and* $p(R'_i) \leq \varphi_i(R)$, *then* $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$;
if $p(R_i) > \varphi_i(R)$ *and* $p(R'_i) \geq \varphi_i(R)$, *then* $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$.

Proof. We consider only the case when there exist $R \in \mathcal{R}_R^n$, $i \in N$, and $R'_i \in \mathcal{R}_R$ such that $p(R_i) < \varphi_i(R)$ and $p(R'_i) \leq \varphi_i(R)$. The other case can be treated symmetrically.

By contradiction, suppose that $\varphi_i(R'_i, R_{-i}) \neq \varphi_i(R)$. There are two cases.

CASE 1. $\varphi_i(R) < \varphi_i(R'_i, R_{-i})$.

Since $p(R'_i) \leq \varphi_i(R)$, $p_i(R'_i) \leq \varphi_i(R) < \varphi_i(R'_i, R_{-i})$. Therefore, $\varphi_i(R) P'_i \varphi_i(R'_i, R_{-i})$. Agent i can manipulate φ at (R'_i, R_{-i}) via R_i, contradicting strategy-proofness.

CASE 2. $\varphi_i(R'_i, R_{-i}) < \varphi_i(R)$.

If $p(R_i) \leq \varphi_i(R'_i, R_{-i}) < \varphi_i(R)$, then $\varphi_i(R'_i, R_{-i}) P_i \varphi_i(R)$, contradicting strategy-proofness. Thus, let $p(R_i) > \varphi_i(R'_i, R_{-i})$. By the convexity of the option set $\sigma_i(R_{-i})$ (Lemma 4), there exists $R''_i \in \mathcal{R}_R$ such that $\varphi_i(R''_i, R_{-i}) = p(R_i)$. Then, $\varphi_i(R''_i, R_{-i}) P_i \varphi_i(R)$, contradicting strategy-proofness.

We obtain contradictions in both cases. Therefore, $\varphi_i(R'_i, R_{-i}) = \varphi_i(R)$.

Q. E. D.

Ching (1994) proves his characterization by using the properties of own peak monotonicity and uncompromisingness on the single-peaked domain. We have established the same properties on a minimally rich domain. Using these properties, we can prove Theorem 1 as Ching (1994) does.

Proof of Theorem 1.

It is trivial that the uniform rule satisfies strategy-proofness, symmetry, and efficiency on a minimally rich domain. To prove the uniqueness, let φ be a strategy-proof, symmetry, and efficient rule on a minimally rich domain $\mathcal{R}_R^n$. We show that for all $R \in \mathcal{R}_R^n$, $\varphi(R) = U(R)$.

Let $R \in \mathcal{R}_R^n$. There are three cases.

CASE 1. $\sum_{i \in N} p(R_i) = M$.

By Fact (i), for all $i \in N$, $\varphi_i(R) = p(R_i)$. By the definition of the uniform rule, for all $i \in N$, $U_i(R) = p(R_i)$. Thus, since for all $i \in N$, $\varphi_i(R) = U_i(R)$, it follows that $\varphi(R) = U(R)$.

CASE 2. $\sum_{i \in N} p(R_i) < M$.

Without loss of generality, we rename the agents so that $p(R_1) \leq \cdots \leq p(R_n)$. There are two cases.

CASE 2-1. $R_1 = \cdots = R_n$.

Then by Fact (i) and symmetry, for all $i \in N$, $\varphi_i(R) = M/n$. By the definition of the uniform rule, for all $i \in N$, $U_i(R) = M/n$. Therefore, $\varphi(R) = U(R)$.

CASE 2-2. $R \neq (R_1, \cdots, R_1)$.

This means that at least one agent has a different preference from other agents. By contradiction, suppose that $\varphi(R) \neq U(R)$.

Step 1. By Fact (i) and feasibility, there exists $k \in N$ such that $p(R_k) \leq \varphi_k(R) < U_k(R)$. Let $R'_k \in \mathcal{R}_R$ be such that $p(R'_k) = p(R_1)$. Suppose that agent k changes his preference from R_k to R'_k. Since $p(R'_k) \leq p(R_k)$, by the own peak monotonicity of φ (Lemma 1), $\varphi_k(R'_k, R_{-k}) \leq \varphi_k(R)$. Since $p(R_k) < U_k(R)$ and $p(R'_k) \leq U_k(R)$, the uncompromisingness of the uniform rule implies that $U_k(R'_k, R_{-k}) = U_k(R)$. Thus $\varphi_k(R'_k, R_{-k}) < U_k(R'_k, R_{-k})$. Then if $(R'_k, R_{-k}) = (R_1, \cdots, R_1)$, by Case 2-1, for all

$i \in N$, $\varphi_i(R'_k, R_{-k}) = U_i(R'_k, R_{-k}) = M/n$. Therefore, $\varphi_k(R'_k, R_{-k}) = U_k(R'_k, R_{-k})$. This is a contradiction. If $(R'_k, R_{-k}) \neq (R_1, \cdots, R_1)$, we proceed to Step 2.

Step 2. Since $\varphi_k(R'_k, R_{-k}) < U_k(R'_k, R_{-k})$, and since $\sum_{j \in N} p(R_j) < M$ and $p(R'_k) \leq p(R_k)$ together imply $p(R'_k) + \sum_{j \neq k} p(R_j) < M$, it follows from Fact (i) and feasibility that there exists $l \in N \setminus \{k\}$ such that $p(R_l) \leq U_l(R'_k, R_{-k}) < \varphi_l(R'_k, R_{-k})$. Let $R'_l \in \mathcal{R}_R$ be such that $p(R'_l) = p(R_1)$, $R'_{kl} = (R'_k, R'_l)$, and $R_{-kl} = (R_i)_{i \in N \setminus \{k,l\}}$. Suppose that agent l changes his preference from R_l to R'_l. Since $p(R'_l) \leq p(R_l)$, the own peak monotonicity of the uniform rule implies that $U_l(R'_{kl}, R_{-kl}) \leq U_l(R'_k, R_{-k})$. Since $p(R_l) < \varphi_l(R'_k, R_{-k})$ and $p(R'_l) \leq \varphi_l(R'_k, R_{-k})$, by the uncompromisingness of φ (Lemma 5), $\varphi_l(R'_{kl}, R_{-kl}) = \varphi_l(R'_k, R_{-k})$. Thus $U_l(R'_{kl}, R_{-kl}) < \varphi_l(R'_{kl}, R_{-kl})$. Then if $(R'_{kl}, R_{-kl}) = (R_1, \cdots, R_1)$, by Case 2-1, for all $i \in N$, $\varphi_i(R'_{kl}, R_{-kl}) = U_i(R'_{kl}, R_{-kl}) = M/n$. Therefore $U_l(R'_{kl}, R_{-kl}) = \varphi_l(R'_{kl}, R_{-kl})$. This is a contradiction. If $(R'_{kl}, R_{-kl}) \neq (R_1, \cdots, R_1)$, we go back to Step 1.

R_k or R_l might be R_1. But both of them are not R_1. Therefore, repeating Step 1 and Step 2, we replace the preference of at least one new agent by R_1. Since there is only a finite number of agents, we can reach the contradiction at Step 1 or Step 2. Hence, $\varphi(R) = U(R)$.

CASE 3. $M < \sum_{i \in N} p(R_i)$.

The same reasoning as in Case 2 can be applied to this case to show $\varphi(R) = U(R)$.

Hence, for all $R \in \mathcal{R}_R^n$, $\varphi(R) = U(R)$.

Q. E. D.

3. 2. *Proof of Theorem 2*

The basic structure of the proof of Theorem 2 is similar to that of Ching and Serizawa's (1998) proof. However since Ching and Serizawa (1998) require the domain to include the single-peaked domain while Theorem 2 only requires the domain to include a minimally rich domain, there are crucial differences between the two proofs. First, Ching and Serizawa's (1998) proof depends on Ching's (1994) characterization on the single-peaked domain. On the other hand, the proof of Theorem 2 uses Theorem 1 instead. Secondly, since a minimally rich domain is smaller than the single-peaked domain, the domain assumed in Theorem 2 may not include some of the single-peaked preferences that Ching and Serizawa (1998) use in their proof. We overcome this problem by distinguishing between more cases than Ching and Serizawa (1998) does.

Before starting the proof of Theorem 2, we present two useful lemmas. Lemma 6 below says that if a rule on a domain including a minimally rich domain satisfies efficiency, then symmetry implies strong symmetry on the minimally rich domain. This result is equivalent to Lemma 1 in Ching and Serizawa (1998) which was stated for the single-peaked domain.

LEMMA 6. *Let $\mathcal{R}_R \subseteq \mathcal{R}_A \subseteq \mathcal{R}_C$. If a rule φ on $\mathcal{R}_A^n \times \mathbb{R}_{++}$ satisfies symmetry and efficiency, then for all $(R, M) \in \mathcal{R}_A^n \times \mathbb{R}_{++}$ and for all $i, j \in N$ such that $R_i = R_j \in \mathcal{R}_R$, $\varphi_i(R, M) = \varphi_j(R, M)$.*

Proof. Let $(R, M) \in \mathcal{R}_A^n \times \mathbb{R}_{++}$ and i, $j \in N$ be such that $R_i = R_j \in \mathcal{R}_R$. By efficiency, $\varphi_i(R, M) \leq p(R_i)$ and $\varphi_j(R, M) \leq p(R_j)$ or $\varphi_i(R, M) \geq p(R_i)$ and $\varphi_j(R, M) \geq p(R_j)$. Then symmetry and $R_i = R_j \in \mathcal{R}_R$ imply that $\varphi_i(R, M) = \varphi_j(R, M)$.

Q. E. D.

We introduce the notion of "convex" preferences.

Definition 17. A preference $R_0 \in \mathcal{R}$ is *convex* if $p(R_0)$ is an interval $[\underline{p}(R_0), \overline{p}(R_0)]$ and for all $x, y \in \mathbb{R}_+$ such that $[y < x < \underline{p}(R_0)]$ and $[\overline{p}(R_0) < x < y]$, xR_0y.

Let $\mathcal{R}_{CONV} \subseteq \mathcal{R}_C$ be the domain of all convex preferences. We call it the convex domain.

Lemma 7 below is an extension of Fact (i) in the characterization of efficiency. While Fact (i) pertains to a statement on the single-peaked domain, Lemma 7 pertains to the convex domain.

Lemma 7.[10] *Let $\mathcal{R}_{CONV} \subseteq \mathcal{R}_A \subseteq \mathcal{R}_C$. If a rule φ on $\mathcal{R}_A^n \times \mathbb{R}_{++}$ is efficient, then for all $(R, M) \in \mathcal{R}_{CONV}{}^n \times \mathbb{R}_{++}$, the following properties hold.*
(i) *If $M \leq \sum_{i \in N} \underline{p}(R_i)$, for all $j \in N$, $\varphi_j(R, M) \leq \overline{p}(R_j)$.*
(ii) *If $\sum_{i \in N} \underline{p}(R_i) \leq M \leq \sum_{i \in N} \overline{p}(R_i)$, for all $j \in N$, $\underline{p}(R_j) \leq \varphi_j(R, M) \leq \overline{p}(R_j)$.*
(iii) *If $\sum_{i \in N} \overline{p}(R_i) \leq M$, for all $j \in N$, $\underline{p}(R_j) \leq \varphi_j(R, M)$.*

Proof. First, we prove (i). Suppose $M \leq \sum_{i \in N} \underline{p}(R_i)$. By contradiction, suppose that there exists agent $j \in N$ such that $\overline{p}(R_j) < \varphi_j(R, M)$. By feasibility, there is another agent $l \in N$ such that $\varphi_l(R, M) < \underline{p}(R_l)$. Let $\varepsilon_j = \varphi_j(R, M) - \overline{p}(R_j)$, $\varepsilon_l = \underline{p}(R_l) - \varphi_l(R, M)$, and $\varepsilon = \min\{\varepsilon_j, \varepsilon_l\}$. Let $z' = (z'_1, \cdots, z'_n)$ be such that $z'_j = \varphi_j(R, M) - \varepsilon$, $z'_l = \varphi_l(R, M) + \varepsilon$, and $z'_i = \varphi_i(R, M)$ for all $i \in N \backslash \{j, l\}$. Then, z' Pareto dominates $\varphi(R, M)$. This is a contradiction.

A symmetric reasoning applies to prove (iii).

Lastly, we prove (ii). Suppose $\sum_{i \in N} \underline{p}(R_i) \leq M \leq \sum_{i \in N} \overline{p}(R_i)$. By contradiction, suppose that there exists an agent $j \in N$ such that $\varphi_j(R, M) \notin [\underline{p}(R_j), \overline{p}(R_j)]$. Since $\sum_{i \in N} \underline{p}(R_i) \leq M \leq \sum_{i \in N} \overline{p}(R_i)$, there is an allocation $z = (z_1, \cdots, z_n)$ such that for all $i \in N$, $\underline{p}(R_i) \leq z_i \leq \overline{p}(R_i)$ and $\sum_{i \in N} z_i = M$. Since $z_j P_j \varphi_j(R, M)$ and $z_i R_i \varphi_i(R, M)$ for all $i \in N \backslash \{j\}$, z Pareto dominates $\varphi(R, M)$, contradicting efficiency.

Q. E. D.

Proof of Theorem 2. Let $\mathcal{R}_R \subseteq \mathcal{R}_A \subseteq \mathcal{R}_C$. Suppose that a rule φ on $\mathcal{R}_A^n \times \mathbb{R}_{++}$ is strategy-proof, symmetric, and efficient. There are two steps.

[10] Ching and Serizawa (1998) state a very similar remark on *efficient rules* on the convex domain, namely:

If a rule φ on $\mathcal{R}_{CONV}^n$ is efficient, then for all $(R, M) \in \mathcal{R}_{CONV}^n \times \mathbb{R}_{++}$, the following properties hold;
(i) If $M \leq \sum_{i \in N} \underline{p}(R_i)$, then for all $j \in N$, $\varphi_j(R, M) \leq \underline{p}(R_j)$.
(ii) If $\sum_{i \in N} \underline{p}(R_i) \leq M \leq \sum_{i \in N} \overline{p}(R_i)$, then for all $j \in N$, $\underline{p}(R_j) \leq \varphi_j(R, M) \leq \overline{p}(R_j)$.
(iii) If $\sum_{i \in N} \overline{p}(R_i) \leq M$, then for all $j \in N$, $\overline{p}(R_j) \leq \varphi_j(R, M)$.

However, (i) and (iii) of their remark are false.

To see that (i) is false, we have only to construct an example that there are $(R, M) \in \mathcal{R}_{CONV}^n \times \mathbb{R}_{++}$, an efficient allocation $z \in Z(M)$ and $j \in N$ such that $M \leq \sum_{i \in N} \underline{p}(R_i)$ and $\varphi_j(R, M) > \underline{p}(R_j)$. Let $R_0 \in \mathcal{R}_{CONV}$ be such that $x_0 < y_0 < \underline{p}(R_0) < \overline{p}(R_0)$; $x' I_0 y'$ for all $x', y' \in [x_0, y_0]$; $y' P_0 x'$ for all $x' < y'$ such that $x', y' \in (y_0, \underline{p}(R_0)]$; and $\overline{p}(R_0) - y_0 \leq y_0 - x_0$. Let $\varepsilon > 0$ be such that $x_0 < y_0 - \varepsilon$ and $\underline{p}(R_0) < y_0 + \varepsilon \leq \overline{p}(R_0)$. Let $n = 2$, $M = 2y_0$, and $R_1 = R_2 = R_0$. Then, the allocation $z = (y_0 - \varepsilon, y_0 + \varepsilon)$ is efficient, $M \leq \sum_{i \in N} \underline{p}(R_i)$, but $\varphi_2(R, M) > \underline{p}(R_2)$.

Similarly, (iii) of Ching and Serizawa's (1998) remark is false. Although Ching and Serizawa's (1998) remark is used in their proof, their proof still works. For our Lemma 7 is enough for their proof. See Step 2 of the proof of Theorem 2.

Step 1. First, we show that $\mathcal{R}_A \subseteq \mathcal{R}_{CONV}$. By contradiction, suppose that there exists a preference $R_0 \in \mathcal{R}_A \backslash \mathcal{R}_{CONV}$. Then there are three points $x_0 < y_0 < z_0$ such that $x_0 P_0 y_0$ and $z_0 P_0 y_0$. Let

$$x_0^* = \begin{cases} \max\{x_0' \in [x_0, y_0] | x_0' I_0 x_0\} & \text{if } z_0 R_0 x_0 \\ \max\{x_0' \in [x_0, y_0] | x_0' I_0 z_0\} & \text{otherwise,} \end{cases} \quad \text{and}$$

$$z_0^* = \begin{cases} \min\{z_0' \in [y_0, z_0] | z_0' I_0 x_0\} & \text{if } z_0 R_0 x_0 \\ \min\{z_0' \in [y_0, z_0] | z_0' I_0 z_0\} & \text{otherwise.} \end{cases}$$

Since R_0 is continuous, x_0^* and z_0^* are well-defined. By definition, $x_0 \leq x_0^* < y_0 < z_0^* \leq z_0$, $x_0^* I_0 z_0^*$, and $z_0^* P_0 x_0'$ for all $x_0' \in (x_0^*, z_0^*)$. Let $M = nz_0^*$. Let $R_0' \in \mathcal{R}_R$ be such that $p(R_0') \in (z_0^*, (M - x_0^*)/(n-1))$ and $(M - x_0^*)/(n-1) P_0' z_0^*$. Let $R' = (R_1', \cdots, R_n')$ be such that for all $i \in N$, $R_i' = R_0'$. By Lemma 6, $\varphi(R', M) = (z_0^*, \cdots, z_0^*)$.

Let $R_1 = R_0$. We consider the allocation when agent $1 \in N$ changes his preference from R_1' to R_1. There are two cases.

Case 1. $\varphi_1(R_1, R_{-1}', M) = \varphi_1(R', M)$.

In this case, since $\varphi_1(R_1, R_{-1}', M) = z_0^*$, $\sum_{i \neq 1} \varphi_i(R_1, R_{-1}', M) = (n-1) z_0^*$. By Lemma 6, for all $i \in N \backslash \{1\}$, $\varphi_j(R_1, R_{-1}', M) = z_0^*$. Consider the allocation $(x_0^*, (M - x_0^*)/(n-1), \cdots, (M - x_0^*)/(n-1))$. Since $x_0^* I_1 z_0^*$ and $(M - x_0^*)/(n-1) P_j' z_0^*$ for all $j \in N \backslash \{1\}$, the allocation $(x_0^*, (M - x_0^*)/(n-1), \cdots, (M - x_0^*)/(n-1))$ Pareto dominates $\varphi(R_1, R_{-1}', M) = (z_0^*, \cdots, z_0^*)$, contradicting efficiency.

Case 2. $\varphi_1(R_1, R_{-1}', M) \neq \varphi_1(R', M) (= z_0^*)$.

There are also four subcases. Let $r \in \mathbb{R}_+$ be such that $r I_0' z_0^*$ and $r \neq z_0^*$. If there does not exist such r, we omit Case 2-1 below.

Case 2-1. $r \leq \varphi_1(R_1, R_{-1}', M)$.

Let $R_1'' \in \mathcal{R}_R$ be such that $p(R_1'') \geq \varphi_1(R_1, R_{-1}', M)$. Corollary 1 tells us that for all $i \in N$, $\varphi_i(R_1'', R_{-1}', M) = U_i(R_1'', R_{-1}', M)$. Since $M = nz_0^*$, $z_0^* < p(R_1'')$, and $z_0^* < p(R_i')$ for all $i \in N$, it follows that $M < p(R_1'') + \sum_{i \neq 1} p(R_i')$. Thus by the definition of the uniform rule, $\varphi_i(R_1'', R_{-1}', M) = z_0^*$ for all $i \in N$. Since $\varphi_1(R_1, R_{-1}', M) P_1'' \varphi_1(R_1'', R_{-1}', M)$, agent 1 manipulates φ at (R_1'', R_{-1}', M) via R_1, contradicting strategy-proofness.

Case 2-2. $\varphi_1(R_1, R_{-1}', M) \in (z_0^*, r)$.

By the definition of r, $\varphi_1(R_1, R_{-1}', M) P_1' \varphi_1(R', M)$. Thus agent 1 manipulates φ at (R', M) via R_1, contradicting strategy-proofness.

Case 2-3. $x_0^* < \varphi_1(R_1, R_{-1}', M) < z_0^*$.

Since $\varphi_1(R', M) = z_0^*$, $\varphi_1(R', M) P_1 \varphi_1(R_1, R_{-1}', M)$. Thus agent 1 manipulates φ at (R_1, R_{-1}', M) via R_1', contradicting strategy-proofness.

Case 2-4. $\varphi_1(R_1, R_{-1}', M) \leq x_0^*$.

Let $R_1'' \in \mathcal{R}_R$ be such that $p(R_1'') \leq \varphi_1(R_1, R_{-1}', M)$. Corollary 1 tells us that $\varphi(R_1'', R_{-1}', M) = U(R_1'', R_{-1}', M)$. Since $p(R_1'') \leq x_0^*$ and for all $i \in N$, $z_0^* < p(R_i') < (M - x_0^*)/(n-1)$, $p(R_1'') + \sum_{i \neq 1} p(R_i') < M$. By the definition of the uniform rule, $\lambda(R_1'', R_{-1}', M) \leq z_0^*$. Otherwise, since for all $i \in N \backslash \{1\}$, $z_0^* < \lambda(R_1'', R_{-1}', M) \leq U_i(R_1'', R_{-1}', M)$, it follows that $M < \sum_{i \in N} U_i(R_1'', R_{-1}', M) = M$. It is a contradiction. Therefore, $\varphi_1(R_1'', R_{-1}', M) = M - (n-1) p(R_0')$ and for all $i \in N \backslash \{1\}$, $\varphi_i(R_1'', R_{-1}', M) = p(R_i')$. Thus, since $p(R_0') \in (z_0^*, (M - x_0^*)/(n-1))$, $\varphi_1(R_1'', R_{-1}', M) \in (x_0^*, z_0^*)$. Since $p(R_1'') \leq \varphi_1(R_1, R_{-1}', M) < \varphi_1(R_1'', R_{-1}', M)$, $\varphi_1(R_1, R_{-1}', M) P_1'' \varphi_1(R_1'', R_{-1}', M)$. Agent 1 manipulates φ at (R_1'', R_{-1}', M) via R_1, contradicting strategy-proofness.

Since we obtain contradictions in all cases, we know that $\mathcal{R}_A \subseteq \mathcal{R}_{CONV}$.

Step 2. We further show that $\mathcal{R}_A \subseteq \mathcal{R}_P$. By contradiction, suppose that there exists a preference $R_0 \in \mathcal{R}_A \backslash \mathcal{R}_P$. By Step 1, we know that $R_0 \in \mathcal{R}_{CONV} \backslash \mathcal{R}_P$. Suppose, without loss of generality, that there exist two points x_0, $y_0 \in \mathbb{R}_{++}$ such that $\overline{p}(R_0) < x_0 < y_0$, for all $x_0' \in [\overline{p}(R_0), x_0)$, $x_0' P_0 x_0$, and for all $x_0' \in [x_0, y_0]$, $x_0' I_0 x_0$. Let $M = nx_0$ and $R = (R_0, \cdots, R_0)$. If $\varphi(R, M) \neq (x_0, \cdots, x_0)$, there exist two agents j and k, such that $\varphi_j(R, M) < x_0 < \varphi_k(R, M)$. Without loss of generality, let $j = 1$ and $k = 2$. Since $\sum_{i \in N} \overline{p}(R_i) < M$, Lemma 7 guarantees that for all $i \in N$, $\underline{p}(R_0) \leq \varphi_i(R, M)$. Since $\varphi_1(R, M) \in [\underline{p}(R_0), x_0)$ and $x_0 < \varphi_2(R, M)$, it follows that $\varphi_1(R, M) P_1 x_0$ and $x_0 R_1 \varphi_2(R, M)$. By transitivity, $\varphi_1(R, M) P_1 \varphi_2(R, M)$, contradicting symmetry. Therefore $\varphi(R, M) = (x_0, \cdots, x_0)$.

Let $\varepsilon > 0$ be such that $x_0 - \varepsilon \in (\overline{p}(R_0), x_0)$ and $x_0 + \varepsilon \in (x_0, y_0)$. Then we consider the allocation $(x_0 - \varepsilon, x_0 + \varepsilon, x_0, \cdots, x_0)$. Since $(x_0 - \varepsilon) P_1 x_0$ and $(x_0 + \varepsilon) R_2 x_0$, $(x_0 - \varepsilon, x_0 + \varepsilon, x_0, \cdots, x_0)$ Pareto dominates $\varphi(R, M) = (x_0, \cdots, x_0)$, contradicting efficiency. Therefore, $\mathcal{R}_A \subseteq \mathcal{R}_P$.

To complete the proof of Theorem 2, we need to identify a rule satisfying strategy-proofness, symmetry, and efficiency on the single-plateaued domain. Ching and Serizawa (1998) extended the uniform rule to the single-plateaued domain, which satisfies the three requirements. The extended uniform rule, represented by $\overline{U} = (\overline{U}_1, \cdots, \overline{U}_n)$, is defined as follows:

For all $(R, M) \in \mathcal{R}_P^n \times \mathbb{R}_{++}$ and all $i \in N$,

$$\overline{U}_i(R, M) = \begin{cases} \min\{\underline{p}(R_i), \lambda(R, M)\} & \text{if } M \leq \sum_{j \in N} \underline{p}(R_j) \\ \min\{\underline{p}(R_i) + \lambda(R, M), \overline{p}(R_i)\} & \sum_{j \in N} \underline{p}(R_j) < M < \sum_{j \in N} \overline{p}(R_j) \\ \max\{\overline{p}(R_i), \lambda(R, M)\} & \sum_{j \in N} \overline{p}(R_j) \leq M \end{cases}$$

where $\lambda(R, M)$ solves $\sum_{j \in N} \overline{U}_j(R, M) = M$.

4. CONCLUDING REMARKS

We established that *the uniform rule is the unique strategy-proof, symmetric, and efficient rule on a minimally rich domain.* Weymark (1999) characterizes the uniform rule when discontinuous single-peaked preference are admitted. He shows that even if agents have discontinuous single-peaked preferences, the uniform rule is still the unique rule satisfying strategy-proofness, anonymity, and efficiency. Minimally rich domains can be generalized so as to contain discontinuous single-peaked preference. Therefore, there is an open question: Does our characterization of the uniform rule still hold on such generalized minimally rich domains?

We established that *there is a unique maximal domain including a minimally rich domain for strategy-proofness, symmetry, and efficiency, and it is the single-plateaued domain.* To identify the maximal domain, we defined a rule as a function of a preference profile and the amount of the good, M. Thus, our result of the maximal domain does not apply when M is fixed and a rule is a function of the preference profile only. Therefore, our maximal domain result does not exclude the possibility that when M is fixed, there exist rules satisfying strategy-proofness, symmetry, and efficiency on larger domain than the single-plateaued domain. Massó and Neme (2001) obtain maximal domain for rules for which the amount M is fixed. However, they adopt the stronger version of symmetry, and assume that domains include the single-peaked domain. Therefore, another open question is: When the amount of the good is fixed, what domain is a maximal domain including a minimally rich domain for strategy-proofness, symmetry, and efficiency?

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