Chew Soo Hong, and Shigehiro Serizawa

Supplementary Online Material of "Characterizing the Vickrey Combinatorial Auction by Induction"

October 27, 2005

In this note, we provide the discussions and the proofs that we have omitted in "Characterizing the Vickrey Combinatorial Auction by Induction." In Section A, we prove Theorem 3. In Section B, we analyze the case in which commodities are homogeneous.

Section A. Proof of Theorem 3

In this section, we prove Theorem 3. Efficiency and individual rationality of the Vickrey allocation rule follow directly from its definition. The usual argument implies that the Vickrey allocation rule is strategy-proof as well.

It remains to establish the uniqueness of a strategy-proof, efficient, and individually rational rule on V. Let f be a rule that is strategy-proof, efficient, and individually rational on V. We prove that f is the Vickrey allocation rule. It follows directly from efficiency that

$$f_A(v) \in \arg\max\{\sum_{i\in N} v^i(B^i) : B \in \mathcal{A}\}$$

for any $v \in V$. It suffices to prove $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$ for any $i \in N$ and any $v \in V$. Given $v \in V$ and $A^i \subseteq M$, we denote

$$\bar{\sigma}^{-i}(v; A^i) = \max\{\sum_{j\neq i} v^j(B^i) : B \in \mathcal{A} \text{ and } B^i = A^i\}.$$

We employ the following facts.

Fact 1: For any $v \in V$ and any $i \in N$, $f_t^i(v) \leq v^i(f_A^i(v))$.

Proof: Fact 1 follows from individual rationality. ■

Fact 2: For any $v \in V$, any $i \in N$, and any $\hat{v}^i \in V^i$, if $v^i(f_A^i(\hat{v}^i, v^{-i})) \geq v^i(f_A^i(v))$ and $\hat{v}^i(f_A^i(\hat{v}^i, v^{-i})) \leq \hat{v}^i(f_A^i(v))$, then $f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v)$.

Proof: Let $v \in V$, $i \in N$, and $\hat{v}^i \in V^i$ be such that $v^i(f_A^i(\hat{v}^i, v^{-i})) \geq v^i(f_A^i(v))$ and $\hat{v}^i(f_A^i(\hat{v}^i, v^{-i})) \leq \hat{v}^i(f_A^i(v))$. If $f_t^i(\hat{v}^i, v^{-i}) < f_t^i(v)$, then $u^i(f^i(\hat{v}^i, v^{-i})) = v^i(f_A^i(\hat{v}^i, v^{-i})) - f_t^i(\hat{v}^i, v^{-i}) > v^i(f_A^i(v)) - f_t^i(v) = u^i(f^i(v))$, contradicting strategy-proofness. If $f_t^i(\hat{v}^i, v^{-i}) > f_t^i(v)$, then $\widehat{u}^i(f^i(v)) = \hat{v}^i(f_A^i(v)) - f_t^i(v) > \hat{v}^i(f_A^i(\hat{v}^i, v^{-i})) - f_t^i(\hat{v}^i, v^{-i}) = \widehat{u}^i(f^i(\hat{v}^i, v^{-i}))$, also contradicting strategy-proofness. ■

Fact 3: For any $v \in V$, any $i \in N$, and any $A \in \mathcal{A}$, if $A^i \supseteq f_A^i(v)$,

$$\sum_{j\neq i} v^j(A^j) \leq \bar{\sigma}^{-i}(v; A^i) \leq \sigma^{-i}(v).$$

Proof: Let $v \in V$, $i \in N$, and $A \in \mathcal{A}$ be such that $A^i \supseteq f_A^i(v)$. By the definition of $\bar{\sigma}^{-i}(v; A^i)$,

$$\sum_{j\neq i} v^j(A^j) \leq \max\{\sum_{j\neq i} v^j(B^i) : B \in \mathcal{A} \text{ and } B^i = A^i\} = \bar{\sigma}^{-i}(v; A^i).$$

By $A^i \supseteq f_A^i(v)$, $\bar{\sigma}^{-i}(v; A^i) \leq \bar{\sigma}^{-i}(v; f_A^i(v))$, and by efficiency, $\bar{\sigma}^{-i}(v; f_A^i(v)) = \sigma^{-i}(v)$. ■

Let $v \in V$ and $i \in N$. We shall show that $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$ by induction on the cardinality $\# f_A^i(v)$ of $f_A^i(v)$ and the cardinality of the items from which agent i with $v^i \in V_a^i$ obtains positive value. If $\# f_A^i(v) = 0$ or if $v^i \in V_a^i$ and agent i obtains positive value from no items in $f_A^i(v)$, then Fact 1 implies $f_t^i(v) = g_t^i(v) = 0$. Thus, we start with the case that one of such cardinalities is one.

STEP 1: Assume that (i) $\# f_A^i(v) = 1$ or (ii) $v^i \in V_a^i$ and there is one item k such that $k \in f_A^i(v)$ and for any $k' \in M \backslash \{k\}$, $v^i(\{k'\}) = 0$. Without loss of generality, assume that (i) $f_A^i(v) = \{1\}$ or (ii) $v^i \in V_a^i$, $1 \in f_A^i(v)$, and for any $k \in M \backslash \{1\}$, $v^i(\{k\}) = 0$. Suppose $f_t^i(v) \neq \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. We derive a contradiction in each of the following three cases.

Case 1: $f_t^i(v) > v^i(f_A^i(v))$. This contradicts Fact 1.

Case 2: $f_t^i(v) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Let $\hat{v}^i \in V_a^i$ be such that

$$f_t^i(v) < \hat{v}^i(\{1\}) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v) \tag{1}$$

and

$$\hat{v}^i(\{k\}) = 0, \forall k \neq 1. \tag{2}$$

Note that whether (i) or (ii) holds, $1 \in f_A^i(v)$, and efficiency implies $\sigma^{-i}(v) = \overline{\sigma}^{-i}(v; f_A^i(v))$. Also note that $1 \in f_A^i(v)$ implies $\overline{\sigma}^{-i}(v; f_A^i(v)) \leq \overline{\sigma}^{-i}(v; \{1\})$. Suppose $\overline{\sigma}^{-i}(v; f_A^i(v)) < \overline{\sigma}^{-i}(v; \{1\})$. Then, $f_A^i(v) \supsetneq \{1\}$, and so (ii) holds. Let $A \in \mathcal{A}$ be such that $A^i = \{1\}$ and $\sum_{j \neq i} v^j(A^j) = \overline{\sigma}^{-i}(v; \{1\})$. Then, since (ii) implies $v^i(f_A^i(v)) = v^i(\{1\})$, $\sum_{j \in N} v^j(A^j) = v^i(\{1\}) + \overline{\sigma}^{-i}(v; \{1\}) > \sum_{j \in N} v^j(f_A^j(v))$, and so $f_A(v)$ is not efficient for v. This is a contradiction. Therefore, whether (i) or (ii) holds, $\sigma^{-i}(v) = \overline{\sigma}^{-i}(v; \{1\})$.

For any $A \in \mathcal{A}$, if $1 \in A^i$, then since $\sigma^{-i}(v) = \overline{\sigma}^{-i}(v; \{1\})$ implies $\sum_{j \neq i} v^j(A^j) \leq \sigma^{-i}(v)$, it follows from (2) and the RHS of (1) that

$$\hat{v}^i(A^i) + \sum_{j \neq i} v^j(A^j) \leq \hat{v}^i(\{1\}) + \sigma^{-i}(v) < \overline{\sigma}^{-i}(v).$$

Thus, any commodity allocation A with $1 \in A^i$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies that $1 \notin f_A(\hat{v}^i, v^{-i})$.

Fact 1, in conjunction with (2) and $1 \notin f_A(\hat{v}^i, v^{-i})$, implies that $f_t(\hat{v}^i, v^{-i}) = 0$. It follows from the LHS of (1) that

$$\hat{u}^i(f^i(v)) = \hat{v}^i(\{1\}) - f_t^i(v) > 0 = \hat{u}^i(f^i(\hat{v}^i, v^{-i})).$$

This contradicts strategy-proofness.

Case 3: $\overline{\sigma}^{-i}(v) - \sigma^{-i}(v) < f_t^i(v) \leq v^i(f_A^i(v))$. Let $\hat{v}^i \in V_a^i$ be such that

$$\overline{\sigma}^{-i}(v) - \sigma^{-i}(v) < \hat{v}^i(\{1\}) < f_t^i(v), \tag{3}$$

and

$$\hat{v}^i(\{k\}) = 0, \forall k \neq 1. \tag{4}$$

For any $A \in \mathcal{A}$, if $1 \notin A^i$, then since $\sum_{j \neq i} v^j(A^j) \leq \overline{\sigma}^{-i}(v)$, it follows from (4) and the LHS of (3) that

$$\hat{v}^i(A^i) + \sum_{j \neq i} v^j(A^j) \leq \overline{\sigma}^{-i}(v) < \hat{v}^i(\{1\}) + \sigma^{-i}(v) = \hat{v}^i(f_A^i(v)) + \sum_{j \neq i} v^j(f_A^j(v)).$$

Thus, any commodity allocation A with $1 \notin A^i$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies $1 \in f_A^i(\hat{v}^i, v^{-i})$.

In case of (i), since $1 \in f_A^i(\hat{v}^i, v^{-i})$ implies $v^i(f_A^i(\hat{v}^i, v^{-i})) \geq v^i(f_A^i(v))$, and since (4) implies $\hat{v}^i(f_A^i(v)) = \hat{v}^i(f_A^i(\hat{v}^i, v^{-i}))$, Fact 2 implies that $f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v)$. In case of (ii), since $1 \in f_A^i(\hat{v}^i, v^{-i})$ implies $v^i(f_A^i(\hat{v}^i, v^{-i})) = v^i(f_A^i(v))$, and since (4) implies $\hat{v}^i(f_A^i(v)) \geq \hat{v}^i(f_A^i(\hat{v}^i, v^{-i}))$, Facts 2 also implies that $f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v)$. Thus, by (4) and the RHS of (3),

$$f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v) > \hat{v}^i(\{1\}) = \hat{v}^i(f_A^i(\hat{v}^i, v^{-i})).$$

This contradicts Fact 1.

STEP 2: Let $m' \leq m$. As induction hypothesis, assume that if (i) $\# f_A^i(v) \leq m' - 1$ or if (ii) $v^i \in V_a^i$ and there is $M' \subseteq M$ such that $\# M' \leq m' - 1$, $M' \subseteq f_A^i(v)$, and for any $k \notin M'$, $v^i(\{k\}) = 0$, then $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. We show that if (i) $\# f_A^i(v) = m'$, or if (ii) $v^i \in V_a^i$ and there is $M' \subseteq M$ such that $\# M' \leq m'$, $M \subseteq f_A^i(v)$, and for any $k \notin M'$, $v^i(\{k\}) = 0$, then $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Without loss of generality, assume that (i) $f_A^i(v) = M' = \{1, \ldots, m'\}$ or (ii) $v^i \in V_a^i$, $M' = \{1, \ldots, m'\} \subseteq f_A^i(v)$, and for any $k \notin M'$, $v^i(\{k\}) = 0$.

Note that whether (i) or (ii) holds, $M' \subseteq f_A^i(v)$, and efficiency implies $\sigma^{-i}(v) = \overline{\sigma}^{-i}(v; f_A^i(v))$. Also note that $M' \subseteq f_A^i(v)$ implies $\overline{\sigma}^{-i}(v; f_A^i(v)) \leq \overline{\sigma}^{-i}(v; M')$. Suppose $\overline{\sigma}^{-i}(v; f_A^i(v)) < \overline{\sigma}^{-i}(v; M')$. Then, $f_A^i(v) \supsetneq M'$, and so (ii) holds. Let $A \in \mathcal{A}$ be such that $A^i = M'$ and $\sum_{j \neq i} v^j(A^j) = \overline{\sigma}^{-i}(v; M')$. Then, since (ii) implies $v^i(f_A^i(v)) = v^i(M')$, $\sum_{j \in N} v^j(A^j) = v^i(M') + \overline{\sigma}^{-i}(v; M') > \sum_{j \in N} v^j(f_A^j(v))$, and so $f_A(v)$ is not efficient for v. This is a contradiction. Therefore, whether (i) or (ii) holds, we have:

$$\bar{\sigma}^{-i}(v; M') = \sigma^{-i}(v). \tag{5}$$

We derive a contradiction in each of the following two cases.

Case 1: $f_t^i(v) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Note

$$f_t^i(v) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \overline{\sigma}^{-i}(v) < \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \sigma^{-i}(v).$$

Let $\hat{v}^i \in V_a^i$ be such that

$$f_t^i(v) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \overline{\sigma}^{-i}(v) < \hat{v}^i(m') < \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \sigma^{-i}(v), \tag{6}$$

$$\hat{v}^i(\{k\}) > \overline{\sigma}^{-i}(v), \forall k \in \{1, \ldots, m'-1\}, \tag{7}$$

and

$$\hat{v}^i(\{k\}) = 0, \forall k \notin M'. \tag{8}$$

Together with (7), efficiency implies $k \in f_A^i(\hat{v}^i, v^{-i})$ for any $k \in \{1, \ldots, m'-1\}$. Thus, it follows from (5), the RHS of (6), $\hat{v}^i \in V_a^i$, and (8) that for any $A \in \mathcal{A}$, if $A^i \supseteq M'$,

$$\begin{aligned}\hat{v}^i(A^i) + \sum_{j \neq i} v^j(A^j) &\leq \sum_{k=1}^{m'} \hat{v}^i(\{k\}) + \bar{\sigma}^{-i}(v; M'\}) \\ &= \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) + \hat{v}^i(\{m\}) + \sigma^{-i}(v) \\ &< \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}).\end{aligned}$$

Therefore, any commodity allocation A with $A^i \supseteq \{1, \ldots, m'\}$ is not efficient for $(\hat{v}^i, v^{-i})$. Thus, efficiency implies that $f_A^i(\hat{v}^i, v^{-i}) \supseteq \{1, \ldots, m'-1\}$ but $m \notin f_A^i(\hat{v}^i, v^{-i})$.

Accordingly, since we have $\sigma^{-i}(\hat{v}^i, v^{-i}) = \bar{\sigma}^{-i}((\hat{v}^i, v^{-i}); \{1, \ldots, m'-1\}) = \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\})$ similarly to (5), the induction hypothesis implies that

$$\hat{u}^i(f^i(\hat{v}^i, v^{-i})) = \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) - [\overline{\sigma}^{-i}(\hat{v}^i, v^{-i}) - \bar{\sigma}^{-i}((\hat{v}^i, v^{-i}); \{1, \ldots, m'-1\})].$$

On the other hand, it follows from the LHS of (6) that

$$\begin{aligned}\hat{u}^i(f^i(v)) &= \sum_{k=1}^{m'} \hat{v}^i(\{k\}) - f_t^i(v) \\ &> \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) - [\overline{\sigma}^{-i}(v) - \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\})] = \hat{u}^i(f^i(\hat{v}^i, v^{-i})).\end{aligned}$$

This contradicts strategy-proofness.

Case 2: $f_t^i(v) > \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Note

$$f_t^i(v) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \overline{\sigma}^{-i}(v) > \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \sigma^{-i}(v).$$

Let $\hat{v}^i \in V_a^i$ be such that

$$f_t^i(v) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \overline{\sigma}^{-i}(v) > \hat{v}^i(\{m'\}) > \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) - \sigma^{-i}(v), \quad (9)$$

$$\hat{v}^i(\{k\}) > \overline{\sigma}^{-i}(v), \forall k \in \{1, \ldots, m'-1\}, \quad (10)$$

and,

$$\hat{v}^i(\{k\}) = 0, \forall k \notin M'. \quad (11)$$

Together with (10), efficiency implies that $k \in f_A^i(\hat{v}^i, v^{-i})$ for any $k \in \{1, \ldots, m'-1\}$. Thus, $f_A^i(\hat{v}^i, v^{-i}) \supseteq \{1, \ldots, m'-1\}$. Moreover, for any $A \in \mathcal{A}$, it follows from (5), (11), the RHS of (9), and $\hat{v}^i \in V_a^i$ that if $A^i \supseteq \{1, \ldots, m'-1\}$ but $m' \notin A^i$, then

$$\begin{aligned}\hat{v}^i(A^i) + \sum_{j \neq i} v^j(A^j) &\leq \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) + \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}) \\ &< \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) + \hat{v}^i(\{m'\}) + \bar{\sigma}^{-i}(v; M') \\ &= \sum_{k=1}^{m'} \hat{v}^i(\{k\}) + \bar{\sigma}^{-i}(v; M').\end{aligned}$$

Thus, any commodity allocation A with $A^i \supseteq \{1, \ldots, m'-1\}$ but $m' \notin A^i$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies that $f_A^i(\hat{v}^i, v^{-i}) \supseteq M'$.

In case of (i), since $f_A^i(\hat{v}^i, v^{-i}) \supseteq M'$ implies $v^i(f_A^i(\hat{v}^i, v^{-i})) \geq v^i(f_A^i(v))$, and since (11) implies $\hat{v}^i(f_A^i(v)) = \hat{v}^i(f_A^i(\hat{v}^i, v^{-i}))$, Fact 2 implies that $f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v)$. In case of (ii), since $f_A^i(\hat{v}^i, v^{-i}) \supseteq M'$ implies $v^i(f_A^i(\hat{v}^i, v^{-i})) = v^i(f_A^i(v))$, and since (11) implies $\hat{v}^i(f_A^i(v)) \geq \hat{v}^i(f_A^i(\hat{v}^i, v^{-i}))$, Facts 2 also implies that $f_t^i(\hat{v}^i, v^{-i}) = f_t^i(v)$. Therefore,

$$\hat{u}^i(f^i(\hat{v}^i, v^{-i})) = \sum_{k=1}^{m'} \hat{v}^i(\{k\}) - f_t^i(v).$$

Let $\widetilde{v}^i \in V_a^i$ be such that

$$\widetilde{v}^i(\{k\}) > \overline{\sigma}^{-i}(v), \forall k \in \{1, \ldots, m'-1\}, \tag{12}$$

and

$$\widetilde{v}^i(\{k\}) = 0, \forall k \notin \{1, \ldots, m'-1\}. \tag{13}$$

Then together with (12), efficiency implies that $k \in f_A^i(\widetilde{v}^i, v^{-i})$ for any $k \in \{1, \ldots, m'-1\}$. Thus, $f_A^i(\widetilde{v}^i, v^{-i}) \supseteq \{1, \ldots, m'-1\}$. Accordingly, since by (13), we have $\sigma^{-i}(\widetilde{v}^i, v^{-i}) = \bar{\sigma}^{-i}((\widetilde{v}^i, v^{-i}); \{1, \ldots, m'-1\}) = \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\})$ similarly to (5), the induction hypothesis implies

$$f_t^i(\widetilde{v}^i, v^{-i}) = \overline{\sigma}^{-i}(v) - \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\}).$$

It follows from the LHS of (9) and $\hat{v}^i \in V_a^i$ that

$$\begin{aligned}\hat{u}^i(f^i(\widetilde{v}^i, v^{-i})) &= \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) - [\overline{\sigma}^{-i}(v) - \bar{\sigma}^{-i}(v; \{1, \ldots, m'-1\})] \\ &> \sum_{k=1}^{m'-1} \hat{v}^i(\{k\}) - [(f_t^i(v) - \hat{v}^i(m')] \\ &= \sum_{k=1}^{m'} \hat{v}^i(\{k\}) - f_t^i(v) \\ &= \hat{u}^i(f^i(\hat{v}^i, v^{-i})).\end{aligned}$$

This contradicts strategy-proofness.

We have established the uniqueness of the strategy-proof, efficient, and individually rational allocation rule on V. ■

Section B. Homogenous commodities

In this section, we analyze the cases where commodities are homogeneous.

There are m homogenous commodities to be allocated among the n agents. Denote $M = \{0, 1, \ldots, m\}$. The set of feasible commodity allocations is:

$$\mathcal{A} = \{a = (a^1, \ldots, a^n) \in M^n : \sum_{i \in N} a^i = m\},$$

where a^i refers to the number of the homogenous commodity agent i receives. Agent i's utility is given by $u^i(a^i, t^i) = v^i(a^i) - t^i$, where v^i is a value function from M to $\mathbb{R}_+$ such that $v^i(0) = 0$.

We consider the following types of value functions.

Monotonicity. For any a^i and b^i with $a^i \leq b^i$, $v^i(a^i) \leq v^i(b^i)$.
Strictly monotonicity. For any a^i and b^i with $a^i < b^i$, $v^i(a^i) < v^i(b^i)$.
Nonincreasing marginal utility. For any a^i, $v^i(a^i + 2) - v^i(a^i + 1) \leq v^i(a^i + 1) - v^i(a^i)$.

The definition of the Vickrey allocation rule reduces to the allocation rule g such that for any $v \in V$,

$$g_a(v) = \arg\max\{\sum_{i \in N} v^i(b^i) : b \in \mathcal{A}\}$$

and, for any $i \in N$, $g_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$,

$$\text{where} \quad \sigma^{-i} = \sum_{j \neq i} v^j(g_a^j(v)) \text{ and } \overline{\sigma}^{-i} = \max\{\sum_{j \neq i} v^i(b^i) : b \in \mathcal{A}\}.$$

Other notations are the same as our paper. The concepts such as efficiency, strategy-proofness, individual rationality are also defined in the same way.

Theorem 4: *(i) The Vickrey allocation rule the unique allocation that is strategy-proof, efficient, and individually rational on the class V_{sh} of strictly monotonic value functions.*

(ii) The Vickrey allocation rule the unique allocation rule that is strategy-proof, efficient, and individually rational on the class V_{sd} of strictly monotonic value functions with nonincreasing marginal utility.

We prove Theorem 4 (i) first, and then explain how the same proof method can be applied to prove Theorem 4 (ii).

Proof of Theorem 4 (i). It is straightforward to see that the Vickrey allocation rule satisfies strategy-proofness, efficiency, and individual rationality on V_{sh}. We now establish the uniqueness of the strategy-proof, efficient, and individually rational allocation rule on V_{sh}.

Let f be a rule that is strategy-proof, efficient, and individually rational on V_{sh}. We prove that $f = g$. It follows from efficiency that $f_a = g_a$. Thus, it suffices to prove $f_t = g_t$, that is, $f_t^i = g_t^i$ for any $i \in N$. Given $v \in V$ and $a^i \subseteq M$, we denote

$$\bar{\sigma}^{-i}(v; a^i) = \max\{\sum_{j \neq i} v^j(b^i) : b \in \mathcal{A} \text{ and } b^i = a^i\}.$$

We employ the following facts.

Fact 1: For any $v \in V_{sh}$ and any $i \in N$, $f_t^i(v) \leq v^i(f_a^i(v))$.

Fact 2: For any $v \in V_{sh}$, any $i \in N$, and $\widehat{v}^i \in V_{sh}^i$, if $f_a^i(\widehat{v}^i, v^{-i}) = f_a^i(v)$, then $f_t^i(\widehat{v}^i, v^{-i}) = f_t^i(v)$.

Fact 3: For any $v \in V_{sh}$, any $i \in N$, and $a \in \mathcal{A}$, if $a^i = f_a^i(v)$, then $\sum_{j \neq i} v^j(a^i) \leq \bar{\sigma}^{-i}(v; a^i) = \sigma^{-i}(v)$.

Fact 1 follows from individual rationality. Facts 2 follows from strategy-proofness. In Fact 3, the inequality follows from the definition of $\bar{\sigma}^{-i}(v; a^i)$ and the equality follows from efficiency. We establish that $f_t^i(v) = g_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$ for any $v \in V_{sh}$ and any $i \in N$. Let $v \in V_{sh}$ and $i \in N$. We shall show by induction on $f_a^i(v)$ that $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. If $f_a^i(v) = 0$, Fact 1 implies $f_t^i(v) = 0 = g_t^i(v)$. Thus, we start with the case that $f_a^i(v) = 1$.

STEP 1: Assume that $f_a^i(v) = 1$. Suppose $f_t^i(v) \neq \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. We derive a contradiction in each of the following three cases.

Case 1: $f_t^i(v) > v^i(f_a^i(v))$. This contradicts Fact 1.

Case 2: $f_t^i(v) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Let $\widehat{v}^i \in V_{sh}^i$ be such that

$$f_t^i(v) < \widehat{v}^i(1) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v), \tag{1}$$

and

$$\forall s \geq 2, \widehat{v}^i(s) - \widehat{v}^i(s-1) < \min_{j \neq i} \min\{v^j(s') - v^j(s'-1) : s' \geq 1\}. \tag{2}$$

We show $f_a^i(\widehat{v}^i, v^{-i}) = 0$. By (2), efficiency implies $f_a^i(\widehat{v}^i, v^{-i}) \leq 1$. For any $a \in \mathcal{A}$, if $a^i = 1$, then since Fact 3 implies $\sum_{j \neq i} v^j(a^j) \leq \sigma^{-i}(v)$, it follows from the RHS of (1) that

$$\widehat{v}^i(a^i) + \sum_{j \neq i} v^j(a^j) \leq \widehat{v}^i(1) + \sigma^{-i}(v) < \overline{\sigma}^{-i}(v).$$

Thus, any allocation $a \in$ with $a^i = 1$ is not efficient. Therefore, efficiency implies $f_a(\widehat{v}^i, v^{-i}) = 0$.

Fact 1, in conjunction with $f_a(\hat{v}^i, v^{-i}) = 0$, implies that $f_t(\hat{v}^i, v^{-i}) = 0$. Thus, it follows from the LHS of (1) that $\hat{u}^i(f^i(v)) = \hat{v}^i(\{1\}) - f_t^i(v) > 0 = \hat{u}^i(f^i(\hat{v}^i, v^{-i}))$. This contradicts strategy-proofness.

Case 3: $\overline{\sigma}^{-i}(v) - \sigma^{-i}(v) < f_t^i(v) \leq v^i(f_a^i(v))$. Let $\widehat{v}^i \in V_{sh}^i$ be such that

$$\overline{\sigma}^{-i}(v) - \sigma^{-i}(v) < \widehat{v}^i(1) < f_t^i(v), \tag{3}$$

and

$$\forall s \geq 2, \widehat{v}^i(s) - \widehat{v}^i(s-1) < \min_{j \neq i} \min\{v^j(s') - v^j(s'-1) : s' \geq 1\}. \tag{4}$$

We show $f_a^i(\widehat{v}^i, v^{-i}) = 1$. Together with (4), efficiency implies $f_a^i(\widehat{v}^i, v^{-i}) \leq 1$. For any $a \in \mathcal{A}$, if $a^i = 0$, then it follows from the LHS of (3) that

$$\widehat{v}^i(a^i) + \sum_{j \neq i} v^j(a^j) = 0 + \sum_{j \neq i} v^j(a^j) \leq \overline{\sigma}^{-i}(v) < \widehat{v}^i(1) + \sigma^{-i}(v) = \widehat{v}^i(f_a^i(v)) + \sum_{j \neq i} v^j(f_a^j(v)).$$

Thus, any commodity allocation a with $a^i = 0$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies $f_a^i(\widehat{v}^i, v^{-i}) = 1$.

Fact 2, in conjunction with $f_a^i(\hat{v}^i, v^{-i}) = 1$ and the RHS of (3), implies that $f_t^i(\widehat{v}^i, v^{-i}) = f_t^i(v) > \widehat{v}^i(f_a^i(\widehat{v}^i, v^{-i}))$. This contradicts Fact 1.

STEP 2: Let $m' \leq m$. As induction hypothesis, assume that if $f_a^i(v) \leq m' - 1$, then $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. We show that if $f_a^i(v) = m'$, then $f_t^i(v) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. Let $f_a^i(v) = m'$. Then Fact 3 implies:

$$\sigma^{-i}(v) = \overline{\sigma}^{-i}(v; m'). \tag{5}$$

Suppose $f_t^i(v) \neq \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. We derive a contradiction in each of the following two cases.

Case 1: $f_t^i(v) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$. By $f_t^i(v) < \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$, we have

$$f_t^i(v) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)] < \overline{\sigma}^{-i}(v; m' - 1) - \sigma^{-i}(v).$$

Let $\widehat{v}^i \in V_{sh}^i$ be such that

$$f_t^i(v) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)] < \widehat{v}^i(m') - \widehat{v}^i(m' - 1) < \overline{\sigma}^{-i}(v; m' - 1) - \sigma^{-i}(v), \tag{6}$$

$$\forall s \in \{1, \ldots, m' - 1\}, \widehat{v}^i(s) - \widehat{v}^i(s - 1) > \overline{\sigma}^{-i}(v), \tag{7}$$

and

$$\forall s \geq m' + 1, \widehat{v}^i(s) - \widehat{v}^i(s - 1) < \min_{j \neq i} \min\{v^j(s') - v^j(s' - 1) : s' \geq 1\}. \tag{8}$$

We show here that $f_A^i(\hat{v}^i, v^{-i}) = m' - 1$. Together with (8), efficiency implies that $f_a^i(\widehat{v}^i, v^{-i}) \leq m'$. Together with (7) and $\hat{v}^i \in V_{sa}^i$, efficiency also implies $f_a^i(\widehat{v}^i, v^{-i}) \geq m' - 1$. Thus, $f_a^i(\widehat{v}^i, v^{-i}) = m' - 1$ or $f_a^i(\widehat{v}^i, v^{-i}) = m'$. Moreover, for any $a \in \mathcal{A}$, it follows from (5) and the RHS of (6) that if $a^i = m'$, then

$$\hat{v}^i(a^i) + \sum_{j \neq i} v^j(a^j) \leq \hat{v}^i(m') + \bar{\sigma}^{-i}(v; m') < \hat{v}^i(m' - 1) + \bar{\sigma}^{-i}(v; m' - 1).$$

Thus, any commodity allocation a with $a^i = m'$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies that $f_a^i(\widehat{v}^i, v^{-i}) = m' - 1$.

The induction hypothesis, in conjunction with $f_A^i(\hat{v}^i, v^{-i}) = m' - 1$, implies that $f_t^i(\widehat{v}^i, v^{-i}) = \overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)$. Therefore

$$\widehat{u}^i(f^i(\widehat{v}^i, v^{-i})) = \widehat{v}^i(m' - 1) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)].$$

On the other hand, it follows from the LHS of (6) and Fact 2 that

$$\begin{aligned}
\widehat{u}^i(f(v)) &= \widehat{v}^i(m') - f_t^i(v) \\
&> \widehat{v}^i(m' - 1) + (f_t^i(v) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)]) - f_t^i(v) \\
&= \widehat{v}^i(m' - 1) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)] \\
&= \widehat{u}^i(f^i(\widehat{v}^i, v^{-i})).
\end{aligned}$$

This contradicts strategy-proofness.

Case 2: $\overline{\sigma}^{-i}(v) - \sigma^{-i}(v) < f_t^i(v)$. By $f_t^i(v) > \overline{\sigma}^{-i}(v) - \sigma^{-i}(v)$, we have

$$f_t^i(v) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)] > \overline{\sigma}^{-i}(v; m' - 1) - \sigma^{-i}(v).$$

Let $\widehat{v}^i \in V_{sh}^i$ be such that

$$f_t^i(v) - [\overline{\sigma}^{-i}(v) - \overline{\sigma}^{-i}(v; m' - 1)] > \widehat{v}^i(m') - \widehat{v}^i(m' - 1) > \overline{\sigma}^{-i}(v; m' - 1) - \sigma^{-i}(v), \quad (9)$$

$$\forall s \in \{1, \ldots, m' - 1\}, \widehat{v}^i(s) - \widehat{v}^i(s - 1) > \overline{\sigma}^{-i}(v), \quad (10)$$

and

$$\forall s > m' + 1, \widehat{v}^i(s) - \widehat{v}^i(s - 1) < \min_{j \neq i} \min\{v^j(s') - v^j(s' - 1) : s' \geq 1\}. \quad (11)$$

We show here that $f_a^i(\hat{v}^i, v^{-i}) = m'$. Together with (11), efficiency implies that $f_a^i(\widehat{v}^i, v^{-i}) \leq m'$. Together with (10), efficiency also implies that $f_a^i(\widehat{v}^i, v^{-i}) \geq m' - 1$. Thus, $f_a^i(\widehat{v}^i, v^{-i})$ equals to either $m' - 1$ or m'. Moreover, for any $a \in \mathcal{A}$, it follows from (5) and the RHS of (9) that if $a^i = m' - 1$, then

$$\hat{v}^i(a^i) + \sum_{j \neq i} v^j(a^j) \leq \hat{v}^i(m' - 1) + \bar{\sigma}^{-i}(v; m' - 1) < \hat{v}^i(m') + \bar{\sigma}^{-i}(v; m').$$

Thus, any commodity allocation a with $a^i = m' - 1$ is not efficient for $(\hat{v}^i, v^{-i})$. Therefore, efficiency implies that $f_a^i(\widehat{v}^i, v^{-i}) = m'$.

Fact 2, in conjunction with $f_a^i(\hat{v}^i, v^{-i}) = m'$, implies that $f_t^i(\widehat{v}^i, v^{-i}) = f_t^i(v)$. Thus, $\widehat{u}^i(f^i(\widehat{v}^i, v^{-i})) = \widehat{v}^i(m') - f_t^i(v)$.

Let $\widetilde{v}^i \in V_{sh}^i$ be such that

$$\forall s \in \{1, \ldots, m' - 1\}, \widetilde{v}^i(s) - \widetilde{v}^i(s - 1) > \overline{\sigma}^{-i}(v) \quad (12)$$

and

$$\forall s \geq m', \widetilde{v}^i(s) - \widetilde{v}^i(s - 1) < \min_{j \neq i} \min\{v^j(s') - v^j(s' - 1) : s' \geq 1\}. \quad (13)$$

Then together with (12) and (13), efficiency implies $f_a^i(\widetilde{v}^i, v^{-i}) = m' - 1$, so that $f_t^i(\widetilde{v}^i, v^{-i}) = \overline{\sigma}^{-i}(v) - \sigma^{-i}(m' - 1)$ by the induction hypothesis. Thus, it follows from the LHS of (9) that

$$\begin{aligned} \widehat{u}^i(f^i(\widetilde{v}^i, v^{-i})) &= \widehat{v}^i(m' - 1) - [\overline{\sigma}^{-i}(v) - \sigma^{-i}(m' - 1)] \\ &> \widehat{v}^i(m' - 1) - [(f_t^i(v) - \widehat{v}^i(m') + \widehat{v}^i(m' - 1)] \\ &= \widehat{v}^i(m') - f_t^i(v) \\ &= \widehat{u}^i(f^i(\widehat{v}^i, v^{-i})). \end{aligned}$$

This contradicts strategy-proofness.

We have established the uniqueness of the strategy-proof, efficient, and individually rational allocation rule on V_{sh}, and completed the proof of Theorem 4 (i). ■

Proof of Theorem 4 (ii). Note that Facts 1, 2, and 3 in the proof of Theorem 4 (i) hold on the class V_{sd} of strictly monotonic value functions with nonincreasing marginal utility. In the proof of Theorem 4 (i), besides the original value profile v being selected from V_{sh}, we use candidate value functions $\hat{v}^i$ and $\widetilde{v}^i$ from V_{sd}, to derive contradictions in Cases 2 and 3 of Step 1 and Cases 1 and 2 of Step 2. In case the original value profile v is in V_{sd}, such value functions $\hat{v}^i$ and $\widetilde{v}^i$ can be selected from V_{sd} to derive contradictions. Therefore, the method of proof of Theorem 4 (i) can be applied to demonstrate the requisite uniqueness on V_{sd}. ■