Discussion Paper No. 648

STRATEGY-PROOF AND ANONYMOUS ALLOCATION RULES OF INDIVISIBLE GOODS: A NEW CHARACTERIZATION OF VICKREY ALLOCATION RULE

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March 2006

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March 3, 2006

Keywords: strategy-proofness, anonymity, indivisible good, Vickrey allocation rule

We consider situations where a society allocates a finite units of an indivisible good among agents, and each agent receives at most one unit of the good. For example, imagine that a government allocates a fixed number of licences to private firms, or imagine that a government distributes equally divided lands to households. We show that the Vickrey allocation rule is the unique allocation rule satisfying strategy-proofness, anonymity and individual rationality.

1. Introduction

We consider situations where a society allocates a finite units of an indivisible good among agents, and each agent receives at most one unit of the good. For example, imagine that a government allocates a fixed number of licences to private firms, or imagine that a government distributes equally divided lands to households.[1] A number of allocation rules, including several forms of auction, are proposed for various social purposes such as efficiency, revenue maximization, etc. For the purpose of efficiency, one rule has a remarkable feature. It is "the Vickrey allocation rule". First, the Vickrey allocation rule allocates the goods to agents evaluating the good highest. ("Efficiency") Second, the Vickrey allocation rule extracts true information on agents' valuations from them. ("Strategy-Proofness") Thirdly, the Vickrey allocation rule induces agents' voluntary participation. ("Individual Rationality") And most importantly, as proved by Holmstrom (1979), the Vickrey allocation rule is the unique rule satisfying these three properties. It is well known that the Vickrey allocation rule also satisfies a property of impartiality called "anonymity". In this paper, we characterize the Vickrey allocation rule by focusing on anonymity instead of efficiency. Our characterization emphasizes that the Vickrey allocation rule also has a remarkable feature for the purpose of impartiality.

An *allocation rule* is generally formulated as a function from the set of agents' valuations on the good to the feasible set. An allocation rule is *efficient* if it allocates the goods to agents evaluating the good highest. Given an allocation rule, since agents' private valuations are not known to the others, there may be incentives for agents to misrepresent their values in order to manipulate the final outcomes to their favor. As a result, the actual outcomes may not constitute a socially desirable allocation relative to agents' true valuations. Therefore, allocation rules need to be immune to such strategic misrepresentation in order to securely attain a desirable allocation for agents' true valuations. If an allocation rule is immune to such strategic behavior, that is, if it is a dominant strategy for each agent to announce his true valuations, then the allocation rule is said to be *strategy-proof*. A condition of *individual rationality* is also imposed on allocation rules to induce agents' voluntary participation; it says that an allocation rule never assigns an allocation which makes some agent worse off than he would be if he receives no good and pays nothing. It is important to know what allocation rules satisfy efficiency, strategy-proofness, and individual rationality. The *Vickrey allocation rule* is the rule such that agents with m highest valuations of the goods receive the goods and pay the $(m+1)$-th valuation, and other agents pay nothing. Holmstrom (1979)[2] establishes that *the Vickrey allocation rule is the unique rule satisfying efficiency, strategy-proofness, and individual rationality.* This result emphasizes the distinguished importance of the Vickrey allocation rule for the purpose of efficiency.

However, society members are often more sensitive to impartiality than efficiency. In such environments, governments need to make more of impartiality than efficiency. Anonymity is a condition of impartiality in the sense that it requires allocation rules to treat agents equally from the viewpoint of agents who are ignorant of their own values

[1] The vehicle ownership licence in Singapore is also an example, where the ownership licences are distributed among residents through auction.

[2] Similar characterizations of Groves rules in public good models are previously established by Green and Laffont (1977), and Walker (1978). However, the characterizations of the these two articles cannot be applied to allocation rules of indivisible goods since they assume that the class of admissible preferences include preferences which are not admissible in the model of indivisible goods allocation.

or identities. An allocation rule is *anonymous* if when the valuations of two agents are switched, their net gain under the rule are also switched. In this article, we establish that *the Vickrey allocation rule is the unique rule satisfying strategy-proofness, anonymity, and individual rationality.* (Theorem 2)

There is also literature analyzing the fairness of the Vickrey allocation rule. Most of such literature focus on "envy-freeness." An allocation rule is *envy-free* if no agent prefers another agent's allocation to his own. Svensson (1983) shows that envy-freeness of indivisible goods allocation implies efficiency. This result, together with Holmstrom's (1979), implies that *the Vickrey allocation rule is also the unique rule satisfying strategy-proofness, envy-freeness, and individual rationality.* Many authors such as Papai (2003), Ohseto (2005), Sakai (2005), etc., apply Svensson's (1983) result to characterize the Vickrey allocation rule.

In this article, we do not impose envy-freeness on allocation rules, and so we cannot apply Svensson's (1983) result. Instead, we show that strategy-proofness, anonymity, and individual rationality together imply efficiency (Proposition). This result, together with Holmstrom's (1979), implies our characterization of the Vickrey allocation rule (Theorem 2). Focusing on anonymity, our characterization emphasizes that the Vickrey allocation rule also has a remarkable feature for the purpose of impartiality, and complements Holmstrom's (1979).

Section 2 sets up the model, defines basic notions, and states main results. Section 3 provides the proof of Proposition.

Section 2 Model and Main Result

The set of agents is $N = \{1, \ldots, N\}$. There are m units of an indivisible good. An *item allocation* is a $n-$tuple $x = (x_1, \ldots, x_n)$ such that $\sum x_i = m$, where for each $i \in N$, x_i is the units of the good agent i receives.[3] We assume that $x_i = 0$ or $x_i = 1$ for each $i \in N$, that is, agents can receive at most one unit of the good. Denote the set of item allocations by X, that is, $X = \{x = (x_1, \ldots, x_n) \in \{0,1\}^n : \sum x_i = m\}$. For each $i \in N$, denote agent i's payment by $p_i \in \mathbb{R}_+$. We assume that payments are nonnegative. The feasible set is $Z = X \times \mathbb{R}^n_+$. An allocation is an element $z = (x, p) = (x_1, \ldots, x_n; p_1, \ldots, p_n)$ of Z, and agent i's allocation is $z_i = (x_i, p_i)$.

Each agent $i \in N$ has a quasi-linear utility function $u_i : \{0,1\} \times \mathbb{R}_+ \to \mathbb{R}$, that is, there is a value $v_i \in \mathbb{R}_+$ such that for all $(x_i, p_i) \in \{0,1\} \times \mathbb{R}_+$, $u_i(x_i, p_i) = v_i \cdot x_i - p_i$. Denote the set of agent i's such values by V_i.

Let $V = V_1 \times \cdots \times V_n$. A value profile is an element $v = (v_1, \ldots, v_n) \in V$. Given $v = (v_1, \ldots, v_n) \in V$, $N' \subseteq N$, and $i \in N$, $v_{N'}$ denotes $(v_j)_{j \in N'}$ and v_{-i} denotes $(v_j)_{j \in N \setminus \{i\}}$. Given $v = (v_1, \ldots, v_n) \in V$ and $\widehat{v}_i \in V_i$, $(\widehat{v}_i, v_{-i})$ denotes the value profile $(v_1, \ldots, v_{i-1}, \widehat{v}_i, v_{i+1}, \ldots, v_n)$. Similarly, given $v \in V$, $N' \subseteq N$, and $\widehat{v}_{N'} \in V_{N'} \equiv \prod_{i \in N'} V_i$, $(\widehat{v}_{N'}, v_{-N'})$ denotes the value profile generated from v by replacing the values of the set N' of agents by $\widehat{v}_{N'}$.

An *allocation rule* is a function f from V to Z. Given an allocation rule f and a value profile $v \in V$, we denote $f(v) = (x(v), p(v))$, where $x(v) = (x_1(v), \ldots, x_n(v))$ and $p(v) = (p_1(v), \ldots, p_n(v))$ respectively denote the outcome item allocation and payments of f for v, and we also denote $f_i(v) = (x_i(v), p_i(v))$.

We introduce several conditions of allocation rules. The first one is "efficiency". It says that allocation rules maximize the total value.

[3] We assume that all the units of the good are allocated to agents.

DEFINITION: An allocation rule f is *efficient* if for all $v \in V$, $x(v) \in \arg\max\{\sum_{i \in N} v_i \cdot x_i : x \in X\}$.

Next, we introduce impartiality conditions. "Equal treatment of equals" says that allocation rules give equal utility levels to agents with the same value. "Anonymity" says that when the valuations of two agents are switched, their utility levels are also switched. This condition requires rules to treat agents equally from the viewpoints of agents who are ignorant of their own values or identities.

DEFINITION: An allocation rule f *equally treats equals* if for all $v \in V$,

$$v_i = v_j \Longrightarrow u_i(f_i(v)) = u_j(f_j(v)).$$

DEFINITION: An allocation rule f is *anonymous* if for all $v \in V$, all $i \in N$, all $j \in N$, all $\widehat{v}_i \in V_i$, and all $\widehat{v}_j \in V_j$,

$$[\widehat{v}_i = v_j \& \widehat{v}_j = v_i] \Longrightarrow [\widehat{u}_i(f_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}})) = u_j(f_j(v)) \& \widehat{u}_j(f_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}})) = u_i(f_i(v))],$$

where $\widehat{u}_i(x_i, p_i) = \widehat{v}_i \cdot x_i - p_i$ and $\widehat{u}_j(x_j, p_j) = \widehat{v}_j \cdot x_j - p_j$.

REMARK: Anonymity implies equal treatment of equals.[4]

"Envy-freeness" says that no agent prefers another agent's allocation to his own. In contrast to anonymity, this condition compares agents' welfare from the viewpoints of agents whose identifies and values are specified.

DEFINITION: An allocation rule f is *envy-free* if for all $v \in V$, all $i \in N$ and all $j \in N$, $u_i(f_i(v)) \geq u_i(f_j(v))$.

"Strategy-proofness" is one of strongest incentive compatibility conditions. It says that to announce one's true value is a dominant strategy.

DEFINITION: An allocation rule f is *strategy-proof* if for all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, $u_i(f_i(v)) \geq u_i(f_i(\widehat{v}_i, v_{-i}))$.

"Individual rationality" induces agents to participate voluntarily by guaranteeing that an allocation rule never assigns the outcome that makes some agent worse off than his status quo $u_i(0,0) = 0$.

DEFINITION: An allocation rule f is *individually rational* if for all $v \in V$ and all $i \in N$, $u_i(f_i(v)) \geq 0$.

Given a value profile $v \in V$, we rank agents' values, and denote the agent with the first highest value by $i(v,1)$, the agent with the second highest value by $i(v,2)$, and so on. Ties are broken arbitrarily. Under the Vickrey allocation rule defined below, agents with m highest values receive the goods and pay the $(m+1)-$th highest value, $v^{i(v,m+1)}$, and other agents pay nothing.

DEFINITION: A *Vickrey allocation rule* is an allocation rule $f^* = (x^*(\cdot), p^*(\cdot))$ such that

$$(1)\ \forall v \in V, \forall i \in N, x_i^*(v) = \begin{cases} 1 & \text{if } v_i > v^{i(v,m)} \\ 1 & \text{if } i = i(v,m), \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{and (2) } \forall i \in N,\ p_i^*(v) = \begin{cases} v^{i(v,m+1)} & \text{if } x_i^*(v) = 1, \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

[4]This implication is intensively employed in the proof of Proposition below.

To be precise, the Vickrey allocation rule is not unique, since the way to break tie in ranking agents' values is not unique. In other words, there are as many Vickrey allocation rules as the ways to break ties. However, in the view of Remark below, we treat Vickrey allocation rules as if they were unique.

REMARK: Let f^* and $\widehat{f}^*$ be Vickrey allocation rules. Then
(i) $\sum_{i \in N} x_t^{*i}(u) = \sum_{i \in N} \widehat{x}_t^{*i}(u)$ for any $u \in U$.
(ii) $u^i(f^{*i}(u)) = u^i(\widehat{f}^{*i}(u))$ for any $u \in U$ and any $i \in N$.

Theorem 1 below says that the Vickrey allocation rule is the unique rule satisfying strategy-proofness, efficiency, and individual rationality.

THEOREM 1 (HOLMSTROM, 1979): *An allocation rule satisfies strategy-proofness, efficiency, and individual rationality if and only if it is the Vickrey allocation rule.*

Proposition below says that the condition of anonymity, together with strategy-proofness and individual rationality, implies efficiency. We prove Proposition in Section 3.

PROPOSITION: *If an allocation rule satisfies strategy-proofness, anonymity and individual rationality, it also satisfies efficiency.*

As Svensson (1983) shows, envy-freeness alone implies efficiency. However, as Example 1 illustrates, anonymity alone does not imply efficiency.

EXAMPLE 1: Let $m = 1$ and $n = 2$. Let f be the allocation rule such that if $v_1 \leq v_2$, $f(v) = (1, 0; v_2, 0)$, and if $v_1 > v_2$, $f(v) = (0, 1; 0, v_1)$. Then, f is anonymous, but not efficient. f also violates strategy-proofness, and individual rationality.

It is well known that the Vickrey allocation rule satisfies the three axioms of strategy-proofness, anonymity and individual rationality. Thus, Theorem 1 and Proposition together imply Theorem 2 below. Theorem 2 says that the Vickrey allocation rule is the unique rule satisfying strategy-proofness, anonymity, and individual rationality.

THEOREM 2: *An allocation rule satisfies strategy-proofness, anonymity and individual rationality if and only if it is the Vickrey allocation rule.*

The three examples below illustrate that the three axioms in the Theorem 2 are all indispensable.

EXAMPLE 2: Let $m = 1$ and $n = 2$. Let f be the allocation rule such that if $v_1 \geq 1$, $f(v) = (1, 0; 1, 0)$, and if $v_1 < 1$, $f(v) = (0, 1; 0, 0)$. Then, f satisfies strategy-proofness, and individual rationality, but not anonymity. f is not the Vickrey allocation rule.

EXAMPLE 3: Let $m = 1$ and $n = 2$. Let f be the allocation rule such that if $v_1 \geq v_2$, $f(v) = (1, 0; v_1, 0)$, and if $v_1 < v_2$, $f(v) = (0, 1; 0, v_2)$. Then, f satisfies anonymity, and individual rationality, but not strategy-proofness. f is not the Vickrey allocation rule.

EXAMPLE 4: Let $m = 1$ and $n = 2$. Let f be the allocation rule such that if $v_1 \geq v_2$, $f(v) = (1, 0; 2v_2, v_1)$, and if $v_1 < v_2$, $f(v) = (0, 1; v_2, 2v_1)$. Then, f satisfies strategy-proofness, and anonymity, but not individual rationality. f is not the Vickrey allocation rule.

SECTION 3 PROOFS

We devote this section to the proof of Proposition.

3.1 Lemmas

In this subsection, we state and prove several lemmas as preliminary results for the proof of Proposition.

LEMMA 1: Let f be an individually rational allocation rule. For all $v \in V$ and all $i \in N$, if $x_i(v) = 0$, then $p_i(v) = 0$ and $u_i(f_i(v)) = 0$.

Lemma 1 directly follows from individual rationality. Thus, we omit the proof of Lemma 1.

LEMMA 2: Let f be a strategy-proof allocation rule. For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(\widehat{v}_i, v_{-i}) = x_i(v)$, $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$.

PROOF: Let $v \in V$, $i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(\widehat{v}_i, v_{-i}) = x_i(v)$. If $p_i(\widehat{v}_i, v_{-i}) > p_i(v)$, $\widehat{u}_i(f_i(v)) < \widehat{u}_i(f_i(\widehat{v}_i, v_{-i}))$, and if $p_i(\widehat{v}_i, v_{-i}) < p_i(v)$, $u_i(f_i(v)) < u_i(f_i(\widehat{v}_i, v_{-i}))$. Both cases contradict strategy-proofness. Thus $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$. □

LEMMA 3: Let f be a strategy-proof and individually rational allocation rule. For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(v) = 1$ and $\widehat{v}_i > v_i$, then $x_i(\widehat{v}_i, v_{-i}) = 1$, and $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$.

PROOF: Let $v \in V$, $i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(v) = 1$ and $\widehat{v}_i > v_i$. Suppose that $x_i(\widehat{v}_i, v_{-i}) = 0$. Then by Lemma 1, $\widehat{u}_i(f_i(\widehat{v}_i, v_{-i})) = 0$. By individual rationality, $p_i(v) \leq v_i < \widehat{v}_i$. Thus, $\widehat{u}_i(f_i(v)) = \widehat{v}_i - p_i(v) > 0 = \widehat{u}_i(f_i(\widehat{v}_i, v_{-i}))$. This is a contradiction to strategy-proofness. Therefore, $x_i(\widehat{v}_i, v_{-i}) = 1 = x_i(v)$, and by Lemma 2, $p_i(\widehat{v}_i, v_{-i}) = p_i(v)$. □

LEMMA 4: Let f be a strategy-proof and individually rational allocation rule. For all $v \in V$, all $i \in N$, and all $\widehat{v}_i \in V_i$, if $x_i(v) = 0$ and $\widehat{v}_i < v_i$, then $x_i(\widehat{v}_i, v_{-i}) = 0$ and $p_i(\widehat{v}_i, v_{-i}) = 0$.

PROOF: Let $v \in V$, $i \in N$, and $\widehat{v}_i \in V_i$ be such that $x_i(v) = 0$ and $\widehat{v}_i < v_i$. Suppose that $x_i(\widehat{v}_i, v_{-i}) = 1$. By Lemma 1, $u_i(f_i(v)) = 0$. By individually rationality, $p_i(\widehat{v}_i, v_{-i}) \leq \widehat{v}_i < v_i$. Thus, $u_i(f_i(\widehat{v}_i, v_{-i})) = v_i - p_i(\widehat{v}_i, v_{-i}) > 0 = u_i(f_i(v))$. This is a contradiction to strategy-proofness. Therefore, $x_i(\widehat{v}_i, v_{-i}) = 0$, and by Lemma 1, $p_i(\widehat{v}_i, v_{-i}) = 0$. □

LEMMA 5: Let f be a strategy-proof, anonymous and individually rational allocation rule. Let $v \in V$, $i \in N$, $\widehat{v}_i \in V_i$, $v_0 > 0$ and $N' \subseteq N$ be such that $x_i(v) = 1$, $v_i < v_0 = \widehat{v}_i$, and $v_j = v_0$ for all $j \in N'$. Then $x_i(\widehat{v}_i, v_{-i}) = x_j(\widehat{v}_i, v_{-i}) = 1$ for all $j \in N'$.

PROOF: By Lemma 3, $x_i(\widehat{v}_i, v_{-i}) = 1$ and $p_i(\widehat{v}_i, v_{-i}) = p_i(v) \leq v_i < v_0$. Thus, since $v_j = v_0 = \widehat{v}_i$ for all $j \in N'$, it follows from anonymity that $u_j(\widehat{v}_i, v_{-i}) = \widehat{u}_i(\widehat{v}_i, v_{-i}) \geq \widehat{v}_i - v_i > 0$ for all $j \in N'$. Therefore, $x_j(\widehat{v}_i, v_{-i}) = 1$ for all $j \in N'$. □

LEMMA 6: Let f be a strategy-proof, anonymous and individually rational allocation rule. Let $v \in V$, $i \in N$, $\widehat{v}_i \in V_i$, $v_0 \geq 0$ and $N' \subseteq N$ be such that $x_i(v) = 0$, $v_i > v_0 = \widehat{v}_i$, and $v_j = v_0$ for all $j \in N'$. Then $x_i(\widehat{v}_i, v_{-i}) = x_j(\widehat{v}_i, v_{-i}) = 0$ for all $j \in N'$.

PROOF: By Lemma 1 and $x_i(v) = 0$, $u_i(v) = 0$. By Lemma 4, $x_i(v) = 0$ and $\widehat{v}_i < v$, we have $x_i(\widehat{v}_i, v_{-i}) = 0$, and so by Lemma 1, $\widehat{u}_i(\widehat{v}_i, v_{-i}) = 0$. Thus, since $v_j = v_0 = \widehat{v}_i$ for all $j \in N'$, it follows from anonymity that $u_j(\widehat{v}_i, v_{-i}) = \widehat{u}_i(\widehat{v}_i, v_{-i}) = 0$ for all $j \in N'$. Therefore, for all $j \in N'$, if $x_j(\widehat{v}_i, v_{-i}) = 1$, $p_j(\widehat{v}_i, v_{-i}) = v_0$.

Since $x_i(\widehat{v}_i, v_{-i}) = 0$, suppose that there is $j \in N'$ such that $x_j(\widehat{v}_i, v_{-i}) = 1$. Let $\widehat{v}_j = v_i$. Then, since $v_i > v_0 = v_j$, Lemma 3 implies that $x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$, $p_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = p_j(\widehat{v}_i, v_{-i}) = v_0 < v_i = \widehat{v}_j$, and $\widehat{u}_j(f_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}})) = \widehat{v}_j - v_0 > 0$. Since $u_i(f_i(v)) = 0$,

$\widehat{v}_i = v_0 = v_j$, and $\widehat{v}_j = v_i$, this is a contradiction to anonymity. Hence, for all, $j \in N'$, $x_j(\widehat{v}_i, v_{-i}) = 0$. □

3.2 Proof of Proposition

In this subsection, we prove Proposition. Let f be an allocation rule satisfying strategy-proofness, anonymity and individual rationality. Let $v \in V$. We show that $f(v)$ is efficient. Without loss of generality, assume $v_1 \geq v_2 \geq \cdots \geq v_m = \cdots = v_{n'} > v_{n'+1} \geq v_{n'+2} \geq \cdots \geq v_n$.[5]

Case 1: $n' = n$**.** Note that $f(v)$ is efficient if for all $i \in N$, $v^i > v^m$ implies $x_i(v) = 1$. Suppose that there is $i \in N$ such that $v_i > v_m$ and $x_i(v) = 0$. By $v_i > v_m$, $i < m$. Let $\widehat{v}_i = v_m$. Then, by Lemma 6, $x_i(\widehat{v}_i, v_{-i}) = x_m(\widehat{v}_i, v_{-i}) = x_{m+1}(\widehat{v}_i, v_{-i}) = \cdots = x_n(\widehat{v}_i, v_{-i}) = 0$. Thus, the number of agents who receive the good at the value profile $(\widehat{v}_i, v_{-i})$ is less than or equal to $m-1$. Since there are m units of the good, this is a contradiction. Thus $f(v)$ is efficient.

Case 2: $n' < n$**.** Note that $f(v)$ is efficient if (A) for all $i \in N$, $v_i > v_m$ implies $x_i(v) = 1$, and (B) for all $i \in N$, $x_i(v) = 1$ implies $i \leq n'$.

The basic idea to prove Condition (A) is similar to that of Case 1, but it involves a more complicate procedure. If there is an agent i such that $v_i > v_m$ and $x_i(v) = 0$, then we lower his value to $\widehat{v}_i = v_m$ and obtain $x_i(\widehat{v}_i, v_{-i}) = x_m(\widehat{v}_i, v_{-i}) = x_{m+1}(\widehat{v}_i, v_{-i}) = \cdots = x_{n'}(\widehat{v}_i, v_{-i}) = 0$ by Lemma 6. However, this is not yet a contradiction. For there may be an agent j such that $v_j < v_m$, and $x_j(\widehat{v}_i, v_{-i}) = 1$. Note that $v_i > v_m$ implies $i < m$, and that $v_j < v_m$ implies $j > n'$. Here we raise the value of agent j to $\widehat{v}_j = v_m$, and obtain $x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_m(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_{m+1}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = \cdots = x_{n'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$ by Lemma 5. Since there are only m units of the goods, it follows that there is agent $i' < m$ such that $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. $i' < m$ implies $v_{i'} \geq v_m$. If $v_{i'} = v_m$, Lemma 5 implies $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$, contradicting to $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. Thus, $v_{i'} > v_m$. Then, we lower i''s value to $\widehat{v}_{i'} = v_m$ again, and obtain $x_{i'}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_i(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_j(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_m(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = x_{m+1}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = \cdots = x_{n'}(\widehat{v}_i, \widehat{v}_{i'}, \widehat{v}_j, v_{-\{i,i',j\}}) = 0$ by Lemma 6. We repeat this procedure until we get a contradiction. To prove Condition (B), we also repeat a similar procedure. During the repetitions, Claim below holds.

Claim: Let $l \in \{0, 1, \ldots, m-1\}$, $N_1 \subset N$, $N_2 \subset N$, $N_3 \subset N$, $N_4 \subset N$, $\widehat{v}_{N_1} \in V_{N_1}$, and $\widehat{v}_{N_4} \in V_{N_4}$ be such that
(1) $N_1 \cup N_2 = \{1, \ldots, m-1\}$ and $N_1 \cap N_2 = \varnothing$,
(2) $\#N_1 = l$, and for all $j \in N_1$, $\widehat{v}_j = v_m$,
(3) $N_3 = \{m, \ldots, n'\}$
(4) $N_4 \subseteq N \backslash \{1, \ldots, n'\}$, $\#N_4 = l$, and for all $j \in N_4$, $\widehat{v}_j = v_m$.

We consider the value profile $(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, v_{N \backslash (N_1 \cup N_2 \cup N_3 \cup N_4)})$, the value profile generated from v by lowering the values of N_1 to v_m and raising those of N_4 to v_m.[6]

If there is $i \in N_2$ such that $v_i > v_m$ and $x_i(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, v_{N \backslash (N_1 \cup N_2 \cup N_3 \cup N_4)}) = 0$, and if we lower i's value to $\widehat{v}_i = v_m$, then
(i) for all $j \in N_1 \cup N_3 \cup N_4 \cup \{i\}$,

$$x_j(\widehat{v}_{N_1}, \widehat{v}_i, v_{N_2 \backslash \{i\}}, v_{N_3}, \widehat{v}_{N_4}, v_{N \backslash (N_1 \cup N_2 \cup N_3 \cup N_4)}) = 0,$$

[5] n' may eqaul to m or n. In case of $n' = m$, $v_1 \geq v_2 \geq \cdots \geq v_m > v_{m+1} \geq v_{m+2} \geq \cdots \geq v_n$. In case of $n' = n$, $v_1 \geq v_2 \geq \cdots \geq v_m = v_{m+1} = \cdots = v_n$.

[6] In case of $N_1 = \{1, \ldots, l\}$, $N_2 = \{l+1, \ldots, m-1\}$, and $N_4 = \{n'+1, \ldots, n'+l\}$,

and (ii) there is $j \in N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)$ such that for $\widehat{v}_j = v_m$, and for all $k \in N_1 \cup N_3 \cup N_4 \cup \{i, j\}$,

$$x_k(\widehat{v}_{N_1}, \widehat{v}_i, v_{N_2\backslash\{i\}}, v_{N_3}, \widehat{v}_{N_4}, \widehat{v}_j, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4 \cup \{j\})}) = 1.$$

If there is $i \in N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)$ such that $x_i(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)}) = 1$, and if we raise agent i's value to $\widehat{v}_i = v_m$, then
(iii) for all $j \in N_1 \cup N_3 \cup N_4 \cup \{i\}$,

$$x_j(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, \widehat{v}_i, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4 \cup \{i\})}) = 1,$$

and (iv) there is $j \in N_2$ such that for $\widehat{v}_j = v_m$, and for all $k \in N_1 \cup N_3 \cup N_4 \cup \{i, j\}$,

$$x_k(\widehat{v}_{N_1}, \widehat{v}_j, v_{N_2\backslash\{j\}}, v_{N_3}, \widehat{v}_{N_4}, \widehat{v}_i, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4 \cup \{i\})}) = 0.$$

Proof of Claim: Assume that there is $i \in N_2$ such that $v_i > v_m$ and $x_i(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)}) = 0$, and $\widehat{v}_i = v_m$. Then, Conclusion (i) follows from Lemma 6. Note that Conclusion (i) implies that the number of agents in $N_1 \cup N_2 \cup N_3 \cup N_4$ who receive the good at the value profile $(\widehat{v}_{N_1}, \widehat{v}_i, v_{N_2\backslash\{i\}}, v_{N_3}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)})$ is less than or equal to $(m-1)-l-1 = m-l-2$. Thus, since there are m units of the good, there is $j \in N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)$ such that $x_j(\widehat{v}_{N_1}, \widehat{v}_i, v_{N_2\backslash\{i\}}, v_{N_3}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)}) = 1$. Since $j \in N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)$ implies $v_j < v_m$, Lemma 5 implies Conclusion (ii).

Similarly, Conclusion (iii) follows from Lemma 5, and Conclusion (iv) follows from Conclusion (iii) and Lemma 6. □

Proof of Condition (A): Suppose that there is $i \in N$ such that $v_i > v_m$ and $x_i(v) = 0$. We derive a contradiction. Note that $v_i > v_m$ implies that $m \geq 2$ and $i < m$.

Case 2-A-1: $m - 1 \geq n - n'$**.** By applying Conclusion (ii) of Claim, we can pick up an agent j from $\{n'+1, \ldots, n\}$, set $\widehat{v}_i = \widehat{v}_j = v_m$, and have $x_k(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$ for all $k \in \{m, \ldots, n'\}$. Since there are only m units of the good, there is $i' \in \{1, \ldots, m-1\}\backslash\{i\}$ such that $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. By $i' \leq m-1$, $v_{i'} \geq v_m$. If $v_{i'} = v_m$, Lemma 5 implies $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 1$, contradicting to $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$. Thus, $v_{i'} > v_m$ and we can apply Conclusion (ii) of Claim again. Apply Conclusion (ii) of Claim $(n-n')$ times in this way. As a result, we have: $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_4)}) = 1$ for all $k \in N_1 \cup N_3 \cup N_4$, where $N_1 \subseteq \{1, \ldots, m-1\}$, $\#N_1 = n - n'$, $N_3 = \{m, \ldots, n'\}$, $N_4 = \{n'+1, \ldots, n\}$, $\widehat{v}_{N_1} = (v_m, \ldots, v_m)$, and $\widehat{v}_{N_4} = (v_m, \ldots, v_m)$.[7] If $m - 1 = n - n'$, then $N_1 \cup N_3 \cup N_4 = N$, and so we have already a

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$(\widehat{v}_{N_1}, v_{N_2}, v_{N_3}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_2 \cup N_3 \cup N_4)})$ is the value profile such that

$$\begin{aligned}
&(\underbrace{\widehat{v}_1, \ldots, \widehat{v}_l}_{N_1}, \underbrace{v_{l+1}, \ldots, v_{m-1}}_{N_2}, \underbrace{v_m, \ldots, v_{n'}}_{N_3}, \underbrace{\widehat{v}_{n'+1}, \ldots, \widehat{v}_{n'+l}}_{N_4}, v_{n'+l+1}, \ldots, v_n) \\
= &(\underbrace{v_m, \ldots, v_m}_{N_1}, \underbrace{v_{l+1}, \ldots, v_{m-1}}_{N_2}, \underbrace{v_m, \ldots, v_m}_{N_3}, \underbrace{v_m, \ldots, v_m}_{N_4}, v_{n'+l+1}, \ldots, v_n).
\end{aligned}$$

[7] In case of $N_1 = \{1, \ldots, n-n'\}$, $(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1 \cup N_4)})$ is the value profile such that

$$\begin{aligned}
&(\underbrace{\widehat{v}_1, \ldots, \widehat{v}_{n-n'}}_{N_1}, v_{n-n'+1}, \ldots, v_{m-1}, \underbrace{v_m, \ldots, v_{n'}}_{N_3}, \underbrace{\widehat{v}_{n'+1}, \ldots, \widehat{v}_n}_{N_4}) \\
= &(\underbrace{v_m, \ldots, v_m}_{N_1}, v_{n-n'+1}, \ldots, v_{m-1}, \underbrace{v_m, \ldots, v_m}_{N_3}, \underbrace{v_m, \ldots, v_m}_{N_4}).
\end{aligned}$$

contradiction. Thus, $m-1 > n-n'$. Since $n > n'$ and there are only m units of the good, there is $k \in N\backslash(N_1 \cup N_3 \cup N_4)$ such that $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)}) = 0$. We can show $v_k > v_m$ similarly to the case of i'. Let $\widehat{v}_k = v_m$. Then, by Conclusion (i) of Claim, for all $h \in N_1 \cup N_3 \cup N_4$, $x_h(\widehat{v}_{N_1}, \widehat{v}_{N_4}, \widehat{v}_k, v_{N\backslash(N_1\cup N_4\cup\{k\})}) = x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, \widehat{v}_k, v_{N\backslash(N_1\cup N_4\cup\{k\})}) = 0$. The number of agents who receive the good at the value profile $(\widehat{v}_{N_1}, \widehat{v}_{N_4}, \widehat{v}_k, v_{N\backslash(N_1\cup N_4\cup\{k\})})$ is less than or equal to $n - [(n-n') + 1 + (n'-m+1) + (n-n')] = (m-2) - (n-n')$. By $n > n'$, this is a contradiction.

Case 2-A-2: $m-1 < n-n'$. Similarly to Case 2-A-1, apply Conclusion (ii) of Claim $(m-1)$ times. As a result, we have: $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)}) = 1$ for all $k \in N_1 \cup N_3 \cup N_4$, where $N_1 = \{1, \ldots, m-1\}$, $N_3 = \{m, \ldots, n'\}$, $N_4 \subseteq \{n'+1, \ldots, n\}$, $\#N_4 = m-1$, $\widehat{v}_{N_1} = (v_m, \ldots, v_m)$, and $\widehat{v}_{N_4} = (v_m, \ldots, v_m)$.[8] Since there are only m units of the good and $m \geq 2$, this is a contradiction.

Since we get a contradiction whether $m-1 \geq n-n'$ or $m-1 < n-n'$, Condition (A) holds.

Proof of Condition (B): Suppose that there is $i \in N$ such that $x_i(v) = 1$ and $i \in \{n'+1, \ldots, n\}$. We derive a contradiction. By $i > n'$, $v_i < v_m$. If $m = 1$, then by applying Conclusion (i) of Claim, we derive a contradiction. For Conclusion (i) of Claim implies that for all $j \in \{m, \ldots, n'\} \cup \{i\}$, $x_j(\widehat{v}_i, v_{-i}) = 1$, where $\widehat{v}_i = v_m$. Thus, let $m \geq 2$ hereafter.

Case 2-B-1: $m-1 \geq n-n'$. By applying Conclusion (iv) of Claim, we can pick up an agent j from $\{1, \ldots, m-1\}$, set $\widehat{v}_i = \widehat{v}_j = v_m$, and have $x_k(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_i(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = x_j(\widehat{v}_i, \widehat{v}_j, v_{-\{i,j\}}) = 0$ for all $k \in \{m, \ldots, n'\}$. Since there are m units of the good, there is $i' \in \{n'+1, \ldots, n\}\backslash\{i\}$ such that $x_{i'}(\widehat{v}_i, \widehat{v}_j, v_{-\{i,k\}}) = 1$. By $i' > n'$, $v_{i'} < v_m$. Thus, we can apply Conclusion (iv) of Claim again. Apply Conclusion (iv) of Claim $(n-n')$ times in this way. As a result, we have: $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)}) = 0$ for all $k \in N_1 \cup N_3 \cup N_4$, where $N_1 \subseteq \{1, \ldots, m-1\}$, $\#N_1 = n-n'$, $N_3 = \{m, \ldots, n'\}$, $N_4 = \{n'+1, \ldots, n\}$, $\widehat{v}_{N_1} = (v_m, \ldots, v_m)$, and $\widehat{v}_{N_4} = (v_m, \ldots, v_m)$. The number of agents who receive the good at the preference profile $(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)})$ is less than or equal to $n - [(n-n') + (n'-m+1) + (n-n')] = (m-1) - (n-n')$. By $n > n'$, this is a contradiction.

Case 2-B-2: $m-1 < n-n'$. Similarly to Case 2-B-1, apply Conclusion (iv) of Claim $(m-1)$ times. As a result, we have: $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)}) = 0$ for all $k \in N_1 \cup N_3 \cup N_4$, where $N_1 = \{1, \ldots, m-1\}$, $N_3 = \{m, \ldots, n'\}$, $N_4 \subseteq \{n'+1, \ldots, n\}$, $\#N_4 = m-1$, $\widehat{v}_{N_1} = (v_m, \ldots, v_m)$, and $\widehat{v}_{N_4} = (v_m, \ldots, v_m)$. Since there are m units of the good, there is $k \in N\backslash(N_1 \cup N_3 \cup N_4)$ such that $x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)}) = 1$. By $k > n'$, $v_k < v_m$. Let $\widehat{v}_k = v_m$. Then, by Conclusion (iii) of Claim, for all $h \in N_1 \cup N_3 \cup N_4$, $x_h(\widehat{v}_{N_1}, \widehat{v}_{N_4}, \widehat{v}_k, v_{N\backslash(N_1\cup N_4\cup\{k\})}) = x_k(\widehat{v}_{N_1}, \widehat{v}_{N_4}, \widehat{v}_k, v_{N\backslash(N_1\cup N_4\cup\{k\})}) = 1$. Since there are only m units of the good, this is a contradiction.

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[8] In case of $N_4 = \{n'+1, \ldots, n'+m-1\}$, $(\widehat{v}_{N_1}, \widehat{v}_{N_4}, v_{N\backslash(N_1\cup N_4)})$ is the value profile such that

$$\begin{aligned} & (\underbrace{\widehat{v}_1, \ldots, \widehat{v}_{m-1}}_{N_1}, \underbrace{v_m, \ldots, v_{n'}}_{N_3}, \underbrace{\widehat{v}_{n'+1}, \ldots, \widehat{v}_{n'+m-1}}_{N_4}, v_{n'+m}, \ldots, v_n) \\ = \; & (\underbrace{v_m, \ldots, v_m}_{N_1}, \underbrace{v_m, \ldots, v_m}_{N_3}, \underbrace{v_m, \ldots, v_m}_{N_4}, v_{n'+m}, \ldots, v_n). \end{aligned}$$

Since we get a contradiction whether $m-1 \geq n-n'$ or $m-1 < n-n'$, Condition (B) holds.

Since Conditions (A) and (B) both hold, $f(v)$ is efficient. We have completed the proof of Proposition.

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