Discussion Paper No. 652

ON THE EQUIVALENCE OF G– WEAK AND –STRONG CORES IN THE MARRIAGE PROBLEM

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March 2006
Revised July 2006

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On the equivalence of $\mathcal{G}$-weak and -strong cores in the marriage problem*

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6 July 2006

Abstract

In the marriage problem (two-sided one-to-one matching problem), it is well-known that the weak core, the strong core and the set of stable matchings are all equivalent. This paper generalizes the above observation considering the $\mathcal{G}$-weak core and the $\mathcal{G}$-strong core. These are core concepts in which blocking power is restricted to the coalitions belonging to the prescribed class of coalitions $\mathcal{G}$. I give a necessary and sufficient condition that $\mathcal{G}$ should satisfy for the equivalence of the $\mathcal{G}$-weak core and the $\mathcal{G}$-strong core.

JEL Classification— C71, C78.

Keywords— marriage problem, $\mathcal{G}$-weak core, $\mathcal{G}$-strong core.

0 Introduction

In the marriage problem (Gale and Shapley, 1964), it is well-known that the weak core, the strong core and the set of stable matchings are all equivalent (Roth and Sotomayor, 1990). This paper generalizes this observation considering the "$\mathcal{G}$-weak core" and the "$\mathcal{G}$-strong core." These are core concepts in which blocking power is restricted to the coalitions belonging to the prescribed class of coalitions $\mathcal{G}$. I give a theorem from which one obtains a necessary and sufficient condition that $\mathcal{G}$ should satisfy for the equivalence of the $\mathcal{G}$-weak core and the $\mathcal{G}$-strong core.

0.1 Preliminary definitions

A **marriage problem** is a list $(M, W, (R^i)_{i \in M \cup W})$. Here M is the set of "men," and W is the set of "women." Assume that M and W are both nonempty and finite. $(R^i)_{i \in M \cup W}$ is a preference profile. For $i \in M$ (W, resp.), R^i is assumed to be a linear ordering (i.e. complete, transitive and anti-symmetric binary relation)

*I am grateful to Eiichi Miyagawa for motivating this research. All errors are my own responsibility.

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over $W \cup \{i\}$ ($M \cup \{i\}$, resp.). As usual, P^i (I^i, resp.) denotes the asymmetric (symmetric, resp.) part of R^i. Let $\mathcal{P}$ denote the set of preference profiles. A **matching** is a function $\mu : M \cup W \to M \cup W$ satisfying

$$\begin{aligned} i \in M &\Rightarrow \mu(i) \in W \cup \{i\}, \\ i \in W &\Rightarrow \mu(i) \in M \cup \{i\}, \\ i \in M \cup W &\Rightarrow \mu(\mu(i)) = i. \end{aligned}$$

Let $\mathcal{M}$ denote the set of matchings.

Let $\mu, \nu \in \mathcal{M}$. Let $S \subset M \cup W$ with $S \neq \emptyset$. Then we say that μ **weakly dominates** ν **via** S if

$$\begin{aligned} \forall i \in S, \ \mu R^i \nu; \\ \exists j \in S : \mu P^j \nu; \\ \mu(S) = S. \end{aligned}$$

And we say that μ **strongly dominates** ν **via** S if

$$\begin{aligned} \forall i \in S, \ \mu P^i \nu; \\ \mu(S) = S. \end{aligned}$$

When a matching μ dominates (weakly or strongly) some other matching ν via S, equivalently, we say "S blocks ν by μ (with weak or strong domination)."

Let $\mathcal{G}$ be a nonempty class of coalitions. Then the **$\mathcal{G}$-weak core** is the set of all matchings that are not strongly dominated by any other matchings via any coalitions in $\mathcal{G}$. Similarly, the **$\mathcal{G}$-strong core** is the set of all matchings that are not weakly dominated by any other matchings via any coalitions in $\mathcal{G}$. For the case where $\mathcal{G} = 2^{M \cup W} \setminus \{\emptyset\}$, these cores are simply called the **weak core** and the **strong core**, respectively. Since in the sequel the components M, W are fixed, these $\mathcal{G}$-cores are regarded as functions of preference profiles. Let us denote the $\mathcal{G}$-weak core by $w\mathcal{C}^{\mathcal{G}}$, and the $\mathcal{G}$-strong core by $s\mathcal{C}^{\mathcal{G}}$. And let $w\mathcal{C}$ and $s\mathcal{C}$ denote the weak core and the strong core, respectively.

Let V denote the set of "pairs." Here a "pair" means a coalition consisting of one man and one woman. That is, $V := \{S \subset M \cup W \mid \#S = 2, \ \#(S \cap M) = 1\}$. A **stable matching** is a matching that cannot be blocked (with strong domination) by any pair or individual. Thus the set of stable matchings is nothing but the $\mathcal{G}$-weak core with $\mathcal{G} = \{\{i\} \mid i \in M \cup W\} \cup V$.

0.2 Motivation

There are two observations that have motivated the present study. The first one is the following well-known result.

Theorem 0 (Roth and Sotomayor, 1990) *Let $\mathcal{G}$ be $\{\{i\} \mid i \in M \cup W\} \cup V$. Then on $\mathcal{P}$, the following four core concepts are all equivalent:*
(i) $\mathcal{G}$-strong core,
(ii) $\mathcal{G}$-weak core,

(iii) weak core,
(iv) strong core.

Given the theorem in the above, one may suspect that the equivalence similar to as stated in this theorem holds true for **arbitrary** $\mathcal{G}$. However, this is not the case. (A counter-example is Example 1 in the below.) Then we are motivated to ask for what $\mathcal{G}$ one obtains such equivalence.

Example 1 Let M be $\{1,3\}$ and W be $\{2,4\}$. Let $\mathcal{G}$ consist of only one element, namely $M \cup W$. Assume that each individual has the following preference at R.

$$R^1 : (2,1,4),$$
$$R^2 : (1,2,3),$$
$$R^3 : (4,3,2),$$
$$R^4 : (3,4,1).$$

Consider a matching μ such that $\mu(1) = 1$, $\mu(2) = 2$ and $\mu(3) = 4$, and another matching ν such that $\nu(1) = 2$ and $\nu(3) = 4$. Then ν weakly dominates μ via $M \cup W$. But no matchings strongly dominate μ via $M \cup W$. Thus $w\mathcal{C}^{\mathcal{G}}(R) \neq s\mathcal{C}^{\mathcal{G}}(R)$. $\square$

The second observation that has motivated this study is some results from implementation theory. In Sönmez (1997), it has been proved that in the context of the marriage problem, the direct mechanism induced by any solution that is individually rational and Pareto efficient implements the **$\mathcal{G}$-weak core** in $\mathcal{G}$-proof Nash equilibrium.[1] This result unifies several preceding results in implementation by direct mechanisms in the marriage problem such as Alcalde (1996), Ma (1995) and Shin and Suh (1996).

On the other hand, Takamiya (2005) proved that the direct mechanism induced by any solution that is individually rational and Pareto efficient implements the **$\mathcal{G}$-strong core** in **strict** $\mathcal{G}$-proof Nash equilibrium, under the additional assumption that $\mathcal{G}$ is monotonic.[2] This result is proved in the context of a general class of allocation problems (as defined in Sönmez (1999)) that contains the marriage problem as a special case.

Given these results, one would naturally ask when these two results overlap with each other. Then, one is motivated to ask for what $\mathcal{G}$, the $\mathcal{G}$-weak core and the $\mathcal{G}$-strong core are equivalent to each other.

[1] In a game in strategic-form, a strategy profile is said to be "$\mathcal{G}$-proof Nash equilibrium" if no coalition belonging to $\mathcal{G}$ can deviate from that strategy profile with every player in the coalition being strictly better off. "Strict $\mathcal{G}$-proof Nash equilibrium" is defined in the same way except that a group deviation succeeds merely with at least one player being strictly better off and the other players not being worse off. These concepts originate in Kalai, Postlewaite and Roberts (1979).

[2] Here $\mathcal{G}$ is said to be "monotonic" if for any coalition S in $\mathcal{G}$, any supercoalition of S belongs to $\mathcal{G}$.

1 Results

1.1 Main theorem and corollaries

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be classes of coalitions. In the sequel, for example, "$w\mathcal{C}^{\mathcal{G}_2} \subset s\mathcal{C}^{\mathcal{G}_1}$" is a shorthand notation for "$\forall R \in \mathcal{P}, w\mathcal{C}^{\mathcal{G}_2}(R) \subset s\mathcal{C}^{\mathcal{G}_1}(R)$."

Condition A For any $S \in \mathcal{G}_1$, both of the following hold true:
(i) $(\{m, w\} \subset S \ \& \ \{m, w\} \in V) \Rightarrow \{m, w\} \in \mathcal{G}_2$, and
(ii) $[i \in S \ \& \ ((i \in M \ \& \ W \not\subset S) \vee (i \in W \ \& \ M \not\subset S))] \Rightarrow \{i\} \in \mathcal{G}_2$.

Then the following is the main theorem of this paper.

Theorem 1 *$\mathcal{G}_1$ and $\mathcal{G}_2$ satisfy Condition A if, and only if, it holds true that $w\mathcal{C}^{\mathcal{G}_2} \subset s\mathcal{C}^{\mathcal{G}_1}$.*

Let $\mathcal{G}$ be a class of coalitions. Then by replacing $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathcal{G}$ in Condition A above, one obtains a necessary and sufficient condition for the equivalence of the $\mathcal{G}$-weak core and the $\mathcal{G}$-strong core. This is the most important corollary to Theorem 1.

Condition A$'$ For any $S \in \mathcal{G}$, both of the following hold true:
(i) $(\{m, w\} \subset S \ \& \ \{m, w\} \in V) \Rightarrow \{m, w\} \in \mathcal{G}$, and
(ii) $[i \in S \ \& \ ((i \in M \ \& \ W \not\subset S) \vee (i \in W \ \& \ M \not\subset S))] \Rightarrow \{i\} \in \mathcal{G}$.

Corollary 1 *$\mathcal{G}$ satisfies Condition A$'$ if, and only if, it holds true that $w\mathcal{C}^{\mathcal{G}} = s\mathcal{C}^{\mathcal{G}}$.*

The following presents some other corollaries that directly follow from Theorem 1 in the above. Let us denote $\mathcal{F} = \{\{i\} \mid i \in M \cup W\} \cup V$.

Corollary 2 *For any class of coalitions $\mathcal{G}$, $w\mathcal{C}^{\mathcal{F}} \subset s\mathcal{C}^{\mathcal{G}}$ and $s\mathcal{C}^{\mathcal{F}} \subset w\mathcal{C}^{\mathcal{G}}$.*

Corollary 3 *If $\mathcal{F} \subset \mathcal{G}$, then $s\mathcal{C}^{\mathcal{F}} = w\mathcal{C}^{\mathcal{F}} = s\mathcal{C}^{\mathcal{G}} = w\mathcal{C}^{\mathcal{G}} = w\mathcal{C} = s\mathcal{C}$.*

Clearly, the well-known Theorem 0 (Roth and Sotomayor, 1990) mentioned in Section 0.2 follows from Corollary 3. The following is somewhat less obvious.

Corollary 4 *$\mathcal{F}$ is the unique $\subset$-minimal element in the set of classes of coalitions $\{\mathcal{G} \mid s\mathcal{C}^{\mathcal{G}} = w\mathcal{C}^{\mathcal{G}} = s\mathcal{C} = w\mathcal{C}\}$.*

Remark In all these results, the assumption of linear preferences is essential. Things would be totally different when weak preferences are included. This deserves independent research.

1.2 Proof of Theorem 1

Proof of the "only if" part. Suppose that Condition A is satisfied and that there exist some $R \in \mathcal{P}$ and $\mu \in \mathcal{M}$ such that $\mu \in w\mathcal{C}^{\mathcal{G}_2}(R)$ and $\mu \notin s\mathcal{C}^{\mathcal{G}_1}(R)$. Then there exists some $S \in \mathcal{G}_1$ that blocks μ with weak domination, that is, for some $\nu \in \mathcal{M}$,

$$\forall i \in S,\ \nu(i) R^i \mu(i), \tag{1}$$
$$\exists j \in S : \nu(j) P^j \mu(j), \tag{2}$$
$$\nu(S) = S. \tag{3}$$

Then fix $j \in S$ that satisfies (2) above. And denote $k = \nu(j)$. Then there are two cases that have to be considered:

(i) If $k \neq j$, then $k \in S$ by (3). Then (i) of Condition A implies $\{j, k\} \in \mathcal{G}_2$. And $\nu(k) P^k \mu(k)$ since all preferences are linear orderings. Thus ν strongly dominates μ via $\{j, k\}$. This yields $\mu \notin w\mathcal{C}^{\mathcal{G}_2}(R)$, a contradiction.

(ii) If $k = j$, then j having improved strictly from μ to ν means that j was matched with some other individual in μ. Let us call this individual l. Then, without loss of generality, assume $j \in M$. This implies $l \in W$. Let us consider the two cases in the following:

(ii-1) First, assume $W \not\subset S$. In this case, (ii) of Condition A implies $\{j\} \in \mathcal{G}_2$. And clearly, ν strongly dominates μ via $\{j\}$. Then it follows $\mu \notin w\mathcal{C}^{\mathcal{G}_2}(R)$, a contradiction.

(ii-2) Next, assume $W \subset S$. This implies $l \in S$. Then since linear preferences are assumed, it must be $\nu(l) P^l \mu(l)$. Let us denote $h = \nu(l)$. Then let us consider two cases, namely $h = l$ and $h \neq l$: First, assume $h = l$. Then $\nu(\{j, l\}) = \{j, l\}$. Then it follows that ν strongly dominates μ via $\{j, l\}$. And by (i) of Condition A, $\{j, l\} \in \mathcal{G}_2$. Thus $\mu \notin w\mathcal{C}^{\mathcal{G}_2}(R)$, a contradiction. Second, assume $h \neq l$. This case is identical to the case (i) in the above. □

Proof of the "if" part. Suppose that Condition A is not satisfied. Then I want to show that there are some preference profiles R for which $w\mathcal{C}^{\mathcal{G}_2}(R) \not\subset s\mathcal{C}^{\mathcal{G}_1}(R)$. There are two cases to consider.

(i) Suppose that (i) of Condition A is not satisfied. Then there exist a pair $\{m, w\}$ and a coalition S such that

$$S \in \mathcal{G}_1, \tag{4}$$
$$\{m, w\} \subset S, \tag{5}$$
$$\{m, w\} \notin \mathcal{G}_2. \tag{6}$$

Consider a preference profile $R \in \mathcal{P}$ that satisfies

$$R^m : (w, m, \cdots), \tag{7}$$
$$R^w : (m, w, \cdots), \tag{8}$$
$$(i \in M \cup W \ \&\ i \neq m, w) \Rightarrow R^i : (i, \cdots). \tag{9}$$

Consider two matchings μ and ν such that

$$i \in M \cup W \Rightarrow \mu(i) = i; \tag{10}$$

$$\nu(m) = w, \tag{11}$$

$$(i \in M \cup W \ \& \ i \neq m, w) \Rightarrow \nu(i) = i. \tag{12}$$

Clearly, ν strongly dominates μ via $\{m, w\}$. Note that no other matching strongly dominates μ via any coalition. Then since $\{m, w\} \notin \mathcal{G}_2$ (by (6)), it follows $\mu \in w\mathcal{C}^{\mathcal{G}_2}(R)$.

On the other hand, since $\nu(S) = S$ (by (5), (11) and (12)), ν weakly dominates μ via S. And $S \in \mathcal{G}_1$ (by (4)). These imply $\mu \notin s\mathcal{C}^{\mathcal{G}_1}(R)$. Thus $w\mathcal{C}^{\mathcal{G}_2}(R) \not\subset s\mathcal{C}^{\mathcal{G}_1}(R)$.

(ii) Suppose that (ii) of Condition A is not satisfied. Then without loss of generality, we may assume that there exist $m \in M$ and a coalition S such that

$$m \in S \in \mathcal{G}_1, \tag{13}$$

$$W \not\subset S, \tag{14}$$

$$\{m\} \notin \mathcal{G}_2. \tag{15}$$

Now fix one $w \in W$ such that $w \notin S$. (This is possible due to (14).)

Consider a preference profile $R \in \mathcal{P}$ that satisfies

$$R^m : (m, w, \cdots), \tag{16}$$

$$R^w : (m, w, \cdots), \tag{17}$$

$$(i \in M \cup W \ \& \ i \neq m, w) \Rightarrow R^i : (i, \cdots). \tag{18}$$

Consider two matchings μ and ν such that

$$\mu(m) = w, \tag{19}$$

$$(i \in M \cup W \ \& \ i \neq m, w) \Rightarrow \mu(i) = i; \tag{20}$$

$$i \in M \cup W \Rightarrow \nu(i) = i. \tag{21}$$

Clearly, ν strongly dominates μ via $\{m\}$. Note that no other matching strongly dominates μ via any coalition. Then since $\{m\} \notin \mathcal{G}_2$ (by (15)), it follows $\mu \in w\mathcal{C}^{\mathcal{G}_2}(R)$.

On the other hand, since $\nu(S) = S$ (by (21)), ν weakly dominates μ via S. And $S \in \mathcal{G}_1$ (by (13)). These imply $\mu \notin s\mathcal{C}^{\mathcal{G}_1}(R)$. Thus $w\mathcal{C}^{\mathcal{G}_2}(R) \not\subset s\mathcal{C}^{\mathcal{G}_1}(R)$. $\square$

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