Supplementary Note for "Secure Implementation in Economies with Indivisible Objects and Money"

Yuji Fujinaka Takuma Wakayama

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In this supplementary note, we first discuss the tightness of Proposition 1. Next, we study the question of what class of social choice functions are securely implementable in environments where the domain is not minimally rich. Finally, we demonstrate that minimal richness is much weaker than box-shapedness.

1 Independence of Axioms

We will verify the independence of *strategy-proofness* and the *rectangular property*. In what follows, we will exhibit a social choice function that satisfies neither *strategy-proofness* nor the *rectangular property*.

Example (Dropping *strategy-proofness*). Let f be a social choice function such that for each $i \in I$, σ_i is constant, and for each $v \in V$,

$$m_1(v) = v_1,$$
$$m_i(v) = -\frac{v_1}{n-1} \quad \text{for each } i \neq 1.$$

Then, the social choice function satisfies the *rectangular property*, but violates *strategy-proofness*. ■

Example (Dropping the *rectangular property*). Let f be a social choice function such that for each $i \in I$, σ_i is constant, and for each $v \in V$,

$$m_n(v) = v_1,$$
$$m_i(v) = v_{i+1} \quad \text{for each } i \neq n.$$

Then, the social choice function satisfies *strategy-proofness*, but violates the *rectangular property*. ■

2 Non-Dominance

We briefly examine the class of securely implementable social choice functions in environments where the domain is not minimally rich.

We now introduce two conditions of social choice functions: (1) *dual dominance* proposed by Saijo (1987) and (2) *non-dominance*, which is a new notion. To define these conditions formally, we introduce some extra notation. Given $i \in I$, $v_i \in V_i$, and $(t, m_i) \in T \times \mathbb{R}$, we define the lower contour set of v_i at (t, m_i) by $L(v_i, (t, m_i)) \equiv \{(t', m_i') \in T \times \mathbb{R} \colon v_i(t) + m_i \geq v_i(t') + m_i'\}$ and the set of monotonic transformation of v_i at (t, m_i) by $MT(v_i, (t, m_i)) \equiv \{v_i' \in V_i \colon L(v_i, (t, m_i)) \subseteq L(v_i', (t, m_i))\}$. Let $f(V) \equiv \{(\sigma, m) \in T^I \times \mathbb{R}^I \colon f(v) = (\sigma, m)$ for some $v \in V\}$ be the range of f over V.

***Dual Dominance*:** For each $(\sigma', m'), (\sigma'', m'') \in f(V)$, there exists $v', v'', v^* \in V$ such that

(i) $(\sigma', m') = f(v')$ and $(\sigma'', m'') = f(v'')$,

(ii) for each $i \in I$, $v_i^* \in MT(v_i', (\sigma_i', m_i')) \cap MT(v_i'', (\sigma_i'', m_i''))$.

***Non-Dominance*:** For each $(\sigma', m'), (\sigma'', m'') \in f(V)$, each $i \in I$, and each $v', v'' \in V$ with $(\sigma', m') = f(v')$ and $(\sigma'', m'') = f(v'')$, if $MT(v_i', (\sigma_i', m_i')) \cap MT(v_i'', (\sigma_i'', m_i'')) = \emptyset$, there exists $v_i^* \in V_i$ such that

(i) $\Delta v_i^*(\sigma_i'; \sigma_i'') = m_i'' - m_i'$,

(ii) $\Delta v_i'(t; \sigma_i'') \geq \Delta v_i^*(t; \sigma_i'')$ for each $t \in T \setminus \{\sigma_i', \sigma_i''\}$.

Remark. The following statement vacuously holds: If a social choice function satisfies *dual dominance*, then it also satisfies *non-dominance*.

Although *non-dominance* is not a domain condition, it is similar to minimal richness. The reason for referring to *non-dominance* is that it is a condition when there is no profile of valuations that "dominates" v_i' at (σ_i', m_i') and "dominates" v_i'' at (σ_i'', m_i''); i.e., $MT(v_i', (\sigma_i', m_i')) \cap MT(v_i'', (\sigma_i'', m_i'')) = \emptyset$.

Saijo (1987) establishes that *dual dominance* together with *Maskin monotonicity* (Maskin, 1999) implies constancy. Therefore, obviously, it is established that a social choice function satisfying *dual dominance* is securely implementable if and only if it is constant. We prove that such a similar constancy theorem can be established even if *dual dominance* is replaced with *non-dominance*.

Theorem 2. A social choice function f satisfying *non-dominance* is securely implementable if and only if it is constant.

The proof of Theorem 2 is similar to that of Proposition 1 in the paper in terms of structure. Therefore, we omitted it.

In our model, *non-dominance* is a very weak requirement because any reasonable social choice function satisfies the condition. Therefore, from Theorem 2, it follows that any reasonable social choice function can almost never be securely implemented. Therefore, Theorem 2 is a partial answer to the open question mentioned in Section 4 in the paper. It leaves the open question of what class of social choice functions are securely implementable in economies with indivisible objects and money, where the domain is not minimally rich.

3 The Relationship Between Minimal Richness and Box-Shapedness

In our paper, we provide a new domain-richness condition, called *minimal richness*. On the other hand, in a previous version of our paper, we propose *box-shapedness* and characterize the set of secure implementable social choice functions on any box-shaped domain. We below show that minimal richness is much weaker than box-shapedness.

Fact. If a domain V is box-shaped, then it is also minimally rich.

Proof. Suppose that V is a box-shaped domain. Let $i \in I$, $v'_i, v''_i \in V_i$, $t', t'' \in T$, and $M \in \mathbb{R}$ be such that $\Delta v'_i(t';t'') > M > \Delta v''_i(t';t'')$. That is, $v'_i(t') - v'_i(t'') > M > v''_i(t') - v''_i(t'')$. Then, we have either $v'_i(t') > v''_i(t')$ or $v'_i(t'') < v''_i(t'')$. Otherwise, since $v'_i(t') \leq v''_i(t')$ and $v'_i(t'') \geq v''_i(t'')$, it follows that $v'_i(t') - v'_i(t'') \leq v''_i(t') - v''_i(t'')$, which is a contradiction. Without loss of generality, we assume that $v'_i(t') > v''_i(t')$. There are two cases.

Case 1. $v'_i(t') - v'_i(t'') > M \geq v''_i(t') - v'_i(t'')$: Since $v'_i(t') - v'_i(t'') > M \geq v''_i(t') - v'_i(t'')$, we have $v'_i(t') > M + v'_i(t'') \geq v''_i(t')$. Then, since V is a box-shaped domain, there exists $v^*_i \in V_i$ such that

(i) $v^*_i(t') \in [v''_i(t'), v'_i(t')]$,

(ii) $v^*_i(t') = M + v'_i(t'')$,

(iii) $v^*_i(t) = v'_i(t)$ for each $t \in T \setminus \{t'\}$.

Thus, we obtain

$$\begin{aligned} \Delta v^*_i(t';t'') &= M, \\ \Delta v^*_i(t;t'') &= \Delta v'_i(t;t'') \text{ for each } t \in T \setminus \{t', t''\}. \end{aligned}$$

Case 2. $v'_i(t') - v'_i(t'') > v''_i(t') - v'_i(t'') > M > v''_i(t') - v''_i(t'')$: Since $v''_i(t') - v'_i(t'') > M > v''_i(t') - v''_i(t'')$, $v'_i(t'') < v''_i(t') - M < v''_i(t'')$, which implies that $v'_i(t'') < v''_i(t'')$. Since V is a box-shaped domain, there exists $v^*_i \in V_i$ such that

(i) $v^*_i(t'') \in [v'_i(t''), v''_i(t'')]$,

(ii) $v^*_i(t'') = v''_i(t') - M$,

(iii) $v^*_i(t') = v''_i(t')$,

(iv) $v_i^*(t) = v_i'(t)$ for each $t \in T \setminus \{t', t''\}$.

Thus, we obtain

$$\Delta v_i^*(t'; t'') = M,$$
$$\Delta v_i'(t; t'') \geq \Delta v_i^*(t; t'') \text{ for each } t \in T \setminus \{t', t''\}.$$

Hence, V is also minimally rich. □

The next example illustrates that there exists a minimally rich domain that is not box-shaped.

Example. Suppose that $T = \{1, 2, 3\}$ and for each $i \in I$, $V_i(1) = \mathbb{R} \setminus \{1\}$ and $V_i(2) = V_i(3) = \mathbb{R}$. Obviously, V is not box-shaped. To observe that V is minimally rich, let $i \in I$, $v_i', v_i'' \in V_i$, $t', t'' \in T$, and $M \in \mathbb{R}$ be such that $\Delta v_i'(t'; t'') > M > \Delta v_i''(t'; t'')$.

Case 1. $t' \neq 1$ and $t'' \neq 1$: Without loss of generality, we assume that $t' = 2$ and $t'' = 3$. Let v_i^* be such that

(i) $v_i^*(1) \in (-\infty, v_i'(1)] \cap (-\infty, 1)$,

(ii) $v_i^*(2) = M + v_i'(3)$,

(iii) $v_i^*(3) = v_i'(3)$.

It is easy check that $v_i^* \in V_i$, $\Delta v_i^*(2; 3) = M$, and $\Delta v_i'(1; 3) \geq \Delta v_i^*(1; 3)$.

Case 2. $t' = 1$ or $t'' = 1$: Without loss of generality, we assume that $t' = 1$ and $t'' = 2$. Let v_i^* be such that

(i) $v_i^*(1) = v_i'(1)$,

(ii) $v_i^*(2) = v_i'(1) - M$,

(iii) $v_i^*(3) \in (-\infty, v_i'(3) - v_i'(2) + v_i'(1) - M]$.

It is easy check that $v_i^* \in V_i$, $\Delta v_i^*(1; 2) = M$, and $\Delta v_i'(3; 2) \geq \Delta v_i^*(3; 2)$. ■

Therefore, minimal richness is much weaker than box-shapedness.

References

Maskin, E. (1999) "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies* **66**, 23–38.

Saijo, T. (1987) "On Constant Maskin Monotonic Social Choice Functions," *Journal of Economic Theory* **42**, 382–386.