Supplementary Note for "Coalitionally Strategy-Proof Rules in Allotment Economies with Homogeneous Indivisible Goods"

Kentaro Hatsumi* and Shigehiro Serizawa†

First Version: May 29, 2007
This Version: January 13, 2009

In this supplementary note, we provide the proof of Fact 1 and detailed explanations for Example 4, 5, and 6. The *only if* part of Fact 1 is proved by Sasaki (1997) and the *if* part is by Kureishi (2000). However, both papers have not been published. Thus, we provide the proof for completeness. In the explanations for Examples 4, 5, and 6, we provide tables that make it easy to check especially coalitionally strategy-proof conditions.

Fact 1. [Sasaki 1997, Kureishi 2000] A marginal distribution profile $p \in P$ satisfies Pareto-efficiency with respect to u if and only if it satisfies same-sideness with respect to u and at most binary.

Proof of Fact 1. First, we show the *only if* part of the fact, and then, prove the *if* part.

(A) *only if* part. Let $u \in U^n$. Let $p \in P$ be a marginal distribution profile satisfying Pareto-efficiency with respect to u.

We show that (i) p satisfies same-sideness with respect to u, and (ii) it also satisfies at most binary.

(i) Suppose, on the contrary, p does not satisfy same-sideness with respect to u.

Without loss of generality, assume $\sum_{i \in N} b(u_i) > k$ since the other case is treated symmetry. Since p violates same-sideness, there exist $i \in N$ such that for some $x \in (b(u_i), k]$, $p_i(x) > 0$. Without loss of generality, assume $i = 1$.

By feasibility, there also exists $j \in N \backslash \{1\}$ such that for some $y \in [0, b(u_j))$, $p_j(y) > 0$ and the following $p' \in P$ is well defined. Without loss of generality, assume $j = 2$.

Let $p' \in P$ be such that

$$\begin{cases} p'_1(x) = p_1(x) - \epsilon, p'_1(x-1) = p_1(x-1) + \epsilon, \text{ for all } z \in K \backslash \{x-1, x\}, p'_1(z) = p_1(z), \\ p'_2(y) = p_2(y) - \epsilon, p'_2(y+1) = p_2(y+1) + \epsilon, \text{ for all } z \in K \backslash \{y, y+1\}, p'_2(z) = p_2(z), \\ \text{for all } h \in N \backslash \{1, 2\} \text{ and all } v \in K, p'_h(v) = p_h(v). \end{cases}$$

Then, single-peakedness implies that $E(p'_1, u_1) > E(p_1, u_1)$, $E(p'_2, u_2) > E(p_2, u_2)$, and for all $j \in N \backslash \{1, 2\}$, $E(p'_j; u_j) = E(p_j; u_j)$. It contradicts Pareto-efficiency of p.

Thus, we have that p satisfies same-sideness with respect to u.

(ii) Suppose, on the contrary, p does not satisfy at most binary.

*Graduate School of Economics, Osaka University. 1-7, Machikaneyama, Toyonaka, 560-0043, Japan. E-Mail: egp020hk@mail2.econ.osaka-u.ac.jp

†Institute of Social and Economic Research, Osaka University. 6-1, Mihogaoka, Ibaraki, 567-0047, Japan. E-Mail: serizawa@iser.osaka-u.ac.jp

Without loss of generality, assume $\sum_{i \in N} b(u_i) > k$ since the other case is treated symmetry. Since p violates at most binary, there exists $i \in N$ such that for some $x, y \in K$ with $x + 1 < y$, $p_i(x) > 0$ and $p_i(y) > 0$. Without loss of generality, assume $i = 1$.

By feasibility, there also exists $j \in N \backslash \{1\}$ such that for some $z, w \in K$ with $z < w$, $p_j(z) > 0$ and $p_j(w) > 0$ and the following $p' \in P$ is well defined. Without loss of generality, assume $j = 2$. Since p satisfies same-sideness by (i), $x, y \in [0, b(u_1)]$ and $z, w \in [0, b(u_2)]$.

Let $p' \in P$ be such that

$$\begin{cases} p'_1(x) = p_1(x) - \epsilon, p'_1(x+1) = p_1(x+1) + \epsilon, p'_1(y) = p_1(y) - \epsilon, p'_1(y-1) = p_1(y-1) + \epsilon, \\ \text{for all } v \in K \backslash \{x, x+1, y-1, y\}, p'_1(v) = p_1(v), \\ p'_2(z) = p_2(z) - \epsilon, p'_2(z+1) = p_2(z+1) + \epsilon, p'_2(w) = p_2(w) - \epsilon, p'_2(w-1) = p_2(w-1) + \epsilon, \\ \text{for all } v \in K \backslash \{z, x+1, w-1, w\}, p'_2(v) = p_2(v), \\ \text{for all } j \in N \backslash \{1, 2\} \text{ and all } v \in K, p'_j(v) = p_j(v). \end{cases}$$

Then, single-peakedness and risk-averseness imply that $E(p'_1, u_1) > E(p_1, u_1)$, $E(p'_2, u_2) \geq E(p_2, u_2)$, and for all $h \in N \backslash \{1, 2\}$, $E(p'_h; u_h) = E(p_h; u_h)$. It contradicts Pareto-efficiency of p.

Thus, we have that p satisfies at most binary.

(B) *if* part. Let $u \in U^n$. Let $p \in P$ be a marginal distribution profile satisfying same-sideness with respect to u and at most binary. We show that p is Pareto-efficient with respect to u. Without loss of generality, assume $\sum_{i \in N} b(u_i) \geq k$, since the other case is treated symmetry.

Suppose, on the contrary, p does not satisfy Pareto-efficiency with respect to u. Then, there exists $p' \in P$ such that for all $i \in N$, $E(p'_i; u_i) \geq E(p_i, u_i)$, and for some $j \in N$, $E(p'_j; u_j) > E(p_j; u_j)$.

For all $i \in N$, let $\lambda_i \in \mathbb{R}_+$ be such that $\lambda_i = \sum_{x \in K} p'_i \cdot x$, and $x_i \in K$ be such that $\lambda_i \in [x_i, x_i + 1)$. Note that $[1 - (\lambda_i - x_i)] \cdot x_i + (\lambda_i - x_i) \cdot (x_i + 1) = \lambda$. Therefore, by feasibility, $\sum_{i \in N} \{[1 - (\lambda_i - x_i)] \cdot x_i + (\lambda_i - x_i) \cdot (x_i + 1)\} = k$.

Also note that single-peakedness and risk-averseness imply that for all $i \in N$,

$$[1 - (\lambda_i - x_i)] \cdot u_i(x_i) + (\lambda_i - x_i) \cdot u_i(x_i + 1) \geq E(p'_i; u_i). \tag{1}$$

Since p satisfies both same-sideness with respect to u and at most binary, for all $i \in N$, there exists $y_i \in K$ such that $y_i + 1 \leq b(u_i)$ and $p_i(y_i) + p_i(y_i + 1) = 1$.

Then, together with the assumption that p' Pareto-dominates p, (1) implies that for all $i \in N$,

$$[1 - (\lambda_i - x_i)] \cdot u_i(x_i) + (\lambda_i - x_i) \cdot u_i(x_i + 1) \geq p_i(y_i) \cdot u_i(y_i) + p_i(y_i + 1) \cdot u_i(y_i + 1), \tag{2}$$

and for some $j \in N$,

$$[1 - (\lambda_j - x_j)] \cdot u_j(x_j) + (\lambda_j - x_j) \cdot u_j(x_j + 1) > p_j(y_j) \cdot u_j(y_j) + p_j(y_j + 1) \cdot u_j(y_j + 1). \tag{3}$$

Then, by single-peakedness, (2) implies that for all $i \in N$,

$$x_i > y_i \text{ or } [x_i = y_i \text{ and } 1 - (\lambda_i - x_i) \leq p_i(y_i)]$$

and (3) implies that for some $j \in N$,

$$x_j > y_j \text{ or } [x_j = y_j \text{ and } 1 - (\lambda_j - x_j) < p_i(y_i)].$$

Thus, we have that $[1 - (\lambda_i - x_i)] \cdot x_i + (\lambda_i - x_i) \cdot (x_i + 1) \geq p_i(y_i) \cdot y_i + p_i(y_i + 1) \cdot (y_i + 1)$, and for some $j \in N$, $[1 - (\lambda_j - x_j)] \cdot x_j + (\lambda_j - x_j) \cdot (x_j + 1) > p_j(y_j) \cdot y_j + p_j(y_j + 1) \cdot (y_j + 1)$.

The summing up implies that $\sum_{i \in N} \{[1 - (\lambda_i - x_i)] \cdot x_i + (\lambda_i - x_i) \cdot (x_i + 1)\} > \sum_{i \in N} \{(p_i(y_i) \cdot y_i + p_i(y_i + 1) \cdot (y_i + 1)\}$. However, since both sides equal k by feasibility, it is a contradiction.

Hence, we have that p is Pareto-efficient with respect to u. □

Example 4. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as below:
If $u \in U^3$ is such that for one agent, say i, $b(u_i) = 1$ and for any other agent $j \in N \backslash \{i\}$, $b(u_j) = 0$, then, (i) in the case of $u_i(1) - u_i(0) \geq u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(1) = \frac{18}{20}, f_i(u)(2) = \frac{2}{20} \\ f_j(u)(0) = \frac{11}{20}, f_j(u)(1) = \frac{9}{20} \end{cases}$$

and (ii) in the case of $u_i(1) - u_i(0) < u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(0) = \frac{2}{20}, f_i(u)(1) = \frac{18}{20} \\ f_j(u)(0) = \frac{9}{20}, f_j(u)(1) = \frac{11}{20}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, although the probabilistic rule f satisfies coalitional strategy-proofness, respect for unanimity, and strong symmetry, it is not the uniform probabilistic rule.

We show that f satisfies respect for unanimity, strong symmetry, and coalitional strategy-proofness. However, since it is obvious that f satisfies respect for unanimity and strong symmetry, we check only coalitional strategy-proofness.

Table 1 attached to the end of this note is designed to check coalitional strategy-proofness.

For each agent $i \in N$, the utility function of agent i is classified into four types; the type of utility functions with $b(u_i) = 0$, the type of utility functions such that $b(u_i) = 1$ and (i) $u_i(1) - u_i(0) \geq u_i(1) - u_i(2)$, the type of utility function such that $b(u_i) = 1$ and (ii) $u_i(1) - u_i(0) < u_i(1) - u_i(2)$, and the type of utility functions with $b(u_i) = 2$. Since there are three agents in this example, utility profiles are classified into $64 (= 4^3)$ types. Note that the rule f assigns the same marginal distribution profile to utility profiles of the same type.

Table 1 describes how the rule f assigns to each type of utility profile a marginal distribution profile. Table 1 comprises four matrices. Each matrix corresponds to a type of agent 3's utility function; The first matrix corresponds to the type of agent 3's utility function with $b(u_3) = 0$, the second corresponds to the type of agent 3's utility function such that $b(u_3) = 1$ and (i) $u_3(1) - u_3(0) \geq u_3(1) - u_3(2)$, the third corresponds to the type of agent 3's utility function such that $b(u_3) = 1$ and (ii) $u_3(1) - u_3(0) < u_3(1) - u_3(2)$, and the fourth corresponds to the type of agent 3's utility function such that $b(u_3) = 2$. The last four rows of each matrix correspond to the types of agent 1's utility function. The row dented "0" in the first column corresponds to the type of agent 1's utility function with $b(u_1) = 0$, the row dented "1 case (i)" in the first column corresponds to the type of agent 1's utility function such that $b(u_1) = 1$ and (i) $u_1(1) - u_1(0) \geq u_1(1) - u_1(2)$, and so on. Similarly, the last four columns of each matrix correspond to the types of agent 2's utility function.

Marginal distribution profiles are described as 3×3 matrices nested in the corresponding cells of the four matrices. For example, the matrix

$$\begin{pmatrix} 11/20 & 9/20 & 0 \\ 0 & 18/20 & 2/20 \\ 11/20 & 9/20 & 0 \end{pmatrix}$$

is nested in the cell corresponding to the utility profiles such that $b(u_1) = 0$, $b(u_2) = 1$,(i) $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$, and $b(u_3) = 0$. The first rows of the nested matrices corresponds to agent 1's allocations, the second rows to agent 2's allocations, and the third rows to agent 3's allocations. The first columns of the nested matrices corresponds to the probability that agents receive 0 units, the second columns to the probability that agents receive 1 units, and the third columns to the probability that agents receive 2 units. For example, the nested matrix

$$\begin{pmatrix} 11/20 & 9/20 & 0 \\ 0 & 18/20 & 2/20 \\ 11/20 & 9/20 & 0 \end{pmatrix}$$

implies that agent 1 receives 0 unit with probability $\frac{11}{20}$, 1 unit with probability $\frac{9}{20}$, and 2 units with probability 0; agent 2 receives 0 unit with probability 0, 1 unit with probability $\frac{18}{20}$, and 2 units with probability $\frac{2}{20}$; and agent 3 receives 0 unit with probability $\frac{11}{20}$, 1 unit with probability $\frac{9}{20}$, and 2 units with probability 0. The bold fonts of the nested matrices indicates that the marginal distribution profiles are different from those assigned by the uniform probabilistic rule, while the regular font shows that the marginal distribution profiles coincide with those assigned by the uniform probabilistic rule.

We use Table 1 to show that the rule f in Example 4 is coalitional strategy-proof. Let $u \in U^n$, $N' \subseteq N$, and $\hat{u}_{N'} \in U^{N'}$. We need to show that whenever there is $i \in N'$ such that $E(f_i(\hat{u}_{N'}, u_{-N'}); u_i) > E(f_i(u); u_i)$, there exists $j \in N'$ such that $E(f_j(u); u_j) > E(f_j(\hat{u}_{N'}, u_{-N'}); u_j)$. However, it is sufficient to show that (a) or (b) below holds:

(a) for any $i \in N'$, $E(f_i(u); u_i) \geq E(f_i(\hat{u}_{N'}, u_{-N'}); u_i)$,

(b) there exists $i \in N'$ such that $E(f_i(u); u_i) > E(f_i(\hat{u}_{N'}, u_{-N'}); u_i)$.

Since it is routine to show that (a) or (b) holds for all possible $u \in U^n$, we check only utility profiles I) and II) below for demonstration purposes:

I) the utility profile u such that $b(u_1) = 0$, $b(u_2) = 1$, (i) $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$, and $b(u_3) = 0$,

II) the utility profile u such that $b(u_1) = 2$, $b(u_2) = 1$, (i) $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$, and $b(u_3) = 0$.

In the first utility profile, we emphasize the role of the inequality $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$. In the second, we explain that by coalitional strategy-proofness of the uniform probabilistic rule, we can omit checking many possible cases of $\hat{u}_{N'}$.

I) First, consider the utility profile u such that $b(u_1) = 0$, $b(u_2) = 1$, (i) $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$, and $b(u_3) = 0$. Then, $f(u)$ is represented by a nested matrix of bold font

$$\begin{pmatrix} 11/20 & 9/20 & 0 \\ 0 & 18/20 & 2/20 \\ 11/20 & 9/20 & 0 \end{pmatrix}.$$

If $b(\hat{u}_1) = 0$, $b(\hat{u}_2) = 1$, (i) $\hat{u}_2(1) - \hat{u}_2(0) \geq \hat{u}_2(1) - \hat{u}_2(2)$, and $b(\hat{u}_3) = 0$, then since $f(\hat{u}_{N'}, u_{-N'}) = f(u)$, (a) holds. Thus, in the following, we omit checking the case where $b(\hat{u}_1) = 0$, $b(\hat{u}_2) = 1$, (i) $\hat{u}_2(1) - \hat{u}_2(0) \geq \hat{u}_2(1) - \hat{u}_2(2)$, and $b(\hat{u}_3) = 0$.

Case 1: $\#N' = 1$ In this case, we show that (a) holds.

Case 1-1: $N' = \{1\}$. If $b(\hat{u}_1) \neq 0$, then since $b(u_1) = 0$,

$$E(f_1(u); u_1) = \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \geq u_1(1) = E(f_1(\hat{u}_1, u_{-1}); u_1).$$

Case 1-2: $N' = \{2\}$. If $b(\hat{u}_2) = 0$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &\geq \tfrac{1}{3} \cdot u_2(0) + \tfrac{2}{3} \cdot u_2(1) = E(f_2(\hat{u}_2, u_{-2}); u_2). \end{aligned}$$

If $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &\geq \tfrac{2}{20} \cdot u_2(0) + \tfrac{18}{20} \cdot u_2(1) = E(f_2(\hat{u}_2, u_{-2}); u_2). \end{aligned}$$

If $b(\hat{u}_2) = 2$, then since $b(u_2) = 1$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &\geq u_2(2) = E(f_2(\hat{u}_2, u_{-2}); u_2). \end{aligned}$$

Case 1-3: $N' = \{3\}$. If $b(\hat{u}_3) \neq 0$, since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_3) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &\geq u_3(1) = E(f_3(\hat{u}_2, u_{-2}); u_2). \end{aligned}$$

Case 2: $\#N' = 2$ In this case we check that (b) holds.
Case 2-1: $N' = \{1, 2\}$.
Case 2-1-1: $b(\hat{u}_1) = 0$.
If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{1}{3} \cdot u_1(0) + \tfrac{2}{3} \cdot u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{9}{20} \cdot u_1(0) + \tfrac{11}{20} \cdot u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) = 2$, then since $b(u_2) = 1$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_1(1) + \tfrac{2}{20} \cdot u_1(2) \\ &> u_2(2) = E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

Case 2-1-2: $b(\hat{u}_1) = 1$ and $\hat{u}_1(1) - \hat{u}_1(0) \geq \hat{u}_1(1) - \hat{u}_1(2)$.
If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{18}{20} \cdot u_1(1) + \tfrac{2}{20} \cdot u_1(2) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 2-1-3: $b(\hat{u}_1) = 1$ and $\hat{u}_1(1) - \hat{u}_1(0) < \hat{u}_1(1) - \hat{u}_1(2)$.
If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{2}{20} \cdot u_1(0) + \tfrac{18}{20} \cdot u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 2-1-4: $b(\hat{u}_1) = 2$.
If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(2) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 2-2: $N' = \{1, 3\}$.
Case 2-2-1: $b(\hat{u}_1) = 0$.
If $b(\hat{u}_3) \neq 0$, then since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_1) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &> u_3(1) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

Case 2-2-2: $b(\hat{u}_1) = 1$.
If $b(\hat{u}_3) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_3) \neq 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{1}{3} \cdot u_1(0) + \tfrac{2}{3} \cdot u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 2-2-3: $b(\hat{u}_1) = 2$.
If $b(\hat{u}_3) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_3) \neq 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> \tfrac{1}{3} \cdot u_1(0) + \tfrac{2}{3} \cdot u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 2-3: $N' = \{2, 3\}$.
Case 2-3-1: $b(\hat{u}_3) = 0$.
If $b(\hat{u}_2) = 0$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{1}{3} \cdot u_2(0) + \tfrac{2}{3} \cdot u_2(2) = E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

If $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{2}{20} \cdot u_2(0) + \tfrac{18}{20} \cdot u_2(1) = E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

If $b(\hat{u}_2) = 2$, then since $b(u_2) = 1$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_3(0) + \tfrac{2}{20} \cdot u_3(1) \\ &> u_3(1) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

Case 2-3-2: $b(\hat{u}_3) = 1$ and $\hat{u}_3(1) - \hat{u}_3(0) \geq \hat{u}_3(1) - \hat{u}_3(2)$.
If $b(\hat{u}_2) = 0$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{11}{20} \cdot u_2(0) + \tfrac{9}{20} \cdot u_2(1) = E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_3) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &> u_3(1) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

Case 2-3-3: $b(\hat{u}_3) = 1$ and $u_3(1) - u_3(0) < u_3(1) - u_3(2)$.
If $b(\hat{u}_2) = 0$, then since $b(\hat{u}_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{9}{20} \cdot u_2(0) + \tfrac{11}{20} \cdot u_2(1) = E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_3) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &> u_3(1) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

Case 2-3-4: $b(\hat{u}_3) = 2$.
If $b(\hat{u}_2) = 0$, then since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_3) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &> u_3(2) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

If $b(\hat{u}_2) \neq 0$, then since $b(u_3) = 0$,

$$\begin{aligned} E(f_3(u); u_3) &= \tfrac{11}{20} \cdot u_3(0) + \tfrac{9}{20} \cdot u_3(1) \\ &> u_3(1) = E(f_3(\hat{u}_{N'}, u_{-N'}); u_3). \end{aligned}$$

Case 3: $N' = N$ In this case, we show that (b) holds. Note that it is sufficient to show that there is $i \in N$ such that $\hat{u}_i \neq u_i$ and $E(f_i(u); u_i) > E(f_i(\hat{u}_{N'}, u_{-N'}); u_i)$. Owing to Case 2-1, we can omit checking the case where $b(\hat{u}_3) = 0$.

Case 3-1: $b(\hat{u}_3) = 1$.

Owing to Case 2-3-2, we can omit checking the case where $b(\hat{u}_1) = 0$. Owing to Case 2-2, we can also omit checking the case where $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) \geq \hat{u}_2(1) - \hat{u}_2(2)$.

If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, or if $b(\hat{u}_2) = 2$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{2}{3} \cdot u_2(0) + \tfrac{1}{3} \cdot u_2(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 3-2: $b(\hat{u}_3) = 2$.

Owing to Case 2-3-4, we can omit checking the case where $b(\hat{u}_1) = 0$. Owing to Case 2-2, we can also omit checking the case where $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) \geq \hat{u}_2(1) - \hat{u}_2(2)$.

If $b(\hat{u}_2) = 0$, then since $b(u_1) = 0$,

$$\begin{aligned} E(f_1(u); u_1) &= \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &> u_1(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, or if $b(\hat{u}_2) = 2$, then since $b(u_2) = 1$ and $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$,

$$\begin{aligned} E(f_2(u); u_2) &= \tfrac{18}{20} \cdot u_2(1) + \tfrac{2}{20} \cdot u_2(2) \\ &> \tfrac{2}{3} \cdot u_2(0) + \tfrac{1}{3} \cdot u_2(1) = E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

II) Next, consider the utility profile u such that $b(u_1) = 2$, $b(u_2) = 1$, (i) $u_2(1) - u_2(0) \geq u_2(1) - u_2(2)$, and $b(u_3) = 0$. Then, $f(u)$ is equal to the marginal distribution profile that the uniform probabilistic rule assigns to u, and it is represented by a nested matrix of regular font

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Thus, since the uniform probabilistic rule is coalitionally strategy-proof, we do not need to check the case where $f(\hat{u}_{N'}, u_{-N'})$ is equal to the marginal distribution profile that the uniform probabilistic rule assigns to $(\hat{u}_{N'}, u_{-N'})$. Therefore, we only need to check the case where $f(\hat{u}_{N'}, u_{-N'})$ is not equal to the marginal distribution profile that the uniform probabilistic rule assigns to $(\hat{u}_{N'}, u_{-N'})$, i.e., the case where $f(\hat{u}_{N'}, u_{-N'})$ is represented by nested matrices of bold font. Note that it is sufficient to show that there is $i \in N'$ such that $\hat{u}_i \neq u_i$ and $E(f_i(u); u_i) > E(f_i(\hat{u}_{N'}, u_{-N'}); u_i)$.

Case 1: $b(\hat{u}_3) = 0$.

If $b(\hat{u}_1) = 0$, and $b(\hat{u}_2) = 1$ and $\hat{u}_2 = u_2$, then since $b(u_1) = 2$,

$$\begin{aligned} E(f_1(u); u_1) &= u_1(1) \\ &> \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &= E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_1) = 0$, and $b(\hat{u}_2) = 1$ and $\hat{u}_2(1) - \hat{u}_2(0) < \hat{u}_2(1) - \hat{u}_2(2)$, then since $b(u_1) = 2$,

$$\begin{aligned} E(f_1(u); u_1) &= u_1(1) \\ &> \tfrac{9}{20} \cdot u_1(0) + \tfrac{11}{20} \cdot u_1(1) \\ &= E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

If $b(\hat{u}_1) = 1$ and $\hat{u}_1(1) - \hat{u}_1(0) \geq \hat{u}_1(1) - \hat{u}_1(2)$ and $b(\hat{u}_2) = 0$, then since $b(u_2) = 1$,

$$\begin{aligned} E(f_2(u); u_2) &= u_2(1) \\ &> \tfrac{11}{20} \cdot u_2(0) + \tfrac{9}{20} \cdot u_2(1) \\ &= E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

If $b(\hat{u}_1) = 1$ and $\hat{u}_1(1) - \hat{u}_1(0) < \hat{u}_1(1) - \hat{u}_1(2)$ and $b(\hat{u}_2) = 0$, then since $b(u_2) = 1$,

$$\begin{aligned} E(f_2(u); u_2) &= u_2(1) \\ &> \tfrac{9}{20} \cdot u_2(0) + \tfrac{11}{20} \cdot u_2(1) \\ &= E(f_2(\hat{u}_{N'}, u_{-N'}); u_2). \end{aligned}$$

Case 2: $b(\hat{u}_3) = 1$ and $\hat{u}_3(1) - \hat{u}_3(0) \geq \hat{u}_3(1) - \hat{u}_3(2)$.

In this case, $f(\hat{u}_{N'}, u_{-N'})$ is not equal to the marginal distribution profile that the uniform probabilistic rule assigns to $(\hat{u}_{N'}, u_{-N'})$ only when $b(\hat{u}_1) = b(\hat{u}_2) = 0$. If $b(\hat{u}_1) = b(\hat{u}_2) = 0$, then since $b(u_1) = 2$,

$$\begin{aligned} E(f_1(u); u_1) &= u_1(1) \\ &> \tfrac{11}{20} \cdot u_1(0) + \tfrac{9}{20} \cdot u_1(1) \\ &= E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Case 3: $b(\hat{u}_3) = 1$ and $\hat{u}_3(1) - \hat{u}_3(0) < \hat{u}_3(1) - \hat{u}_3(2)$.

In this case, $f(\hat{u}_{N'}, u_{-N'})$ is not equal to the marginal distribution profile that the uniform probabilistic rule assigns to $(\hat{u}_{N'}, u_{-N'})$ only when $b(\hat{u}_1) = b(\hat{u}_2) = 0$. If $b(\hat{u}_1) = b(\hat{u}_2) = 0$, then since $b(u_1) = 2$,

$$\begin{aligned} E(f_1(u); u_1) &= u_1(1) \\ &> \tfrac{9}{20} \cdot u_1(0) + \tfrac{11}{20} \cdot u_1(1) \\ &= E(f_1(\hat{u}_{N'}, u_{-N'}); u_1). \end{aligned}$$

Note that $f(\hat{u}_{N'}, u_{-N'})$ is always equal to the marginal distribution profile that the uniform probabilistic rule assigns to $(\hat{u}_{N'}, u_{-N'})$ in the case where $b(\hat{u}_3) = 2$. Thus, we do not need to check this case.

Example 5. Let $n = 4$ and $k = 2$. We define a probabilistic rule f as below:
If $u \in U^4$ is such that for one agent, say i, $b(u_i) = 0$, and for any other agent $j \in N \setminus \{i\}$, $b(u_j) \geq 1$, then

$$\begin{cases} f_i(u)(0) = \frac{27}{30}, f_i(u)(1) = \frac{3}{30} \\ f_j(u)(0) = \frac{11}{30}, f_j(u)(1) = \frac{19}{30} \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, although the rule f satisfies the four properties of coalitional strategy-proofness, respect for unanimity, strong symmetry, and peaks-onlyness, it is not the uniform probabilistic rule[1].

We check that f satisfies the properties. Similarly to Example 4, it is obvious that f in Example 5 satisfies respect for unanimity, strong symmetry and peaks-onlyness. We can check that f also satisfies coalitional strategy-proofness by using Table 2. In this example, utility profiles are classified into $81 (= 3^4)$ types. Each type coincides with each peak profile because of peaks-onlyness. The way to use Table 2 for checking coalitional strategy-proofness is the same as that for Table 1 in Example 4. Thus, we omit the detailed explanation.

Example 6. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as below:
For all $u \in U^3$, if $b(u_1) = 2$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(1) = \frac{1}{15}, f_1(u)(2) = \frac{13}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{27}{30}, f_2(u)(1) = f_3(u)(1) = \frac{3}{30}, \end{cases}$$

[1] In the case of $n \leq 3$ and $k = 2$, a rule satisfies coalitional strategy-proofness, respect for unanimity, strong symmetry, and peaks-onlyness if and only if it is the uniform probabilistic rule.

and if $b(u_1) = 1$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(1) = \frac{14}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{7}{15}, f_2(u)(1) = f_3(u)(1) = \frac{8}{15}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, the rule f satisfies strongly coalitional strategy-proofness and same-sideness, even though it violates at most binary.

We show that f satisfies the properties. It is easy to check same-sideness. By Table 3, we can check it also satisfies coalitional strategy-proofness. Since the way to use Table 3 is the same as that for Table 1 in Example 4, we omit the detailed explanation.

Table 1 (1)
the marginal probability distribution profiles induced by f in Example 4.

$b(u_3)$ = 0

	$b(u_2)$	0			1 case (i)			1 case (ii)			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2	0	1	2
0	agent 1	1/3	2/3	0	**11/20**	**9/20**	**0**	**9/20**	**11/20**	**0**	1	0	0
	agent 2	1/3	2/3	0	**0**	**18/20**	**2/20**	**2/20**	**18/20**	**0**	0	0	1
	agent 3	1/3	2/3	0	**11/20**	**9/20**	**0**	**9/20**	**11/20**	**0**	1	0	0
1 case (i)	agent 1	**0**	**18/20**	**2/20**	0	1	0	0	1	0	0	1	0
	agent 2	**11/20**	**9/20**	**0**	0	1	0	0	1	0	0	1	0
	agent 3	**11/20**	**9/20**	**0**	1	0	0	1	0	0	1	0	0
1 case (ii)	agent 1	**2/20**	**18/20**	**0**	0	1	0	0	1	0	0	1	0
	agent 2	**9/20**	**11/20**	**0**	0	1	0	0	1	0	0	1	0
	agent 3	**9/20**	**11/20**	**0**	1	0	0	1	0	0	1	0	0
2	agent 1	0	0	1	0	1	0	0	1	0	0	1	0
	agent 2	1	0	0	0	1	0	0	1	0	0	1	0
	agent 3	1	0	0	1	0	0	1	0	0	1	0	0

$b(u_3)$ = 1, case (i)

	$b(u_2)$	0			1 case (i)			1 case (ii)			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2	0	1	2
0	agent 1	**11/20**	**9/20**	**0**	1	0	0	1	0	0	1	0	0
	agent 2	**11/20**	**9/20**	**0**	0	1	0	0	1	0	0	1	0
	agent 3	**0**	**18/20**	**2/20**	0	1	0	0	1	0	0	1	0
1 case (i)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
1 case (ii)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
2	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 1 (2)
the marginal probability distribution profiles induced by f in Example 4.

$b(u_3)$ = 1, case (ii)

$b(u_1)$	$b(u_2)$ / # of objects	0			1 case (i)			1 case (ii)			2		
		0	1	2	0	1	2	0	1	2	0	1	2
0	agent 1	**9/20**	**11/20**	**0**	1	0	0	1	0	0	1	0	0
	agent 2	**9/20**	**11/20**	**0**	0	1	0	0	1	0	0	1	0
	agent 3	**2/20**	**18/20**	**0**	0	1	0	0	1	0	0	1	0
1 case (i)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
1 case (ii)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
2	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0

$b(u_3)$ = 2

$b(u_1)$	$b(u_2)$ / # of objects	0			1 case (i)			1 case (ii)			2		
		0	1	2	0	1	2	0	1	2	0	1	2
0	agent 1	1	0	0	1	0	0	1	0	0	1	0	0
	agent 2	1	0	0	0	1	0	0	1	0	0	1	0
	agent 3	0	0	1	0	1	0	0	1	0	0	1	0
1 case (i)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
1 case (ii)	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
2	agent 1	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 2	1	0	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0
	agent 3	0	1	0	1/3	2/3	0	1/3	2/3	0	1/3	2/3	0

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 2 (1)
the marginal probability distribution profiles induced by f in Example 5.

$b(u_3) = 0$, $b(u_4) = 0$

	$b(u_2)$	0			1			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2
0	agent 1	1/2	1/2	0	2/3	1/3	0	1	0	0
	agent 2	1/2	1/2	0	0	1	0	0	0	1
	agent 3	1/2	1/2	0	2/3	1/3	0	1	0	0
	agent 4	1/2	1/2	0	2/3	1/3	0	1	0	0
1	agent 1	0	1	0	0	1	0	0	1	0
	agent 2	2/3	1/3	0	0	1	0	0	1	0
	agent 3	2/3	1/3	0	1	0	0	1	0	0
	agent 4	2/3	1/3	0	1	0	0	1	0	0
2	agent 1	0	0	1	0	1	0	0	1	0
	agent 2	1	0	0	0	1	0	0	1	0
	agent 3	1	0	0	1	0	0	1	0	0
	agent 4	1	0	0	1	0	0	1	0	0

$b(u_3) = 0$, $b(u_4) = 1$

	$b(u_2)$	0			1			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2
0	agent 1	2/3	1/3	0	1	0	0	1	0	0
	agent 2	2/3	1/3	0	0	1	0	0	1	0
	agent 3	2/3	1/3	0	1	0	0	1	0	0
	agent 4	0	1	0	0	1	0	0	1	0
1	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 4	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
2	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 4	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 2 (2)

the marginal probability distribution profiles induced by *f* in Example 5.

$b(u_3) = 1$, $b(u_4) = 0$

	$b(u_2)$	0			1			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2
0	agent 1	2/3	1/3	0	1	0	0	1	0	0
	agent 2	2/3	1/3	0	0	1	0	0	1	0
	agent 3	0	1	0	0	1	0	0	1	0
	agent 4	2/3	1/3	0	1	0	0	1	0	0
1	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 4	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
2	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 4	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**

$b(u_3) = 0$, $b(u_4) = 2$

	b(u2)	0			1			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2
0	agent 1	1	0	0	1	0	0	1	0	0
	agent 2	1	0	0	0	1	0	0	1	0
	agent 3	1	0	0	1	0	0	1	0	0
	agent 4	0	0	1	0	1	0	0	1	0
1	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 4	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
2	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 4	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 2 (3)

the marginal probability distribution profiles induced by *f* in Example 5.

$b(u_3) = 2$, $b(u_4) = 0$

$b(u_1)$	$b(u_2)$ / # of objects	0			1			2		
		0	1	2	0	1	2	0	1	2
0	agent 1	1	0	0	1	0	0	1	0	0
	agent 2	1	0	0	0	1	0	0	1	0
	agent 3	0	0	1	0	1	0	0	1	0
	agent 4	1	0	0	1	0	0	1	0	0
1	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 4	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
2	agent 1	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 4	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**

$b(u_3) \geq 1$, $b(u_4) \geq 1$

$b(u_1)$	$b(u_2)$ / # of objects	0			1			2		
		0	1	2	0	1	2	0	1	2
0	agent 1	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 2	1	0	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 3	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
	agent 4	0	1	0	**11/30**	**19/30**	**0**	**11/30**	**19/30**	**0**
1	agent 1	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 2	**27/30**	**3/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 3	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 4	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0
2	agent 1	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 2	**27/30**	**3/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 3	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0
	agent 4	**11/30**	**19/30**	**0**	1/2	1/2	0	1/2	1/2	0

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 3 (1)

the marginal probability distribution profiles induced by *f* in Example 6.

$b(u_3)$ = 0

$b(u_1)$	$b(u_2)$ / # of objects	0: 0	0: 1	0: 2	1: 0	1: 1	1: 2	2: 0	2: 1	2: 2
0	agent 1	1/3	2/3	0	1/2	1/2	0	1/2	1/2	0
	agent 2	1/3	2/3	0	0	1	0	0	1	0
	agent 3	1/3	2/3	0	1/2	1/2	0	1/2	1/2	0
1	agent 1	0	1	0	0	1	0	0	1	0
	agent 2	1/2	1/2	0	0	1	0	0	1	0
	agent 3	1/2	1/2	0	1	0	0	1	0	0
2	agent 1	0	1	0	0	1	0	0	1	0
	agent 2	1/2	1/2	0	0	1	0	0	1	0
	agent 3	1/2	1/2	0	1	0	0	1	0	0

$b(u_3)$ = 1

$b(u_1)$	$b(u_2)$ / # of objects	0: 0	0: 1	0: 2	1: 0	1: 1	1: 2	2: 0	2: 1	2: 2
0	agent 1	1/2	1/2	0	1	0	0	1	0	0
	agent 2	1/2	1/2	0	0	1	0	0	1	0
	agent 3	0	1	0	0	1	0	0	1	0
1	agent 1	0	1	0	**1/15**	**14/15**	**0**	**1/15**	**14/15**	**0**
	agent 2	1	0	0	**7/15**	**8/15**	**0**	**7/15**	**8/15**	**0**
	agent 3	0	1	0	**7/15**	**8/15**	**0**	**7/15**	**8/15**	**0**
2	agent 1	0	1	0	**1/15**	**1/15**	**13/15**	**1/15**	**1/15**	**13/15**
	agent 2	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 3	0	1	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**

bold : the different marginal distribution profiles from those by the uniform probabilistic rule

Table 3 (2)

the marginal probability distribution profiles induced by *f* in Example 6.

$b(u_3) = 2$

	$b(u_2)$	0			1			2		
$b(u_1)$	# of objects	0	1	2	0	1	2	0	1	2
0	agent 1	1/2	1/2	0	1	0	0	1	0	0
	agent 2	1/2	1/2	0	0	1	0	0	1	0
	agent 3	0	1	0	0	1	0	0	1	0
1	agent 1	0	1	0	**1/15**	**14/15**	**0**	**1/15**	**14/15**	**0**
	agent 2	1	0	0	**7/15**	**8/15**	**0**	**7/15**	**8/15**	**0**
	agent 3	0	1	0	**7/15**	**8/15**	**0**	**7/15**	**8/15**	**0**
2	agent 1	0	1	0	**1/15**	**1/15**	**13/15**	**1/15**	**1/15**	**13/15**
	agent 2	1	0	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**
	agent 3	0	1	0	**27/30**	**3/30**	**0**	**27/30**	**3/30**	**0**

bold : the different marginal distribution profiles from those by the uniform probabilistic rule