Discussion Paper No. 686

COALITIONALLY STRATEGY-PROOF RULES IN ALLOTMENT ECONOMIES WITH HOMOGENEOUS INDIVISIBLE GOODS

Kentaro Hatsumi
and
Shigehiro Serizawa

March 2007
Revised July 2008
Revised February 2009

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Coalitionally Strategy-Proof Rules in Allotment Economies with Homogeneous Indivisible Goods*

Kentaro Hatsumi† and Shigehiro Serizawa‡

First Version: January 12, 2006
This Version: January 13, 2009

Abstract: We consider the allotment problem of homogeneous indivisible goods among agents with single-peaked and risk-averse von Neumann-Morgenstern expected utility functions. We establish that *a rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule*. By constructing an example, we show that if same-sideness is replaced by respect for unanimity, this statement does not hold even with the additional requirements of no-envy, anonymity, at most binary, peaks-onlyness and continuity.

Keywords: allotment problem, coalitional strategy-proofness, homogeneous indivisible goods, single-peaked preference, uniform probabilistic rule

JEL Classification Numbers: C72, D71, D81

---

*We would like to thank the associate editor and two anonymous referees whose comments and suggestions significantly improve this paper. We are also grateful to Masaki Aoyagi, Kazuhiko Hashimoto, Hervé Moulin, Hiroo Sasaki, Koji Takamiya, William Thomson, Takuma Wakayama as well as other participants at the Eighth International Meeting of the Society for Social Choice and Welfare in Istanbul, the 2006 Autumn Annual Meeting of the Japanese Economic Association in Osaka, and Yokohama National University Seminar of Economic Theory for their helpful comments. We acknowledge financial supports from the Japan Society for the Promotion of Science via the Research Fellowship for Young Scientist (Hatsumi) and the Grant-in-Aid for Scientific Research (Serizawa).

†Graduate School of Economics, Osaka University. 1-7, Machikaneyama, Toyonaka, 560-0043, Japan. E-Mail: egp020hk@mail2.econ.osaka-u.ac.jp

‡Institute of Social and Economic Research, Osaka University. 6-1, Mihogaoka, Ibaraki, 567-0047, Japan. E-Mail: serizawa@iser.osaka-u.ac.jp

1 Introduction

Situations exist that can be interpreted as allotment problems of goods without any disposal. For example, consider the situation where the manager of a firm assigns staff overtime with a fixed wage. Each staff member has her own ideal overtime working hours: some staff may not want any overtime, others who need extra spending money may want to do some but not too much overtime. The manager has the power to assign all staff overtime work in a just proportion. We can interpret this situation as an allotment problem of overtime work for the manager.

Sprumont (1991) initiates an axiomatic analysis of this kind of allotment problem. He analyzes a model in which there is a perfectly divisible good and allocation rules are deterministic. In this model, he assumes that agents have "single-peaked" preferences over their consumption levels, and characterizes "the uniform rule"[1]. A preference is *single-peaked* if there is some ideal point called a "peak", and welfare is strictly decreasing on either direction away from the peak. *The uniform rule* is the rule such that agents are allowed to choose their preferred consumption subject to a common bound, which is chosen to satisfy feasibility. Sprumont (1991) shows that the uniform rule is a unique allocation rule satisfying strategy-proofness, Pareto-efficiency, and anonymity. *Strategy-proofness* is a frequently employed incentive compatibility property. It requires that it is a weakly dominant strategy for each agent to represent her true preference. *Anonymity* requires that the name of each agent does not matter for the outcome allocation. Many subsequent studies follow Sprumont (1991) in analyzing the uniform rule from different perspectives by employing various axioms[2].

In the real world, the goods to be allocated are not often perfectly divisible. In the above example of assigning overtime work, working time is often institutionally restricted to hour units, even though time is perfectly divisible. If the number of units is sufficiently large, the assumption of a perfectly divisible good may approximate the situation. However, if the number of units is small, it is doubtful.

Based on the works by Sasaki (1997) and Kureishi (2000), Ehlers and Klaus (2003) investigate the problem of probabilistically allocating finite units of homogeneous indivisible goods. They show that when agents have single-peaked preferences with a probabilistic extension[3], "the uniform prob-

[1] The *uniform rule* is first considered by Benassy (1982) for the analysis of fixed price economies.

[2] For example, Thomson (1994a, 1994b, 1995), Barberà, Jackson and Neme (1997), Ching and Serizawa (1998), Massó and Neme (2001, 2004, 2007), Chun (2006), Kesten (2006), Klaus (2006), and Mizobuchi and Serizawa (2006).

[3] Ehlers and Klaus (2003) define preferences with a probabilistic extension as follows. A marginal (probability) distribution is weakly preferred to another if the first marginal distribution assigns the upper contour set of each consumption level at least the same probability as that assigned by the second. This preference drops completeness over the

abilistic rule", a probabilistic variant of the uniform rule, is a unique rule satisfying strategy-proofness, Pareto-efficiency, and no-envy[4]. They also show that when agents have single-peaked and "risk-averse" utility functions satisfying the von Neumann-Morgenstern expected utility property, the uniform probabilistic rule is a unique rule satisfying strategy-proofness, Pareto-efficiency, and "symmetry"[5]. The latter is the counterpart of Ching (1994)'s result showing that anonymity in the uniqueness result of Sprumont (1991) can be weaken to symmetry in the deterministic model. Ehlers and Klaus (2003) demonstrate that the latter result is obtained as a corollary of Ching (1994) especially due to *at most binary* property, which is a property implied by Pareto-efficiency and requires that each agent has strictly positive probabilities over at most two adjacent consumption levels.

This probabilistic model of allotment economies developed by Ehlers and Klaus (2003) is very important and interesting. However, research in this area is relatively underdeveloped[6]. The target of the present paper is to provide richer knowledge on this probabilistic model, and to investigate it using other interesting axioms other than strategy-proofness.

In this paper, we focus on the property of "coalitional strategy-proofness". *Coalitional strategy-proofness* is a stronger concept than strategy-proofness, and requires that no coalition can increase the utilities of all members at the same time. In situations where agents are likely to cooperate in misrepresenting their preferences, the property of coalitional strategy-proofness is beneficial for making agents reveal their true preferences without worrying about cooperative manipulation. Our main result is that when agents possess single-peaked and risk-averse von Neumann-Morgenstern expected utility functions, *the uniform probabilistic rule is a unique probabilistic allocation rule satisfying coalitional strategy-proofness, "same-sideness", and "strong symmetry"*[7]. *Same-sideness* in this probabilistic model is a weaker efficiency property than Pareto-efficiency, and requires that if the indivisible goods are in excess demand, that is, if the sum of agents' peaks is greater than the endowment, then no agent receives an amount more than her peak with positive probability, and conversely, if the goods are in excess supply, then no agent receive an amount less than her peak with positive probability. Thus, our result is an alternative characterization of the uniform probabilistic rule in Ehlers and Klaus (2003).

---

set of marginal distributions.

[4]*No-envy* requires that no agent strictly prefers the consumption of any other agent to her own.

[5]*Symmetry* requires that whenever two agents have the same preferences, they receive the indifferent consumptions.

[6]As far as we know, Kureishi and Mizukami (2007) is the only other research article on this environment. They show that symmetry in the second result of Ehlers and Klaus (2003) can be replaced by "equal probability for the best" property.

[7]*Strong symmetry* requires that whenever two agents have the same preferences, the goods are distributed to them by the same probability distribution.

We also show by constructing an example that if same-sideness is replaced by "respect for unanimity"[8], which is a weaker axiom than same-sideness, the uniqueness of the uniform probabilistic rule is violated even with "strong coalitional strategy-proofness"[9], no-envy, anonymity and at most binary. Furthermore, we find that even with the additional requirements of "peaks-onlyness"[10] and "continuity"[11], the uniqueness of the uniform probabilistic rule does not hold. That is, the uniform probabilistic rule is not a unique allocation rule satisfying peaks-onlyness and continuity in addition to the above requirements.

In his recent work, Serizawa (2006) shows that in Sprumont's (1991) deterministic model of a perfectly divisible good, the uniform rule is a unique rule satisfying "effectively pairwise strategy-proofness", respect for unanimity and symmetry. *Effectively pairwise strategy-proofness* is a property that rules are strategy-proof and that if a pair of agents can make both agents better off by pairwise manipulation, one of them has an incentive to betray her partner. It requires rules to be immune only to unilateral manipulations and to pairwise manipulations in which agents have no incentives to betray, while coalitional strategy-proofness requires rules to be immune to all coalitional manipulations. In this point, the former is much weaker than the latter. Our results provide a remarkable comparison to Serizawa's (2006) result, and demonstrate that when dropping the property of same-sideness, the probabilistic assignment problem is much different from Sprumont's (1991) deterministic problem.

This paper is organized as follows. Section 2 describes the model and the results. Section 3 concludes the paper. Technical discussions including the proofs of facts and the theorem are in the Appendix.

2 The model and the results

There are $k \in \mathbb{Z}_{++}$[12] units of homogeneous indivisible goods. We consider the problem of allotting k units of the goods to a set of agents $N = \{1, \cdots, n\}$. A coalition is a subset N' of N. Given a coalition $N' \subseteq N$ and an agent $i \in N$, we denote the coalition $N \backslash N'$ by $-N'$, and the coalition $N \backslash \{i\}$ by $-i$. Let $K = \{0, 1, \cdots, k\}$, which is the set of consumption levels. We call $a = (x_1, \cdots, x_n) \in K^n$ a *feasible allocation* if $\sum_{i \in N} x_i = k$. Let A

[8] *Respect for unanimity* requires that if the sum of agents' peaks equals the endowment, all agents receive their peak consumption.

[9] *Strong coalitional strategy-proofness* requires that by coalitional manipulation, no coalition can increase the utility of any member in the coalition without decreasing the utility of some other member.

[10] *Peaks-onlyness* requires that the outcome allocation depends only on the peak profile.

[11] *Continuity* requires that small changes in the utility profile cause only small changes in the outcome allocation.

[12] $\mathbb{Z}_{++}$ is the set of positive integers and $\mathbb{Z}_+$ is the set of nonnegative integers.

denote the set of all feasible allocations.

A (probability) distribution over A is interpreted as a lottery on A. For $A = \{a^1, \cdots, a^{|A|}\}$[13], we denote such a distribution over A by $[\tilde{p}^1 \circ a^1, \cdots, \tilde{p}^{|A|} \circ a^{|A|}]$ where for all $l \in \{1, \cdots, |A|\}$, $\tilde{p}^l \in [0,1]$ is the probability of a^l, and $\sum_{l=1}^{|A|} \tilde{p}^l = 1$. For convenience, to express a distribution, we only write feasible allocations a^l that occur with a strictly positive probability $\tilde{p}^l > 0$. For example, instead of $[\frac{1}{2} \circ a^1, \frac{1}{2} \circ a^2, 0 \circ a^3, \cdots, 0 \circ a^{|A|}]$, we write $[\frac{1}{2} \circ a^1, \frac{1}{2} \circ a^2]$. Let $\tilde{P}$ denote the set of all distributions over A.

Let P_i denote the set of all marginal (probability) distributions for $i \in N$ over her allotments in K, induced by all $\tilde{p} \in \tilde{P}$. Each agent $i \in N$ only cares for her marginal distribution $p_i \in P_i$ on K. Given $p_i \in P_i$ and $K' \subseteq K$, $p_i(K')$ denotes the probability that the marginal distribution p_i places over K'. If $K' = \{x\}$, we write simply $p_i(x)$ instead of $p_i(K')$ to refer to the probability that agent i receives x units through the marginal distribution p_i.

Each agent $i \in N$ has a utility function $u_i : K \to \mathbb{R}$ that satisfies the (von Neumann-Morgenstern) expected utility property. Given a marginal distribution $p_i \in P_i$, we denote the expected utility by

$$E(p_i; u_i) = \sum_{x \in K} p_i(x) \cdot u_i(x).$$

It is well known that any affine transformation of an expected utility function represents the same preference relation over P_i. Therefore, we normalize any utility function u_i as $u_i(0) = 0$ and $||u_i(1) - u_i(0)|| = 1$[14]. We assume that the utility functions possess the following two properties.

Definition. A utility function u_i is *single-peaked* if there exists a unique peak $b(u_i) \in K$ such that for all $x, y \in K$ with $x > y \geq b(u_i)$ or $b(u_i) \geq y > x$, $u(y) > u(x)$.

Definition. A utility function u_i is *risk-averse* if for all $x \in K \backslash \{0, k\}$, $u_i(x) - u_i(x-1) > u_i(x+1) - u_i(x)$.

Let U denote the class of all single-peaked and risk-averse von Neumann-Morgenstern utility functions[15]. Let U^n denote the set of all von Neumann-Morgenstern utility profiles $u = (u_i)_{i \in N}$ such that for all $i \in N$, $u_i \in U$. Given a coalition $N' \subseteq N$, let $u_{N'}$ denote a partial utility profile of the coalition $(u_i)_{i \in N'}$ such that for all $i \in N'$, $u_i \in U$. $(\hat{u}_{N'}, u_{-N'})$ represents the utility profile such that agent $i \in N'$ has $\hat{u}_i$ and agent $j \in -N'$ has u_j.

[13] $|A|$ is the number of feasible allocations.

[14] For $a \in \mathbb{R}$, $||a||$ represents the absolute value of a.

[15] If a utility function exhibits risk-aversion, then it is *weakly single-peaked*, *i.e.*, there exist at most two adjacent peaks $b(u_i), b(u_i) + 1 \in K$, and for all $x, y \in K$, if $x > y \geq b(u_i) + 1$ or $b(u_i) \geq y > x$, then $u(y) > u(x)$. However, (strict) single-peakedness and risk-averseness are independent.

Note that the two distributions need not be equal, even though their marginal distributions are the same, as illustrated by Example 1 below.

Example 1 (Ehlers and Klaus, 2003)**.** Let $N = \{1, 2, 3\}$, $k = 9$, $\tilde{p} = [\frac{1}{3} \circ (3, 6, 0), \frac{1}{3} \circ (0, 3, 6), \frac{1}{3} \circ (6, 0, 3)]$, and $\tilde{p}' = [\frac{1}{3} \circ (3, 0, 6), \frac{1}{3} \circ (6, 3, 0), \frac{1}{3} \circ (0, 6, 3)]$. Let p_i and p'_i be the marginal distributions for $i \in N$ induced by $\tilde{p}$ and $\tilde{p}'$. Then, for all $i \in N$, $p_i = p'_i$, but $\tilde{p} \neq \tilde{p}'$. $\Diamond$

If two distributions $\tilde{p}, \tilde{p}' \in \tilde{P}$ have the same marginal distribution profile, *i.e.*, $p_i = p'_i$ for all $i \in N$, then $\tilde{p}$ and $\tilde{p}'$ are equivalent from the viewpoint of agents. Thus, we focus on marginal distribution profiles instead of distributions on A. A marginal distribution profile $p = (p_1, \cdots, p_n) \in \prod_{i \in N} P_i$ is *feasible* if there is a probability distribution $\tilde{p} \in \tilde{P}$ such that for all $i \in N$, p_i is induced by $\tilde{p}$. We denote by P the set of all feasible marginal distribution profiles.

We define two efficiency properties of the marginal distribution profiles. The first is "Pareto-efficiency", one of the most common efficiency properties in economics. This requires that there are no other feasible marginal distributions where all agents are weakly better-off and some are strictly better-off. The other is "same-sideness." This requires that if the indivisible goods are in excess demand, that is, if the sum of agents' peaks is greater than the endowment, then no agent receives an amount more than her peak with positive probability, and conversely, if the goods are in excess supply, then no agent receive an amount less than her peak with positive probability. Same-sideness has the following merits as an alternative efficiency property. First, it is a necessary condition for Pareto-efficiency. Second, it is much simpler and easier to check. Moreover, violating same-sideness results in the imposition of amounts more than wished on agents when the goods are in excess demand. Such allocations can hardly be justified[16].

Definition. A marginal distribution profile $p \in P$ satisfies *Pareto-efficiency* with respect to $u \in U^n$ if there is no $p' \in P$ such that for all $i \in N$, $E(p'_i; u_i) \geq E(p_i; u_i)$ and for some $j \in N$, $E(p'_j; u_j) > E(p_j; u_j)$.

Definition. A marginal distribution profile $p \in P$ satisfies *same-sideness* with respect to $u \in U^n$ if $\sum_{i \in N} b(u_i) \geq k$ implies that for all $i \in N$, $p_i([0, b(u_i)]) = 1$, and $\sum_{i \in N} b(u_i) \leq k$ implies that for all $i \in N$, $p_i([b(u_i), k]) = 1$.

In the deterministic model with a perfectly divisible good, Pareto-efficiency is equivalent to same-sideness. In this probabilistic model, Pareto-efficiency implies same-sideness, however, Example 2 illustrates that the inverse implication is not true.

[16] Amorós (2002) investigates the allotment economies with two perfectly divisible goods where same-sideness is a weaker condition of Pareto-efficiency, and employs same-sideness in his main characterization. Amorós (2002) refers to same-sideness as "condition E".

Example 2. Let $N = \{1, 2\}$ and $k = 2$. Let $u \in U^2$ be such that $u_1 = u_2$, $u_1(0) = 0$, $u_1(1) = 1$ and $u_1(2) = \frac{3}{2}$. Let $p \in P$ be such that $p_i(0) = p_i(2) = \frac{1}{2}$ for $i = 1, 2$ and $p' \in P$ be such that $p'_i(1) = 1$ for $i = 1, 2$.

Then, $E(p_i; u_i) = \frac{3}{4}$ and $E(p'_i; u_i) = 1$ for $i = 1, 2$. p is same-sided with respect to u but is not Pareto-efficient with respect to u. $\diamondsuit$

The next property called "at most binary" states that each agent has strictly positive probabilities over at most two adjacent elements of K. Fact 1 below implies that this property has an important role in the model.

Definition. A marginal distribution profile $p \in P$ satisfies *at most binary* if for all $i \in N$, there exists $x \in K \backslash \{k\}$ such that $p_i(x) + p_i(x + 1) = 1$.

Fact 1 (Sasaki 1997, Kureishi 2000)**.** A marginal distribution profile $p \in P$ satisfies Pareto-efficiency with respect to u if and only if it satisfies same-sideness with respect to u and at most binary[17].

A *probabilistic (allocation) rule* is a function $f : U^n \to P$. Given a probabilistic rule f, $u \in U^n$ and $i \in N$, $f_i(u)$ denotes the marginal distribution of agent i when the utility profile is u under the rule f. Given $K' \subseteq K$, $f_i(u)(K')$ denotes the probability that $f_i(u)$ places over K', and if $K' = \{x\}$, $f_i(u)(x)$ denotes $f_i(u)(K')$.

We introduce several properties of f. The first properties relate to the efficiency of f.

Definition. A probabilistic rule f satisfies *Pareto-efficiency* if for all $u \in U^n$, $f(u)$ is Pareto-efficient with respect to u.

Definition. A probabilistic rule f satisfies *same-sideness* if for all $u \in U^n$, $f(u)$ is same-sided with respect to u.

Definition. A probabilistic rule f satisfies *at most binary* if for all $u \in U^n$, $f(u)$ satisfies at most binary.

Definition. A probabilistic rule f satisfies *respect for unanimity* if for all $u \in U^n$ such that $\sum_{i \in N} b(u_i) = k$, and all $i \in N$, $f_i(u)(b(u_i)) = 1$.

Note that a probabilistic rule f satisfies Pareto-efficiency if and only if it satisfies same-sideness and at most binary, due to Fact 1. In addition, note that same-sideness implies respect for unanimity.

The following three properties relate to the incentive compatibility for agents to reveal their true utility functions. "Strategy-proofness" requires that no agent can increase her utility by manipulating her revealed utility. "Coalitional strategy-proofness" is a stronger condition; it requires that no

[17]The *only if* part of Fact 1 is given by Sasaki (1997) and the *if* part is by Kureishi (2000). Unfortunately, these two are rarely promulgated. Therefore, we reconstruct the proof in an online supplementary note (Hatsumi and Serizawa 2009).

coalition can increase the utilities of all members at the same time. Further, "strong coalitional strategy-proofness" requires that no coalition can increase the utility of any member in the coalition via coalitional manipulation without decreasing the utility of some other member in the coalition.

Definition. A probabilistic rule f satisfies *strategy-proofness* if for all $u \in U^n$, for all $i \in N$, and all $\hat{u}_i \in U$, $E(f_i(u); u_i) \geq E(f_i(\hat{u}_i, u_{-i}); u_i)$.

Definition. A probabilistic rule f satisfies *coalitional strategy-proofness* if for all $u \in U^n$, all $N' \subseteq N$, and all $\hat{u}_{N'} \in U^{N'}$, there exists $i \in N'$ such that $E(f_i(u); u_i) \geq E(f_i(\hat{u}_{N'}, u_{-N'}); u_i)$.

Definition. A probabilistic rule f satisfies *strong coalitional strategy-proofness* if for all $u \in U^n$, all $N' \subseteq N$, and all $\hat{u}_{N'} \in U^{N'}$, whenever there is $i \in N'$ such that $E(f_i(\hat{u}_{N'}, u_{-N'}); u_i) > E(f_i(u); u_i)$, there exists $j \in N'$ such that $E(f_j(u); u_j) > E(f_j(\hat{u}_{N'}, u_{-N'}); u_j)$.

In addition, we introduce several properties relating to fairness. "Symmetry" requires that agents with the same utility functions obtain the same expected utilities. "Strong symmetry" requires that agents with the same utility functions have the same marginal distributions. "No-envy" requires that no agent strictly prefers the marginal distribution of any other agent to her own. "Anonymity" requires that the name of each agent does not matter. The relationships between these properties are as follows. No-envy and anonymity are independent. No-envy and strong symmetry are also independent. Anonymity implies strong symmetry. Strong symmetry or no-envy implies symmetry.

Definition. A probabilistic rule f satisfies *symmetry* if for all $u \in U^n$ and all $i, j \in N$ such that $u_i = u_j$, $E(f_i(u); u_i) = E(f_j(u); u_j)$.

Definition. A probabilistic rule f satisfies *strong symmetry* if for all $u \in U^n$ and all $i, j \in N$ such that $u_i = u_j$, $f_i(u) = f_j(u)$.

Definition. A probabilistic rule f satisfies *no-envy* if for all $u \in U^n$ and all $i, j \in N$, $E(f_i(u); u_i) \geq E(f_j(u); u_i)$.

Definition. Let Π^n be the class of all permutations on N. For all $u \in U^n$ and all $\pi \in \Pi^n$, let $u^\pi = (u_{\pi(i)})_{i \in N}$. A probabilistic rule f satisfies *anonymity* if for all $u \in U^n$, all $\pi \in \Pi^n$, and all $i \in N$, $f_{\pi(i)}(u) = f_i(u^\pi)$.

We note Fact 2 below related to the feasibility of the marginal distribution profile. Given that all marginal distribution profiles other than those in Examples 4, 5, and 6 satisfy the conditions of Fact 2, we do not explicitly discuss the feasibilities.

Fact 2. Let $N^a \subseteq N$ and $N^b = -N^a$. Let $n^a \in \mathbb{N} \cup \{0\}$ be the number of agents in N^a and $n^b \in \mathbb{N} \cup \{0\}$ be that in N^b. Let $\{x_i\}_{i \in N^a} \in K^{n^a}$ be such that $\sum_{i \in N^a} x_i \leq k$. Let $\mu \in \mathbb{R}_+$ be such that $\mu = (k - \sum_{i \in N^a} x_i)/n^b$, and $x_\mu \in K$ be such that $\mu \in [x_\mu, x_\mu + 1)$. Let a marginal distribution profile p be such that for all $i \in N^a$, $p_i(x_i) = 1$, and for all $j \in N^b$, $p_j(x_\mu) = 1 - (\mu - x_\mu)$ and $p_j(x_\mu + 1) = \mu - x_\mu$. Then, p is feasible, *i.e.*, $p \in P$.

The proof of Fact 2 is in the appendix.

We introduce the "uniform probabilistic rule". The uniform probabilistic rule assigns agents the marginal distribution profile subject to the common bound depending on each utility profile to satisfy feasibility. Before the formal description of the uniform probabilistic rule, we introduce an example of an algorithm to find the outcome of the uniform probabilistic rule.

Example 3 (Sasaki, 1997)**.** (i) Let $N = \{1, 2, 3, 4\}$ and $k = 15$. Assume $u \in U^4$ is such that $b(u_1) = 1$, $b(u_2) = b(u_3) = 2$, and $b(u_4) = 5$. Note that $\sum_{i \in N} b(u_i) < k$ (excess supply). First, consider that the common bound is $\frac{k}{n} = \frac{15}{4}$. Then, in the case of excess supply, search for agents with peaks larger than the common bound. In this case, it is agent 4. Then, let agent 4 take her peak consumption level 5 with probability 1, and shift the common bound for the remaining three agents to $(k - b(u_4))/(n - 1) = \frac{10}{3}$. Iterate this operation until no remaining agent's peak is larger than the common bound. In this example, since the peaks of agents 1, 2, and 3 are less than $\frac{10}{3}$, the iteration stops at $\frac{10}{3}$. Thus, the common bound finally fixed is $\frac{10}{3}$. The outcome of the uniform probabilistic rule is that agent 4 is assigned $5 (= b(u_4))$ with probability 1, and the remaining agents are assigned 3 with probability $\frac{2}{3}$ and 4 with probability $\frac{1}{3}$ to satisfy feasibility.
(ii) In the excess demand case ($\sum_{i \in N} b(u_i) > k$), the symmetric algorithm of (i) can be applied. Let $N = \{1, 2, 3, 4\}$ and $k = 12$. Assume $u \in U^4$ is such that $b(u_1) = 4$, $b(u_2) = 2$, $b(u_3) = 10$, and $b(u_4) = 3$. The final common bound is calculated as $\frac{7}{2}$. Agents 2 and 4 are assigned their peak levels with probability 1 since their peaks are less than the common bound, and agents 1 and 3 are assigned 3 with probability $\frac{1}{2}$ and 4 with probability $\frac{1}{2}$. $\diamondsuit$

The formal definition of the uniform probabilistic rule is as follows. Let $\lambda : U^n \to \mathbb{R}_+$ be the function such that if $\sum_{i \in N} b(u_i) \geq k$, $\sum_{i \in N} \min\{b(u_i), \lambda(u)\} = k$, and if $\sum_{i \in N} b(u_i) < k$, $\sum_{i \in N} \max\{b(u_i), \lambda(u)\} = k$. λ is the function to determine the common bound. Let $x_\lambda : U^n \to K$ be the function such that if $\sum_{i \in N} b(u_i) \geq k$, $\lambda(u) \in [x_\lambda(u), x_\lambda(u) + 1)$ and if $\sum_{i \in N} b(u_i) < k$, $\lambda(u) \in (x_\lambda(u), x_\lambda(u) + 1]$.

Definition (Sasaki, 1997)**.** The *uniform probabilistic rule* is the probabilistic rule f such that for all $u \in U^n$, the following holds:
(i) If $\sum_{i \in N} b(u_i) > k$ (*excess demand*) , then for all $i \in N$,

$$b(u_i) \leq x_\lambda(u) \Longrightarrow f_i(u)(b(u_i)) = 1, \qquad \text{and}$$

$$b(u_i) \geq x_\lambda(u) + 1 \Longrightarrow \begin{cases} f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u) \\ f_i(u)(x_\lambda(u)) = 1 - (\lambda(u) - x_\lambda(u)). \end{cases}$$

(ii) If $\sum_{i \in N} b(u_i) = k$ (*balanced demand*), then for all $i \in N$, $f_i(u)(b(u_i)) = 1$.
(iii) If $\sum_{i \in N} b(u_i) < k$ (*excess supply*), then for all $i \in N$,

$$b(u_i) \geq x_\lambda(u) + 1 \Longrightarrow f_i(u)(b(u_i)) = 1, \qquad \text{and}$$

$$b(u_i) \leq x_\lambda(u) \Longrightarrow \begin{cases} f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u) \\ f_i(u)(x_\lambda(u)) = 1 - (\lambda(u) - x_\lambda(u)). \end{cases}$$

Notice that in Example 3 (i), $\lambda(u) = \frac{10}{3}$ and $x_\lambda(u) = 3$, and in (ii), $\lambda(u) = \frac{7}{2}$ and $x_\lambda(u) = 3$.

In a previous investigation of the probabilistic model, Sasaki (1997) shows that the uniform probabilistic rule is the only rule satisfying strategy-proofness, Pareto-efficiency, and anonymity. Kureishi (2000) and Ehlers and Klaus (2003) weaken anonymity to symmetry and show the uniqueness of the uniform probabilistic rule satisfying these properties. These results for the probabilistic model are parallel to those of Sprumont (1991) and Ching (1994), respectively, who originally studied a deterministic model in which the goods are perfectly divisible.

In the deterministic model with a perfectly divisible good, Serizawa (2006) recently showed that the uniform rule is the only rule satisfying effectively pairwise strategy-proofness, respect for unanimity, and symmetry. Thus, it is an interesting question as to whether a result parallel to Serizawa (2006) also holds in the probabilistic model. However, Example 4 below illustrates that Serizawa's (2006) uniqueness result does not hold in the probabilistic model even though effectively pairwise strategy-proofness and symmetry are respectively strengthened to strong coalitional strategy-proofness and no-envy or anonymity with an additional requirement of at most binary.

Example 4. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as follows.
If $u \in U^3$ is such that for one agent, say i, $b(u_i) = 1$ and for any other agent $j \in N \backslash \{i\}$, $b(u_j) = 0$, then, (i) in the case of $u_i(1) - u_i(0) \geq u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(1) = \frac{18}{20}, f_i(u)(2) = \frac{2}{20} \\ f_j(u)(0) = \frac{11}{20}, f_j(u)(1) = \frac{9}{20} \end{cases}$$

and (ii) in the case of $u_i(1) - u_i(0) < u_i(1) - u_i(2)$,

$$\begin{cases} f_i(u)(0) = \frac{2}{20}, f_i(u)(1) = \frac{18}{20} \\ f_j(u)(0) = \frac{9}{20}, f_j(u)(1) = \frac{11}{20}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, although the probabilistic rule f satisfies strong coalitional strategy-proofness, respect for unanimity, no-envy, anonymity, and at most binary, it is not the uniform probabilistic rule[18] $\Diamond$

We introduce two additional properties. "Peaks-onlyness" requires that the outcome marginal distribution profile depends only on the peak profile. If a rule satisfies peaks-onlyness, we can reduce the necessary information for a planner to the peak profile. "Continuity" requires that small changes in the utility profile cause only small changes in the outcome allocation.

Definition. A probabilistic rule f satisfies *peaks-onlyness* if for all $u, u' \in U^n$ such that for all $i \in N$, $b(u_i) = b(u'_i)$, $f(u) = f(u')$.

Definition. A probabilistic rule f satisfies *continuity* if for all $u \in U^n$ and any $\epsilon > 0$, there exists $\delta > 0$ such that for all $u' \in U^n$,

$$\begin{aligned} &[\forall i \in N, \forall x \in K, \| \, u_i(x) - u'_i(x) \, \| < \delta] \\ \Longrightarrow &[\forall i \in N, \forall x \in K, \| \, f_i(u)(x) - f_i(u')(x) \, \| < \epsilon]. \end{aligned}$$

In the deterministic model with a perfectly divisible good, these two properties are standard and often obtained from strategy-proofness with auxiliary properties. However, note that the rule in Example 4 does not satisfy peaks-onlyness or continuity, even though it satisfies coalitional strategy-proofness, respect for unanimity, no-envy, anonymity, and at most binary. Thus, in the probabilistic model, these properties do not imply peaks-onlyness or continuity.

In this model, peaks-onlyness implies continuity.

Fact 3. If a probabilistic rule f satisfies peaks-onlyness, then it satisfies continuity.

The proof of Fact 3 is in the appendix.

Example 5 below illustrates that even though we impose peaks-onlyness as well as the previous properties, we cannot characterize the uniform probabilistic rule as a unique rule satisfying such properties. In addition, because of Fact 3, adding continuity to these properties has no effect.

Example 5. Let $n = 4$ and $k = 2$. We define a probabilistic rule f as follows.
If $u \in U^4$ is such that for one agent, say i, $b(u_i) = 0$, and for any other agent $j \in N \backslash \{i\}$, $b(u_j) \geq 1$, then

$$\begin{cases} f_i(u)(0) = \frac{27}{30}, \; f_i(u)(1) = \frac{3}{30} \\ f_j(u)(0) = \frac{11}{30}, \; f_j(u)(1) = \frac{19}{30}. \end{cases}$$

[18] The feasibility is given in the appendix. A detailed explanation is given in an online supplementary note (Hatsumi and Serizawa 2009).

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, although the rule f satisfies the properties of strong coalitional strategy-proofness, respect for unanimity, no-envy, anonymity, at most binary, and peaks-onlyness, it is not the uniform probabilistic rule[19]. $\Diamond$

In the deterministic model, coalitional strategy-proofness is a much stronger requirement than strategy-proofness in that the uniform rule is characterized as a unique rule satisfying coalitional strategy-proofness together with mild auxiliary conditions. On the other hand, in the probabilistic model, as Example 5 illustrates, some rules other than the uniform probabilistic rule satisfy coalitional strategy-proofness and the many auxiliary conditions that characterize the uniform rule in the deterministic model. The existence of such rules makes a remarkable contrast between the probabilistic and deterministic models. Therefore, it is worthwhile to discuss why such rules exist in the probabilistic model.

In the probabilistic model, the peaks of the utility functions and peak profiles are finite. Thus, peaks-only rules such as the uniform probabilistic rule have finite ranges. Note that coalitional strategy-proofness is a system of inequalities of allocations imposed on rules. Because of the finiteness of the range, there is slackness of the system of inequalities, *i.e.*, there is room to change the range of the uniform probabilistic rule without violating these inequalities and the above conditions. That is, in the probabilistic model, we can construct rules from the uniform probabilistic rule by changing the range within such room.

It is also worthwhile to compare the fact discussed above to the results in Ehlers and Klaus (2003). They point out that if we impose at most binary, we can reduce the probabilistic model to the deterministic model since any marginal distribution satisfying at most binary is represented by a nonnegative real number. In addition, they mention that the domain of utility profiles in the probabilistic model is rich enough to apply Ching's (1994) proof to show that the uniform probabilistic rule is a unique rule satisfying strategy-proofness, Pareto-efficiency, and symmetry. However, the above discussion reveals that the domain in the probabilistic model is not sufficiently rich to obtain Serizawa's (2006) parallel result.

Since coalitional strategy-proofness is not so strong in the probabilistic model, to characterize the uniform probabilistic rule by coalitional strategy-proofness, we require a stronger efficiency property than respect for unanimity. Our main characterization employs same-sideness instead of respect for unanimity.

Theorem. *A probabilistic rule f satisfies coalitional strategy-proofness, same-sideness and strong symmetry if and only if it is the uniform probabilistic*

[19]The feasibility is given in the appendix. A detailed explanation is in an online supplementary note (Hatsumi and Serizawa 2009).

rule.

The proof of the theorem is in the appendix.

In this characterization, we do not employ strong coalitional strategy-proofness but the weaker version of coalitional strategy-proofness. Note that since same-sideness is weaker than Pareto-efficiency in the probabilistic model, this characterization is independent of Sasaki (1997), Kureishi (2000), and Ehlers and Klaus (2003).

Although coalitional strategy-proofness is stronger than strategy-proofness, we emphasize that even strong coalitional strategy-proofness and same-sideness do not imply at most binary. This fact is illustrated by Example 6 below.

Example 6. Let $n = 3$ and $k = 2$. We define the probabilistic rule f as follows.
For all $u \in U^3$, if $b(u_1) = 2$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(1) = \frac{1}{15}, f_1(u)(2) = \frac{13}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{27}{30}, f_2(u)(1) = f_3(u)(1) = \frac{3}{30}, \end{cases}$$

and if $b(u_1) = 1$ and $b(u_2) = b(u_3) \geq 1$,

$$\begin{cases} f_1(u)(0) = \frac{1}{15}, f_1(u)(1) = \frac{14}{15} \\ f_2(u)(0) = f_3(u)(0) = \frac{7}{15}, f_2(u)(1) = f_3(u)(1) = \frac{8}{15}. \end{cases}$$

Otherwise, f induces the same marginal distribution profile as the uniform probabilistic rule.

Then, the rule f satisfies strong coalitional strategy-proofness and same-sideness, even though it violates at most binary[20]. $\diamondsuit$

In the deterministic model, an agent's consumption set is one dimensional, and the feasible allocation set is $n - 1$ dimensional. On the other hand, in the probabilistic model, an agent's consumption set, *i.e.*, the set of the marginal distributions is k dimensional, and so the set of the feasible marginal distribution profiles is considerably higher than $n - 1$ even though the feasibility constraint makes its dimension less than $k \cdot (n - 1)$. Furthermore, for any utility function $u_i \in U$, at a marginal distribution with support of more than or equal to three consumption levels, no "Maskin monotonic transformation"[21] of u_i can be taken from U. That is, the technique of Maskin monotonic transformation, which plays an important role in the

[20] The feasibility is given in the appendix. A detailed explanation is given in an online supplementary note (Hatsumi and Serizawa 2009).

[21] A utility function $u_i' \in U$ is *Maskin monotonic transformation* of $u_i \in U$ at a marginal distribution p_i if $p_i' \neq p_i$ and $E(p_i'; u_i') \geq E(p_i; u_i')$ together imply $E(p_i'; u_i) > E(p_i; u_i)$. This property was developed by Maskin (1999).

strategy-proofness literature, cannot be applied to this marginal distribution. Accordingly, analyzing the probabilistic model is much more difficult than analyzing the deterministic model.

As Ehlers and Klaus (2003) explain, when at most binary is assumed, the probabilistic model is reduced to the deterministic model, that is, at most binary makes the probabilistic model tractable. However, Example 6, where agent 1 is assigned a marginal distribution with positive probabilities of all the consumption levels, demonstrates that without at most binary, even if coalitional strategy-proofness and same-sideness are assumed, analyzing the probabilistic model is still difficult.

3 Concluding Remarks

We have established that *a rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule*. This result implies that the uniform probabilistic rule retains a very important role in the probabilistic model of homogeneous indivisible goods when a planner wishes to coalitionally strategy-proof property, similarly to the consequence in the deterministic model. We also show, by constructing examples, that if same-sideness is replaced by respect for unanimity, the statement does not hold even with the additional requirements of no-envy, anonymity, at most binary, peaks-onlyness and continuity. This fact emphasizes the difference between the probabilistic and deterministic models, and suggests that matters of interest remain in the probabilistic model of assigning homogeneous indivisible goods. We anticipate the current paper will encourage further study of this model.

Appendix

Proof of Fact 2. Without loss of generality, assume $N^a = \{1, \cdots, n^a\}$ and $N^b = \{n^a + 1, \cdots, n\}$. Let $y = k - \sum_{i \in N^a} x_i - n^b \cdot x_\mu$. Note that $\frac{y}{n^b} = \mu - x_\mu$. We construct a distribution inducing the marginal distribution profile of the statement.

Consider the distribution below.

$$[\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y}, x_\mu, \cdots, x_\mu),$$

$$\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, x_\mu, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y}, x_\mu, \cdots, x_\mu),$$

$$\cdots,$$

$$\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, x_\mu, \cdots, x_\mu, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y}),$$

$$\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, \underbrace{x_\mu + 1}_{1}, x_\mu, \cdots, x_\mu, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y-1}),$$
$$\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, \underbrace{x_\mu + 1, x_\mu + 1}_{2}, x_\mu, \cdots, x_\mu, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y-2}),$$
$$\cdots,$$
$$\frac{1}{n^b} \circ (x_1, \cdots, x_{n^a}, \underbrace{x_\mu + 1, \cdots, x_\mu + 1}_{y-1}, x_\mu, \cdots, x_\mu, \underbrace{x_\mu + 1}_{1})]$$

This distribution is constructed by assigning allocations with probability $\frac{1}{n^b}$ to each of n^b allocations. Notice that each allocation is feasible. It induces that for all $i \in N^a$, x_i is assigned with probability 1, and for all $j \in N^b$, $x_\mu + 1$ is assigned with probability $\frac{y}{n^b}(= \mu - x_\mu)$ and x_μ with probability $1 - \frac{y}{n^b}$. Thus, we have the statement. □

Proof of Fact 3. We introduce a lemma at first, and then prove the fact.

Lemma 1. For all $u, u' \in U^n$ and all $i \in N$, if $b(u_i) \neq b(u'_i)$, then there exists $x \in K$ such that $\parallel u_i(x) - u'_i(x) \parallel > [u_i(b(u_i)) - u_i(b(u'_i))]/2$.

Proof of Lemma 1. Let $u, u' \in U^n$, $i \in N$ and $b(u_i) \neq b(u'_i)$. First, we show that

$$\begin{aligned} &\parallel u_i(b(u_i)) - u'_i(b(u_i)) \parallel + \parallel u_i(b(u'_i)) - u'_i(b(u'_i)) \parallel \\ &> u_i(b(u_i)) - u_i(b(u'_i)). \end{aligned} \tag{1}$$

Case 1. $u_i(b(u'_i)) \geq u'_i(b(u'_i))$

Note that $u_i(b(u_i)) - u'_i(b(u_i)) > u_i(b(u_i)) - u'_i(b(u'_i)) \geq u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (1) holds.

Case 2. $u_i(b(u_i)) \leq u'_i(b(u_i))$

Note that $u'_i(b(u'_i)) - u_i(b(u'_i)) > u'_i(b(u_i)) - u_i(b(u'_i)) \geq u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (1) holds.

Case 3. $u_i(b(u'_i)) < u'_i(b(u'_i))$ and $u_i(b(u_i)) > u'_i(b(u_i))$

In this case, $\parallel u_i(b(u_i)) - u'_i(b(u_i)) \parallel + \parallel u_i(b(u'_i)) - u'_i(b(u'_i)) \parallel = u_i(b(u_i)) - u'_i(b(u_i)) + u'_i(b(u'_i)) - u_i(b(u'_i)) > u_i(b(u_i)) - u_i(b(u'_i))$. Thus, (1) holds.

By the above three cases, we have (1). Thus $\max\{\parallel u_i(b(u_i)) - u'_i(b(u_i)) \parallel, \parallel u_i(b(u'_i)) - u'_i(b(u'_i)) \parallel\} > [u_i(b(u_i)) - u_i(b(u'_i))]/2$ and we have the statement. □

Let a probabilistic rule f satisfy peaks-onlyness. Let $u \in U^n$ and $\epsilon > 0$. Given $i \in N$, let $y_i = \arg\max_{y \in K \setminus b(u_i)} u_i(y)$. Let $\delta > 0$ be such that for all $i \in N$, $\delta \leq [u_i(b(u_i)) - u_i(y_i)]/2$, and let $u' \in U^n$ be such that for all $i \in N$ and all $x \in K$, $\| u_i(x) - u'_i(x) \| < \delta$. We show that for all $i \in N$ and all $x \in K$, $\| f_i(u)(x) - f_i(u')(x) \| < \epsilon$.

First, we show that for all $i \in N$, $b(u_i) = b(u'_i)$. Suppose there exists $i \in N$ such that $b(u_i) \neq b(u'_i)$, and we derive a contradiction. By the assumption, for all $x \in K$, $\| u_i(x) - u'_i(x) \| < \delta \leq [u_i(b(u_i)) - u_i(y_i)]/2$. By the definition of y_i, $[u_i(b(u_i)) - u_i(y)]/2 \leq [u_i(b(u_i)) - u_i(b(u'_i))]/2$. Thus we have that for all $x \in K$, $\| u_i(x) - u'_i(x) \| < [u_i(b(u_i)) - u_i(b(u'_i))]/2$. It is a contradiction to Lemma 1. Thus, we have that for all $i \in N$, $b(u_i) = b(u'_i)$.

Therefore, peaks-onlyness implies $f(u) = f(u')$, and we have the statement of the fact. □

Proof of the Theorem. It is easy to check the *if* part of the theorem. Here, we show the *only if* part. We first introduce three lemmas. Since the marginal distribution profiles in this section all satisfy the condition of Fact 2, we do not explicitly check the feasibilities.

Lemma 2. For all $u \in U^n$, if $p, p' \in P$ are both Pareto-efficient with respect to u, and for all $i \in N$, $E(p_i; u_i) = E(p'_i; u_i)$, then $p = p'$.

Proof of Lemma 2. Let $u \in U^n$, let $p, p' \in P$ be Pareto-efficient with respect to u, and let $E(p_i; u_i) = E(p'_i; u_i)$ for all $i \in N$. We show $p = p'$.

Suppose, on the contrary, that there exists $i \in N$ such that $p_i \neq p'_i$, and we derive a contradiction.

Since both p and p' are Pareto-efficient with respect to u, Fact 1 implies that p and p' satisfy same-sideness with respect to u and at most binary.

From at most binary, there exist $x \in K$ such that $p_i(x) > 0$ and $p_i(x) + p_i(x+1) = 1$, and $y \in K$ such that $p'_i(y) > 0$ and $p'_i(y) + p'_i(y+1) = 1$[22].

Case 1: $x \neq y$.

Without loss of generality, assume $x > y$. If $\sum_{i \in N} b(u_i) \geq k$, by same-sideness, $y < y+1 \leq x < x+1 \leq b(u_i)$[23]. Then by single-peakedness and $p'_i(y) > 0$, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) > p'_i(y) \cdot u(y) + p'_i(y+1) \cdot u(y+1) = E(p_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

If $\sum_{i \in N} b(u_i) < k$, by same-sideness, $b(u_i) \leq y < y+1 \leq x$. Then by single-peakedness and $p'_i(y) > 0$, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) < p'_i(y) \cdot u(y) + p'_i(y+1) \cdot u(y+1) = E(p'_i; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p'_i; u_i)$.

[22] In the case of $x = k$, $p_i(x) = 1$. Similarly, in the case of $y = k$, $p_i(y) = 1$.

[23] If $x = b(u_i)$, then same-sideness implies $p_i(x) = 1$ and $p_i(x+1) = 0$ even though $b(u_i) < x + 1$. Thus, the proof still works.

CASE 2: $x = y$.

Without loss of generality, assume $p_i(x) > p_i'(x)$. If $\sum_{i \in N} b(u_i) \geq k$, then by same-sideness and single-peakedness, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) < p_i'(x) \cdot u(x) + p_i'(x+1) \cdot u(x+1) = E(p_i'; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p_i'; u_i)$.

If $\sum_{i \in N} b(u_i) < k$, then by same-sideness and single-peakedness, $E(p_i; u_i) = p_i(x) \cdot u(x) + p_i(x+1) \cdot u(x+1) > p_i'(y) \cdot u(y) + p_i'(y+1) \cdot u(y+1) = E(p_i'; u_i)$. It is a contradiction to the assumption $E(p_i; u_i) = E(p_i'; u_i)$.

From Cases 1 and 2, we have $p = p'$. □

Lemma 3. Let f be a rule satisfying coalitional strategy-proofness and symmetry. For all $u \in U^n$ such that $u_1 = \cdots = u_n$ and all $u' \in N$ such that for all $i \in N$, $b(u_i') = b(u_i)$, if $f(u)$ is Pareto-efficient with respect to u, then $f(u) = f(u')$.

Proof of Lemma 3. Let $u, u' \in U^n$ be such that $u_1 = \cdots = u_n$, for all $i \in N$, $b(u_i') = b(u_i)$, and $f(u)$ is Pareto-efficient with respect to u. We show $f(u) = f(u')$ by mathematical induction.

STEP A: If $u_1' = \cdots = u_n'$, then $f(u) = f(u')$.

By symmetry, $E(f_1(u); u_1) = \cdots = E(f_n(u); u_n)$ and $E(f_1(u'); u_1') = \cdots = E(f_n(u'); u_n')$. Since $f(u)$ is Pareto-efficient with respect to u, Fact 1 implies that $f(u)$ satisfies same-sideness with respect to u and at most binary. Since $b(u_i) = b(u_i')$ for all $i \in N$, $f(u)$ also satisfies same-sideness with respect to u'. Thus, $f(u)$ is Pareto-efficient with respect to u'.

If for some $j \in N$, $E(f_j(u); u_j') < E(f_j(u'); u_j')$, then by symmetry, for all $i \in N$, $E(f_i(u); u_i') < E(f_i(u'); u_i')$. It contradicts Pareto-efficiency of $f(u)$ with respect to u'. Thus, for all $i \in N$, $E(f_i(u); u_i') \geq E(f_i(u'); u_i')$.

If for some $j \in N$, $E(f_j(u); u_j') > E(f_j(u'); u_j')$, then by symmetry, for all $i \in N$, $E(f_i(u); u_i') > E(f_i(u'); u_i')$. Then, the coalition of all agents N with profile u' manipulates the rule via u and increases the utilities of all members. It is a contradiction to coalitional strategy-proofness.

Therefore, for all $i \in N$, $E(f_i(u); u_i') = E(f_i(u'); u_i')$. By Lemma 2, $f(u) = f(u')$.

STEP B: Let $h \in N$. Assume that if $u_1' = \cdots = u_h'$, $f(u) = f(u')$. Then, if $u_1' = \cdots = u_{h-1}'$, $f(u) = f(u')$.

Let $u' \in U^n$ be such that $u_1' = \cdots = u_{h-1}'$. Then by symmetry, $E(f_1(u'); u_1') = \cdots = E(f_{h-1}(u'); u_{h-1}')$. Thus, if for some $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u_i') > E(f_i(u'); u_i')$, then for all $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u_i') > E(f_i(u'); u_i')$. Then, the coalition $\{1, \cdots, h-1\}$ with $u_{\{1,\cdots,h-1\}}'$ manipulates the rule via $\hat{u}_{\{1,\cdots,h-1\}}$ such that for all $i \in \{1, \cdots, h-1\}$, $\hat{u}_i = u_h'$. Then, any $i \in \{1, \cdots, h-1\}$ obtains $f_i(u)$ and increases her utility by the induction hypothesis. It is a contradiction to coalitional strategy-proofness. Therefore, for all $i \in \{1, \cdots, h-1\}$, $E(f_i(u); u'_i) \leq E(f_i(u'); u'_i)$.

If, for some $j \in \{h, \cdots, n\}$, $E(f_j(u); u'_j) > E(f_j(u'); u'_j)$, then j with u'_j manipulates the rule via $\hat{u}_j = u'_1$ and obtains $f_j(u)$ by the induction hypothesis. It is a contradiction to strategy-proofness. Thus, for all $j \in \{h, \cdots, n\}$, $E(f_j(u); u'_j) \leq E(f_j(u'); u'_j)$.

Therefore, for all $i \in N$, $E(f_i(u); u'_i) \leq E(f_i(u'); u'_i)$. Similarly to Step A, We can show that $f(u)$ is Pareto-efficient with respect to u'. Thus, for all $i \in N$, $E(f_i(u); u'_i) = E(f_i(u'); u'_i)$. Therefore, by Lemma 2, $f(u) = f(u')$.

From Step A and B, we have the statement of the lemma. □

Lemma 4. If f satisfies same-sideness, then it respects unanimity.

Proof of Lemma 4. By same-sideness, $\sum_{i \in N} b(u_i) = k$ implies that for all $i \in N$, $f_i(u)([0, b(u_i)]) = 1$ and $f_i(u)([b(u_i), k]) = 1$. Thus for all $i \in N$, $f_i(u)(b(u_i)) = 1$. □

We prove the theorem by five steps. Hereafter, let f be a rule satisfying coalitional strategy-proofness, same-sideness, and strong symmetry.

Step 1. For all $u \in U^n$ such that $\sum_{i \in N} b(u_i) = k$ and all $i \in N$, $f_i(u)(b(u_i)) = 1$.

Proof of Step 1. By Lemma 4, the statement is directly implied. □

Step 2. Let $x \in K$ be such that $\frac{k}{n} \in [x, x+1)$. Let $u \in U^n$ be such that for all $i \in N$, $b(u_i) = x$. Then for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x+1) = \frac{k}{n} - x$.

Proof of Step 2. For all $z \in K$ such that $x + 2 \leq z \leq k$, let $r_z(u_i) \in \mathbb{R}$ be such that $r_z(u_i) \cdot [u_i(x) - u_i(x+1)] = u_i(x+1) - u_i(z)$. Note that by single-peakedness of u_i with $b(u_i) = x$, for all $z \in K$ such that $x + 2 \leq z \leq k - 1$, $r_z(u_i) < r_{z+1}(u_i)$. By single-peakedness and risk-averseness, we also have that for all $z \in K$ such that $x + 2 \leq z \leq k - 1$,

$$0 < r_z(u_i) - [z - (x+1)] < r_{z+1}(u_i) - [(z+1) - (x+1)]. \quad (2)$$

[Figure 1 enters around here.]

Let $p \in P$ be such that for all $i \in N$, $p_i(x) = x + 1 - \frac{k}{n}$ and $p_i(x+1) = \frac{k}{n} - x$. We show $f(u) = p$.

Case A: $n \cdot x = k$.

By Step 1, for all $i \in N$, $f_i(u)(x) = 1$ and $f_i(u)(x+1) = 0$. Thus the statement holds.

Case B: $n \cdot x < k$

Step B-1. First, we consider the case where $u_1 = \cdots = u_n$. By same-sideness, for all $i \in N$, $f_i(u)([x,k]) = 1$. Let $u' \in U^n$ be such that $b(u'_1) = \cdots = b(u'_{k-nx}) = x+1$ and $b(u'_{k-nx+1}) = \cdots = b(u'_n) = x$. Then, we have $\sum_{i \in N} b(u'_i) = k$. Thus, by Step 1, for all $i \in \{1, \cdots, k-nx\}$, $f_i(u')(x+1) = 1$ and for all for all $i \in \{k-nx+1, \cdots, k\}$, $f_i(u')(x) = 1$.

Thus, coalitional strategy-proofness and symmetry imply that for all $i \in \{1, \cdots, k-nx\}$, $E(f_i(u); u_i) \geq E(f_i(u'); u_i) = u_i(x+1)$. By symmetry, for all $i \in N$, $E(f_i(u); u_i) \geq u_i(x+1)$. Note that

$$\begin{aligned}
&E(f_i(u); u_i) \geq u_i(x+1) \\
\iff &\sum_{z \in [x,k]} f_i(u)(z) \cdot u_i(z) \geq u_i(x+1) \qquad \text{(by same-sideness)} \\
\iff &\sum_{z \in [x,k]} f_i(u)(z) \cdot [u_i(z) - u_i(x+1)] \geq 0 \\
\iff &f_i(u)(x) \cdot [u_i(x) - u_i(x+1)] \\
&\quad - \sum_{z \in [x+2,k]} f_i(u)(z) \cdot [u_i(x+1) - u_i(z)] \geq 0. \qquad (3)
\end{aligned}$$

By using the notation r, we rewrite (3) as: for all $i \in N$,

$$f_i(u)(x) - \sum_{z \in [x+2,k]} f_i(u)(z) \cdot r_z(u_i) \geq 0. \qquad (4)$$

We show that for all $i \in N$, $f_i(u)([x+2,k]) = 0$ by mathematical induction.

Step B-1-1: For all $i \in N$, $f_i(u)(k) = 0$.

Suppose, on the contrary, for some $j \in N$, $f_i(u)(k) > 0$. Then, by strong symmetry and $u_1 = \cdots u_n$, for all $i \in N$, $f_i(u)(k) > 0$. We derive a contradiction.

Let $\hat{u} \in U^n$ be such that $\hat{u}_1 = \cdots = \hat{u}_n$, for all $i \in N$, $b(\hat{u}_i) = x$, and $r_{x+2}(\hat{u}_i) > \frac{1}{f_i(u)(k)}$. By strong symmetry, $f_1(\hat{u}) = \cdots = f_n(\hat{u})$.

Suppose for some $j \in N$, $f_j(\hat{u})([x+2,k]) \geq f_j(u)(k)$. Then,

$$\begin{aligned}
&f_j(\hat{u})(x) - \sum_{z \in [x+2,k]} f_j(\hat{u})(z) \cdot r_z(\hat{u}_j) \\
\leq\ &f_j(\hat{u})(x) - \sum_{z \in [x+2,k]} f_j(\hat{u})(z) \cdot r_{x+2}(\hat{u}_j) \\
&\text{(by } r_{x+2}(\hat{u}_j) \leq r_z(\hat{u}_j) \text{ for all } z \in [x+2,k])
\end{aligned}$$

$$
\begin{aligned}
&= f_j(\hat{u})(x) - f_j(\hat{u})([x+2,k]) \cdot r_{x+2}(\hat{u}_j) \\
&\leq f_j(\hat{u})(x) - f_j(u)(k) \cdot r_{x+2}(\hat{u}_j) \\
&< f_j(\hat{u})(x) - 1 \qquad (\text{by } r_{x+2}(\hat{u}_j) > \frac{1}{f_j(u)(k)}) \\
&\leq 0.
\end{aligned}
$$

It is a contradiction since $\hat{u} \in U^n$ also has to satisfy (4). Thus, for all $i \in N$,

$$
\sum_{z \in [x+2,k]} f_i(\hat{u})(z) < f_i(u)(k). \tag{5}
$$

By feasibility, $\sum_{i \in N} \sum_{z \in K} f_i(\hat{u})(z) \cdot z = \sum_{i \in N} \sum_{z \in K} f_i(u)(z) \cdot z = k$. Thus, by strong symmetry, for all $i \in N$,

$$
\sum_{z \in K} f_i(\hat{u})(z) \cdot z = \sum_{z \in K} f_i(u)(z) \cdot z = \frac{k}{n}.
$$

Then, by same-sideness, $\sum_{z \in [x,k]} f_i(\hat{u})(z) \cdot z = \sum_{z \in [x,k]} f_i(u)(z) \cdot z$. Note that

$$
\begin{aligned}
& \sum_{z \in [x,k]} f_i(\hat{u})(z) \cdot z = \sum_{z \in [x,k]} f_i(u)(z) \cdot z \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z = -[f_i(\hat{u})(x+1) - f_i(u)(x+1)] \cdot (x+1) \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z \\
& = -\{[1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(\hat{u})(z)] - [1 - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(u)(z)]\} \cdot (x+1) \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot z \\
& = \{ \sum_{z \in [x,k] \setminus \{x+1\}} f_i(\hat{u})(z) - \sum_{z \in [x,k] \setminus \{x+1\}} f_i(u)(z)\} \cdot (x+1) \\
\iff & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)] = 0 \\
\iff & \sum_{z \in [x+2,k]} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot [z - (x+1)] = f_i(\hat{u})(x) - f_i(u)(x). \qquad (6)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
& E(f_i(\hat{u}); u_i) - E(f_i(u); u_i) \\
= & \sum_{z \in [x,k] \setminus \{x+1\}} [f_i(\hat{u})(z) - f_i(u)(z)] \cdot u_i(z)
\end{aligned}
$$

$$
\begin{aligned}
&+\{[1-\sum_{z\in[x,k]\setminus\{x+1\}} f_i(\hat{u})(z)]-[(1-\sum_{z\in[x,k]\setminus\{x+1\}} f_i(u)(z)]\}\cdot u_i(x+1)\\
&=\sum_{z\in[x,k]\setminus\{x+1\}}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot[u_i(z)-u_i(x+1)]\\
&=\{\sum_{z\in[x+2,k]}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot[z-(x+1)]\}\cdot[u_i(x)-u_i(x+1)]\\
&\quad+\sum_{z\in[x+2,k]}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot[u_i(z)-u_i(x+1)] \qquad \text{(by (6))}\\
&=\{\sum_{z\in[x+2,k]}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot[z-(x+1)]\}\cdot[u_i(x)-u_i(x+1)]\\
&\quad+\sum_{z\in[x+2,k]}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot\{-r_z(u_i)\cdot[u_i(x)-u_i(x+1)]\}\\
&\qquad\text{(by the definition of } r)\\
&=[u_i(x)-u_i(x+1)]\cdot\{\sum_{z\in[x+2,k]}[f_i(\hat{u})(z)-f_i(u)(z)]\cdot[z-(x+1)-r_z(u_i)]\}\\
&=[u_i(x)-u_i(x+1)]\\
&\quad\cdot\{\sum_{z\in[x+2,k]}[f_i(u)(z)-f_i(\hat{u})(z)]\cdot[r_z(u_i)-\{z-(x+1)\}]\} \qquad (7)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\sum_{z\in[x+2,k]}[f_i(u)(z)-f_i(\hat{u})(z)]\cdot[r_z(u_i)-\{z-(x+1)\}]\\
&=[f_i(u)(k)-f_i(\hat{u})(k)]\cdot[r_k(u_i)-\{k-(x+1)\}]\\
&\quad+\sum_{z\in[x+2,k-1]}[f_i(u)(z)-f_i(\hat{u})(z)]\cdot[r_z(u_i)-\{z-(x+1)\}]\\
&\geq[f_i(u)(k)-f_i(\hat{u})(k)]\cdot[r_k(u_i)-\{k-(x+1)\}]\\
&\quad-\sum_{z\in[x+2,k-1]}f_i(\hat{u})(z)\cdot[r_z(u_i)-\{z-(x+1)\}]\\
&\qquad\text{(by (2), for all } z\in[x+2,k-1], r_z(u_i)-\{z-(x+1)\}>0)\\
&\geq[f_i(u)(k)-f_i(\hat{u})(k)]\cdot[r_k(u_i)-\{k-(x+1)\}]\\
&\quad-\sum_{z\in[x+2,k-1]}f_i(\hat{u})(z)\cdot[r_k(u_i)-\{k-(x+1)\}] \qquad \text{(by (2))}\\
&=[f_i(u)(k)-\sum_{z\in[x+2,k]}f_i(\hat{u})(z)]\cdot[r_k(u_i)-\{k-(x+1)\}]\\
&>0 \qquad \text{(by (5) and (2).)} \qquad (8)
\end{aligned}
$$

Then (7) and (8) together imply that for all $i\in N$, $E(f_i(\hat{u});u_i)-E(f_i(u);u_i)>0$. It is a contradiction to coalitional strategy-proofness. Thus, for all $i\in N$,

$f_i(u)(k) = 0$.

STEP B-1-2. Let $y \in K$ be such that $y \geq x + 2$. Assume that for all $i \in N$, $f_i(u)([y + 1, k]) = 0$. Then, for all $i \in N$, $f_i(u)([y, k]) = 0$.

By same-sideness and the induction hypothesis, for all $i \in N$, $f_i([x, y]) = 1$. Then we apply a similar argument to Step B-1-1 by replacing k with y, and we have that for all $i \in N$, $f_i(u)([y, k]) = 0$.

Now, we have for all $u \in U^n$ such that $u_1 = \cdots = u_n$ and $b(u_i) = x$, $f_i(u)([x, x + 1]) = 1$. Symmetry and feasibility imply that for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x + 1) = \frac{k}{n} - x$, *i.e.*, $f(u) = p$.

STEP B-2. Note that for all $u \in U^n$ such that for all $i \in N$, $b(u_i) = x$, p is Pareto-efficient with respect to u. Thus by Lemma 3 and Step B-1, for all $u \in U^n$ such that for all $i \in N$, $b(u_i) = x$, $f(u) = p$. We finish Case B.

From Cases A and B, the statement is established. □

Step 3. Let $x \in K$ be such that $\frac{k}{n} \in [x, x + 1)$. Let $u \in U^n$ be such that $b(u_1) = \cdots = b(u_n)$. Then for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x + 1) = \frac{k}{n} - x$.

Proof of Step 3. STEP A. First, we consider the case where $u_1 = \cdots = u_n$.

Let $p \in P$ be such that for all $i \in N$, $p_i(x) = x + 1 - \frac{k}{n}$ and $p_i(x + 1) = \frac{k}{n} - x$. Then, p satisfies same-sideness with respect to u and at most binary. Thus, it is Pareto-efficient with respect to u.

Let $\hat{u} \in U^n$ be such that for all $i \in N$, $b(\hat{u}_i) = x$. Then, by Step 2, $f(\hat{u}) = p$. Since p is Pareto-efficient with respect to u, and symmetry implies $E(f_1(u); u_1) = \cdots = E(f_n(u); u_n)$, it follows that for all $i \in N$, $E(f_i(\hat{u}); u_i) = E(p_i; u_i) \geq E(f_i(u); u_i)$. If $E(f_i(\hat{u}); u_i) > E(f_i(u); u_i)$, it is a contradiction to coalitional strategy-proofness. Thus, for all $i \in N$, $E(f_i(u); u_i) = E(f_i(\hat{u}); u_i) = E(p_i, u_i)$.

Therefore, by Lemma 2, $f(u) = p$.

STEP B. From Lemma 3 and Step A, for all $u \in U^n$ such that $b(u_1) = \cdots = b(u_n)$, we have $f(u) = p$. □

Step 4. Let $x \in K$ be such that $\frac{k}{n} \in [x, x + 1)$. Let $u \in U^n$ be such that for all $i \in N$, $b(u_i) \leq x$, or for all $i \in N$, $b(u_i) \geq x + 1$. Then for all $i \in N$, $f_i(u)(x) = x + 1 - \frac{k}{n}$ and $f_i(u)(x + 1) = \frac{k}{n} - x$.

Proof of Step 4. Assume that for all $i \in N$, $b(u_i) \leq x$, since the other case can be treated symmetrically. Let $p \in P$ be such that for all $i \in N$, $p_i(x) = x + 1 - \frac{k}{n}$ and $p_i(x + 1) = \frac{k}{n} - x$. We prove $f(u) = p$ by mathematical induction.

Step A: If $u_1 = \cdots = u_n$, then $f(u) = p$.

The statement is from Step 3.

Step B: Let $h \in N$. Assume that for all $u' \in U^n$ such that for all $u'_1 = \cdots = u'_h$, we have $f(u) = p$. Then, if u is such that $u_1 = \cdots = u_{h-1}$, we have $f(u) = p$.

Let $u_1 = \cdots u_{h-1}$. By symmetry, $E(f_1(u); u_1) = \cdots = E(f_{h-1}(u); u_{h-1})$. Thus, if, for some $j \in \{1, \cdots, h-1\}$, $E(p_j; u_j) > E(f_j(u); u_j)$, then for all $i \in \{1, \cdots, h-1\}$, $E(p_i; u_i) > E(f_i(u); u_i)$. Then, coalition $\{1, \cdots, h-1\}$ with $u_{\{1,\cdots,h-1\}}$ manipulates the rule via $\hat{u}_{\{1,\cdots,h-1\}}$ such that for all $i \in \{1, \cdots, h-1\}$, $\hat{u}_i = u_h$, and any $i \in \{1, \cdots, h-1\}$ obtains p_i by the induction hypothesis and increases her utility. It is a contradiction to coalitional strategy-proofness. Therefore, for all $i \in \{1, \cdots, h-1\}$, $E(p_i; u_i) \leq E(f_i(u); u_i)$.

On the other hand, if, for some $j \in \{h, \cdots, n\}$, $E(p_j; u_j) > E(f_j(u); u_j)$, then j with u_j manipulates the rule via $\hat{u}_j = u_1$. Then, j obtains p_j by induction hypothesis and increases her utility. It is a contradiction to strategy-proofness. Therefore, for all $j \in \{h, \cdots, n\}$, $E(p_j; u_j) \leq E(f_j(u); u_j)$.

Thus, for all $i \in N$, $E(p_i; u_i) \leq E(f_i(u); u_i)$. Since p satisfies same-sideness with respect to u and at most binary, Fact 1 implies that p is Pareto-efficient with respect to u. Therefore, for all $i \in N$, $E(f_i(u); u_i) = E(p_i; u_i)$. By Lemma 2, $f(u) = p$.

By Step A and Step B, we have the statement of this step. □

Step 5. (i) For all $u \in U^n$, if $\sum_{i \in N} b(u_i) < k$, then for all $i \in N$ such that $b(u_i) \geq x_\lambda(u) + 1$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$.
(ii) For all $u \in U^n$, if $\sum_{i \in N} b(u_i) > k$, then for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \geq x_\lambda(u) + 1$, $f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$.

Proof of Step 5. Given $u \in U^n$, let $\overline{N}(u) = \{i \in N : b(u_i) \geq x_\lambda(u) + 1\}$ and $\underline{N}(u) = \{i \in N : b(u_i) \leq x_\lambda(u)\}$. In addition, let $\overline{n}(u)$ be the number of agents in $\overline{N}(u)$, and $\underline{n}(u)$ be the number of agents in $\underline{N}(u)$. Note that $\overline{N}(u) \cup \underline{N}(u) = N$.

Without loss of generality, let $u \in U^n$ be such that $\sum_{i \in N} b(u_i) < k$, since the other case is symmetrically proved. We prove this step by mathematical induction on $\overline{n}(u)$.

Step A: For all $u \in U^n$, if $\overline{n}(u) = 0$, then for all $i \in \underline{N}(u) = N$, $f_i(u)(x_\lambda(u) + 1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$.

In this case, $\lambda(u) = \frac{k}{n}$. Thus by Step 4, the statement is directly implied.

STEP B: Let $l \in N \backslash \{0, n\}$[24]. Assume that for all $u \in U^n$, if $\overline{n}(u) \leq l-1$, then for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u) + 1 - \lambda(u)$ and $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$. Then for all $u \in U^n$, if $\overline{n}(u) = l$, for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$ and for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u) + 1 - \lambda(u)$ and $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$.

Let $\overline{n}(u) = l$. First, we show that for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$. Here, we start a new mathematical induction within Step B. Note that by same-sideness, we have that for all $i \in \overline{N}(u)$, $f_i(u)([b(u_i), k]) = 1$.

STEP B-I: For all $i \in \overline{N}(u)$, $f_i(u)(k) = 0$.

Suppose, on the contrary, that for some $i \in \overline{N}(u)$, $f_i(u)(k) > 0$. We derive a contradiction.

Let $u_i' \in U$ be such that $b(u_i') = 0$. Then, $f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i})) = x_\lambda(u_i', u_{-i}) + 1 - \lambda(u_i', u_{-i})$ and $f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i}) + 1) = \lambda(u_i', u_{-i}) - x_\lambda(u_i', u_{-i})$ by the induction hypothesis of Step B. Note that by the definition of x_λ, $b(u_i) \geq x_\lambda(u_i', u_{-i}) + 1$[25].

[Figure 2 enters around here.]

Since $f_i(u)(k) > 0$, there is $\hat{u}_i \in U$ such that $b(\hat{u}_i) = b(u_i)$ and $\hat{u}_i(0) > \{1 - f_i(u)(k)\} \cdot \hat{u}_i(b(u_i)) + f_i(u)(k) \cdot \hat{u}_i(b(u_i)+1)$. Suppose $f_i(\hat{u}_i, u_{-i})([b(u_i)+1, k]) \geq f_i(u)(k)$. Then, by same-sideness, we have that

$$f_i(\hat{u}_i, u_{-i})(b(u_i)) \leq 1 - f_i(u)(k). \tag{9}$$

Then,

$$\begin{aligned}
&E(f_i(u_i', u_{-i}); \hat{u}_i) \\
&= f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i})) \cdot \hat{u}_i(x_\lambda(u_i', u_{-i})) \\
&\quad + f_i(u_i', u_{-i})(x_\lambda(u_i', u_{-i}) + 1) \cdot \hat{u}_i(x_\lambda(u_i', u_{-i}) + 1) \\
&> \hat{u}_i(0) \qquad \text{(by } b(u_i) \geq x_\lambda(u_i', u_{-i}) + 1 \text{ and single-peakedness)} \\
&> \{1 - f_i(u)(k)\} \cdot \hat{u}_i(b(u_i)) + f_i(u)(k) \cdot \hat{u}_i(b(u_i) + 1) \\
&\quad \text{(by the definition of } \hat{u}_i\text{)} \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot \hat{u}_i(b(u_i)) + f_i(\hat{u}_i, u_{-i})([b(u_i)+1, k]) \cdot \hat{u}_i(b(u_i) + 1) \\
&\quad \text{(by (9) and } \hat{u}_i(b(u_i)) > \hat{u}_i(b(u_i)+1)\text{)} \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot \hat{u}_i(b(u_i)) + \sum_{z \in [b(u_i)+1, k]} f_i(\hat{u}_i, u_{-i})(z) \cdot \hat{u}_i(z)
\end{aligned}$$

[24]We need not consider the case of $l = n$ since if $\sum_{i \in N} b(u_i) < k$, $\overline{n}(u)$ cannot be equal to n.

[25]Suppose, on the contrary, that $b(u_i) \leq x_\lambda(u_i', u_{-i})$. Then, $x_\lambda(u_i', u_{-i}) = x_\lambda(u)$ by the definition of x_λ in the case $\sum_{i \in N} b(u_i) < k$. This contradicts the assumption that $b(u_i) \geq x_\lambda(u) + 1$. Thus, $b(u_i) \geq x_\lambda(u_i', u_{-i}) + 1$.

$$\text{(by single-peakedness)}$$
$$= E(f_i(\hat{u}_i, u_{-i}); \hat{u}_i)$$

It is a contradiction to strategy-proofness. Thus

$$f_i(\hat{u}_i, u_{-i})([b(u_i)+1, k]) < f_i(u)(k). \tag{10}$$

Then,

$$\begin{aligned}
& E(f_i(\hat{u}_i, u_{-i}); u_i) \\
&= f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot u_i(b(u_i)) + \sum_{z \in [b(u_i)+1, k]} f_i(\hat{u}_i, u_{-i})(z) \cdot u_i(z) \\
&\geq f_i(\hat{u}_i, u_{-i})(b(u_i)) \cdot u_i(b(u_i)) + f_i(\hat{u}_i, u_{-i})([b(u_i)+1, k]) \cdot u_i(k) \\
&\quad \text{(by single-peakedness)} \\
&> \{1 - f_i(u)(k)\} \cdot u_i(b(u_i)) + f_i(u)(k) \cdot u_i(k) \qquad \text{(by (10))} \\
&\geq \sum_{z \in [b(u_i), k-1]} f_i(u)(z) \cdot u_i(z) + f_i(u)(k) \cdot u_i(k) \qquad \text{(by single-peakedness)} \\
&= E(f_i(u); u_i).
\end{aligned}$$

It is a contradiction to strategy-proofness. Thus, we have $f_i(u)(k) = 0$ for all $i \in \overline{N}(u)$.

STEP B-II: Let $x \in K$ be such that $b(u_i) + 1 \leq x \leq k - 1$. Assume that for all $i \in \overline{N}(u)$, $f_i(u)([x+1, k]) = 0$. Then for all $i \in \overline{N}(u)$, $f_i(u)([x, k]) = 0$.

Suppose, on the contrary, that for some $i \in \overline{N}(u)$, $f_i(u)([x, k]) > 0$, and we derive a contradiction. By $f_i(u)([x+1, k]) = 0$ (induction hypothesis), $f_i(u)([b(u_i), x]) > 0$. Then we apply a similar argument to STEP B-I by replacing k with x, and we have that for all $i \in N$, $f_i(u)([x, k]) = 0$.

Now, we have that for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$. Next, we show that for all $i \in \underline{N}(u)$, $f_i(u)(x_\lambda(u)) = x_\lambda(u) + 1 - \lambda(u)$ and $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$.

Let $k' = k - \sum_{i \in \overline{N}(u)} b(u_i)$. Since for all $i \in \overline{N}(u)$, $f_i(u)(b(u_i)) = 1$, $\sum_{i \in \underline{N}(u)} \sum_{x_i \in K} f_i(u)(x_i) x_i = k'$. Note that $\lambda(u) = \frac{k'}{\underline{n}(u)}$ and for all $i \in \underline{N}(u)$, $b(u_i) < \lambda(u)$.

Then we can use a similar argument to Step 4 by replacing k with k' and $\frac{k}{n}$ by $\frac{k'}{\overline{n}(u)}$. We omit the detailed proof.

By Steps A and B, we have for all $i \in N$ such that $b(u_i) \geq x_\lambda(u) + 1$, $f_i(u)(b(u_i)) = 1$ and for all $i \in N$ such that $b(u_i) \leq x_\lambda(u)$, $f_i(u)(x_\lambda(u)+1) = \lambda(u) - x_\lambda(u)$ and $f_i(u)(x_\lambda(u)) = (x_\lambda(u) + 1) - \lambda(u)$. □

Finally, by Step 1 and Step 5, a probabilistic rule satisfies coalitional strategy-proofness, same-sideness, and strong symmetry if and only if it is the uniform probabilistic rule. □

Feasibilities in Example 4, 5 and 6. Due to Fact 2, it is sufficient to show that in each example, there is a distribution over feasible allocations inducing the marginal distribution profile $f(u)$ when $f(u)$ is different from the outcome of the uniform probabilistic rule.

Example 4. Without loss of generality, let $u \in U^3$ be such that $b(u_1) = 1$ and $b(u_2) = b(u_3) = 0$. In the case of $u_1(1) - u_1(0) \geq u_1(1) - u_1(2)$, $[\frac{2}{20} \circ (2,0,0), \frac{9}{20} \circ (1,1,0), \frac{9}{20} \circ (1,0,1)]$ induces $f(u)$. In the case of $u_1(1) - u_1(0) < u_1(1) - u_1(2)$, $[\frac{2}{20} \circ (0,1,1), \frac{9}{20} \circ (1,1,0), \frac{9}{20} \circ (1,0,1)]$ induces $f(u)$. $\diamondsuit$

Example 5. Without loss of generality, let $u \in U^4$ be such that $b(u_1) = 0$ and for any other agent $j \in N\backslash\{1\}$, $b(u_j) \geq 1$. Then, $[\frac{1}{30} \circ (1,1,0,0), \frac{1}{30} \circ (1,0,1,0), \frac{1}{30} \circ (1,0,0,1), \frac{9}{30} \circ (0,1,1,0), \frac{9}{30} \circ (0,1,0,1), \frac{9}{30} \circ (0,0,1,1)]$ induces $f(u)$. $\diamondsuit$

Example 6. Let $u \in U^n$ be such that $b(u_2) \geq 1$, $b(u_3) \geq 1$. When $b(u_1) = 2$, $[\frac{1}{15} \circ (0,1,1), \frac{1}{30} \circ (1,1,0), \frac{1}{30} \circ (1,0,1), \frac{13}{15} \circ (2,0,0)]$ induces $f(u)$. When $b(u_1) = 1$, $[\frac{1}{15} \circ (0,1,1), \frac{7}{15} \circ (1,1,0), \frac{7}{15} \circ (1,0,1)]$ induces $f(u)$. $\diamondsuit$

References

Amorós, P. (2002) Single-peaked preferences with several commodities. *Social Choice and Welfare* 19: 57–67

Barberà, S., M.O. Jackson, and A. Neme (1997) Strategy-proof allotment rules. *Games and Economic Behavior* 18: 1–21

Benassy, J-P. (1982) *The Economics of Market Disequilibrium.* Academic Press, New York

Ching, S. (1994) An alternative characterization of the uniform rule. *Social Choice and Welfare* 11: 131–136

Ching, S. and S. Serizawa (1998) A maximal domain for the existence of strategy-proof rules. *Journal of Economic Theory* 78: 157–166

Chun, Y. (2006) The separability principle in economies with single-peaked preferences. *Social Choice and Welfare* 26: 239–253

Ehlers, L. and B. Klaus (2003) Probabilistic assignments of identical indivisible objects and uniform probabilistic rules. *Review of Economic Design* 8: 249–268

Hatsumi, K. and S. Serizawa (2009) Supplementary note for "coalitionally strategy-proof rules in allotment economies with homogeneous indivisible goods." *ISER Discussion Paper* 686 (http://www.iser.osaka-u.ac.jp/library/dp/index.html)

Kesten, O. (2006) More on the uniform rule: characterizations without Pareto optimality. *Mathematical Social Sciences* 51: 192–200

Klaus, B. (2006) A note on the separability principle in economies with single-peaked preferences. *Social Choice and Welfare* 26: 255–261

Kureishi, W. (2000) A simple characterization of the randomized uniform rule. Working Paper, Osaka University

Kureishi, W. and H. Mizukami (2007) Equal probability for the best and the assignment of identical indivisible objects. *Economics Bulletin* 4–8: 1–10

Maskin, E. (1999) Nash equilibrium and welfare optimality. *Review of Economic Studies* 66: 23–38

Massó, J. and A. Neme (2001) Maximal domain of preferences in the division problem. *Games and Economic Behavior* 37: 367–387

Massó, J. and A. Neme (2004) A maximal domain of preferences for strategy-proof, efficient, and simple rules in the division problem. *Social Choice and Welfare* 23: 187–206

Massó, J. and A. Neme (2007) Bribe-proof rules in the division problem. *Games and Economic Behavior* 61: 331–343

Mizobuchi, H. and S. Serizawa (2006) Maximal domain for strategy-proof rules in allotment economies. *Social Choice and Welfare* 27: 195–210

Sasaki, H. (1997) Randomized uniform allocation mechanism and single-peaked preferences of indivisible goods. Working Paper, Waseda University

Serizawa, S. (2006) Pairwise strategy-proofness and self-enforcing manipulation. *Social Choice and Welfare* 26: 305–331

Sprumont, Y. (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. *Econometrica* 59: 509–519

Thomson, W. (1994a) Consistent solutions to the problem of fair division when preferences are single-peaked. *Journal of Economic Theory* 63: 219–245

Thomson, W. (1994b) Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. *Social Choice and Welfare* 11: 205–223

Thomson, W. (1995) Population-monotonic solutions to the problem of fair division when preferences are single-peaked. *Economic Theory* 5: 229–246

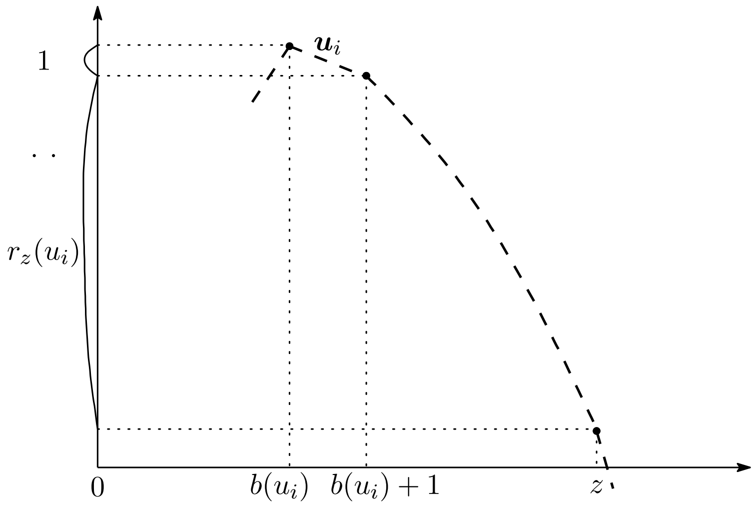

Figure 1. Illustration of $r_z(u_i)$ in the proof of Step 2.

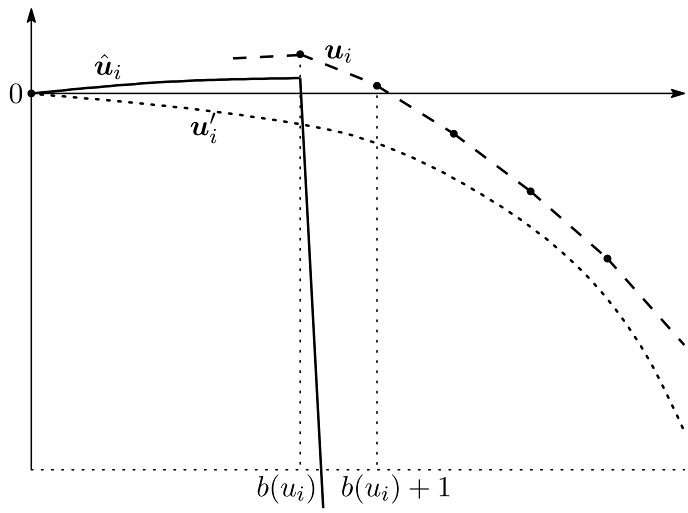

Figure 2. Illustration of u_i, u'_i and $\hat{u}_i$ in the proof of Step B-i of Step 5