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CHEAP TALK WITH AN INFORMED RECEIVER

Junichiro Ishida
Takashi Shimizu

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The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

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Junichiro Ishida* and Takashi Shimizu†

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Abstract

This paper considers a model of strategic information transmission with an imperfectly informed receiver and provides a simple logic by which the receiver's prior knowledge becomes an impediment to efficient communication. We show that the extent of communication is severely limited as the receiver becomes more informed. Moreover, in a simple example with two signals, we show that no information can be conveyed via cheap talk for an arbitrarily small degree of preference incongruence. This result draws sharp contrast to the case with an uninformed receiver which always yields a fully separating equilibrium as long as the preferences are sufficiently congruent.

Keywords: Cheap talk, Informed receiver, Information gap, Preference congruence.

JEL Classification Number: D23; D82.

*Institute of Social and Economic Research, Osaka University, 6-1 Mihogaoka, Ibaraki-shi, Osaka 567-0047 JAPAN (E-mail: jishida@iser.osaka-u.ac.jp)

†Graduate School of Economics, Kobe University, 2-1 Rokkodai-cho Nada-ku, Kobe, 657-8501 JAPAN (E-mail: shimizu@econ.kobe-u.ac.jp)

1 Introduction

Since the seminal work of Crawford and Sobel [5], a substantial amount of attention has been paid to strategic aspects of communication between players with conflicting interests. Despite many notable developments that have been made over the years, however, most existing works focus on the case where the sender who knows the state of nature with precision sends a message to the receiver who knows nothing about it. The latter assumption that the receiver has no information of her own seems particularly restrictive as the receiver often has access to alternative sources of information.

In this paper, we consider a simple model of strategic information transmission to investigate how the receiver's prior information affects the extent of communication. The situation we consider is as follows. The sender observes a signal which imperfectly reflects the true state and then sends a costless message to the receiver. Upon receiving the message, the receiver then "double-checks" it by using her own source of information and chooses some action. Within this environment, we show that the receiver's ability to gain some information on her own, aside from the conflict of interests, can be a major impediment to effective communication. Moreover, in a two-signal example, we show that no information can be conveyed via cheap talk for an arbitrarily small preference incongruence when the receiver's source of information becomes as reliable as the sender's. This result contrasts sharply with the case with an uninformed receiver where there is always an informative equilibrium as long as the degree of preference incongruence is sufficiently small.

There are now a handful of works which examine the nature of strategic information transmission when the receiver, broadly defined, has alternative sources of information. Among them, the paper is more closely related to Morgan and Stocken [15], Galeotti et al. [7], Lai [13] and Moreno de Barreda [6]. The first two consider models with multiple imperfectly informed senders,[1] while the latter two consider cases with an imperfectly informed receiver.[2] While these works do not necessarily concentrate on the questions we pose here, they theoretically have one common feature: the incentive for truthful communication diminishes as

[1] Also see Austen-Smith [1], Gilligan and Krehbiel [8], Krishna and Morgan [12], Battaglini [2], and Li [14] for works with multiple senders. Kawamura [10] considers a setting in which senders have an incentive to "exaggerate" their preferences and show that the use of binary messages is a robust mode of communication. Kawamura [11] also uses a similar setting to analyze how the sample size affects the quality of communication.

[2] Some early attempts to model an informed receiver are provided by Seidmann [16] and Watson [17]. See Chen [3, 4] for more recent works on models with an imperfectly informed receiver.

the receiver accumulates more information.[3] An important feature of these studies is that the action space is continuous. In such an environment, when the receiver becomes more informed, her resultant action necessarily becomes less sensitive to the sender's message, which yields an effect equivalent to an increase in the preference bias. This implies that the original insight of Crawford and Sobel [5] directly applies to each of these cases: in fact, when the signal space is discrete as in Morgan and Stocken [15], truthful communication is always possible as long as the preferences are sufficiently congruent.[4]

2 Model

We consider a simple model of cheap talk to illustrate how the receiver's (imperfect) prior knowledge becomes an impediment to effective communication. There are two players, Player 0 (male) and Player 1 (female), to whom we interchangeably refer as the sender and the receiver, respectively. The game proceeds as follows:

1. Nature randomly draws the state $t \in \{0, 1\}$. Denote $P(t = 1)$ by π.

2. Each player n, $n = 0, 1$, observes a private signal $s_n \in S_n = \{1, \ldots, K_n\}$. Given the realized state, s_0 and s_1 are independently distributed.

3. Upon observing s_0, Player 0 costlessly sends a message $m \in \mathcal{M} = S_0$ to Player 1.

4. Upon observing s_1 and m, Player 1 chooses an action $a \in \{0, 1\}$.[5]

5. Player n receives a payoff $u_n(t, a)$ given by

$$u_n(t, a) = \mathbb{I}(a = t) + b_n \mathbb{I}(a = n),$$

   where $\mathbb{I}$ is the indicator function, and b_n is Player n's private benefit of implementing

[3] In contrast, Ishida and Shimizu [9] consider a setting in which the receiver observes a signal which carries no information with some positive probability. It is shown that there is a case where the receiver's prior knowledge enhances the amount of information conveyed via cheap talk.

[4] In a variant of our model where the action space is continuous while the signal space is discrete, we can show that truthful communication is always possible if the preference bias is sufficiently small.

[5] An essential feature of our model is that the action space is discrete. For practical purposes, there are several interpretations of this structure. One is that the underlying situation is such that optimal actions diverge to extreme points (the case of corner solutions), i.e., when an action is to be taken, it is always optimal to go all the way to the end point. Another possible interpretation is that there are some technical restrictions on the set of feasible actions, so that the decision maker can only take some pre-specified points in the action space.

his or her preferred action.[6] Let $\beta_n := (1 - b_n)/2$, where we assume $b_n \in (0, 1)$ or, equivalently, $\beta_n \in (0, 0.5)$.

Let ℓ_n denote the likelihood ratio where

$$\ell_n(i) = \frac{P(s_n = i|t = 1)}{P(s_n = i|t = 0)}.$$

Without loss of generality, we order the signals by the likelihood ratio:

$$\ell_n(1) \geq \cdots \geq \ell_n(K_n) \quad n = 0, 1. \tag{1}$$

Each player's signal structure is thoroughly characterized by a pair of conditional density functions or, equivalently, a vector of the likelihood ratios derived from them. In what follows, therefore, we identify player n's signal structure with $L_n = (\ell_n(1), ..., \ell_n(K_n))$.

Definition 1 A signal structure L_n is said to be *informative* if the corresponding likelihood ratios satisfy $\ell_n(1) > 1 > \ell_n(K_n)$ and *uninformative* otherwise.[7] We in particular denote $L_n = U$ when Player n's signal structure is uninformative.

Throughout the analysis, we place three restrictions on the sender's signal structure. First, for ease of exposition, we restrict our attention to a class of signal structures which satisfy (1) with strict inequalities.[8] Second, we also assume that none of the sender's signals gives a posterior belief that assigns equal probabilities over states, i.e.,

$$\nexists i \in S_0 \text{ s.t. } P(t = 1|s_0 = i) = 0.5, \text{ or equivalently } \ell_0(i) = \frac{1 - \pi}{\pi}.$$

Third, we assume that it is not possible that the sender has a posterior belief that one state is always more likely irrespective of his signal, i.e.,

$$P(t = 1|s_0 = 1) > 0.5 > P(t = 1|s_0 = K_0), \text{ or equivalently } \ell_0(1) > \frac{1 - \pi}{\pi} > \ell_0(K_0).$$

[6]Note that in the current setup, the preference bias is deterministic in that it is attached to a particular action irrespective of the realized state. While this may appear a departure from the standard setup, this is rather a direct consequence of the fact that the action space is discrete. To be more precise, what ultimately matters in a discrete setup like this is the threshold belief at which a player is indifferent between the two actions. Since all of our subsequent results are characterized by the relationship between the threshold beliefs and the signal structures, two models are strategically equivalent if they lead to the same threshold beliefs, given a pair of preference bias parameters. Given this, we can always construct a strategically equivalent model even if the state space is continuous and the biases are defined in probabilistic terms.

[7]It is easily verified that a signal structure is uninformative if and only if $\ell_n(1) = \cdots = \ell_n(K_n) = 1$.

[8]If there are any two signals with the same likelihood ratio, we can in principle merge them into one signal.

These assumptions guarantee the existence of $i^* \in S_0 \backslash \{K_0\}$ such that

$$P(t=1|s_0=i) \begin{cases} > 0.5 & \text{if } i \leq i^* \\ < 0.5 & \text{if } i > i^*, \end{cases}$$

or equivalently

$$\ell_0(i) \begin{cases} > \frac{1-\pi}{\pi} & \text{if } i \leq i^* \\ < \frac{1-\pi}{\pi} & \text{if } i > i^*. \end{cases}$$

Let $\mathcal{F} \subset \mathbb{R}_+^{K_0}$ denote the set of signal structures that satisfy these assumptions.

As for the receiver's side, note that there would be no point in soliciting information from the sender if her signals were highly informative relative to the sender's.[9] We thus restrict our attention to a set of signal structures that are relatively weak in the following sense

Definition 2 The receiver's signal structure is said to be *relatively weak* if

$$\frac{\ell_0(i^*)}{\ell_0(i^*+1)} > \ell_1(1) \text{ and } \ell_1(K_1) > \frac{\ell_0(i^*+1)}{\ell_0(i^*)} \tag{2}$$

hold. Let $\mathcal{W}(L_0) \subset \mathbb{R}_+^{K_1}$ denote the set of signal structures (with K_1 signals) that satisfy (2) for a given L_0.

Note that $U \in \mathcal{W}(L_0)$, i.e., the uninformative signal structure is relatively weak.

3 Main result

We denote the sender's strategy by $M : S_0 \to \mathcal{M}$, and the receiver's by $A : \mathcal{M} \times S_1 \to \{0, 1\}$. In order to make our contention most succinctly, throughout the analysis, we restrict our attention to when full separation can be achieved in equilibrium. We in particular focus on what we call a fully informative equilibrium, which is a fully separating equilibrium in which communication matters, i.e., an equilibrium in which the sender reveals his private information truthfully and the receiver's choice of action depends on the sender's message with positive probability.[10]

Definition 3 An equilibrium is said to be *fully informative* if

[9]That is, if the receiver's private information is highly accurate by itself, her choice of action depends only on her own information, irrespective of the sender's message, and communication plays no role.

[10]The latter requirement is necessary to exclude truth-telling equilibria in which the receiver's action is independent of the sender's message, and communication hence plays no role.

- it is *truth-telling*, i.e., $M(i) = i$ for any $i \in S_0$, and
- it is *influential*, i.e., there exist $i, i' \in S_0$ and $j \in S_1$ such that $A(i, j) \neq A(i', j)$.

We start our analysis with the receiver's problem over the choice of action. Define $P_{ij} := P(t = 1|s_0 = i, s_1 = j)$, which coincides with the receiver's posterior belief under the sender's truth-telling strategy. Then, the receiver's best response is expressed as follows:[11]

$$A(i, j) = \begin{cases} 1 & \text{if } P_{ij} > \beta_1 \\ 0 & \text{if } P_{ij} < \beta_1. \end{cases} \tag{3}$$

We now shift our attention to the sender's problem. Define

$$\mathcal{S}_1(i, i') = \{j|A(i, j) = 1, A(i', j) = 0\}.$$

It is verified that if the receiver's strategy A satisfies (3), then for any $i, i' \in S_1$, one of the following holds:

(i) $\mathcal{S}_1(i, i') = \mathcal{S}_1(i', i) = \emptyset$.

(ii) $\mathcal{S}_1(i, i') \neq \emptyset$ and $\mathcal{S}_1(i', i) = \emptyset$.

(iii) $\mathcal{S}_1(i, i') = \emptyset$ and $\mathcal{S}_1(i', i) \neq \emptyset$.

Among them, we can show that the sender's incentive matters only in case (ii) when $s_0 = i$. More precisely, one can show that

$$\mathbb{E}\left[u_0(t, A(i, j))|s_0 = i\right] \geq \mathbb{E}\left[u_0(t, A(i', j))|s_0 = i\right]$$

holds if and only if

$$\frac{\sum_{j \in \mathcal{S}_1(i,i')} P_{ij} P(s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)} \geq 1 - \beta_0. \tag{4}$$

These results straightforwardly lead to the conditions that must be satisfied in any truth-telling equilibrium.

Proposition 1 There exists a truth-telling equilibrium if and only if

- (3) holds for any $i \in S_0$ and $j \in S_1$, and

[11] The details of the derivation of the equilibrium conditions are relegated to Appendix A.

- (4) holds for any $i, i' \in S_0$ satisfying $\mathcal{S}_1(i,i') \neq \emptyset$.

Moreover, it is influential if and only if there exist $i, i' \in S_0$ satisfying $\mathcal{S}_1(i,i') \neq \emptyset$.

One implication of the proposition is that there may arise a serious conflict of interests between the sender and the receiver when the resulting posterior belief P_{ij} falls into the range $(\beta_1, 1-\beta_0)$. More precisely, there exists no truth-telling equilibrium only if there exists $i \in S_0$ and $j \in S_1$ satisfying $\beta_1 < P_{ij} < 1 - \beta_0$. Given this fact, we can now make the following statement which is the main result of this paper.

Theorem 1 Fix $L_0 \in \mathcal{F}$. Then, for all $L_1 \in \mathcal{W}(L_0)$, there always exists (b_0, b_1) such that there exists a fully informative equilibrium if and only if $L_1 = U$, i.e., the receiver's signal structure is uninformative.

Proof: We set (b_0, b_1) such that

$$\begin{aligned} \beta_0 &= \frac{1-\pi}{\pi \ell_0(i^*) + 1 - \pi} \\ \beta_1 &= \frac{\pi \ell_0(i^*+1)}{\pi \ell_0(i^*+1) + 1 - \pi}. \end{aligned}$$

Here, (b_0, b_1) is chosen so that the sender is indifferent between the two messages after observing i^*, and the receiver is indifferent between the two actions if the sender's signal is i^*+1, given that the receiver's signal structure is uninformative, i.e., $L_1 = U$. Then, from Proposition 1, there exists a fully informative equilibrium if $L_1 = U$.

On the contrary, suppose that L_1 is informative. Then, there exist $\underline{j}, \overline{j} \in S_1$ such that

$$\ell_1(j) \begin{cases} > 1 & \text{if } j \leq \underline{j} \\ < 1 & \text{if } j \geq \overline{j} \\ = 1 & \text{otherwise.} \end{cases}$$

It follows from (2) that $\{\overline{j}, \ldots, K_1\} \subseteq \mathcal{S}_1(i^*, i^*+1) \subseteq \{\underline{j}+1, \ldots, K_1\}$ and

$$\begin{aligned} &P_{i^*j} \leq 1 - \beta_0 \quad \forall j \in \mathcal{S}_1(i^*, i^*+1), \text{ in particular,} \\ &P_{i^*j} < 1 - \beta_0 \quad \forall j \geq \overline{j}. \end{aligned}$$

It then turns out that (4) is violated, and therefore, by Proposition 1, there exists no fully informative equilibrium. ∎

Theorem 1 states that we can always find a pair (b_0, b_1) such that truthful communication is possible if and only if the receiver has no information of her own, meaning that better information (on the receiver's side) can lead to worse information transmission.[12]

4 A two-signal example

To illustrate the intuition behind our result, consider a simple example in which there are only two signals: $S_0 = S_1 = \{h, l\}$. Suppose that $P(s_n = h \mid t = 1) = P(s_n = l \mid t = 0) = r_n$ where $r_n \in [0.5, 1]$ measures the accuracy of Player n's prior information. For this example, we assume that $\pi = 0.5$, $b_0 = b_1 = b$ and $r_0 > 1 - \beta := (1 + b)/2$, but do not impose the assumption of relatively weak signals.

The receiver is totally uninformed if $r_1 = 0.5$, in which case there always exists a fully informative equilibrium as long as the preference bias is sufficiently small.[13] In contrast, the situation changes rather starkly once we allow for the possibility that the receiver's own signal is also partially informative. Now suppose that the information gap $(r_0 - r_1)$ is positive but sufficiently small such that

$$r_1 > \frac{\beta r_0}{(1 - r_0) + \beta(2r_0 - 1)}. \tag{5}$$

Note that under this condition, we have $P_{hh} > P_{lh} > \beta$ and $1 - \beta > P_{hl} > \beta$, meaning that the second condition in Proposition 1 is violated, and therefore, there is no fully informative equilibrium.[14] A striking fact is that as $r_1 \to r_0$, (5) collapses to $1 > 2\beta$, which is satisfied by any positive b.[15]

An important implication of this result is that the receiver may be better off by remaining uninformed about the current state than by obtaining imperfect information. To see this, suppose that the receiver has a chance to acquire her own information at the beginning. If the receiver chooses to do so, then the accuracy of her signal is given by $r_1 = r \in (0.5, r_0)$; if

[12] It should be noted that the theorem itself does not constitute a direct proof that better information leads to worse information transmission, because it is a non-generic result for this purpose. However, the argument can easily be extended to show that it is not a non-generic phenomenon in our environment (see Theorem 2 in Appendix B).

[13] To verify this, note that when $r_1 = 0.5$, the receiver's best response must be $A(h, j) = 1$ and $A(l, j)$ for $j = h, l$. Given this, one can readily verify that there is no incentive for the sender to make a false recommendation if $r_0 > 1 - \beta$.

[14] One can also show that there is no partially informative equilibrium in which the sender adopts a mixed strategy, meaning that no information can be conveyed via cheap-talk communication under this condition.

[15] To be more precise, one can show that there exists no fully informative equilibrium for any $b > 0$ if $r_1 \geq r_0$.

not, the signal contains no information and $r_1 = 0.5$. If

$$r > \frac{\beta r_0}{(1 - r_0) + \beta(2r_0 - 1)} \geq 0.5,$$

the receiver faces a cumbersome tradeoff: she can receive an informative message from the sender only if she foregoes the chance to acquire her own information. When $r_0 > r$ as we assume throughout, the receiver can raise her payoff by committing to remain uninformed.

To understand this result, one must look at the sender's cost and benefit of misrepresenting information when he observes the signal indicating that his preferred action is unlikely to be optimal, i.e., $s_0 = h$. First, the benefit is relatively clear, as the sender may sway the receiver towards his preferred action by making a false recommendation. This information manipulation comes at a cost, however, because it necessarily entails inefficient use of information. Given the receiver's strategy, a potential loss arises when the receiver observes $s_1 = l$, in which case the sender's message becomes pivotal. That is, the cost of misrepresenting information is determined by how much the sender loses by recommending his preferred action when $s_0 = h$ and $s_1 = l$, which is precisely captured by P_{hl}. As the receiver's information becomes as accurate as the sender's, P_{hl} converges towards one half. At this point, therefore, it is a fair bet, one way or the other, as far as the probability of choosing the right action is concerned. The sender can only gain from lying by the margin of the preference bias, and no information can be conveyed at all even when the preference bias is arbitrarily small.

The key to this result is the fact that the receiver can now overrule the sender's recommendation based on her own information. The problem is that this overruling does not occur randomly: the receiver overrules the sender's recommendation when she believes, judging from her own information, that the recommendation is more likely to be wrong. The receiver thus effectively functions as a gatekeeper to sort out bad information, but this capability, or the lack of commitment not to use her own information, diminishes the sender's incentive to report truthfully. Since the sender knows that he would be corrected whenever he is way off the mark, the salience of the private benefit is magnified, which renders communication less informative.

5 Conclusion

This paper explores the extent to which cheap-talk communication can credibly convey meaningful information when the receiver is partially informed. As it turns out, the receiver's prior knowledge matters and makes a non-trivial difference in the quality of information that can be extracted from the sender. As a general rule, communication becomes less efficient as the receiver becomes more informed. This result yields a critical implication: in order to facilitate communication, it may be advisable for the receiver to refrain herself from acquiring her own information, even if it can be done with a relatively small cost.

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Appendix A: Proof of Proposition 1

The Equilibrium Conditions for the Receiver's Strategy: Given the sender's truth-telling strategy, we have

$$\begin{aligned}
&\mathbb{E}\left[u_1(t,1)|s_0 = i,\ s_1 = j\right] - \mathbb{E}\left[u_1(t,0)|s_0 = i, s_1 = j\right] \\
&= P(t=1|s_0 = i, s_1 = j)(1+b_1) - P(t=0|s_0 = i, s_1 = j)(1-b_1) \\
&= 2\left[P(t=1|s_0 = i,\ s_1 = j) - \beta_1\right].
\end{aligned}$$

This implies that the receiver's best response is given by (3). ∎

The Equilibrium Conditions for the Sender's Strategy:

We first show that $\mathcal{S}_1(i,i') \neq \emptyset \Rightarrow \mathcal{S}_1(i',i) = \emptyset$ necessarily holds as long as the receiver's strategy A satisfies (3). Suppose to the contrary that there exists (i,i',j,j') such that $A(i,j)=1$, $A(i',j)=0$, $A(i,j')=0$, and $A(i',j')=1$. If $i<i'$, then

$$\beta_1 \geq P_{ij'} > P_{i'j'} \geq \beta_1.$$

This is a contradiction. Similarly, if $i > i'$, then

$$\beta_1 \geq P_{i'j} > P_{ij} \geq \beta_1.$$

This is also a contradiction, Therefore, it is verified $\mathcal{S}_1(i,i') \neq \emptyset \Rightarrow \mathcal{S}_1(i',i) = \emptyset$. It then follows that for any $i, i' \in S_0$, one of the following holds in truth-telling equilibrium:

(i) $\mathcal{S}_1(i,i') = \mathcal{S}_1(i',i) = \emptyset$.

(ii) $\mathcal{S}_1(i,i') \neq \emptyset$ and $\mathcal{S}_1(i',i) = \emptyset$.

(iii) $\mathcal{S}_1(i,i') = \emptyset$ and $\mathcal{S}_1(i',i) \neq \emptyset$.

Furthermore, it is clear that $\mathcal{S}_1(i,i') = \mathcal{S}_1(i',i) = \emptyset$ implies $\mathbb{E}\left[u_0(t, A(i,j))|s_0 = i\right] =$

$\mathbb{E}\left[u_0(t, A(i', j))|s_0 = i\right]$. Now suppose $\mathcal{S}_1(i, i') \neq \emptyset$ and $\mathcal{S}_1(i', i) = \emptyset$. Then we have

$$\begin{aligned}
&\mathbb{E}\left[u_0(t, A(i, j))|s_0 = i\right] - \mathbb{E}\left[u_0(t, A(i', j))|s_0 = i\right] \\
&= \sum_{j \in \mathcal{S}_1(i,i')} \left[P(t = 1, s_1 = j|s_0 = i)(1 - b_0) - P(t = 0, s_1 = j|s_0 = i)(1 + b_0)\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\beta_0 - \frac{\sum_{j \in \mathcal{S}_1(i,i')} P(t = 0, s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)}\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\beta_0 - \frac{\sum_{j \in \mathcal{S}_1(i,i')} P(t = 0, s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_0 = i, s_1 = j)}\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\beta_0 - \frac{\sum_{j \in \mathcal{S}_1(i,i')} \frac{P(t=0, s_0=i, s_1=j)}{P(s_0=i, s_1=j)} P(s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_0 = i, s_1 = j)}\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\beta_0 - \frac{\sum_{j \in \mathcal{S}_1(i,i')} (1 - P_{ij}) P(s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_0 = i, s_1 = j)}\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\beta_0 - \frac{\sum_{j \in \mathcal{S}_1(i,i')} (1 - P_{ij}) P(s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)}\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i,i')} P_{ij} P(s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i,i')} P(s_1 = j|s_0 = i)} - (1 - \beta_0)\right].
\end{aligned}$$

Similarly, suppose $\mathcal{S}_1(i,i') = \emptyset$ and $\mathcal{S}_1(i',i) \neq \emptyset$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left[u_0(t, A(i,j))|s_0 = i\right] - \mathbb{E}\left[u_0(t, A(i',j))|s_0 = i\right] \\
&= \sum_{j \in \mathcal{S}_1(i',i)} \left[P(t=0, s_1 = j|s_0 = i)(1+b_0) - P(t=1, s_1 = j|s_0 = i)(1-b_0)\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i',i)} P(t=0, s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)} - \beta_0\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i',i)} P(t=0, s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_0 = i, s_1 = j)} - \beta_0\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i',i)} \frac{P(t=0, s_0=i, s_1=j)}{P(s_0=i, s_1=j)} P(s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_0 = i, s_1 = j)} - \beta_0\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i',i)} (1-P_{ij}) P(s_0 = i, s_1 = j)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_0 = i, s_1 = j)} - \beta_0\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[\frac{\sum_{j \in \mathcal{S}_1(i',i)} (1-P_{ij}) P(s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)} - \beta_0\right] \\
&= 2\left[\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)\right]\left[(1-\beta_0) - \frac{\sum_{j \in \mathcal{S}_1(i',i)} P_{ij} P(s_1 = j|s_0 = i)}{\sum_{j \in \mathcal{S}_1(i',i)} P(s_1 = j|s_0 = i)}\right] \\
&> 0.
\end{aligned}
$$

The last inequality follows from the fact that $j \in \mathcal{S}_1(i',i)$ implies $P_{ij} \leq \beta_1 < 1 - \beta_0$. ∎

Appendix B

Theorem 2 Fix $L_0 \in \mathcal{F}$ and $\tilde{L}_1 \in \mathcal{W}(L_0) \setminus \{U\}$. Then, there exist non-degenerate intervals of b_0 and b_1 such that

- there exists a fully informative equilibrium under $L_1 = U$, while
- there exists no fully informative equilibrium under $L_1 = \tilde{L}_1$.

Proof: Similarly in the proof of Theorem 1, we can show that if

$$
\begin{aligned}
\beta_0 &\geq \frac{1-\pi}{\pi \ell_0(i^*) + 1 - \pi} \\
\beta_1 &\geq \frac{\pi \ell_0(i^*+1)}{\pi \ell_0(i^*+1) + 1 - \pi}.
\end{aligned}
$$

then there exists a fully informative equilibrium under $L_1 = U$.

Also similarly in the proof of Theorem 1, we can show that if

$$P_{i^*K_1} < 1 - \beta_0 < P_{i^*1}$$
$$P_{i^*+1.K_1} < \beta_1 < P_{i^*+1.1}$$
$$P_{i^*+1.1} < 1 - \beta_0$$
$$P_{i^*K_1} > \beta_1,$$

or equivalently

$$\frac{1-\pi}{\pi\ell_0(i^*)\ell_1(K_1)+1-\pi} > \beta_0 > \frac{1-\pi}{\pi\ell_0(i^*)\ell_1(1)+1-\pi}$$
$$\frac{\pi\ell_0(i^*+1)\ell_1(K_1)}{\pi\ell_0(i^*+1)\ell_1(K_1)+1-\pi} < \beta_1 < \frac{\pi\ell_0(i^*+1)\ell_1(1)}{\pi\ell_0(i^*+1)\ell_1(1)+1-\pi}$$
$$\beta_0 < \frac{1-\pi}{\pi\ell_0(i^*+1)\ell_1(1)+1-\pi}$$
$$\beta_1 < \frac{\pi\ell_0(i^*)\ell_1(K_1)}{\pi\ell_0(i^*)\ell_1(K_1)+1-\pi}$$

hold, then there exist no fully informative equilibrium under $L_1 = \tilde{L}_1$.

Combining these results, it is verified that

$$\beta_0 \in \left[\frac{1-\pi}{\pi\ell_0(i^*)+1-\pi}, \min\left\{\frac{1-\pi}{\pi\ell_0(i^*)\ell_1(K_1)+1-\pi}, \frac{1-\pi}{\pi\ell_0(i^*+1)\ell_1(1)+1-\pi}, \frac{1}{2}\right\}\right) \text{ and}$$
$$\beta_1 \in \left[\frac{\pi\ell_0(i^*+1)}{\pi\ell_0(i^*+1)+1-\pi}, \min\left\{\frac{\pi\ell_0(i^*+1)\ell_1(1)}{\pi\ell_0(i^*+1)\ell_1(1)+1-\pi}, \frac{\pi\ell_0(i^*)\ell_1(K_1)}{\pi\ell_0(i^*)\ell_1(K_1)+1-\pi}, \frac{1}{2}\right\}\right),$$

or equivalently

$$b_0 \in \left(\max\left\{\frac{\pi\ell_0(i^*)\ell_1(K_1)-(1-\pi)}{\pi\ell_0(i^*)\ell_1(K_1)+(1-\pi)}, \frac{\pi\ell_0(i^*+1)\ell_1(1)-(1-\pi)}{\pi\ell_0(i^*+1)\ell_1(1)+(1-\pi)}, 0\right\}, \frac{\pi\ell_0(i^*)-(1-\pi)}{\pi\ell_0(i^*)+(1-\pi)}\right] \text{ and}$$
$$b_1 \in \left(\max\left\{\frac{1-\pi-\pi\ell_0(i^*+1)\ell_1(1)}{1-\pi+\pi\ell_0(i^*+1)\ell_1(1)}, \frac{1-\pi-\pi\ell_0(i^*)\ell_1(K_1)}{1-\pi+\pi\ell_0(i^*)\ell_1(K_1)}, 0\right\}, \frac{1-\pi-\pi\ell_0(i^*+1)}{1-\pi+\pi\ell_0(i^*+1)}\right],$$

are the intervals stated in the theorem. It can also be shown that these intervals are non-degenerate. ∎