Discussion Paper No. 852

STRATEGY-PROOFNESS AND EFFICIENCY WITH NONQUASI-LINEAR PREFERENCES: A CHARACTERIZATION OF MINIMUM PRICE WALRASIAN RULE

Shuhei Morimoto
Shigehiro Serizawa

August 2012

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

Strategy-proofness and Efficiency with Nonquasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule*

Shuhei Morimoto[†] and Shigehiro Serizawa[‡]

August 10, 2012

Abstract

We consider the problems of allocating several heterogeneous objects owned by governments to a group of agents and how much agents should pay. Each agent receives at most one object and has nonquasi-linear preferences. Nonquasi-linear preferences describe environments in which large-scale payments influence agents' abilities to utilize objects or derive benefits from them. The "minimum price Walrasian (MPW) rule" is the rule that assigns a minimum price Walrasian equilibrium allocation to each preference profile. We establish that the MPW rule is the unique rule that satisfies the desirable properties of *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *nonnegative payment* on the domain that includes nonquasi-linear preferences. This result does not only recommend the MPW rule based on those desirable properties, but also suggest that governments cannot improve upon the MPW rule once they consider them essential. Since the outcome of the MPW rule coincides with that of the simultaneous ascending (SA) auction, our result explains the pervasive use of the SA auction.

Keywords: minimum price Walrasian equilibrium, simultaneous ascending auction, strategy-proofness, efficiency, heterogeneous objects, nonquasi-linear preferences

JEL Classification Numbers: D44, D71, D61, D82

---

*A preliminary version of this article was presented at the conference honoring Salvador Barberà's 65th birthday at the Universitat Autònoma de Barcelona in June 2011; Frontiers of Market Design at Ascona, Switzerland in May 2012; and the APET 13th annual conference in Taipei, Taiwan in June 2012. We thank participants at those conferences for their valuable comments. We are also grateful to seminar participants at Hitotsubashi, Keio, Kobe, Kyoto, Tohoku, and Waseda universities for their helpful comments. We specially thank Professors A. Kajii, D. Mishra, S. Ohseto, J. Schummer, W. Thomson, and J. Wako for their detailed comments. Remaining errors are ours.

[†]Institute of Social and Economic Research, Osaka University, 6-1, Mihogaoka, Ibaraki, 567-0047, Japan. E-mail: morimoto@iser.osaka-u.ac.jp

[‡]Institute of Social and Economic Research, Osaka University, 6-1, Mihogaoka, Ibaraki, 567-0047, Japan. E-mail: serizawa@iser.osaka-u.ac.jp

1 Introduction

Purpose. Since the 1990s, governments in numerous countries have conducted auctions to allocates a variety of heterogeneous objects or assets including spectrum rights, vehicle ownership licenses, and lands, etc. Although auction revenues sometimes amount to even as large as government annual budgets, the announced goals of many government auctions are rather to allocate objects "efficiently", *i.e.*, to agents who make the most use of them or benefit most from them.[1] Agents making more use of objects or benefiting more are willing to pay higher prices for them, and thus would have more chances to win the objects in auctions. However, large-scale auction payments would influence agents' abilities to utilize objects or benefit from them, thereby complicating efficient allocation. This article analyzes rules that allocate auctioned objects efficiently even when payments are so large that it impairs agents' abilities to utilize them or realize their benefits. We ask *what types of allocation rules can allocate objects efficiently in such environments.*

Main Result. An *allocation rule* (or simply a *rule*) is a function that assigns to each agents' preference profile an *allocation*, which consists of an assignment of objects and agents' payments. Each agent is permitted to receive one object at the most. The *domain* of rules is the class of agents' preference profiles. It is well-known that in this model, there is a minimum price Walrasian equilibrium,[2] and that its allocation coincides with the outcome of the simultaneous ascending (SA) auction.[3] We focus on the rule that assigns a minimum price Walrasian equilibrium allocation to each preference profile. We refer to this rule as the "*minimum price Walrasian (MPW) rule*".

The MPW rule satisfies four desirable properties. The first is *Pareto-efficiency.* An allocation is *Pareto-efficient* if no agent can be better off without making other agents worse off or reducing a government's revenue.[4] Note that *Pareto-efficiency* is evaluated based on agents' preferences. Thus, a *Pareto-efficient* allocation cannot be chosen without information about agents' preferences. Since preferences are private information, agents have incentives to behave strategically to influence the final outcome in their favor. *Strategy-proofness* is an incentive-compatibility property, which gives a strong incentive for each agent to reveal his true preferences. It says that in the normal form game induced by the rule, it is a (weakly) dominant strategy for each agent to reveal his true preference. The MPW rule satisfies *strategy-proofness*,[5] and chooses a *Pareto-efficient* allocation corresponding to the revealed preferences.

Third property is *individual rationality*, that induces agents's voluntary participation. The MPW rule never assigns an allocation that makes an agent worse off than he would be if he had received no object and paid nothing. Fourth is *nonnegative payment.* Under the MPW rule, agents' payments are always nonnegative, that is, governments never subsidize agents.

The primary conclusion of this article is that *only the minimum price Walrasian rule sat-*

[1] For example, frequency auctions in the United States were introduced to promote "efficient and intensive use of the electromagnetic spectrum". See McAfee and McMillan (1996, p.160).

[2] See Demange and Gale (1985).

[3] For example, see Demange, Gale, and Sotomayor (1986).

[4] In our auction model, *Pareto-efficiency* is defined by taking government revenue into account.

[5] In addition, the MPW rule satisfies *group strategy-proofness*, *i.e.,* by misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

isfies strategy-proofness, Pareto-efficiency, individual rationality, and nonnegative payment in environments where large-scale payments influence agents' abilities to utilize the objects or enjoy their benefits (Theorem 5.1). This result does not only recommend the MPW rule based on the four desirable properties, but also implies that no other rules are available options once governments consider the four properties as essential. Since the outcome of the MPW rule coincides with that of the SA auction, the result also supports SA auctions adopted by many governments.

Novelties and technical difficulties. Holmström (1979) establishes a fundamental result relating to our question that applies when agents' benefits from auctioned objects are not influenced by their payments, *i.e.*, agents have "quasi-linear" preferences. He assumes that the domain includes only quasi-linear preference, and shows that *only the Vickrey–Clarke–Groves type (VCG)*[6] *allocation rules satisfy strategy-proofness and Pareto-efficiency.*[7] Preferences are approximately quasi-linear if payments are sufficiently low. However, quasi-linearity is not an appropriate assumption for large-scale auctions. Excessive payment for the auctioned objects may damage bidders' budgets and render effective use of the objects impossible. In fact, in spectrum license auctions and vehicle ownership license auctions, license prices often equal or exceed bidders' annual revenues. Thus, bidders' preferences are nonquasi-linear for such important auctions.[8] As contrasted with Holmström (1979), our result can be applied even to such environments.

Saitoh and Serizawa (2008) investigate a problem similar to ours in the case where the domain includes nonquasi-linear preferences but objects are homogeneous. They generalize VCG-type rules by employing compensating valuations, and characterize the generalized VCG-type rules by the four desirable properties.[9] We stress that when preferences are not quasi-linear, the heterogeneity of objects makes the MPW rule substantially different from the generalized VCG rule. In Section 2, we illustrate the MPW rule for simple cases, and contrast it with the VCG-type rule.

Although the assumption of quasi-linearity neglects the serious effects of large-scale auction payments of auctions in actual practice, it is difficult to investigate the above question without this assumption. Quasi-linearity simplifies the description of *Pareto-efficient* allocations. More precisely, under quasi-linear preferences, a *Pareto-efficient* allocation of objects can be achieved simply by maximizing the sum of realized benefits from objects (agents' net benefits), and hence, *efficient* allocations of objects are independent of how much agents pay. In this sense, Holmström (1979) characterizes only the payment part of *strategy-proof* and *Pareto-efficient* rules. On the other hand, without quasi-linearity, *Pareto-efficient* allocations of objects do depend on payments, and thus are complicated to identify. Moreover, we illustrate this point in Section 2 in more detail. Furthermore, as mentioned earlier, on nonquasi-linear domains, the MPW rule is rather different from the VCG rule, and the former outperforms the latter in terms of our desirable properties. Therefore, the extension of Holmström's (1979) result to nonquasi-linear domains is far from trivial. Needless to say, Holmström's (1979) proof techniques fail when the domain includes nonquasi-linear prefer-

[6]See Clarke (1971), Groves (1973), and Vickrey (1961).

[7]More precisely, Holmström (1979) studies public goods models. When agents have quasi-linear preferences, his result can be applied to the auction model.

[8]Ausubel and Milgrom (2002) also discuss the importance of the analysis under nonquasi-linear preferences.

[9]Sakai (2008) also obtains a result similar to theirs.

ences. It is worthwhile to mention that most standard results of auction theory, such as the Revenue Equivalence Theorem, also depend on assuming quasi-linearity. In this article, we overcome that difficulty.

Related Literature. We relate our results to literature not referenced above. Analyzing a model resembling ours, Miyake (1998) shows that only the MPW rule satisfies *strategy-proofness* among "Walrasian rules".[10] Note that Walrasian rules are a small part of the class of allocation rules satisfying *Pareto-efficiency*, *individual rationality*, and *nonnegative payment*. By developing analytical tools substantially different from Miyake's (1998), we extend his characterization in that we establish the uniqueness of the rules satisfying the desirable properties without confinement to Walrasian rules.

Many authors have analyzed SA auctions in quasi-linear settings. For example, see Ausubel (2004, 2006); Ausubel and Milgrom (2002); de Vries, Schummer, and Vohra (2007); Gul and Stacchetti (2000); and Mishra and Parkes (2007), etc. In nonquasi-linear settings, MPW rules differ from VCG rules, and it is the MPW allocation that coincides with the outcome of the SA auction. Since our result states that only the MPW rule satisfies basic desirable properties, it indicates that their works are more important in nonquasi-linear settings.

Other related literature concerns matching models. The concept of *stability* in matching models is equivalent to Walrasian equilibrium in our model. The "*agent-proposing deferred acceptance algorithm (APDAA)*" in matching models without monetary transfers corresponds to the MPW rule. Alcalde and Barberà (1994) characterize the APDAA rule by *strategy-proofness* among *stable* rules. Kojima and Manea (2010) characterize the APDAA rule without imposing *stability*, but with different properties, which they call *individually rational monotonicity* and *weak Maskin monotonicity*. In a spirit akin to ours, those articles analyze rules satisfying desirable properties.

Organization. The article is organized as follows. Section 2 illustrates the minimum price Walrasian rule, and demonstrates how nonquasi-linear preferences complicate analysis. Section 3 sets up the model and introduces basic concepts formally. Section 4 defines Walrasian equilibria and characterizes them by the concepts of underdemanded and overdemanded sets. Section 5 provides our main result. Section 6 defines the SA auction, and shows that its outcome coincides with the minimum price Walrasian equilibrium allocation. Section 7 gives an overview of the proof of our primary conclusion. Section 8 concludes. All the formal proofs appear in the Appendix.

2 An illustration of Minimum Price Walrasian Rule with Nonquasi-linear preferences

In this section, we illustrate the minimum price Walrasian (MPW) rule in the simplest cases when three agents (agents 1, 2, and 3) have varied preferences and there are only one or two objects. In addition, we contrast the MPW rule with the Vickrey–Clarke–Groves (VCG) rule to demonstrate their difference.

Case I: Quasi-linear domain. When agents have quasi-linear preferences, each agent's

[10] A "Walrasian rule" is the rule that assigns a Walrasian equilibrium allocation to each preference profile.

valuation of each object is independent of his payment, and the outcome of the MPW rule coincides with that of the VCG rule. Under the two rules, the objects are allocated efficiently (*i.e.*, the sum of agents' valuations is maximized), and each agent pays the social opportunity cost of allocating to him the object he receives. It is known that this rule is a unique rule satisfying *efficiency*, *strategy-proofness*, and *individual rationality* (Holmström, 1979; Chew and Serizawa, 2007).

Case II: Nonquasi-linear domain (one-object case). When preferences are not quasi-linear, agents' valuations of objects are not defined independently of their payments. However, when there is only one object, the MPW rule still coincides with a simple generalization of the VCG rule based on compensating valuations from the origin of an agent's consumption space,[11] which we call "the VCG rule from $\mathbf{0}$".

Consider the case where agents preferences, R_1, R_2, and R_3 are depicted in Figure 1, where R_i ($i = 1, 2, 3$) denotes agent i's preference, and z_i denotes i's consumption point assigned by the MPW rule. Denote the highest and second highest compensating valuations from the origin $\mathbf{0}$ among agents by $CV^1(\mathbf{0})$ and $CV^2(\mathbf{0})$, respectively. Under the two rules, the agent with the highest compensating valuation $CV^1(\mathbf{0})$ receives the object and pays $CV^2(\mathbf{0})$.

This rule can easily be extended to the case with n agents and m homogeneous objects. In this case, agents with m highest compensating valuations receive the objects and pay the $(m + 1)$-th highest compensating valuation. While objects are homogeneous, this is also a unique rule satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *nonnegative payment* (Saitoh and Serizawa, 2008; Sakai, 2008).

[Figure 1 about here]

Case III: Nonquasi-linear domain (two-object case, 1). We now illustrate the outcome of the MPW rule when there are two heterogeneous objects, A and B. Consider the case where agents' preferences, R_1, R_2, and R_3 are depicted in Figure 2. Denote agent i's ($i = 1, 2, 3$) compensating valuation of object x ($x = A, B$) from the origin $\mathbf{0}$ by $CV_i(x; \mathbf{0})$. Compensating valuations are ranked as $CV_1(A; \mathbf{0}) > CV_3(A; \mathbf{0}) > CV_2(A; \mathbf{0})$ and $CV_2(B; \mathbf{0}) > CV_3(B; \mathbf{0}) > CV_1(B; \mathbf{0})$. In this case, under the MPW rule, agent 1 receives object A and pays $CV_3(A; \mathbf{0})$, and agent 2 receives object B and pays $CV_3(B; \mathbf{0})$. Note that each object is allocated to the agent with the highest compensating valuation from the origin at the price established by the second highest compensating valuation. This outcome still coincides with the outcome of the VCG rule from $\mathbf{0}$ when it is applied to the two objects independently.

[Figure 2 about here]

Case IV: Nonquasi-linear domain (two-object case, 2). We next consider the case where agents' preferences are depicted in Figure 3. The compensating valuations from the origin are ranked as $CV_2(A; \mathbf{0}) > CV_1(A; \mathbf{0}) > CV_3(A; \mathbf{0})$ and $CV_1(B; \mathbf{0}) > CV_2(B; \mathbf{0}) > CV_3(B; \mathbf{0})$. Denote agent i's ($i = 1, 2, 3$) compensating valuation of object x ($x = A, B$) from his consumption point $z_i = (x_i, p_i)$ by $CV_i(x; z_i)$, where x_i is the object that agent i receives and p_i is his payment. In this case, the outcome of the MPW rule is as follows: Agent 1

[11] In our model, the origin of an agent's consumption space means that he receives no object and pays nothing. Let $\mathbf{0}$ denote the origin of an agent's consumption space.

receives object A and pays $CV_3(A;\mathbf{0})$, *i.e.*, the price p^A of object A is $CV_3(A;\mathbf{0})$. This agent 1's consumption point is depicted as z_1 in Figure 3. Agent 2 receives object B and pays $CV_1(B;z_1)$, *i.e.*, the price p^B of object B is $CV_1(B;z_1)$. This agent 2's consumption point is depicted as z_2 in Figure 3.

Let's see why this is the outcome of the MPW rule. First, note that for each agent $i = 1,2,3$, z_i is maximal for R_i in the budget set $\{\mathbf{0},(A,p^A),(B,p^B)\}$. Thus, the above outcome is a Walrasian equilibrium. Next, let $(\hat{p}^A,\hat{p}^B)$ be a Walrasian equilibrium price. If $\hat{p}^A < p^A$, then, all agents prefer (A,p^A) to $\mathbf{0}$, that is, all three agents demand A or B or both. In that case, one agent cannot receive an object he demands, contradicting Walrasian equilibrium. Therefore, $\hat{p}^A \geq p^A$. If $\hat{p}^B < p^B$, both agents 1 and 2 strictly prefer (B,p^B) to $\mathbf{0}$ and (A,p^A). In that case, agent 1 or 2 cannot receive the object he demands, contradicting Walrasian equilibrium. Therefore, $\hat{p}^B \geq p^B$. Hence, the above outcome is of the MPW rule.

Moreover, it is easy to see that the above outcome is that of the SA auction. While the price p^A of object A is lower than $CV_3(A;\mathbf{0})$, no agent exits, and therefore the auction does not stop. Thus, in the outcome, $p^A \geq CV_3(A;\mathbf{0})$. Similarly, $p^B \geq CV_3(B;\mathbf{0})$. If $p^B < CV_1(B;z_1)$, then since $p^A \geq CV_3(A;\mathbf{0})$, agents 1 and 2 both continue bidding on B. Thus, $p^B \geq CV_1(B;z_1)$. When $p^A = CV_3(A;\mathbf{0})$ and $p^B = CV_1(B;z_1)$, agent 3 exits, and agents 1 and 2 demand objects A and B, respectively. Then, the auction stops.

It is worthwhile to demonstrate that agent 2's compensating valuation of object A from the origin is highest; however, he does not receive A, and that the price of object B is not any agent's compensating valuation of object B from the origin. Accordingly, the MPW outcome does not coincide with the VCG rule from $\mathbf{0}$. Additionally, we demonstrate that *efficient* allocations of objects cannot be obtained simply by maximizing the sum of agents' compensating valuations from the origin in this case.

[Figure 3 about here]

Case V: Nonquasi-linear domain (two-object case, 3). Finally, we consider the case where agents' preferences are depicted in Figure 4. The compensating valuations from the origin are ranked as $CV_1(A;\mathbf{0}) > CV_3(A;\mathbf{0}) > CV_2(A;\mathbf{0})$ and $CV_1(B;\mathbf{0}) > CV_2(B;\mathbf{0}) > CV_3(B;\mathbf{0})$. In this case, the outcome of the MPW rule is as follows: Agent 1 receives object A and pays $CV_3(A;\mathbf{0})$, *i.e.*, the price p^A of object A is $CV_3(A;\mathbf{0})$. This agent 1's consumption point is depicted as z_1 in Figure 4. Agent 2 receives object B and pays $CV_1(B;z_1)$, *i.e.*, the price p^B of object B is $CV_1(B;z_1)$. This agent 2's consumption point is depicted as z_2 in Figure 4. In this case, it is agent 1's preference that decided whether agent 2 or 3 receives an object. In Figure 4, agent 1 prefers $(A, CV_3(A;\mathbf{0}))$ to $(B, CV_2(B;\mathbf{0}))$, and agent 2 receives an object. However, if agent 1 prefers $(B, CV_2(B;\mathbf{0}))$ to $(A, CV_3(A;\mathbf{0}))$, agent 3 instead receives an object.

Similar to above Case IV, it is easy to see why this allocation is the outcome of the MPW rule, and coincides with the outcome of the SA auction. As in Case IV, the price of object B is not any agent's compensating valuation of object B from the origin, the MPW outcome does not coincide with the VCG rule from $\mathbf{0}$, and *efficient* allocation of objects cannot be obtained simply by maximizing the sum of agents' compensating valuations from the origin.

[Figure 4 about here]

In the above five cases, we contrasted the MPW rule with the VCG rule. Outcomes of the two rules coincide in Cases I, II and III, but not in Cases IV and V. The VCG rule above employs only a small part of the information about agents' preferences (*i.e.,* "compensating valuations from the origin"). On the other hand, the MPW rule employs other information (*i.e.,* "compensating valuations from various points"). As we show in the remainder of this article, only the MPW rule satisfies *strategy-proofness*, *Pareto-efficiency*, *individual rationality*, and *nonnegative payment* on the domain including nonquasi-linear preferences. Thus, the information about compensating valuations from various points is necessary to design rules satisfying the above four properties on this domain.

As Demange, Gale, and Sotomayor (1986), etc., discuss and we show formally in Section 6, the outcome of the SA auction always coincides with the minimum price Walrasian equilibrium allocation.

3 The Model and Definitions

There are n agents and m objects, where $2 \leq n < \infty$ and $1 \leq m < \infty$. We denote the set of agents by $N \equiv \{1, \ldots, n\}$, and the set of objects by $M \equiv \{1, \ldots, m\}$. Let $L \equiv \{0\} \cup M$. Each agent is permitted to receive one object at most. We denote the object that agent $i \in N$ receives by $x_i \in L$. Object 0 is referred as the "null object", and $x_i = 0$ means that agent i receives no object. We denote the money that agent i pays by $t_i \in \mathbb{R}$. For each $i \in N$, agent i's **consumption set** is $L \times \mathbb{R}$, and agent i's **(consumption) bundle** is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

Each agent i has a complete and transitive preference relation R_i on $L \times \mathbb{R}$. Let P_i and I_i be the strict and indifference relation associated with R_i, respectively. Given a preference R_i and a bundle $z_i \in L \times \mathbb{R}$, we denote the **upper contour set** and **lower contour set of $\boldsymbol{R_i}$ at $\boldsymbol{z_i}$** by the sets $\boldsymbol{UC(R_i, z_i)} \equiv \{\hat{z}_i \in L \times \mathbb{R} : \hat{z}_i \, R_i \, z_i\}$ and $\boldsymbol{LC(R_i, z_i)} \equiv \{\hat{z}_i \in L \times \mathbb{R} : z_i \, R_i \, \hat{z}_i\}$, respectively. We assume that a preference satisfies the following properties:

Continuity: For each $z_i \in L \times \mathbb{R}$, $UC(R_i, z_i)$ and $LC(R_i, z_i)$ both are closed.

Money monotonicity: For each $x_i \in L$ and each $t_i, \hat{t}_i \in \mathbb{R}$, if $\hat{t}_i < t_i$, then, $(x_i, \hat{t}_i) \, P_i \, (x_i, t_i)$.

Finiteness: For each $t_i \in \mathbb{R}$, each $x_i, \hat{x}_i \in L$, there is $\hat{t}_i \in \mathbb{R}$ such that $(\hat{x}_i, \hat{t}_i) \, R_i \, (x_i, t_i)$.

Let $\mathcal{R}^E$ be the class of continuous, money monotonic, and finite preferences, which we call the "**extended domain**". Given $R_i \in \mathcal{R}^E$, $z_i \in L \times \mathbb{R}$, and $y_i \in L$, we define **compensating valuation $\boldsymbol{CV_i(y_i; z_i)}$ of $\boldsymbol{y_i}$ from $\boldsymbol{z_i}$ for $\boldsymbol{R_i}$** by $(y_i, CV_i(y_i; z_i)) \, I_i \, z_i$. Note that by continuity and finiteness of preferences, $CV_i(y_i; z_i)$ exists, and by money monotonicity, $CV_i(y_i; z_i)$ is unique. The compensating valuation for $\hat{R}_i$ is denoted by $\widehat{CV}_i$.

We introduce another property of preferences.

Desirability of objects: For each $x_i \in M$ and each $t_i \in \mathbb{R}$, $(x_i, t_i) \, P_i \, (0, t_i)$.[12]

Definition 3.1. A preference R_i is **classical** if it satisfies continuity, money monotonicity, finiteness, and desirability of objects.

[12]The following is a weaker condition of desirability of objects. A preference R_i satisfies *weak desirability of objects* if for each $x_i \in M$, $(x_i, 0) \, P_i \, \mathbf{0}$. All the results in this article still hold if desirability of objects is replaced by weak desirability.

Let $\mathcal{R}^C$ be the class of classical preferences, which we call the "**classical domain**". Note that $\mathcal{R}^C \subsetneq \mathcal{R}^E$.

Definition 3.2. A preference R_i is **quasi-linear** if there is a valuation function $v_i : L \to \mathbb{R}_+$ such that $v_i(0) = 0$, and for each $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$, and each $\hat{z}_i \equiv (\hat{x}_i, \hat{t}_i) \in L \times \mathbb{R}$,

$$z_i \, R_i \, \hat{z}_i \iff v_i(x_i) - t_i \geq v_i(\hat{x}_i) - \hat{t}_i.$$

We denote the class of quasi-linear preferences by $\mathcal{R}^Q$, which we call the "quasi-linear domain".

An **object allocation** is an n-tuple $(x_1, \ldots, x_n) \in L^n$ such that for each $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$, that is, any two agents do not receive the same object. Let X be the set of object allocations. A **(feasible) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n)$ of bundles such that $(x_1, \ldots, x_n) \in X$. Let Z be the set of feasible allocations. We denote the object allocation and agents' payments under an allocation $\hat{z}$ by $\hat{x} \equiv (\hat{x}_1, \ldots, \hat{x}_n)$ and $\hat{t} \equiv (\hat{t}_1, \ldots, \hat{t}_n)$, respectively.

Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R} \subseteq \mathcal{R}^E$. A **preference profile** is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \setminus N'}$.

An allocation rule, or simply a **rule**, on $\mathcal{R}^n$ is a function f from $\mathcal{R}^n$ to Z. Given a rule f and a preference profile $R \in \mathcal{R}^n$, we denote agent i's assignment of objects under f at R by $f_i^x(R)$ and i's payment under f at R by $f_i^t(R)$, and we write

$$f_i(R) \equiv (f_i^x(R), f_i^t(R)), \;\; f(R) \equiv (f_1(R), \ldots, f_n(R)), \;\; \text{and} \;\; f_{-i}(R) \equiv f_j(R)_{j \in N \setminus \{i\}}.$$

We introduce basic properties of rules. The efficiency condition defined below takes the auctioneer's preference into account and assumes that he is indifferent to the auctioned objects, that is, he is only interested in his revenue. An allocation $z \in Z$ is **Pareto-efficient for $\boldsymbol{R} \in \boldsymbol{\mathcal{R}^n}$** if there is no feasible allocation $\hat{z} \in Z$ such that

$$\text{(i)} \sum_{i \in N} \hat{t}^i \geq \sum_{i \in N} t^i, \text{ (ii) for each } i \in N, \hat{z}_i \, R_i \, z_i, \text{ and (iii) for some } j \in N, \hat{z}_j \, P_i \, z_j.$$

For each $R \in \mathcal{R}^n$, let $P(R)$ be **the set of Pareto-efficient allocations for $\boldsymbol{R}$**.

Efficiency: For each $R \in \mathcal{R}^n$, $f(R) \in P(R)$.

Individual rationality defined below requires that a rule should never assign an allocation which makes some agent worse off than he would be if he had received no object and paid nothing. *Nonnegative payment* requires that the payment of agents always should be nonnegative.

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \, R_i \, \mathbf{0}$.

Nonnegative payment: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i^t(R) \geq 0$.

The two properties below are of incentive-compatibility. The first says that by misrepresenting his preferences, no agent should obtain an assignment that he prefers.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}_i \in \mathcal{R}$, $f_i(R) \, R_i \, f_i(\hat{R}_i, R_{-i})$.

The second is a stronger property: by misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

Group strategy-proofness: For each $R \in \mathcal{R}^n$ and each $\hat{N} \subseteq N$, there is no $\hat{R}_{\hat{N}} \in \mathcal{R}^{\#\hat{N}}$ such that for each $i \in \hat{N}$, $f_i(\hat{R}_{\hat{N}}, R_{-\hat{N}})\, P_i\, f_i(R)$.[13]

4 Minimum Price Walrasian Equilibrium

We define "Walrasian equilibrium" and "minimum price Walrasian equilibrium" in this model. As Demange, Gale, and Sotomayor (1986), etc., explain, and we show in Section 6, the minimum price Walrasian equilibria coincide with the outcomes of SA auctions. Let $\mathcal{R} \subseteq \mathcal{R}^E$ in this section. All results in this section also hold on the classical domain $\mathcal{R}^C$.

Given a price vector $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$ of m objects, the **budget set** (or **available set**) is defined as $B(p) \equiv \{(x, p^x) : x \in L\}$, where $p^x = 0$ if $x = 0$. Given $R_i \in \mathcal{R}$ and $p \in \mathbb{R}^m_+$, agent i's **demand set** $D(R_i, p)$ is defined as $D(R_i, p) \equiv \{x \in L : \forall\, y \in L,\ (x, p^x)\, R_i\, (y, p^y)\}$.

Next is the definition of "Walrasian equilibrium".

Definition 4.1. Let $R \in \mathcal{R}^n$. A pair $(z, p) \in Z \times \mathbb{R}^m_+$ of feasible allocation and price vector is a **Walrasian equilibrium for *R*** if it satisfies the following two conditions:

$$\text{(WE-i) for each } i \in N,\ x_i \in D(R_i, p) \text{ and } t_i = p^{x_i},$$
$$\text{(WE-ii) for each } x \in M,\ \text{ if for each } i \in N, x_i \neq x,\ \text{ then, } p^x = 0.$$

Condition (WE-i) says that each agent receives the object he demands, and pays its price. Condition (WE-ii) says that an object's price is zero if it is not assigned.

Fact 4.1. *For each $R \in \mathcal{R}^n$, there is a Walrasian equilibrium for R.*

Fact 4.1 is already proven in the literature. For example, see Alkan and Gale (1990). Our model is a special case of their model. In Section 6, we give an alternative proof of the existence of Walrasian equilibrium as Proposition 6.1 by using the SA auction.

Given $R \in \mathcal{R}^n$, let $W(R)$ be **the set of Walrasian equilibrium allocations** for R, that is, $z \in W(R)$ if and only if there is a price vector $p \in \mathbb{R}^m_+$ such that the pair (z, p) is a Walrasian equilibrium for R. Fact 4.2 below is so-called First Welfare Theorem.

Fact 4.2. *Let $R \in \mathcal{R}^n$ and $z \in W(R)$. Then, z is Pareto-efficient for R.*[14]

Fact 4.3 below says that for each preference profile, there is a minimum price Walrasian equilibrium.

[13] Let $\#A$ denote the cardinality of set A.

[14] To see this, suppose that $z \equiv (z_1, \ldots, z_n)$ is not *Pareto-efficient for R*. Then, there is $\hat{z} \equiv (\hat{z}_1, \ldots, \hat{z}_n)$ such that

$$\text{(i) } \sum_{i \in N} \hat{t}^i \geq \sum_{i \in N} t^i,\ \text{(ii) for each } i \in N,\ \hat{z}_i\, R_i\, z_i,\ \text{(iii) for some } j \in N,\ \hat{z}_j\, P_j\, z_j.$$

Since $z \in W(R)$, there is a price vector $p \in \mathbb{R}^m_+$ such that (z, p) is a Walrasian equilibrium for R. Then, by (ii) and (WE-i), for each $i \in N$, $\hat{t}_i \leq p^{\hat{x}_i}$. By (iii) and (WE-i), $\hat{t}_j < p^{\hat{x}_j}$. Thus, $\sum_{i \in N} \hat{t}_i < \sum_{i \in N} p^{\hat{x}_i} = \sum_{i \in N} t_i$. This contradicts (i).

Fact 4.3 (Demange and Gale, 1985). *Let $R \in \mathcal{R}^n$. There is a Walrasian equilibrium $(z_{\min}, p_{\min}) \in Z \times \mathbb{R}^m_+$ for R such that, for each price vector $p \in \mathbb{R}^m_+$, if there is $z \in Z$ such that the pair (z, p) is a Walrasian equilibrium for R, then for each $x \in M$, $p^x_{\min} \leq p^x$.*[15]

Given $R \in \mathcal{R}^n$, let $W_{\min}(R)$ be **the set of the minimum price Walrasian equilibrium allocations for R**. That is, $z \in W_{\min}(R)$ if and only if there is $p_{\min} \in \mathbb{R}^m_+$ such that the pair $(z, p_{\min})$ is a minimum price Walrasian equilibrium for R. By Facts 4.1 and 4.3, for each $R \in \mathcal{R}^n$, the set $W_{\min}(R)$ is nonempty. Although the correspondence $W_{\min}$ is set valued, but it is *essentially single-valued*. That is, for each $R \in \mathcal{R}^n$, each pair $z, z' \in W_{\min}(R)$, and each $i \in N$, $z_i \, I_i \, z'_i$.[16] We denote the minimum Walrasian equilibrium price for R by $p_{\min}(R)$.

Next, we introduce the concepts of "overdemanded set" and "underdemanded set" (Demange, Gale, and Sotomayor, 1986; Mishra and Talman, 2010). We relate these concepts to Walrasian equilibria.

Definition 4.2. A set $M' \subseteq M$ of objects is **(weakly) overdemanded** at p for R if

$$\#\{i \in N : D(R_i, p) \subseteq M'\} \ (\geq) > \#M'.$$

A set $M' \subseteq M$ of objects is **(weakly) underdemanded** at p for R if

$$[\forall x \in M', p^x > 0] \implies \#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} \ (\leq) < \#M'.$$

Fact 4.4 below is shown by Mishra and Talman (2010) under the assumption that preferences are quasi-linear. However, their proof does not depend on this assumption.

Fact 4.4 (Mishra and Talman, 2010). *Let $R \in \mathcal{R}^n$. A price vector p is a Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is underdemanded at p for R.*

Theorem 4.1 below is a characterization of the minimum price Walrasian equilibrium by means of the concepts of overdemanded and weakly underdemanded sets. Mishra and Talman (2010) first obtain the same conclusion on the quasi-linear domain. We emphasize, in contrast to Fact 4.4, that Mishra and Talman's (2010) proof crucially depends on the quasi-linearity. It relies on a simple fact that when preferences are quasi-linear, if a set M' of objects is weakly underdemanded at a Walrasian equilibrium (z, p), then all the prices of M' can be slightly lowered by the same amount while maintaining the Walrasian equilibrium conditions (WE-i) and (WE-ii). However, it is not true when preferences are not quasi-linear. Theorem 4.1 below is a novel result in that point.

Theorem 4.1 is the key to obtaining all the important results introduced in the subsequent sections, such as Theorem 5.1 in Section 5 and Proposition 6.1 in Section 6. As mentioned earlier, we obtain the existence of Walrasian equilibrium as a byproduct of Proposition 6.1. Thus, this theorem is also a key to the existence of Walrasian equilibrium.

Theorem 4.1. *Let $R \in \mathcal{R}^n$. A price vector p is a minimum Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at p for R.*

[15] They also show that for each preference profile, there is a maximum price Walrasian equilibrium.

[16] An allocation $z' \in Z$ *is obtained by an indifferent permutation from* $z \in Z$ if there is a permutation π on N such that for all $i \in N$, $z'_i = z_{\pi(i)}$ and $z'_i \, I_i \, z_i$ (Tadenuma and Thomson, 1991). Note that for each pair $z, z' \in W_{\min}(R)$, z' is obtained by an indifferent permutation from z.

The following structures of the minimum price Walrasian equilibrium are obtained as a corollary of Theorem 4.1. Corollary 4.1 says that if the number of objects is greater than or equal to the number of agents, the price of some objects is 0. Corollary 4.2 says that each object bearing a positive price is connected by agents' demands to the null object or to an object with a price of 0.

Corollary 4.1 (Existence of Free Object). *Let $m \geq n$, $R \in \mathcal{R}^n$, and $z \in W_{\min}(R)$. Then, there is $i \in N$ such that $p^{x_i}_{\min}(R) = 0$.*

Corollary 4.2 (Demand Connectedness).[17] *Let $R \in \mathcal{R}^n$ and (z, p) be a minimum Walrasian equilibrium price for R. For each $x \in M$ with $p^x > 0$, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $x_{i_1} = 0$ or $p^{x_{i_1}} = 0$, (ii) $x_{i_K} = x$, and (iii) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}, x_{i_{k+1}}\} \subseteq D(R_{i_k}, p)$.*

Proofs of Theorem 4.1 and Corollaries 4.1 and 4.2 appear in the Appendix.

5 Main Results

In this section, we provide a characterization of the minimum price Walrasian equilibrium by means of the properties of rules.

Let $\mathcal{R} \subseteq \mathcal{R}^E$. Let g be a rule such that for each $R \in \mathcal{R}^n$, $g(R) \in W_{\min}(R)$. Then, g is called a *selection from the minimum price Walrasian equilibrium*, which we call a **minimum price Walrasian rule**.

5.1 Properties of the Minimum Price Walrasian Rule

We discuss the properties of the minimum price Walrasian rule. Let g be a minimum price Walrasian rule on $\mathcal{R}^n$. First, by Fact 4.2, for each $R \in \mathcal{R}^n$, $g(R)$ is *Pareto-efficient for R*. Let $R \in \mathcal{R}^n$. Then, there is a price vector $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$ such that for each $i \in N$, (a) $g_i(R) \in B(p)$, and (b) for each $\hat{z}_i \in B(p)$, $g_i(R)\, R_i\, \hat{z}_i$. Let $i \in N$. Note that, for each $x \in M$, $p^x \geq 0$, and $B(p) = \{(0,0), (1, p^1), (2, p^2), \ldots, (m, p^m)\}$. Thus, by (a), $g^t_i(R) \geq 0$, and by (b), $g_i(R)\, R_i\, \mathbf{0}$. Therefore, the minimum price Walrasian rules satisfy *efficiency*, *individual rationality*, and *nonnegative payment*.

Fact 5.1 below was first shown by Demange and Gale (1985). By using Theorem 4.1 in Section 4, we show this fact more directly in the Appendix.

Fact 5.1 (Demange and Gale, 1985). *The minimum price Walrasian rules are group strategy-proof.*

5.2 Characterizations

In this subsection, we focus on the analysis in the case where each agent has a classical preference and the number of agents exceeds the number objects. Remember that all results established in Section 4 also hold in this case. Theorem 5.1 below is a main conclusion of this article, a characterization of the minimum price Walrasian rule.

Theorem 5.1. *Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule f on $\mathcal{R}^n$ satisfies strategy-proofness, efficiency, individual rationality, and nonnegative payment if and only if it is a minimum price Walrasian rule: for each $R \in \mathcal{R}^n$, $f(R) \in W_{\min}(R)$.*

[17] This structure is discussed by Demange, Gale, and Sotomayor (1986) and Miyake (1998).

Since the minimum price Walrasian rules are *group strategy-proof*, we obtain the following as a corollary of Theorem 5.1.

Corollary 5.1. *Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule f on $\mathcal{R}^n$ satisfies group strategy-proofness, efficiency, individual rationality, and nonnegative payment if and only if it is a minimum price Walrasian rule.*

Proof of Theorem 5.1 is in the Appendix. In addition, we give an overview of the proof in Section 7.

5.3 Independence of the Axioms

The *only if* part of Theorem 5.1 fails if we drop any of the four axioms. The following examples establish the independence of the axioms in Theorem 5.1.

Example 1 (Dropping *strategy-proofness*). Let f be a rule that chooses a "maximum" price Walrasian equilibrium allocation for each preference profile. Then, the rule f satisfies *efficiency*, *individual rationality*, and *nonnegative payment*, but not *strategy-proofness*.

Example 2 (Dropping *efficiency*). Let f be the rule such that for each preference profile, each agent receives no object and pays nothing. Then, the rule f satisfies *strategy-proofness*, *individual rationality*, and *nonnegative payment*, but not *efficiency*.

Next, we introduce variants of Walrasian equilibrium, one with "entry fee". Let $R \in \mathcal{R}^n$ and $t_0 \in \mathbb{R}$. A pair $(z, p) \in Z \times \mathbb{R}^{m+1}$ of feasible allocation and price vector is a *Walrasian equilibrium with "entry fee t_0" for R* if (i) $p^0 = t_0$ and for each $x \in M$, $p^x \geq t_0$, (ii) for each $i \in N$ and each $y \in L$, $(x_i, p^{x_i})\, R_i\, (y, p^y)$ and $t_i = p^{x_i}$, and (iii) for each $x \in M$, if for each $i \in N$, $x_i \neq x$, then $p^x = t_0$. Note that, by Facts 4.1 and 4.3, for each preference profile and each entry fee t_0, there is a minimum price Walrasian equilibrium with entry fee t_0. Moreover, we remark that, by Fact 5.1, for each entry fee t_0, any selection from the minimum price Walrasian equilibrium with entry fee t_0 is *(group) strategy-proof*.

Example 3 (Dropping *individual rationality*). Let $t_0 > 0$. Let f be a rule that chooses a minimum price Walrasian equilibrium with positive entry fee t_0 for each preference profile. Then, the rule f satisfies *strategy-proofness*, *efficiency*, and *nonnegative payment*, but not *individual rationality*.

Example 4 (Dropping *nonnegative payment*). Let $t_0 < 0$. Let f be a rule that chooses a minimum price Walrasian equilibrium with negative entry fee t_0 for each preference profile. Then, the rule f satisfies *strategy-proofness*, *efficiency*, and *individual rationality*, but not *nonnegative payment*.

6 Simultaneous Ascending Auction

In this section, we define a class of simultaneous ascending auctions, and show that they achieve the minimum price Walrasian equilibrium. Let $\mathcal{R} \subseteq \mathcal{R}^E$.

Definition 6.1. Given $R \in \mathcal{R}^n$ and $p \in \mathbb{R}^m_+$, a set $M' \subseteq M$ of objects is a **minimal overdemanded set** at p for R if M' is overdemanded at p for R, and there is no $M'' \subsetneq M'$ such that M'' is overdemanded at p.

Under a (continuous time) "simultaneous ascending auction", in each time, each bidder submits his demand at a current price vector, and the prices of the objects in a minimal overdemanded set are raised at a speed at least $d > 0$.

Definition 6.2. A **simultaneous ascending (SA) auction** is a function $\hat{p}$ from $\mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathcal{R}^n$ to $\mathbb{R}^m_+$ such that
(i) for each $p \in \mathbb{R}^m_+$, each $R \in \mathcal{R}^n$, and each $x \in M$, $\hat{p}^x(0, p, R) \equiv 0$,
(ii) there is $d > 0$ such that for each $t \in \mathbb{R}_+$, each $p \in \mathbb{R}^m_+$, each $R \in \mathcal{R}^n$, and each $x \in M$,
  (ii-a) $d\hat{p}^x(t, p, R)/dt \geq d$ if x is in a minimal overdemanded set at p for R, and
  (ii-b) $d\hat{p}^x(t, p, R)/dt = 0$ otherwise.

Remark 6.1. For each $R \in \mathcal{R}^n$, an SA auction $\hat{p}$ generates a **price path** $p(\cdot)$ such that for each $x \in M$ and each $t \in \mathbb{R}_+$,

$$p^x(t) = \int_0^t \frac{d\hat{p}^x(s, p(s), R)}{ds} ds.$$

Proposition 6.1. *For each preference profile, the price path generated by any simultaneous ascending auction converges to the minimum Walrasian equilibrium price in a finite time.*

The proof is in the Appendix. Proposition 6.1 says that for each $R \in \mathcal{R}^n$, the price path $p(\cdot)$ generated by an SA auction has a **final time** T such that for each $t \geq T$, $p(t) = p(T) = p_{\min}(R)$, and at the final price $p(T)$, each agent receives an object from his demand. Moreover, this proposition shows the existence of Walrasian equilibrium.

7 Overview of the proof of Theorem 5.1

We give an overview of the proof of Theorem 5.1. Since *if* part of the theorem follows from the discussion in Subsection 5.1, we explain the proof of *only if* part of the theorem.

As we emphasized in Introduction, without quasi-linearity of preferences, *efficient* allocations of objects depend on payments. Thus, it is difficult to identify the object allocations of the rules satisfying our desirable properties without knowing their payments. On the other hand, it is also difficult to identify the payments of the rules satisfying our properties without knowing their object allocations. In this section, we discuss how we overcome those dual difficulties.

Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. The proof consists of the following four parts.

PART 1. We show the four preliminary results below, which are repeatedly used in the proof.

Lemma 5.1 below says that *under individual rationality and nonnegative payment, whenever an agent does not receive any object, then the payment of the agent should be zero.*

Lemma 5.1. *Let f be a rule that satisfies individual rationality and nonnegative payment on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f^x_i(R) = 0$. Then, $f^t_i(R) = 0$.*

Lemma 5.2 says that *under efficiency, individual rationality, and nonnegative payment, each object should be assigned to someone.*

Lemma 5.2. *Let f be a rule that satisfies efficiency, individual rationality, and nonnegative payment on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $x \in M$. Then, there is $i \in N$ such that $f_i^x(R) = x$.*

The next lemma says that *given an allocation, if there is a pair $\{i, j\}$ of agents such that i prefers his own assignment to j's one, but j prefers i's assignment to his own, and the difference between j's payment and i's compensating valuation (CV) of j's assignment of objects from i's assignment is less than the difference between i's payment and j's CV of i's assignment of objects from j's assignment, then there must be a Pareto-improvement.*

Lemma 5.3. *Let $R \in \mathcal{R}^n$, $i, j \in N$, and $z \in Z$ with $x_i \neq 0$. Assume that (a) $0 \leq t_j - CV_i(x_j; z_i) < CV_j(x_i; z_j) - t_i$. Then, there is $\hat{z} \in Z$ that Pareto-dominates z at R.*

Given a bundle $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$ with $x_i \neq 0$, let $\mathcal{R}_{NCV}(z_i)$ be the set of preferences $\hat{R}_i \in \mathcal{R}$ such that for each $y \in L \setminus \{x_i\}$, $\widehat{CV}_i(y; z_i) < 0$, that is, for each object except for x_i, the compensating valuation of $\hat{R}_i$ from z_i is negative. We refer to the preferences in $\mathcal{R}_{NCV}(z_i)$ as "z_i-favoring".

Lemma 5.4 says that *under strategy-proofness and nonnegative payment, given a preference profile R, for each agent who is assigned an object, if the agent's preference is changed to a preference that is $f_i(R)$-favoring, then his assignment remains the same.*

Lemma 5.4. *Let f be a rule that satisfies strategy-proofness and nonnegative payment on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) \neq 0$. Let $\hat{R}_i \in \mathcal{R}_{NCV}(f_i(R))$. Then, $f_i(\hat{R}_i, R_{-i}) = f_i(R)$.*

PART 2. We establish Proposition 5.1 below, which says that *for each preference profile, the allocation chosen by the rule f satisfying strategy-proofness, efficiency, individual rationality,* and *nonnegative payment* on $\mathcal{R}^n$ *should (weakly) dominate the minimum price Walrasian equilibrium allocations.* This proposition implies that under the rule satisfying our properties, the payment of each agent is at most the minimum Walrasian price. Thus, Proposition 5.1 derives stringent upper bounds of outcome payments of the rules even without knowing their object allocations. It is a crucial step to overcome the dual difficulties in the proof of Theorem 5.1.

Proposition 5.1.[18] *Let f be a rule satisfying strategy-proofness, efficiency, individual rationality, and nonnegative payment on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$ and $z \in W_{\min}(R)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.*

To prove Proposition 5.1, we introduce some additional notations and three lemmas. Given $R \in \mathcal{R}^n$, $x \in M$, and $z \in (L \times \mathbb{R})^n$, let $\pi^x(R) \equiv (\pi_1^x(R), \ldots, \pi_n^x(R))$ be the permutation on N such that $CV_{\pi_n^x(R)}(x; z_{\pi_n^x(R)}) \leq \ldots \leq CV_{\pi_1^x(R)}(x; z_{\pi_1^x(R)})$. That is, $\pi_n^x(R)$ is the agent with the lowest compensating valuation of object x from z, $\pi_{n-1}^x(R)$ is the agent with the second lowest compensating valuation of object x from z, and so on. For each $k \in N$, let $C^k(R, x; z) \equiv CV_{\pi_k^x(R)}(x; z_{\pi_k^x(R)})$. That is, $C^k(R, x; z)$ is *the k-th highest compensating valuation (CV) of object x from z.* We simply write $C^k(R, x; (\mathbf{0}, \ldots, \mathbf{0}))$ as $C^k(R, x)$.

Hereafter, we maintain the assumption that f is a rule on $\mathcal{R}^n$, and that the rule f satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *nonnegative payment.*

The next lemma says that *if an agent receives object x, then his payment is not less than the $(m+1)$-th highest CV of object x from the origin.* Thus, the $(m+1)$-th highest CV of

[18] This result also holds for any Walrasian equilibrium allocation z.

each object from the origin is a lower bound for the payment of the agent who obtains the object.

Lemma 5.5. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$ be such that $f_i^x(R) = x$. Then, $f_i^t(R) \geq C^{m+1}(R, x)$.*

By using Lemma 5.5, we obtain Lemma 5.6 below, which says that *if an agent receives object x, then his CV for object x from the origin is not less than the m-th highest CV of object x from the origin.* Lemma 5.6 says that an agent cannot be assigned an object x by the rule unless $CV_i(x; \mathbf{0}) \geq C^m(R, x)$. For each object, this lemma restricts the candidates of agents who obtain the object without knowing payments.

Lemma 5.6. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$ be such that $f_i^x(R) = x$. Then, $CV_i(x; \mathbf{0}) \geq C^m(R, x)$.*

Lemma 5.6 implies that if for any object other than x, an agent's CV from the origin is less than the m-th highest, then he never receives an object other than x. Whether or not an agent receives object x depends on his CV of object x from the origin. It is straightforward from *efficiency* that if an agent has the highest CV of object x from the origin, he receives object x. Lemma 5.7 below gives a weaker sufficient condition that agent i receives object x.

Given $R \in \mathcal{R}^N$, let $Z^{IR}(R)$ be the set of *individually rational* allocations, that is, $Z^{IR}(R) \equiv \{z \in Z : \text{ for each } i \in N, z_i \, R_i \, \mathbf{0}\}$.

Lemma 5.7. *Let $R \in \mathcal{R}^n$, $x \in M$, and $i \in N$ be such that for each $y \in M \setminus \{x\}$, $CV_i(y; \mathbf{0}) < C^m(R, y)$. Let $z \in Z^{IR}(R)$, $CV_i(x; \mathbf{0}) > C^1(R_{-i}, x; z)$, and $f_j(R) \, R_j \, z_j$ for each $j \in N \setminus \{i\}$. Then, $f_i^x(R) = x$.*

Now, we present an informal sketch of the proof of Proposition 5.1, but only for Case V in Section 2 (Figure 4), to explain intuitions in the simplest way.

Sketch of proof of Proposition 5.1. (Figure 5) By *individual rationality*, $f_3(R) \, R_3 \, z_3$. Let $i \in N \setminus \{3\}$. Without loss of generality, let $i = 1$. By contradiction, suppose that $z_1 \, P_1 \, f_1(R)$. Then, since $t_1 < CV_1(A; f_1(R))$, there is a preference $\hat{R}_1$ that satisfies (1-a): $\widehat{CV}_1(B; \mathbf{0}) < C^3(R, B)$ and (1-b): $t_1 < \widehat{CV}_1(A; \mathbf{0}) < CV_1(A; f_1(R))$.

Step 1[19]: We show $z_2 \, P_2 \, f_2(\hat{R}_1, R_{2,3})$. Suppose that $f_2(\hat{R}_1, R_{-1}) \, R_2 \, z_2$. By *individual rationality*, $f_3(\hat{R}_1, R_{2,3}) \, R_3 \, z_3$. By (1-a), $\widehat{CV}_1(B; \mathbf{0}) < C^2(\hat{R}_1, R_{2,3}, B)$. By (1-b) and $z \in W_{\min}(R)$, $\widehat{CV}_1(A; \mathbf{0}) > C^1(R_{2,3}, B; z)$. Since $z \in Z^{IR}(\hat{R}_1, R_{2,3})$, Lemma 5.7 implies $f_1^x(\hat{R}_1, R_{2,3}) = A$, and so, by *individual rationality*, $f_1^t(\hat{R}_1, R_{2,3}) \leq \widehat{CV}_1(A; \mathbf{0})$. However, by (1-b): $\widehat{CV}_1(A; \mathbf{0}) < CV_1(A; f_1(R))$, $f_1(\hat{R}_1, R_{2,3}) \, P_1 \, f_1(R)$, which contradicts *strategy-proofness*.

Step 2[20]: We derive a contradiction to conclude that $f_1(R) \, R_1 \, z_1$. It follows from Step 1 that $t_2 < CV_2(B; f_2(\hat{R}_1, R_{2,3}))$, and so, there is a preference $\hat{R}_2$ that satisfies (2-a): $\widehat{CV}_2(A; \mathbf{0}) < C^3(\hat{R}_1, R_{2,3}, A)$ and (2-b): $t_2 < \widehat{CV}_2(B; \mathbf{0}) < CV_2(B; f_2(\hat{R}_1, R_{2,3}))$. For each $i = 1, 2, 3$, let $\hat{z}_i \equiv \mathbf{0}$. Then, by *individual rationality*, (2-a), and (2-b), the assumptions of Lemma 5.7 hold for the profile $(\hat{R}_{1,2}, R_3)$. Then, by Lemma 5.7, $f_2^x(\hat{R}_{1,2}, R_3) = B$, and thus, by *individual*

[19] This step corresponds to Step 2-1 of Proof of Proposition 5.1 in the Appendix.

[20] This step corresponds to Step 2-2 of Proof of Proposition 5.1 in the Appendix. Here we derive a contradiction in a simpler way by using the assumption that $m = 2$.

rationality, $f_2^t(\hat{R}_{1,2}, R_3) \leq \widehat{CV}_2(B; \mathbf{0})$. However, by (2-b): $\widehat{CV}_2(B; \mathbf{0}) < CV_2(B; f_2(\hat{R}_1, R_{2,3}))$, $f_2(\hat{R}_{1,2}, R_3)\, P_2\, f_2(\hat{R}_1, R_{2,3})$, which contradicts *strategy-proofness.*

[Figure 5 about here]

When there are more than two objects, by applying the similar argument, in STEP 2, we show that there is $i \neq 1, 2$ such that $z_i\, P_i\, f_i(\hat{R}_{1,2}, R_{-1,2})$. Repeating this argument m times inductively, we can also obtain a similar contradiction as in STEP 2. □

PART 3. To prove Theorem 5.1, we introduce more four lemmas. The important steps of PART 3 are to prove Lemma 5.9 and Lemma 5.11 below.

Let f be a rule satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *non-negative payment* on $\mathcal{R}^n$. Given a Walrasian equilibrium allocation z, let $\mathcal{R}^I(z)$ be the set of preferences $R_i \in \mathcal{R}$ such that for each $i, j \in N$, $z_i\, I_i\, z_j$, that is, all the assignments under z are indifferent. We refer to the preferences in $\mathcal{R}^I(z)$ as "z-indifferent".

The next lemma says that *given a minimum price Walrasian equilibrium* (z^*, p)*, if a group of agents change their preferences to* z^**-indifferent preferences, then, for the new preference profile, (a)* z^* *is also a minimum price Walrasian equilibrium allocation, (b) the allocation chosen by the rule* f *(weakly) dominates* z^**, and (c) an agent who does not obtain any object demands the null object at the price* p*.*

Lemma 5.8. *Let* $R \in \mathcal{R}^n$*,* $z^* \in W_{\min}(R)$*,* p *be the price vector associated with* z^**,* $N' \subseteq N$*,* $\hat{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$*, and* $\hat{R} \equiv (\hat{R}_{N'}, R_{-N'})$*. Then, (a)* $z^* \in W_{\min}(\hat{R})$*, (b) for each* $i \in N$*,* $f_i(\hat{R})\, \hat{R}_i\, z_i^*$ *and (c) for each* $i \in N$*, if* $f_i^x(\hat{R}) = 0$*, then* $0 \in D(\hat{R}_i, p)$*.*

Given $p \in \mathbb{R}^m_{++}$ and $R \in \mathcal{R}^n$, let $N(R, p)$ denote the set of demanders of the non-null objects at the price p, that is, $N(R, p) \equiv \{i \in N : D(R_i, p) \cap M \neq \emptyset\}$.

As discussed in Section 4, an important structure of the minimum price Walrasian equilibria is demand connectedness (Corollary 4.2). Lemma 5.9 below implies that the rule f possesses a similar structure, although in a limited pattern. It is an important step to derive the minimum price Walrasian equilibrium allocations from the desirable properties. Lemma 5.9 says that *given a minimum price Walrasian equilibrium* (z^*, p) *and a preference profile such that a group* N' *of agents have* z^**-indifferent preferences, if (9-i) the payments of the agents outside* N' *are not less than the price* p*, and (9-ii) each agent in* N' *receives an object, then (9-a) each agent demanding only the null object at the price* p *receives the null object, and (9-b) an object obtained by a* z^**-indifferent agent is connected to the null object by the demands of non* z^**-indifferent agents.*

Lemma 5.9. *Let* $R \in \mathcal{R}^n$*,* $z^* \in W_{\min}(R)$*, and* p *be the price vector associated with* z^**. Let* $N' \subseteq N$ *with* $1 \leq \#N' \leq m$*,* $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$*,* $\bar{R} \equiv (\bar{R}_{N'}, R_{-N'})$ *and* $N'' \equiv N(R, p) \setminus N'$*. Assume that (9-i) for each* $i \in N \setminus N'$*, and each* $x \in M$*, if* $f_i^x(\bar{R}) = x$*, then* $f_i^t(\bar{R}) \geq p^x$*, and (9-ii) for each* $j \in N'$*,* $f_j^x(\bar{R}) \neq 0$*. Then,*
(9-a) for each $j \notin N(R, p) \cup N'$*,* $f_j^x(\bar{R}) = 0$*, and*
(9-b) there is a sequence $\{i_k\}_{k=1}^K$ *of* K *distinct agents such that (i)* $K \in \{2, \ldots, m+1\}$*, (ii)* $f_{i_1}^x(\bar{R}) = 0$*, (iii) for each* $k \in \{1, \ldots, K-1\}$*,* $i_k \in N''$*, and* $i_K \in N'$*, and (iv) for each* $k \in \{1, \ldots, K-1\}$*,* $\{f_{i_k}^x(\bar{R}), f_{i_{k+1}}^x(\bar{R})\} \subseteq D(R_{i_k}, p)$*.*

See Figure 6 for an illustration of (9-b).

[Figure 6 about here]

In the proof of Lemma 5.9, we intensively use Theorem 4.1, which is a characterization result of the minimum price Walrasian equilibrium by the concepts of overdemanded and weakly underdemanded sets introduced in Section 4.

We give an informal sketch of the proof of Lemma 5.9. Although we sketch the proof only for two objects case, it can be easily generalized to any finite objects case. In the Appendix, we give a formal proof of Lemma 5.9 by using induction.

Sketch of proof of Lemma 5.9 for two objects case. First, we show (9-a). Suppose that for some $j \notin N(R,p) \cup N'$, $x \equiv f_j^x(\bar{R}) \neq 0$. Since agent j demands only the null object at the price p, *individual rationality* implies $f_j^t(\bar{R}) \leq CV_j(x;\mathbf{0}) < p^x$. This contradicts (9-i).[21]

We turn to the proof of (9-b). Since $n > m$, at least one agent receives the null object. By Lemma 5.8-(a), z^* is also a minimum price Walrasian equilibrium for $\bar{R}$. Then, by Theorem 4.1, no weakly underdemanded set exists at p for $\bar{R}$. Thus, at least one agent who obtains the null object demands the non-null objects at p under $\bar{R}$. By (9-ii), no z^*-indifferent agent receives the null object. Thus, $N_1'' \equiv \{i_1 \in N'' : f_{i_1}^x(\bar{R}) = 0\} \neq \emptyset$.

Let $D_1 \equiv [\bigcup_{i \in N_1''} D(R_i,p)] \setminus \{0\}$. Since $\emptyset \neq N_1'' \subseteq N(R,p)$, $D_1 \neq \emptyset$. By Lemma 5.2, for each $x \in D_1$, there is $i(x) \in N \setminus N_1''$ such that $f_{i(x)}^x(\bar{R}) = x$.

Assume that some agents in N' receive the object in D_1, *i.e.,* for some $x_1 \in D_1$, $i(x_1) \in N'$. Since $x_1 \in D_1$, there is $i_1 \in N_1''$ such that $x_1 \in D(R_{i_1},p)$. Let $i_2 \equiv i(x_1)$. Then, $\{i_1, i_2\}$ satisfies conditions (i), (ii), and (iii) of (9-b). By $f_{i_1}^x(\bar{R}) = 0$ and Lemma 5.8-(c), (iv) of (9-b) also holds. Thus, (9-b) holds in this case.

Next, we assume that (9-b-1): no agent in N' receives the object in D_1, *i.e.,* for each $x \in D_1$, $i(x) \notin N'$. By (9-ii), $M \setminus D_1 \neq \emptyset$. Let $N_2'' \equiv \{j \in N'' \setminus N_1'' : \exists\, i \in N_1'' \text{ s.t. } f_j^x(\bar{R}) \in D(R_i,p)\}$. By (9-b-1) and (9-a), for each $x \in D_1$, $i(x) \in N'' \setminus N_1''$. Thus, $N_2'' \neq \emptyset$. Since no two agents receive the same object, $\#D_1 = \#N_2''$. Then, we can show that (9-b-2): there is $j \in N_2''$ who demands the object in $M \setminus D_1$, *i.e.,* $D(R_j,p) \cap (M \setminus D_1) \neq \emptyset$.[22]

Let $D_2 \equiv [\bigcup_{i \in N_2''} D(R_i,p)] \setminus (D_1 \cup \{0\})$. By (9-b-2), $D_2 \neq \emptyset$. By Lemma 5.2, for each $x \in D_2$, there is $i(x) \in N \setminus (N_1'' \cup N_2'')$ such that $f_{i(x)}^x(\bar{R}) = x$.

Note that, in two objects case, some agents in N' receive the object in D_2, *i.e.,* for some $x_2 \in D_2$, $i(x_2) \in N'$.[23] Let $i_3 \equiv i(x_2)$. Since $x_2 \in D_2$, there is $i_2 \in N_2''$ such that $x_2 \in D(R_{i_2},p)$. Thus, $f_{i_3}^x(\bar{R}) \in D(R_{i_2},p)$. Since $i_2 \in N_2''$, there is $i_1 \in N_1''$ such that

[21] Note that the proof of (9-a) does not depend on the assumption that $m = 2$.

[22] To see this, suppose that for each $j \in N_2''$, $D(R_j,p) \cap (M \setminus D_1) = \emptyset$. Then,

$$\begin{aligned} \#\left\{j \in N : D(\bar{R}_j,p) \cap (M \setminus D_1) \neq \emptyset\right\} &= \#N' + \#N'' - \#N_1'' - \#N_2'' \\ &= \#M - \#D_1 \\ &= \#M \setminus D_1, \end{aligned}$$

where the first equality follows from $\#\left\{j \in N : D(\bar{R}_j,p) \cap M \neq \emptyset\right\} = \#N' + \#N''$, and for each $k \in \{1,2\}$ and each $i \in N_k''$, $D(R_i,p) \cap (M \setminus D_1) = \emptyset$, and the second from $\#N' + \#N'' - \#N_1'' = m$ and $\#D_1 = \#N_2''$.

Thus, the set $M \setminus D_1$ is weakly underdemanded at p for $\bar{R}$. However, by Lemma 5.8-(a) and Theorem 4.1, there is no weakly underdemanded set at p for $\bar{R}$. This is a contradiction.

[23] To see this, suppose that for each $x \in D_2$, $i(x) \notin N'$. Then, by (9-b-1), for each $x \in D_1 \cup D_2$, $i(x) \notin N'$. Since $D_1 \neq \emptyset$, $D_2 \neq \emptyset$ and $D_1 \cap D_2 = \emptyset$, in two objects case, we have $D_1 \cup D_2 = M$. However, by $N' \neq \emptyset$ and (9-ii), there is $j \in N'$ such that $f_j^x(\bar{R}) \neq 0$. Thus, for some $x \in D_1 \cup D_2$, $i(x) \in N'$, which is a contradiction.

$f^x_{i_2}(\bar{R}) \in D(R_{i_1}, p)$. We show that $\{i_1, i_2, i_3\}$ satisfies conditions (i), (ii), (iii), and (iv) in (9-b) (see Figure 7).

Note that, by $f^x_{i_1}(\bar{R}) = 0$ and Lemma 5.8-(c), $\{f^x_{i_1}(\bar{R}), f^x_{i_2}(\bar{R})\} \subseteq D(R_{i_1}, p)$. Finally, we show that $f^x_{i_2}(\bar{R}) \in D(R_{i_2}, p)$. By contradiction, suppose that $f^x_{i_2}(\bar{R}) \notin D(R_{i_2}, p)$. Let $y \equiv f^x_{i_2}(\bar{R})$. Since $x^*_{i_2} \in D(R_{i_2}, p)$, $z^*_{i_2} \, P_{i_2} \, (y, p^y)$. By Lemma 5.8-(b), $f_{i_2}(\bar{R}) \, R_{i_2} \, z^*_{i_2}$. Thus, $f_{i_2}(\bar{R}) \, R_{i_2} \, z^*_{i_2} \, P_{i_2} \, (y, p^y)$, which implies $f^t_{i_2}(\bar{R}) < p^y$. This contradicts (9-i). Thus, (9-b) also holds in this case.

[Figure 7 about here]

When there are more than two objects, we next consider the case where no agents in N' receives the object in D_2. Applying a similar argument repeatedly, we can also show (9-b) in Lemma 5.9 for more general cases. □

Lemma 5.10 below says that *when an agent i receives object x and his CV of the null object from his assignment is negative, for each agent $j \neq i$, if j's CV of object x from the origin is greater than the difference between i's payment and i's CV of the null object from his assignment, then agent j receives an object.*

Lemma 5.10. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$ be such that $f^x_i(R) = x$ and $CV_i(0; f_i(R)) < 0$. Let $j \in N \setminus \{i\}$. Assume that (10-i) $-CV_i(0; f_i(R)) < CV_j(x; \mathbf{0}) - f^t_i(R)$. Then, $f^x_j(R) \neq 0$.*

Next, we explain Lemma 5.11 below, which says that *given a minimum price Walrasian equilibrium (z^*, p) and a preference profile such that a group N' of agents have z^*-indifferent preferences, if (11-i) for "any" z^*-indifferent preferences of the group N', the payments of the agents outside N' are not less than the price p, then the payments of the agents in N' are not less than the price p.* Thus, although in a limited pattern, this lemma derives stringent lower bounds of outcome payments of the rules even without knowing their object allocations.

Lemma 5.11. *Let $R \in \mathcal{R}^n$, $z^* \in W_{\min}(R)$, and p be the price vector associated with z^*. Let $N' \subsetneq N$. Assume that (11-i) for each $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$, each $i \in N \setminus N'$, and each $x \in M$, if $f^x_i(\bar{R}_{N'}, R_{-N'}) = x$, then $f^t_i(\bar{R}_{N'}, R_{-N'}) \geq p^x$. Let $\hat{R}_{N'} \in \mathcal{R}^I(z^*)^{\#N'}$. Then, for each $i \in N'$ and each $x \in M$, if $f^x_i(\hat{R}_{N'}, R_{-N'}) = x$, then $f^t_i(\hat{R}_{N'}, R_{-N'}) \geq p^x$.*

In the proof of Lemma 5.11, we derive a contradiction by showing that whenever the payment of a z^*-indifferent agent is less than the price p, there is another allocation that Pareto-dominates the allocation chosen by the rule. To guarantee the existence of such Pareto-improvements, we apply Lemma 5.9.

Let us explain how Lemma 5.9 works in the proof. Let $\hat{R} \equiv (\hat{R}_{N'}, R_{-N'})$ be a preference profile such that the agents in N' have z^*-indifferent preferences $\hat{R}_{N'}$. Suppose that the payment of a z^*-indifferent agent i who obtains object x is less than the price p^x of object x. Let $\bar{R}_i$ be an "$f_i(\hat{R})$-favoring" and "z^*-indifferent" preference such that the difference between i's payment and i's compensating valuation of the null object from i's assignment is less than the price p^x, *i.e.*, (11-ii) $f^t_i(\hat{R}) - \overline{CV}_i(0; f_i(\hat{R})) < p^x$. Let $\bar{R} \equiv (\bar{R}_i, \hat{R}_{N' \setminus \{i\}}, R_{-N'})$. Then, since $\bar{R}_i$ is $f_i(\hat{R})$-favoring, Lemma 5.4 implies $f_i(\bar{R}) = f_i(\hat{R})$. Since $\bar{R}_i$ is z^*-indifferent, the preferences $\bar{R}_{N'}$ of the group N' are also in $\mathcal{R}^I(z^*)^{\#N'}$. Then, (9-i) in Lemma 5.9 follows from (11-i). Note that for each $j \in N' \setminus \{i\}$, $\widehat{CV}_j(x; \mathbf{0}) = p^x$. Thus, by (11-ii) and Lemma 5.10, for each $j \in N' \setminus \{i\}$, $f^x_j(\bar{R}) \neq 0$. Moreover, by $f_i(\bar{R}) = f_i(\hat{R})$, $f^x_i(\bar{R}) = x \neq 0$. Thus, (9-ii) in Lemma 5.9 also holds.

Then, by lemma 5.9, there is a sequence $\{i_k\}_{k=1}^K$ of distinct agents satisfying conditions (i), (ii), (iii), and (iv) in (9-b). By (11-i) and Proposition 5.1, for each $k < K$, $f^x_{i_k}(\bar{R}) = p^{x(k)}$, where $x(k) \equiv f^x_{i_k}(\bar{R})$. For simplicity, we focus on the case where (a) $K = 4$, (b) $i = i_K$, and (c) for each $k \in \{1, 2, 3, 4\}$, agent i_k's assignment under f at $\bar{R}$ is depicted in Figure 8.

[Figure 8 about here]

Let z' be the allocation such that for each $k \in \{1, 2\}$, agent i_k obtains i_{k+1}'s assignment under f at $\bar{R}$, i_3 and i_4 receive z'_{i_3} and z'_{i_4} depicted in Figure 8 respectively, and the other agents receive their own assignments under f at $\bar{R}$. Then, since $f^t_{i_1}(\bar{R}) = 0$, $f^t_{i_2}(\bar{R}) = p^1$, and $f^t_{i_3}(\bar{R}) = p^2$, agent i_3 prefers z'_{i_3} to his own assignment $f_{i_3}(\bar{R})$, but all the other agents are indifferent between the two assignments. Thus, z' is a Pareto-improvement for the allocation under f at $\bar{R}$. Applying a similar argument, we can also show the existence of such Pareto-improvements for more general cases.

PART 4. We complete the proof of Theorem 5.1, that is, we show that if a rule f satisfies *strategy-proofness*, *efficiency*, *individual rationality*, and *nonnegative payment* on $\mathcal{R}^n$, then, for each preference profile, the allocation chosen by the rule f is a minimum price Walrasian equilibrium allocation.

Sketch of proof of Theorem 5.1. We present an informal sketch of the proof of Theorem 5.1. Let R be a preference profile, and let (z^*, p) be a minimum price Walrasian equilibrium associated with R.

Let $\bar{R}$ be a profile of z^*-indifferent preferences. Then, for each object, the $(m+1)$-th highest CV from the origin is equal to the price p. Thus, by Lemma 5.5, for each object x, the payment of an agent who obtains object x is not less than the price p^x. We replace the preferences in $\bar{R}$ by the original preferences in R one by one, and inductively show that for each object x, the payment of an agent who obtains x is not less than the price p^x.

STEP 1: We replace the preference $\bar{R}_i$ in $\bar{R}$ of an agent i by his original preference R_i. Then, if agent i obtains an object x at the new profile $(R_i, \bar{R}_{-i})$, then $f^t_i(R_i, \bar{R}_{-i}) \geq p^x$. For otherwise since $\bar{R}_i$ is z^*-indifferent, $f_i(R_i, \bar{R}_{-i}) \, \bar{P}_i \, f_i(\bar{R})$, contradicting *strategy-proofness*. Then, Lemma 5.11 implies that the payments of the remaining agents are also not less than the price p.

STEP 2: We replace the preference $\bar{R}_j$ in $(R_i, \bar{R}_{-i})$ of an agent $j \neq i$ by his original preferences R_j. Then, if agent i obtains an object x at the new profile $(R_{i,j}, \bar{R}_{-i,j})$, then $f^t_i(R_{i,j}, \bar{R}_{-i,j}) \geq p^x$. For otherwise since $\bar{R}_i$ is z^*-indifferent, Step 1 implies $f_i(R_{i,j}, \bar{R}_{-i,j}) \, \bar{P}_i \, f_i(R_j, \bar{R}_{-j})$, contradicting *strategy-proofness*. Similarly, if agent j obtains an object x at the new profile $(R_{i,j}, \bar{R}_{-i,j})$, then $f^t_j(R_{i,j}, \bar{R}_{-i,j}) \geq p^x$. Then, Lemma 5.11 implies that the payments of the remaining agents are also not less than the price p.

$\vdots$

Repeating this argument inductively, we conclude that, under the original preference profile R, the payment of each agent is not less than the minimum Walrasian equilibrium price p. Together with Proposition 5.1, this implies that each agent receives an assignment of objects in his demand set at the price p and pays its price. Thus, (WE-i) in Definition 4.1 holds. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, the minimum Walrasian equilibrium price of each object is positive. Lemma 5.2 implies that each object is assigned to someone under the rule f. Thus, (WE-ii) in Definition 4.1 also holds. Since p is the minimum Walrasian equilibrium price for R, we conclude that $f(R) \in W_{\min}(R)$. $\square$

8 Concluding Remarks

In this article, we considered the problem of allocating several heterogeneous objects among a group of agents and how much agents should pay. Each agent is permitted to receive one object at most and has "nonquasi-linear" preferences. First, we extended the results of Mishra and Talman (2010) on the quasi-linear domain to the domains including nonquasi-linear preferences, that is, we established that *on the extended domain, a price vector is a minimum Walrasian equilibrium price if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at the price (Theorem 4.1).* Next, in the case where the number of agents exceeds the number of objects, we established that *on the domain of classical preferences, the minimum price Walrasian rule is a unique rule that satisfies strategy-proofness, efficiency, individual rationality, and nonnegative payment (Theorem 5.1).*

Since the minimum price Walrasian equilibrium allocations can be achieved by conducting the simultaneous ascending auctions (Proposition 6.1; Demange, Gale, and Sotomayor, 1986; etc.), our results provide an answer to the question: "what types of auction rules are desirable for large scale auctions?", that is, the simultaneous ascending auctions should be employed when agents' preferences are not necessary quasi-linear.

Appendix: Proofs

A.1 Proofs for Section 4 (Theorem 4.1, and Corollaries 4.1 and 4.2)

Let $\mathcal{R} \subseteq \mathcal{R}^E$. To prove Theorem 4.1, we introduce the concept of "truncation"of a preference, and show a remark, two lemmas, and a fact below.

Given $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the $\boldsymbol{d_i}$**-truncation of** $\boldsymbol{R_i}$ is the preference $\hat{R}_i$ such that for each $z_i \in M \times \mathbb{R}$, $\widehat{CV}_i(0; z_i) = CV_i(0; z_i) + d_i$. Given $R \in \mathcal{R}^n$, the $\boldsymbol{d}$**-truncation of** $\boldsymbol{R}$ is the preference profile $\hat{R}$ such that for each $i \in N$, $\hat{R}_i$ is the d_i-truncation of R_i.

Remark 4.1. Let $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and $\hat{R}_i$ be the d_i-truncation of R_i. Then, for each $z_i, \hat{z}_i \in M \times \mathbb{R}$, $z_i \, R_i \, \hat{z}_i$ if and only if $z_i \, \hat{R}_i \, \hat{z}_i$.

Lemma 4.1. *Let $R \in \mathcal{R}^n$ and (z, p) be a Walrasian equilibrium for R. Let $\hat{R}$ be the d-truncation of R such that for each $i \in N$ with $x_i \neq 0$, $d_i \leq -CV_i(0; z_i)$, and for each $i \in N$ with $x_i = 0$, $d_i \geq 0$. Then, (z, p) is also a Walrasian equilibrium for $\hat{R}$.*

Proof of Lemma 4.1. Since (z, p) is a Walrasian equilibrium for R, (z, p) satisfies (WE-i) and (WE-ii) for R. Since (WE-ii) is independent of preferences, we show only (WE-i) for $\hat{R}$, that is, that for each $i \in N$ and each $y \in L$, $(x_i, p^{x_i}) \, \hat{R}_i \, (y, p^y)$. Let $i \in N$ and $y \in L$.

First, consider the case where $x_i \neq 0$. If $y \neq 0$, then by Remark 4.1, $(x_i, p^{x_i}) \, \hat{R}_i \, (y, p^y)$. If $y = 0$, then by $d_i \leq -CV_i(0; z_i)$, $(x_i, p^{x_i}) \, \hat{R}_i \, \mathbf{0} = (y, p^y)$.

Next, consider the case where $x_i = 0$. If $y = 0$, then by $(y, p^y) = \mathbf{0} = (x_i, p^{x_i})$, $(x_i, p^{x_i}) \, \hat{R}_i \, (y, p^y)$. If $y \neq 0$, then by $(x_i, p^{x_i}) \, R_i \, (y, p^y)$ and $d_i \geq 0$, $(x_i, p^{x_i}) \, \hat{R}_i \, (y, p^y)$. $\square$

Lemma 4.2. *Let $i \in N$, $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and $\hat{R}_i \in \mathcal{R}$ be the d_i-truncation of R_i. Let $p, q \in \mathbb{R}^m_+$ $x \in M$, and $y \in L$ be such that $x \in D(R_i, p)$ and $y \in D(\hat{R}_i, q)$.*
(i) If $q^x < p^x$ and $y \in M$, then, $(y, q^y) \, P_i \, (x, p^x)$ and $q^y < p^y$.
(ii) If $q^x < p^x$ and $d_i \leq -CV_i(0; (x, p^x))$, then, $y \in M$, $(y, q^y) \, P_i \, (x, p^x)$, and $q^y < p^y$.

Proof of Lemma 4.2.
Proof of (i). Let $q^x < p^x$ and $y \in M$. By $y \in D(\hat{R}_i, q)$, $(y, q^y) \, \hat{R}_i \, (x, q^x)$. Since $\hat{R}_i$ is the d_i-truncation of R_i, it follows from Remark 4.1 that $(y, q^y) \, R_i \, (x, q^x)$. Thus,

$$(y, q^y) \, R_i \, (x, q^x) \, P_i \, (x, p^x) \, R_i \, (y, p^y),$$

where the second preference relation follows from $q^x < p^x$, and the third from $x \in D(R_i, p)$. Thus, $(y, q^y) \, P_i \, (x, p^x)$. Also, $(y, q^y) \, P_i \, (y, p^y)$ implies that $q^y < p^y$.
Proof of (ii). Let $q^x < p^x$ and $d_i \leq -CV_i(0; (x, p^x))$. Then, $\widehat{CV}_i(0; (x, p^x)) \leq 0$, and so $(x, p^x) \, \hat{R}_i \, \mathbf{0}$. Thus,

$$(y, q^y) \, \hat{R}_i \, (x, q^x) \, \hat{P}_i \, (x, p^x) \, \hat{R}_i \, \mathbf{0},$$

where the first preference relation follows from $y \in D(\hat{R}_i, q)$, and the second from $q^x < p^x$. Then, $(y, q^y) \, \hat{P}_i \, \mathbf{0}$ implies that $y \in M$. Thus, by (i) of Lemma 4.2, $(y, q^y) \, P_i \, (x, p^x)$ and $q^y < p^y$. □

Fact 4.5 (Roth and Sotomayor, 1990). *Let $R \in \mathcal{R}^n$, and let $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \geq 0$. Then, $p_{\min}(\hat{R}) \leq p_{\min}(R)$.*

We now proceed to prove Theorem 4.1.

Proof of Theorem 4.1. We first show *if* part of Theorem 4.1. Then, we prove *only if* part.
Proof of "IF" part. Assume that no set of objects is overdemanded and no set of objects is weakly underdemanded at p for R. Then, by Fact 4.4, p is a Walrasian equilibrium price. Suppose that there is a Walrasian equilibrium price q such that $q \leq p$ and $q \neq p$. Without loss of generality, assume that for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, where $M' \equiv \{1, \ldots, m'\}$ and $1 \leq m' \leq m$.

Since M' is not weakly underdemanded at p for R, there is $N' \subseteq N$ such that $\#N' > \#M'$ and for each $i \in N'$, $D(R_i, p) \cap M' \neq \emptyset$. For each $i \in N'$, let $y_i \in D(R_i, p) \cap M'$. Since for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, it follows that for each $i \in N'$ and each $x \notin M'$, $(y_i, q^{y_i}) \, P_i \, (y_i, p^{y_i}) \, R_i \, (x, p^x) = (x, q^x)$. Thus, for each $i \in N'$, $D(R_i, q) \subseteq M'$. By $\#N' > \#M'$, this implies that M' is overdemanded at q. Since q is a Walrasian equilibrium price, by Fact 4.4, this is a contradiction.

Proof of "ONLY IF" part. Let p be the minimum Walrasian equilibrium price for R. Then, by Fact 4.4, no set of objects is overdemanded and no set of objects is underdemanded at p for R. We show that no set of objects is weakly underdemanded at p for R. Suppose that there is a set M' of objects that is weakly underdemanded at p for R, that is, for each $x \in M'$, $p^x > 0$, and $\#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} \leq \#M'$. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Without loss of generality, assume that M' is minimum among the weakly underdemanded sets at p for R, that is, no proper subset of M' is weakly underdemanded at p. Since p is a Walrasian equilibrium price, there is an allocation $z \in Z$ such that for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$. Since no set of objects is underdemanded at p for R, $\#N' = \#M'$. Without loss of generality, let $M' \equiv \{1, \ldots, m'\}$ and $N' \equiv \{1, \ldots, m'\}$.

Step 1. *For each $i \in N'$, $x_i \in M'$.*

Proof of Step 1. Since for each $x \in M'$, $p^x > 0$, it follows from (WE-ii) that for each $x \in M'$, there is $i(x) \in N'$ such that $x_{i(x)} = x$. Then, by $\#N' = \#M'$, for each $i \in N'$, $x_i \in M'$. $\square$

For each $x \in M'$, let $q^x \equiv \max\{\{CV_j(x; z_j) : j \in N \setminus N'\} \cup \{0\}\}$. Then, for each $x \in M'$, $q^x < p^x$.[24] Let $\hat{R}_{m'+1} \in \mathcal{R}$ be such that for each $x \in M'$, if $q^x > 0$, $\widehat{CV}_{m'+1}(x; \mathbf{0}) = q^x$, and if $q^x = 0$, $\widehat{CV}_{m'+1}(x; \mathbf{0}) \in (0, p^x)$. Consider the economy E' with objects M', agents $N'' \equiv N' \cup \{m' + 1\}$, and their preference profile $(R_{N'}, \hat{R}_{m'+1})$. Let $\bar{z}_{m'+1} \equiv \mathbf{0}$ and $\bar{z}_{N''} = (z_{N'}, \bar{z}_{m'+1})$.

Step 2. *$(\bar{z}_{N''}, p^{M'})$ is a minimum price Walrasian equilibrium of the economy E'.*

Proof of Step 2. Let $(\tilde{z}_{N''}, \tilde{p})$ be the minimum price Walrasian equilibrium of E'. Since $(\bar{z}_{N''}, p^{M'})$ is a Walrasian equilibrium of E', $\tilde{p} \leq p^{M'}$. Let $M^- \equiv \{x \in M' : \tilde{p}^x < p^x\}$. We show that $M^- = \emptyset$. Suppose that $M^- \neq \emptyset$. Let $N^- \equiv \{i \in N' : D(R_i, p) \cap M^- \neq \emptyset\}$.

Step 2.1. *For each $i \in N^-$, $\tilde{x}_i \in M^-$.*

Proof of Step 2.1. Let $i \in N^-$. Then, there is $x \in D(R_i, p) \cap M^-$. Thus, $x \in M'$ and $\tilde{p}^x < p^x$. Since $(\tilde{z}_{N''}, \tilde{p})$ is a Walrasian equilibrium for $(R_{N'}, \hat{R}_{m'+1})$, $\tilde{x}_i \in D(R_i, \tilde{p})$. Then, Lemma 4.2-(ii) implies that $\tilde{x}_i \in M'$ and $\tilde{p}^{\tilde{x}_i} < p^{\tilde{x}_i}$. Thus, $\tilde{x}_i \in M^-$. $\square$

Step 2.2. $M^- = M'$, $N^- = N'$, and $\#M^- = \#N^-$.

Proof of Step 2.2. Since no two agents in N^- receive the same object, Step 2.1 implies $\#M^- \geq \#N^-$.

Suppose $M^- \neq M'$. Then, since $M^- \subsetneq M'$ and M' is minimum among the weakly underdemanded sets at p for R, M^- is not weakly underdemanded at p for $(R_{N'}, \hat{R}_{m'+1})$. Thus, since for each $x \in M^-$, $p^x > 0$, we have $\#N^- \geq \#M^- + 1$. This contradicts $\#M^- \geq \#N^-$. Thus, $M^- = M'$.

By the definition of N^-, $M^- = M'$ implies $N^- = N'$.

Since M' is weakly underdemanded, $\#N' = \#M'$. By the above results, $\#M^- = \#M' = \#N' = \#N^-$. $\square$

Step 2.3. *For each $x \in M'$, $\tilde{p}^x \geq q^x$.*

Proof of Step 2.3. Suppose that there is $x \in M'$ such that $\tilde{p}^x < q^x$. Then, by $\tilde{x}_{m'+1} \in D(\hat{R}_{m'+1}, \tilde{p})$ and $\tilde{p}^x < \widehat{CV}_{m'+1}(x; \mathbf{0})$, $\tilde{x}_{m'+1} \in M'$. By $M^- = M'$ and $N^- = N'$ (Step 2.2), Step 2.1 implies that for each $i \in N'$, $\tilde{x}_i \in M'$. Since there are only m' objects in M', this is a contradiction. $\square$

Let $(\hat{z}, \hat{p}) \in Z \times \mathbb{R}^m_+$ be such that $\hat{z}_{N'} = \tilde{z}_{N'}$, $\hat{z}_{-N'} = z_{-N'}$, $\hat{p}^{M'} = \tilde{p}$, and $\hat{p}^{-M'} = p^{-M'}$.

Step 2.4. *$(\hat{z}, \hat{p})$ is a Walrasian equilibrium of the original economy with objects M, agents N, and preference profile R.*

Proof of Step 2.4. By Step 2.3, for each $y \in M'$, $\tilde{p}^y \geq q^y$. Let $h \in N \setminus N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\hat{x}_h, \hat{p}^{\hat{x}_h}) = (x_h, p^{x_h}) \, R_h \, (y, p^y) = (y, \hat{p}^y),$$

[24]To see this, suppose that for some $x \in M'$, $q^x \geq p^x$. Then, there is $j \in N \setminus N'$ such that $(x, p^x) \; R_j \; z_j$. Since $x_j \in D(R_j, p)$, $x \in D(R_j, p)$. Thus, $j \in N'$. This contradicts $j \in N \setminus N'$.

where the preference relation follows from $x_h \in D(R_h, p)$, and if $y \in M'$, then

$$(\hat{x}_h, \hat{p}^{\hat{x}_h}) = (x_h, p^{x_h})\, R_h\, (y, q^y)\, R_h\, (y, \hat{p}^y),$$

where the first preference relation follows from the definition of q^y, and the last from $\hat{p}^y = \tilde{p}^y \geq q^y$. Thus, for each $h \in N \setminus N'$, $\hat{x}_h \in D(R_h, \hat{p})$.

Let $h \in N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\hat{x}_h, \hat{p}^{\hat{x}_h}) = (\tilde{x}_h, \tilde{p}^{\tilde{x}_h})\, R_h\, (x_h, \tilde{p}^{x_h})\, R_h\, (x_h, p^{x_h})\, R_h\, (y, p^y) = (y, \hat{p}^y),$$

where the first preference relation follows from $\tilde{x}_h \in D(R_h, \tilde{p})$, the second from $\tilde{p} \leq p^{M'}$, and the third from $x_h \in D(R_h, p)$, and if $y \in M'$, then

$$(\hat{x}_h, \hat{p}^{\hat{x}_h}) = (\tilde{x}_h, \tilde{p}^{\tilde{x}_h})\, R_h\, (y, \tilde{p}^y) = (y, \hat{p}^y),$$

where the preference relation follows from $\tilde{x}_h \in D(R_h, \tilde{p})$. Thus, for each $h \in N'$, $\hat{x}_h \in D(R_h, \hat{p})$.

Since (z, p) and $(\tilde{z}_{N''}, \tilde{p})$ both satisfy (WE-ii), $(\hat{z}, \hat{p})$ also satisfies (WE-ii). Thus, $(\hat{z}, \hat{p})$ is a Walrasian equilibrium for R. □

Remember that p is the minimum Walrasian equilibrium price for R. However, since $M^- \neq \emptyset$, $\hat{p} \leq p$ and $\hat{p} \neq p$. This is a contradiction. Thus, $M^- = \emptyset$. This completes the proof of Step 2.

Without loss of generality, let $x_1 \equiv 1, \ldots, x_{m'} \equiv m'$. Denote by Π the set of the permutations of M' and by $\{x(k)\}_{k=1}^{m'}$ its generic element. Given $\{x(k)\}_{k=1}^{m'} \in \Pi$, let $\{i(k)\}_{k=1}^{m'}$ be such that

$$x_{i(1)} = x(1),\ x_{i(2)} = x(2),\ \ldots\ , x_{i(m')} = x(m'),$$

and $\{t(k)\}_{k=1}^{m'}$ be such that

$$t(1) \leq \widehat{CV}_{m'+1}(x(1); \mathbf{0}),\ t(2) \equiv CV_{i(1)}(x(2); z_0(1)),\ \ldots\ , t(m') \equiv CV_{i(m'-1)}(x(m'); z_0(m'-1)),$$

where for each $k \in \{1, \ldots, m'\}$, $z_0(k) \equiv (x(k), t(k))$. We call such a pair $\{z_0(k), i(k)\}_{k=1}^{m'}$ an **assignment sequence**. See Figure A.1 for an illustration of assignment sequence.

[Figure A.1 about here]

Step 3. *There is $b < p^1$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$ constructed as above, and for k with $x(k) = 1$, $t(k) < b$.*

Proof of Step 3. For any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$, since $t(1) \leq q^{x(1)} < p^{x(1)}$, the following holds inductively: for each $k \geq 2$,

$$(x(k), t(k))\ R_{i(k-1)}\ z_0(k-1)\ P_{i(k-1)}\ (x(k-1), p^{x(k-1)})\ R_{i(k-1)}\ (x(k), p^{x(k)}),$$
$$\text{and } t(k) < p^{x(k)},$$

where the first preference relation follows from $t(k) = CV_{i(k-1)}(x(k); z_0(k-1))$, the second from $t(k-1) < p^{x(k-1)}$, and the third from $x(k-1) \in D(R_{i(k-1)}, p)$. Since the cardinality of

Π is finite ($m'!$), there is $b < p^1$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$, and for k with $x(k) = 1$, $t(k) < b$. □

Let $\hat{R}_1$ be such that (i) $\hat{R}_1$ is the d_1-truncation of R_1, and (ii) $b < \widehat{CV}_1(x_1; \mathbf{0}) < p^1$.[25] Consider the economy $\widehat{E}$ with objects M', agents $N'' \equiv \{1, \ldots, m'+1\}$, and their preference profile $(\hat{R}_1, \hat{R}_{m'+1}, R_{N'\setminus\{1\}})$. Let $(\hat{z}, \hat{p})$ be a minimum price Walrasian equilibrium of the economy $\widehat{E}$.

Step 4. $\hat{x}_1 \neq 0$.

Proof of Step 4. Suppose that $\hat{x}_1 = 0$. We use Claim 4.1 below. It implies that m' agents (agents $2, \ldots, m'+1$) receive m' different objects in $M'\setminus\{x_1\}$. By $\#M' = m'$, this is a contradiction. Thus, proving Claim 4.1 completes Proof of Step 4.

Claim 4.1. *The following sequences $\{i(k)\}$ and $\{z_0(k) \equiv (x(k), t(k))\}$, $k = 1, \ldots, m'$, can be constructed:*

$$x(1) \equiv \hat{x}_{m'+1},\ x_{i(1)} = x(1),\ \text{and } t(1) \equiv \hat{p}^{x(1)},\ \text{and}$$
$$\forall k \in \{2, \ldots, m'\}, \quad x(k) \equiv \hat{x}_{i(k-1)},\ x_{i(k)} = x(k),\ \text{and } t(k) \equiv CV_{i(k-1)}(x(k); z_0(k-1)).$$

Furthermore, for each $k \in \{1, \ldots, m'\}$, $x(k) \neq 0$, $x(k) \neq x_1$, $\hat{p}^{x(k)} \leq t(k)$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Proof of Claim 4.1. We prove by induction.
Part I. First, we show $x(1) \equiv \hat{x}_{m'+1} \neq 0$. Suppose $\hat{x}_{m'+1} = 0$. Then, since two agents (1 and $m'+1$) in N'' receive no object and $\#N'' = \#M' + 1$, there is $x \in M$ such that for each $h \in N''$, $\hat{x}_h \neq x$. By (WE-ii), $\hat{p}^x = 0$. Since $\widehat{CV}_{m'+1}(x; \mathbf{0}) > 0$, $(x, \hat{p}^x)\, \hat{P}_{m'+1}\, \mathbf{0}$. This is a contradiction since $\hat{x}_{m'+1} = 0$ and $(\hat{z}, \hat{p})$ is a Walrasian equilibrium. Thus, $x(1) \neq 0$.

Note that by Step 1, $x(1) \neq 0$ implies that agent $i(1)$ with $x_{i(1)} = x(1)$ uniquely exists. Thus, $x(1)$, $i(1)$, and $t(1)$ are well-defined.

Second, we show that $x(1) \neq x_1$. Suppose that $x(1) = x_1$. Then, by Step 3 and (ii) of $\hat{R}_1$, $\hat{p}^{x(1)} \equiv t(1) < b < \widehat{CV}_1(x_1; \mathbf{0})$, that is, $(x(1), \hat{p}^{x(1)})\, \hat{P}_1\, \mathbf{0}$. Thus, by $\hat{x}_1 = 0$, $\hat{x}_1 \notin D(\hat{R}_1, \hat{p})$. However, since $(\hat{z}, \hat{p})$ is a Walrasian equilibrium of $\widehat{E}$, this is a contradiction. Thus, $x(1) \neq x_1$.

Third, by $x(1) \equiv \hat{x}_{m'+1} \in D(\hat{R}_{m'+1}, \hat{p})$, $\hat{p}^{x(1)} \leq \widehat{CV}_{m'+1}(x(1); \mathbf{0}) < p^{x(1)}$.
Part II (Induction argument). Let $k \in \{2, \ldots, m'\}$. Assume that Claim 4.1 holds until $k-1$. Since $x(k-1) \in D(R_{i(k-1)}, p)$, $\hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p})$, and $\hat{p}^{x(k-1)} < p^{x(k-1)}$, Lemma 4.2-(ii) implies that $x(k) \equiv \hat{x}_{i(k-1)} \neq 0$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Note that by Step 1, $x(k) \neq 0$ implies that agent $i(k)$ with $x_{i(k)} = x(k)$ uniquely exists. Thus, $x(k)$, $i(k)$, and $t(k)$ are well-defined.

If $\hat{p}^{x(k)} > t(k) = CV_{i(k-1)}(x(k); z_0(k-1))$, then

$$(x(k-1), \hat{p}^{x(k-1)})\, R_{i(k-1)}\, z_0(k-1)\, P_{i(k-1)}\, (x(k), \hat{p}^{x(k)}),$$

contradicting $x(k) \equiv \hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p})$. Thus, $\hat{p}^{x(k)} \leq t(k)$.

We show $x(k) \neq x_1$. Suppose that $x(k) = x_1$. Then, by Step 3 and (ii) of $\hat{R}_1$, $\hat{p}^{x(k)} \leq t(k) < b < \widehat{CV}_1(x_1; \mathbf{0})$. Thus, $(x(k), \hat{p}^{x(k)})\, \hat{P}_1\, \mathbf{0}$. Then, by $\hat{x}_1 = 0$, $\hat{x}_1 \notin D(\hat{R}_1, \hat{p})$. However, since $(\hat{z}, \hat{p})$ is a Walrasian equilibrium of $\widehat{E}$, this is a contradiction. Thus, $x(k) \neq x_1$. □

[25] Note that $d_1 > 0$.

Step 5. *We derive a contradiction to conclude that no set of objects is weakly underdemanded at p for R.*

Note that by (i) and (ii) of $\hat{R}_1$, $d_1 > 0$. Since $(\hat{z}, \hat{p})$ is a minimum price Walrasian equilibrium for $(\hat{R}_1, \hat{R}_j, R_{N'\setminus\{1\}})$, Step 2 and Fact 4.5 imply that $\hat{p} \leq p^{M'}$. Note that

$$(\hat{x}_1, \hat{p}^{\hat{x}_1})\, \hat{R}_1\, \mathbf{0}\, \hat{I}_1\, (x_1, \widehat{CV}_1(x_1; \mathbf{0}))\, \hat{P}_1\, (x_1, p^{x_1}),$$

where the first preference relation follows from $\hat{x}_1 \in D(\hat{R}_1, \hat{p})$, the second from the definition of compensating valuation, and the third from (ii) of $\hat{R}_1$.

By Step 1 and 4, $x_1 \neq 0$ and $\hat{x}_1 \neq 0$. Since (i) of $\hat{R}_1$, by Remark 4.1, $(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P_1\, (x_1, p^{x_1})$. Then,

$$(\hat{x}_1, \hat{p}^{\hat{x}_1})\, P_1\, (x_1, p^{x_1})\, R_1\, (\hat{x}_1, p^{\hat{x}_1}),$$

where the second preference relation follows from $x_1 \in D(R_1, p)$. Thus, $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$.

By (i) and (ii) of $\hat{R}_1$, R_1 is the $(-d_1)$-truncation of $\hat{R}_1$ and $-d_1 \leq 0 \leq -\widehat{CV}_1(0; \hat{z}_1)$. Then, Lemma 4.1 implies that $\hat{p}$ is a Walrasian equilibrium price for $(R_{N'}, \hat{R}_j)$. However, by Step 2, $p^{M'}$ is the minimum Walrasian equilibrium price for $(R_{N'}, \hat{R}_j)$. Since $\hat{p} \leq p^{M'}$ and $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$, this is a contradiction. □

Proof of Corollary 4.1. Suppose that for each $i \in N$, $p^{x_i}_{\min}(R) > 0$. Then, for each $i \in N$, $x_i \neq 0$. Let $\bar{M} \equiv \{x_1, \ldots, x_n\}$. Then, $\#\bar{M} \equiv \#N$. Since $\bar{M} = \{i \in N : D(R_i, p) \cap \bar{M} \neq \emptyset\}$, $\bar{M}$ is weakly underdemanded at p for R. This is a contradiction to Theorem 4.1. □

Proof of Corollary 4.2. Let $x \in M$ be such that $p^x > 0$. Then, by (WE-ii) in Definition 4.1, there is $j_1 \in N$ such that $x_{j_1} = x$. By Theorem 4.1, the set $\{x\}$ is demanded at p by at lease two agents, and so, there is $j_2 \in N \setminus \{j_1\}$ such that $x \in D(R_{j_2}, p)$. If $x_{j_2} = 0$ or $p^{x_{j_2}} = 0$, then by letting $i_1 \equiv j_2$ and $i_2 \equiv j_1$, we obtain the desired conclusion. Thus, we assume that $x_{j_2} \neq 0$ and $p^{x_{j_2}} > 0$. Then, the set $\{x_{j_1}, x_{j_2}\}$ is demanded at p by at lease three agents, and so, there is $j_3 \in N \setminus \{j_1, j_2\}$ such that $x \in D(R_{j_3}, p)$. If $x_{j_3} = 0$ or $p^{x_{j_3}} = 0$, then by letting $i_1 \equiv j_3$, $i_2 \equiv j_2$, and $i_3 \equiv j_1$, we obtain the desired conclusion. Thus, we assume that $x_{j_3} \neq 0$ and $p^{x_{j_3}} > 0$. Repeating this argument inductively, there is a sequence $\{j_k\}_{k=1}^K$ of K distinct agents such that (a) $x_{j_K} = 0$ or $p^{x_{j_K}} = 0$, (b) $x_{j_1} = x$, and (c) for each $k \in \{2, \ldots, K\}$, $\{x_{j_k}, x_{j_{k-1}}\} \subseteq D(R_{j_k}, p)$. For each $k \in \{1, \ldots, K\}$, let $i_k \equiv j_{K-(k-1)}$. Then, the desired conclusion follows from (a), (b), and (c). □

A.2 Proofs for Section 5 (Fact 5.1 and Theorem 5.1)

Proof of Fact 5.1. Let $\mathcal{R} \subseteq \mathcal{R}^E$. Let g be a minimum price Walrasian rule on $\mathcal{R}^n$. By contradiction, suppose that there exist $R \in \mathcal{R}^n$, $\hat{N} \subseteq N$, and $\hat{R}_{\hat{N}} \in \mathcal{R}^{\#\hat{N}}$ such that for each $i \in \hat{N}$, $g_i(\hat{R}_{\hat{N}}, R_{-\hat{N}})\, P_i\, g_i(R)$. Let $z \equiv g(R)$ and $\hat{z} \equiv g(\hat{R}_{\hat{N}}, R_{-\hat{N}})$. Let p and $\hat{p}$ be the equilibrium prices associated with z and $\hat{z}$, respectively. Without loss of generality, let $\hat{N} = \{1, \ldots, \hat{n}\}$. Let $M^+ \equiv \{x \in M : 0 < p^x\}$ and $m^+ \equiv \#\hat{M}$. Note that, if $n > m$, then $n > m^+$, and if $n \leq m$, then by Corollary 4.1, $m^+ \leq n - 1 < n$.

In this paragraph, we show that for each $i \in \hat{N}$, $\hat{x}_i \neq 0$, and $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$. Let $i \in \hat{N}$. Note that $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})\, R_i\, \mathbf{0}$, where the first preference relation follows from $g_i(\hat{R}_{\hat{N}}, R_{-\hat{N}})\, P_i\, g_i(R)$, and the second from $x_i \in D(R_i, p)$. Thus, $\hat{x}_i \neq 0$. Also, note that $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (x_i, p^{x_i})\, R_i\, (\hat{x}_i, p^{\hat{x}_i})$, where the last preference relation also follows from $x_i \in D(R_i, p)$. Thus, $(\hat{x}_i, \hat{p}^{\hat{x}_i})\, P_i\, (\hat{x}_i, p^{\hat{x}_i})$ implies that $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$.

Note that, for each $i \in \hat{N}$, since $0 \le \hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$, $\hat{x}_i \in M^+$. Then, if $m^+ < \hat{n}$, more than m^+ agents receive the objects in M^+, which is a contradiction. Thus, assume that $m^+ \ge \hat{n}$. By Theorem 4.1, there is $i' \in N \setminus \hat{N}$ such that $D(R_{i'}, p) \cap \{\hat{x}_1, \ldots, \hat{x}_{\hat{n}}\} \ne \emptyset$. Without loss of generality, let $i' \equiv \hat{n}+1$. Note that $R_{\hat{n}+1}$ itself is its $d_{\hat{n}+1}$-truncation. Thus, by Lemma 4.2-(ii), $\hat{x}_{\hat{n}+1} \ne 0$, and $0 \le \hat{p}^{\hat{x}_{\hat{n}+1}} < p^{\hat{x}_{\hat{n}+1}}$. Thus, $\hat{x}_{\hat{n}+1} \in M^+$. Then, by Theorem 4.1, there is $i'' \in N \setminus \{1, \ldots, \hat{n}+1\}$ such that $D(R_{i''}, p) \cap \{\hat{x}_1, \ldots, \hat{x}_{\hat{n}+1}\} \ne \emptyset$. Without loss of generality, let $i'' \equiv \hat{n}+2$. Note that $R_{\hat{n}+2}$ itself is its $d_{\hat{n}+2}$-truncation. Thus, by Lemma 4.2-(ii), $\hat{x}_{\hat{n}+2} \ne 0$, and $0 \le \hat{p}^{\hat{x}_{\hat{n}+2}} < p^{\hat{x}_{\hat{n}+2}}$. Thus, $\hat{x}_{\hat{n}+2} \in M^+$. Repeat this argument $(m^+ - \hat{n} + 1)$ times. Then, more than m^+ agents receive the objects in M^+. This is a contradiction. □

Next, we prove Theorem 5.1. Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. Let f be a rule satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *nonnegative payment* on $\mathcal{R}^n$.

Part 1: Preliminary results (Proofs of Lemmas 5.1–5.4)

Proof of Lemma 5.1. By *nonnegative payment*, $f_i^t(R) \ge 0$. By *individual rationality*, $f_i^t(R) \le 0$. Thus, $f_i^t(R) = 0$. □

Proof of Lemma 5.2. By contradiction, suppose that for each $i \in N$, $f_i^x(R) \ne x$. Then, by $n > m$, there is $j \in N$ such that $f_j^x(R) = 0$. By Lemma 5.1, $f_j^t(R) = 0$. Let $\hat{z} \in Z$ be such that $\hat{z}_j \equiv (x, 0)$ and for each $i \in N \setminus \{j\}$, $\hat{z}_i \equiv f_i(R)$. Then, since $(x,0)\, P_i\, (0,0)$, $\hat{z}_j\, P_j\, f_j(R)$. Note that for each $i \in N \setminus \{j\}$, $\hat{z}_i\, I_i\, f_i(R)$, and $\sum_{i \in N} \hat{t}_i = \sum_{i \in N} f_i^t(R)$. Thus, $\hat{z}$ Pareto-dominates $f(R)$ at R, which contradicts *efficiency*. □

Proof of Lemma 5.3. Let $d \equiv t_j - CV_i(x_j; z_i)$, and let $\hat{z} \in Z$ be such that $\hat{z}_i \equiv (x_j, t_j - d)$, $\hat{z}_j \equiv (x_i, t_i + d)$, and for each $k \in N \setminus \{i, j\}$, $\hat{z}_k \equiv z_k$. Then, since $\hat{z}_i = (x_j, CV_i(x_j; z_i))$, $\hat{z}_i\, I_i\, z_i$. By (a) and $\hat{z}_j = (x_i, t_i + t_j - CV_i(x_j; z_i))$, $\hat{z}_j\, P_j\, (x_i, CV_j(x_i; z_j))\, I_j\, z_j$. Also, for each $k \in N \setminus \{i, j\}$, $\hat{z}_k\, I_k\, z_k$, and $\sum_{k \in N} \hat{t}_k = t_j - d + t_i + d + \sum_{k \ne i,j} t_k = \sum_{k \in N} t_k$. Thus $\hat{z}$ Pareto-dominates z at R. □

Proof of Lemma 5.4. First, we show that $f_i^x(\hat{R}_i, R_{-i}) = f_i^x(R)$. Suppose not. Let $x \equiv f_i^x(\hat{R}_i, R_{-i})$. By *strategy-proofness*, $f_i(\hat{R}_i, R_{-i})\, \hat{R}_i\, f_i(R)$. Thus, $f_i^t(\hat{R}_i, R_{-i}) \le \widehat{CV}_i(x; f_i(R))$. Since $\hat{R}_i \in \mathcal{R}_{NCV}(f_i(R))$, $\widehat{CV}_i(x; f_i(R)) < 0$. Thus, $f_i^t(\hat{R}_i, R_{-i}) < 0$, which contradicts *nonnegative payment*.

Next, we show that $f_i^t(\hat{R}_i, R_{-i}) = f_i^t(R)$. Suppose that $f_i^t(\hat{R}_i, R_{-i}) < f_i^t(R)$. (The opposite case can be treated symmetrically.) Then, $f_i(\hat{R}_i, R_{-i})\, P_i\, f_i(R)$, which contradicts *strategy-proofness*. □

Part 2: Proof of Proposition 5.1. (Proofs of Lemmas 5.5–5.7 and Proposition 5.1)

Proof of Lemma 5.5. Note that for each $i \in N$, $(x,0)\, P_i\, (0,0)$. Thus, $C^{m+1}(R, x) > 0$. By contradiction, suppose that $f_i^t(R) < C^{m+1}(R, x)$. Let $\hat{R}_i \in \mathcal{R}_{NCV}(f_i(R))$ be such that (i) $-\widehat{CV}_i(0; f_i(R)) < C^{m+1}(R, x) - f_i^t(R)$. Then, by Lemma 5.4, $f_i(\hat{R}_i, R_{-i}) = f_i(R)$.

Since $\#\{j \in N \setminus \{i\} : CV_j(x; \mathbf{0}) \ge C^{m+1}(R, x)\} \ge m$, there is $j \in N \setminus \{i\}$ such that $CV_j(x; \mathbf{0}) \ge C^{m+1}(R, x)$ and $f_j^x(\hat{R}_i, R_{-i}) = 0$. By Lemma 5.1, $f_j^t(\hat{R}_i, R_{-i}) = 0$. By (i) and Lemma 5.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(\hat{R}_i, R_{-i})$ at $(\hat{R}_i, R_{-i})$, which contradicts *efficiency*. □

Proof of Lemma 5.6. By contradiction, suppose that $CV_i(x; \mathbf{0}) < C^m(R, x)$. Then, by Lemma 5.5, $C^{m+1}(R, x) \le f_i^t(R)$. By *individual rationality*, $f_i^t(R) \le CV_i(x; \mathbf{0})$. Then, by

$CV_i(x;\mathbf{0}) \leq C^{m+1}(R,x)$, $f_i^t(R) = CV_i(x;\mathbf{0})$. Since $\#\{j \in N : CV_j(x;\mathbf{0}) \geq C^m(R,x)\} = m$, there is $j \in N \setminus \{i\}$ such that $CV_j(x;\mathbf{0}) \geq C^m(R,x)$ and $f_j^x(R) = 0$. By Lemma 5.1, $f_j^t(R) = 0$. Then, by $CV_i(x;\mathbf{0}) < C^m(R,x) \leq CV_j(x;\mathbf{0})$ and Lemma 5.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts *efficiency*. □

Proof of Lemma 5.7. (Figure A.2) By contradiction, suppose that $f_i^x(R) \neq x$. Then, by Lemma 5.2, there is $j \in N \setminus \{i\}$ such that $f_j^x(R) = x$. Since $f_j(R)\, R_j\, z_j$, $f_j^t(R) \leq CV_j(x;z_j) < CV_i(x;\mathbf{0})$. By $z \in Z^{IR}(R)$, for each $y \in M$, $CV_j(y;z_j) \leq CV_j(y;\mathbf{0})$. Let $\hat{R}_j \in \mathcal{R}_{NCV}(f_j(R))$ be such that (i) $-\widehat{CV}_j(0;f_j(R)) < CV_i(x;\mathbf{0}) - f_j^t(R)$, and (ii) for each $y \in M \setminus \{x\}$, $\widehat{CV}_j(y;\mathbf{0}) = CV_j(y;\mathbf{0})$. Then, by Lemma 5.4, $f_j(\hat{R}_j, R_{-j}) = f_j(R)$. Since $f_j^x(\hat{R}_j, R_{-j}) = x$, $f_i^x(\hat{R}_j, R_{-j}) \neq x$. Next, we show that $f_i^x(\hat{R}_j, R_{-j}) \notin M \setminus \{x\}$. Suppose that there is $y \in M \setminus \{x\}$ such that $f_i^x(\hat{R}_j, R_{-j}) = y$. By (ii), $C^m(\hat{R}_j, R_{-j}, y) = C^m(R,y)$. Since $CV_i(y;\mathbf{0}) < C^m(R,y)$, $CV_i(y;\mathbf{0}) < C^m(\hat{R}_j, R_{-j}, y)$, which contradicts Lemma 5.6. Thus, $f_i^x(\hat{R}_j, R_{-j}) = 0$. By Lemma 5.1, $f_i^t(\hat{R}_j, R_{-j}) = 0$. Then, by (i) and Lemma 5.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(\hat{R}_j, R_{-j})$ at $(\hat{R}_j, R_{-j})$, which contradicts *efficiency*. □

[Figure A.2 about here]

Proof of Proposition 5.1. We only show $f_1(R)\, R_1\, z_1$ since the case of any other agent can be treated in the same way. If $x_1 = 0$, then $z_1 = \mathbf{0}$, and so, by *individual rationality*, $f_1(R)\, R_1\, z_1$. Thus, we assume that $x_1 \neq 0$. Let $N^+ \equiv \{j \in N : x_j \neq 0\}$. Note that $\#N^+ = m$.

By contradiction, suppose that $z_1\, P_1\, f_1(R)$. We prove Claim 5.1 below by induction. (iv-$(k+1)$) of Claim 5.1 induces a contradiction by the finiteness of N^+.

Claim 5.1. *For each $k \geq 0$, there exist a set $N(k+1)$ of $k+1$ distinct agents, say $N(k+1) \equiv \{1, \ldots, k+1\}$, and $\hat{R}_{N(k+1)} \in \mathcal{R}^{k+1}$ such that*
(i-$(k+1)$) $z_{k+1}\, P_{k+1}\, f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})$,
(ii-$(k+1)$) for each $j \in N(k+1)$ and each $y \in M \setminus \{x_j\}$, $\widehat{CV}_j(y;\mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$,
(iii-$(k+1)$) $t_{k+1} < \widehat{CV}_{k+1}(x_{k+1};\mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$, and
(iv-$(k+1)$) $N(k+1) \subsetneq N^+$,
where $N(k) \equiv \{1, \ldots, k\}$.

Proof of Claim 5.1.
Step 1. Let $k = 0$ and $N(1) \equiv 1$. By $z_1\, P_1\, f_1(R)$, (i-1) holds, and so, $t_1 < CV_1(x_1; f_1(R))$. Note that for each $y \in M$, $C^n(R,y) > 0$. Thus, there is $\hat{R}_1 \in \mathcal{R}$ such that (ii-1): for each $y \in M \setminus \{x_1\}$, $\widehat{CV}_1(y;\mathbf{0}) < C^n(R,y)$, and (iii-1): $t_1 < \widehat{CV}_1(x_1;\mathbf{0}) < CV_1(x_1; f_1(R))$.

Note that $\{1\} \subseteq N^+$. Suppose that $\{1\} = N^+$. Since $\#N^+ = m$, $m = 1$. Thus, by $x_1 \neq 0$, for each $j \in N \setminus \{1\}$, $z_j = \mathbf{0}$. Since $z \in W(R)$, for each $j \in N \setminus \{1\}$, $z_j\, R_j\, z_1$, and so, $CV_j(x_1;\mathbf{0}) \leq t_1$. Thus, by (iii-1), $C^1(R_{-1}, x_1; z) \leq t_1 < \widehat{CV}_1(x_1;\mathbf{0})$. By *individual rationality*, for each $j \in N \setminus \{1\}$, $f_j(\hat{R}_1, R_{-1})\, R_j\, \mathbf{0} = z_j$. Since $z \in Z^{IR}(\hat{R}_1, R_{-1})$, Lemma 5.7 implies that $f_1^x(\hat{R}_1, R_{-1}) = x_1$. By *individual rationality*, $f_1^t(\hat{R}_1, R_{-1}) \leq \widehat{CV}_1(x_1;\mathbf{0})$. However, by (iii-1), $f_1^t(\hat{R}_1, R_{-1}) < CV_1(x_1; f_1(R))$. Thus, $f_1(\hat{R}_1, R_{-1})\, P_1\, f_1(R)$, which contradicts *strategy-proofness*. Therefore, (iv-1): $\{1\} \subsetneq N^+$.

Step 2 (Induction argument). Let $k \geq 1$. As induction hypothesis, we assume that there exist a set $N(k) \supseteq N(1)$ of k distinct agents, say $N(k) \equiv \{1, \ldots, k\}$, and $\hat{R}_{N(k)} \in \mathcal{R}^k$ such

that
(i-k) $z_k \, P_k \, f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$,
(ii-k) for each $j \in N(k)$ and each $y \in M \setminus \{x_j\}$, $\widehat{CV}_j(y;\mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$,
(iii-k) $t_k < \widehat{CV}_k(x_k;\mathbf{0}) < CV_k(x_k; f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$, and
(iv-k) $N(k) \subsetneq N^+$.

See Figure A.3 for an illustration of (i-$(k+1)$), (ii-$(k+1)$) and (iii-$(k+1)$) for $k=1$.

[Figure A.3 about here]

By (iv-k), $N^+ \setminus N(k) \neq \emptyset$. The proof consists of the following two steps.

Step 2-1. *There is $k' \in N^+ \setminus N(k)$ such that $z_{k'} \, P_{k'} \, f_{k'}(\hat{R}_{N(k)}, R_{-N(k)})$.*

Proof of Step 2-1. By contradiction, suppose that for each $j \in N^+ \setminus N(k)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)}) \, R_j \, z_j$.

First, we show that $f_k^x(\hat{R}_{N(k)}, R_{-N(k)}) = x_k$. By (ii-$k$), for each $y \in M \setminus \{x_k\}$,

$$\widehat{CV}_k(y;\mathbf{0}) < C^n(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}, y) = C^{n-1}(\hat{R}_{N(k)}, R_{-N(k)}, y) \leq C^m(\hat{R}_{N(k)}, R_{-N(k)}, y).$$

Let $\hat{z} \in Z$ be such that for each $j \in N \setminus N(k)$, $\hat{z}_j \equiv z_j$, and for each $j \in N(k)$, $\hat{z}_j \equiv \mathbf{0}$. Then, $\hat{z} \in Z^{IR}(\hat{R}_{N(k)}, R_{-N(k)})$. By the supposition of Step 2-1, for each $j \in N^+ \setminus N(k)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)}) \, R_j \, z_j \equiv \hat{z}_j$. By *individual rationality*, for each $j \in N(k) \cup (N \setminus N^+)$, $f_j(\hat{R}_{N(k)}, R_{-N(k)}) \; R_j \; \mathbf{0} = \hat{z}_j$.

Since $z \in W(R)$, for each $j \in N \setminus N(k)$, $CV_j(x_k;\hat{z}_j) = CV_j(x_k; z_j) \leq t_k$. By (ii-$k$), for each $j \in N(k) \setminus \{k\}$,

$$\widehat{CV}_j(x_k;\hat{z}_j) = \widehat{CV}_j(x_k;\mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, x_k) \leq C^n(R, x_k) \leq t_k.$$

Thus, by (iii-k), $C^1(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)}, x_k; \hat{z}) \leq t_k < \widehat{CV}_k(x_k;\mathbf{0})$.

Since the assumptions of Lemma 5.7 hold for the profile $(\hat{R}_{N(k)}, R_{-N(k)})$ as above, Lemma 5.7 implies that $f_k^x(\hat{R}_{N(k)}, R_{-N(k)}) = x_k$.

By *individual rationality*, $f_k^t(\hat{R}_{N(k)}, R_{-N(k)}) \leq \widehat{CV}_k(x_k;\mathbf{0})$. However, (iii-$k$) implies that $f_k^t(\hat{R}_{N(k)}, R_{-N(k)}) < CV_k(x_k; f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$.

Thus, $f_k(\hat{R}_{N(k)}, R_{-N(k)}) \, P_k \, f_k(\hat{R}_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$, contradicting *strategy-proofness*. □

Step 2-2. *We complete the proof of Claim 5.1.*

Proof of Step 2-2. Without loss of generality, let $k+1 \equiv k'$ and $N(k+1) \equiv N(k) \cup \{k+1\}$. Then, $N(k+1) \supsetneq N(k)$, and (i-$(k+1)$) follow from $z_{k'} \, P_{k'} \, f_{k'}(\hat{R}_{N(k)}, R_{-N(k)})$. By (i-$(k+1)$), $t_{k+1} < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$. Also, for each $y \in M$, $C^n(\hat{R}_{N(k)}, R_{-N(k)}, y) > 0$. Thus, there is $\hat{R}_{k+1} \in \mathcal{R}$ such that

$$t_{k+1} < \widehat{CV}_{k+1}(x_{k+1};\mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})),$$

and for each $y \in M \setminus \{x_{k+1}\}$, $\widehat{CV}_{k+1}(y;\mathbf{0}) < C^n(\hat{R}_{N(k)}, R_{-N(k)}, y)$. Let $\hat{R}_{N(k+1)} \equiv (\hat{R}_{N(k)}, \hat{R}_{k+1})$. Then, (ii-$(k+1)$) and (iii-$(k+1)$) follow from (ii-$k$).

By (iv-k) and $\{k+1\} \subseteq N^+$, $N(k+1) \subseteq N^+$.

Finally, we show (iv-$(k+1)$): $N(k+1) \subsetneq N^+$. Suppose that $N(k+1) = N^+$. Then, $\#N(k+1) = \#N^+ = m$. Thus, for each $j \in N \setminus N(k+1)$, $z_j = \mathbf{0}$.

By (ii-$(k+1)$), for each $y \in M \setminus \{x_{k+1}\}$,

$$\widehat{CV}_{k+1}(y; \mathbf{0}) < C^n(\hat{R}_{N(k)}, R_{-N(k)}, y) = C^{n-1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}, y) \leq C^m(\hat{R}_{N(k+1)}, R_{-N(k+1)}, y).$$

Let $\hat{z} \in Z$ be such that for each $j \in N$, $\hat{z}_j \equiv \mathbf{0}$. Then, $\hat{z} \in Z^{IR}(\hat{R}_{N(k+1)}, R_{-N(k+1)})$.

By *individual rationality*, for each $j \in N \setminus \{k+1\}$, $f_j(\hat{R}_{N(k+1)}, R_{-N(k+1)})\, R_j\, \mathbf{0} = \hat{z}_j$. Since $z \in W(R)$, for each $j \in N \setminus N(k+1)$, $CV_j(x_{k+1}; \hat{z}_j) = CV_j(x_{k+1}; z_j) \leq t_{k+1}$. By (ii-$(k+1)$), for each $j \in N(k+1) \setminus \{k+1\}$,

$$\widehat{CV}_j(x_{k+1}; \hat{z}_j) = \widehat{CV}_j(x_{k+1}; \mathbf{0}) < C^n(\hat{R}_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, x_{k+1}) \leq C^n(R, x_{k+1}) \leq t_{k+1}.$$

Thus, by (iii-$(k+1)$), $\widehat{CV}_{k+1}(x_{k+1}; \mathbf{0}) > t_{k+1} \geq C^1(\hat{R}_{N(k)}, R_{-N(k+1)}, x_{k+1}; \hat{z})$, and the assumptions of Lemma 5.7 hold for the profile $(\hat{R}_{N(k+1)}, R_{-N(k+1)})$. Lemma 5.7 implies that $f^x_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$.

By *individual rationality*, $f^t_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) \leq \widehat{CV}_{k+1}(x_{k+1}; \mathbf{0})$. However, by (iii-$(k+1)$), $f^t_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)}) < CV_{k+1}(x_{k+1}; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)}))$.

Thus, $f_{k+1}(\hat{R}_{N(k+1)}, R_{-N(k+1)})\; P_{k+1}\; f_{k+1}(\hat{R}_{N(k)}, R_{-N(k)})$, contradicting *strategy-proofness.* □

Part 3: Proofs of Lemmas 5.8–5.11.

Proof of Lemma 5.8. First, we show (a). Let $M' \subseteq M$. Since $z^* \in W_{\min}(R)$, it follows from Theorem 4.1 that (i) $\#\{i \in N : D(R_i, p) \subseteq M'\} \leq \#M'$ and (ii) $\#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} > \#M'$. Note that for each $i \in N'$, $D(\hat{R}_i, p) = L$ and for each $j \in N \setminus N'$, $D(\hat{R}_j, p) = D(R_j, p)$. Thus, for each $i \in N'$, $D(\hat{R}_i, p) \nsubseteq M'$ and $D(\hat{R}_i, p) \cap M' \neq \emptyset$. Then,

$$\#\{i \in N : D(\hat{R}_i, p) \subseteq M'\} \leq \#\{i \in N : D(R_i, p) \subseteq M'\} \leq \#M', \text{ and}$$
$$\#\{i \in N : D(\hat{R}_i, p) \cap M' \neq \emptyset\} \geq \#\{i \in N : D(R_i, p) \cap M' \neq \emptyset\} > \#M'.$$

That is, no set of objects is overdemanded nor weakly underdemanded at p for $\hat{R}$. Thus, (a) follows from Theorem 4.1. Then, (b) also follows from Proposition 5.1.

Finally, we show (c). Let $i \in N$. By contradiction, suppose that $f^x_i(\hat{R}) = 0$ and $0 \notin D(\hat{R}_i, p)$. Then, by Lemma 5.1, $z^*_i\, \hat{P}_i\, \mathbf{0} = f_i(\hat{R})$. This contradicts (b). □

Proof of (9-b) of Lemma 5.9. Let $N''_1 \equiv \{i \in N'' : f^x_i(\bar{R}) = 0\}$. We show that (9-1-b): $N''_1 \neq \emptyset$. Since $N'' \equiv N(R, p) \setminus N'$, $N'' \cup N' = N(R, p)$. Thus, $\#N'' + \#N' \geq \#N(R, p)$. By Lemma 5.8-(a), $z^* \in W_{\min}(\bar{R})$. Thus, by Theorem 4.1, there is no weakly underdemanded set at p for $\bar{R}$, and so, $\#N(R, p) \geq m + 1$. Therefore, $\#N'' + \#N' \geq m + 1$. By (9-ii), for each $j \in N'$, $f^x_j(\bar{R}) \neq 0$. Thus, at least one agent in N'' receives no object, that is, (9-1-b) holds.

Since $N''_1 \subseteq N(R, p)$, for each $i \in N''_1$, $D(R_i, p) \cap M \neq \emptyset$. Thus, by (9-1-b), we have (9-1-d): there is $i_1 \in N''_1$ such that $D(R_{i_1}, p) \cap M \neq \emptyset$.

Let $N(1) \equiv N''_1$ and $D_1 \equiv [\bigcup_{i \in N(1)} D(R_i, p)] \setminus \{0\}$. Given $k \geq 2$, let $N''_k \equiv \{j \in N'' \setminus N(k-1) : f^x_j(\bar{R}) \in D_{k-1}\}$, $N(k) \equiv N(k-1) \cup N''_k$, and $D_k \equiv [\bigcup_{j \in N''_k} D(R_j, p)] \setminus [\bigcup_{j \in N(k-1)} D(R_j, p)]$.

We introduce Claim 5.2 below to show (9-b) inductively. Note that Assumptions (9-$(k-1)$-b) and (9-$(k-1)$-d) of Claim 5.2 follow from (9-1-b) and (9-1-d) when $k=2$, that (9-k-b) implies $N(k) \supsetneq N(k-1)$, and that Assumptions except for (9-$(k-1)$-a*) hold recursively. Thus, for any $k \geq 2$, as long as (9-$(k-1)$-a*) holds, Claim 5.2 is applied and $N(k)$ increases as k increases. Since $N(k) \subseteq N''$, and N'' is finite,[26] there is $k \leq m$ such that (9-k-a*) does not hold. Let k be the first number that violates (9-k-a*) in this iteration.

By (9-k-b), for each $k' \in \{1, \ldots, k\}$, $N''_{k'} \neq \emptyset$. Since (9-k-a*) does not hold, there are $j_k \in N''_k$ and $j_{k+1} \in N'$ such that $f^x_{j_{k+1}}(\bar{R}) \in D(R_{j_k}, p)$. Then,

for each $k' \in \{1, \ldots, k-1\}$, there is $j_{k'} \in N''_{k'}$ such that $f^x_{j_{k'+1}}(\bar{R}) \in D(R_{j_{k'}}, p)$.

To show that the sequence $\{j_{k'}\}_{k'=1}^{k+1}$satisfies (iv) of (9-b), we prove

for each $k' \in \{1, \ldots, k\}$, $f^x_{j_{k'}}(\bar{R}) \in D(R_{j_{k'}}, p)$.

By $j_1 \in N''_1$, $f^x_{j_1}(\bar{R}) = 0$. Then, by Lemma 5.8-(c), $f^x_{j_1}(\bar{R}) \in D(R_{j_1}, p)$. Let $k' \in \{2, \ldots, k\}$. By contradiction, suppose that $f^x_{j_{k'}}(\bar{R}) \notin D(R_{j_{k'}}, p)$. Let $y \equiv f^x_{j_{k'}}(\bar{R})$. Then, by $x^*_{j_{k'}} \in D(R_{i_{k'}}, p)$, $z^*_{j_{k'}}\ P_{i_{k'}}\ (y, p^y)$. By Lemma 5.8-(b), $f_{j_{k'}}(\bar{R})\ R_{j_{k'}}\ z^*_{j_{k'}}$. Thus, $f_{j_{k'}}(\bar{R})\ R_{j_{k'}}\ z^*_{j_{k'}}\ P_{i_{k'}}\ (y, p^y)$, which implies $f^t_{j_{k'}}(\bar{R}) < p^y$. This contradicts (9-i) of Lemma 5.9.

Then, the sequence $\{j_{k'}\}_{k'=1}^{k+1}$ satisfies (i), (ii), (iii), and (iv) of (9-b). Thus, for the rest of the proof of (9-b), we prove Claim 5.2 below.

Claim 5.2. *Let $k \geq 2$. Assume that*
(9-$(k-1)$-a) for each $i \in N(k-2)$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$,[27]
(9-$(k-1)$-b) for each $k' \in \{1, \ldots, k-1\}$, $N''_{k'} \neq \emptyset$,
(9-$(k-1)$-c) for each $k' \in \{2, \ldots, k-1\}$, $\#N''_{k'} = \#D_{k'-1}$,[28]
(9-$(k-1)$-d) there is $i_{k-1} \in N''_{k-1}$ such that $D(R_{i_{k-1}}, p) \cap [M \setminus \bigcup_{k' \leq k-2} D_{k'}] \neq \emptyset$,[29] *and*
(9-$(k-1)$-a) for each $i \in N''_{k-1}$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$.*
Then,
(9-k-a) for each $i \in N(k-1)$ and each $j \in N'$, $f^x_j(\bar{R}) \notin D(R_i, p)$,
(9-k-b) for each $k' \in \{1, \ldots, k\}$, $N''_{k'} \neq \emptyset$,
(9-k-c) for each $k' \in \{2, \ldots, k\}$, $\#N''_{k'} = \#D_{k'-1}$, and
(9-k-d) there is $i_k \in N''_k$ such that $D(R_{i_k}, p) \cap [M \setminus \bigcup_{k' \leq k-1} D_{k'}] \neq \emptyset$.

Proof of Claim 5.2. First, (9-k-a) follows from (9-$(k-1)$-a) and (9-$(k-1)$-a*). By (9-$(k-1)$-d), there is $i_{k-1} \in N''_{k-1}$ such that $D(R_{i_{k-1}}, p) \cap [M \setminus \bigcup_{k' \leq k-2} D_{k'}] \neq \emptyset$. Thus, $D_k \neq \emptyset$. By Lemma 5.2, for each $x \in D_k$, there is $i(x) \in N$ such that $f^x_{i(x)}(\bar{R}) = x$. Note that, by (9-a), $i(x) \in N(R, p) \cup N'$. By (9-k-a) and the definition of $N(k-1)$, for each $x \in D_k$, $i(x) \in N'' \setminus N(k-1)$. Thus, $N''_k \neq \emptyset$. Then, (9-k-b) follows from (9-$(k-1)$-b).

Since $f^x(\bar{R}) \in X$, no two agents receive the same object *i.e.,* for each $x, y \in D_k$ with $x \neq y$, $i(x) \neq i(y)$. Thus, $\#N''_k = \#D_{k-1}$. Then, (9-k-c) also follows from (9-$(k-1)$-c).

Finally, we show (9-k-d). By contradiction, suppose that for each $i \in N''_k$, $D(R_i, p) \cap [M \setminus \bigcup_{k' \leq k-1} D_{k'}] = \emptyset$. See Figure A.4 for an illustration of proof of (9-k-d).

[Figure A.4 about here]

[26]By (9-ii) of Lemma 5.9 and feasibility of object allocation, it should be $\#N'' \leq m$.
[27]Define $N(0) = \emptyset$. When $k=2$, (9-$(k-1)$-a) holds vacantly.
[28]When $k=2$, (9-$(k-1)$-c) holds vacantly.
[29]When $k=2$, (9-$(k-1)$-d) requires that there is $i_1 \in N''_1$ such that $D(R_{i_1}, p) \cap M \neq \emptyset$.

Then,

$$\#\left\{j \in N : D(\bar{R}_j, p) \cap \left[M \setminus \bigcup_{k' \leq k-1} D_{k'}\right] \neq \emptyset\right\} = \#N' + \#N'' - \#N''_1 - \sum_{k'=2}^{k} \#N''_{k'}$$

$$= \#M - \sum_{k'=2}^{k} \#D_{k'-1}$$

$$= \#\left\{M \setminus \bigcup_{k' \leq k-1} D_{k'}\right\},$$

where the first equality follows from $\#\left\{j \in N : D(\bar{R}_j, p) \cap M \neq \emptyset\right\} = \#N' + \#N''$, and for each $k' \in \{1, \ldots, k\}$ and each $i \in N''_{k'}$, $D(R_i, p) \cap \left[M \setminus \bigcup_{k'' \leq k-1} D_{k''}\right] = \emptyset$, and the second from $\#N' + \#N'' - \#N''_1 = m$ and (9-k-c): for each $k' \in \{2, \ldots, k\}$, $\#N''_{k'} = \#D_{k'-1}$.

Therefore, the set $\left[M \setminus \bigcup_{k' \leq k-1} D_{k'}\right]$ is weakly underdemanded at p for $\bar{R}$. However, by Lemma 5.8-(a), $z^* \in W_{\min}(\bar{R})$, and so, by Theorem 4.1, there is no weakly underdemanded set at p for $\bar{R}$. This is a contradiction. □

Proof of Lemma 5.10. Suppose that $f^x_j(R) = 0$. By Lemma 5.1, $f^t_j(R) = 0$. By assumption (10-i), $-CV_i(0; f_i(R)) < CV_j(x; \mathbf{0}) - f^t_i(R)$. Then, by Lemma 5.3, there is $\hat{z} \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts *efficiency*. □

Proof of Lemma 5.11. Let $\hat{R} \equiv (\hat{R}_{N'}, R_{-N'})$. Without loss of generality, let $N' \equiv \{1, 2, \ldots, n'\}$. We only show that if $f^x_1(\hat{R}) = x \in M$, $f^t_1(\hat{R}) \geq p^x$ since we can treat similarly the other agents in N'. Let $f^x_1(\hat{R}) \equiv x \in M$. By contradiction, suppose that $f^t_1(\hat{R}) < p^x$. Let $N'' \equiv N(R, p) \setminus N'$.

Case 1. $\#N' \geq m + 1$.

Since $f^t_1(\hat{R}) < p^x$, there is $\bar{R}_1 \in \mathcal{R}_{NCV}(f_1(\hat{R}))$ such that (ii) $-\overline{CV}_1(0; f_1(\hat{R})) < p^x - f^t_1(\hat{R})$. Then, by Lemma 5.4, $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$. Note that for each $j \in N' \setminus \{1\}$,

$$-\overline{CV}_1(0; f_1(\hat{R})) < p^x - f^t_1(\hat{R}) = \widehat{CV}_j(x; \mathbf{0}) - f^t_1(\hat{R}),$$

where the inequality follows from (ii) and the equality from $\hat{R}_j \in \mathcal{R}^I(z^*)$. Thus, by Lemma 5.10, for each $j \in N' \setminus \{1\}$, $f^x_j(\bar{R}_1, \hat{R}_{-1}) \neq 0$. However, since $\#N' \geq m + 1$, this is a contradiction.

Case 2. $\#N' \leq m$.

First, we show the following step.

Step 1. *Let $S \subseteq N'$, $\bar{R}_S \in \mathcal{R}^I(z^*)^{\#S}$, and $\bar{R} \equiv (\bar{R}_S, \hat{R}_{-S})$. For each $i \in N'$, let $\overline{x}_i \equiv f^x_i(\bar{R})$. Assume that*
(11-1-i) for each $i \in N'$, $\overline{x}_i \neq 0$,
(11-1-ii) for each $i \in S$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$,
(11-1-iii) there is $j \in S$ such that $f^t_j(\bar{R}) < p^{\overline{x}_j}$, and
(11-1-iv) there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m + 1$, (ii*) $f^x_{i_1}(\bar{R}) = 0$, (iii*) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv*) for each $k \in \{1, \ldots, K-1\}$, $\{f^x_{i_k}(\bar{R}), f^x_{i_{k+1}}(\bar{R})\} \subseteq D(\bar{R}_{i_k}, p)$.*
Then, (11-a) $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$, and (11-b) $i_K \notin S$.

Proof of Step 1.

Proof of (11-a). If $i_K = j$, $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$ follows from (11-1-iii). Thus, let $i_K \neq j$. By Lemma 5.8-(b), $f^t_{i_K}(\bar{R}) \leq p^{\overline{x}_{i_K}}$. By contradiction, suppose that $f^t_{i_K}(\bar{R}) = p^{\overline{x}_{i_K}}$. Let $z' \in Z$ be such that

$$\begin{aligned}
&z'_j \equiv (0, \overline{CV}_j(0; f_j(\bar{R}))),\\
&z'_{i_K} \equiv (\overline{x}_j, f^t_j(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R}))),\\
&\text{for each } k \in \{1, \ldots, K-1\},\ z'_{i_k} \equiv f_{i_{k+1}}(\bar{R}), \text{ and}\\
&\text{for each } i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\}),\ z'_h \equiv f_h(\bar{R}).
\end{aligned}$$

See Figure A.5 for the illustration of z'.

[Figure A.5 about here]

We show that z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$.

By the definition of $\overline{CV}_j(0; f_j(\bar{R}))$, $z'_j \, \bar{I}_j \, f_j(\bar{R})$.

Note that

$$z'_{i_K} \, \bar{P}_{i_K} \, (\overline{x}_j, p^{\overline{x}_j}) \, \bar{I}_{i_K} \, f_{i_K}(\bar{R}),$$

where the first preference relation follows from $z'_{i_K} \equiv (\overline{x}_j, f^t_j(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R})))$, (11-1-iii): $f^t_j(\bar{R}) < p^{\overline{x}_j}$, and (11-1-ii): $-\overline{CV}_j(0; f_j(\bar{R})) < p^{\overline{x}_j} - f^t_j(\bar{R})$, and the indifference relation from $f^t_{i_K}(\bar{R}) = p^{\overline{x}_{i_K}}$ and $i_K \in N'$, which implies $\bar{R}_{i_K} \in \mathcal{R}^I(z^*)$.

Lemma 5.8-(b) and (11-i) imply that for each $k \in \{1, \ldots, K-1\}$, $f^t_{i_k}(\bar{R}) = p^{x_{i_k}}$. Thus, by (11-1-iv)-(iv*), for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} = f_{i_{k+1}}(\bar{R}) \, \bar{I}_{i_k} \, f_{i_k}(\bar{R})$.

For each $i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\})$, by $z'_i \equiv f_i(\bar{R})$, $z'_i \, \bar{I}_i \, f_i(\bar{R})$.

Note that

$$\begin{aligned}
\sum_{i \in N} t'_i &= \overline{CV}_j(0; f_j(\bar{R})) + f^t_j(\bar{R}) - \overline{CV}_j(0; f_j(\bar{R})) + \sum_{k=1}^{K-1} f^t_{i_{k+1}}(\bar{R}) + \sum_{i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\})} f^t_i(\bar{R})\\
&= f^t_j(\bar{R}) + \sum_{k=2}^{K} f^t_{i_k}(\bar{R}) + \sum_{i \in N \setminus (\{i_k\}_{k=1}^K \cup \{j\})} f^t_i(\bar{R})\\
&= \sum_{i \in N} f^t_i(\bar{R}),
\end{aligned}$$

where the last equality follows from (11-1-iv)-(ii*): $f^t_{i_1}(\bar{R}) = 0$. Thus, z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$, which contradicts *efficiency*. □

Proof of (11-b). By contradiction, suppose that $i_K \in S$. By (11-1-i) and (11-1-iv)-(iii*), $\overline{x}_{i_K} \neq 0$. By Step 1-(11-a), $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$. Let $z' \in Z$ be such that

$$\begin{aligned}
&z'_{i_K} \equiv (0, \overline{CV}_{i_K}(0; f_{i_K}(\bar{R}))),\\
&z'_{i_{K-1}} \equiv (\overline{x}_{i_K}, f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})))\\
&\text{for each } k \in \{1, \ldots, K-2\},\ z'_{i_k} \equiv f_{i_{k+1}}(\bar{R}), \text{ and}\\
&\text{for each } i \in N \setminus \{i_k\}_{k=1}^K,\ z'_i \equiv f_i(\bar{R}).
\end{aligned}$$

See Figure A.6 for the illustration of z'.

[Figure A.6 about here]

We show that z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$.

By the definition of $\overline{CV}_j(0; f_j(\bar{R}))$, $z'_{i_K}\, \bar{I}_{I_K}\, f_{i_K}(\bar{R})$.

Lemma 5.8-(b) and (11-i) imply that for each $k \in \{1, \ldots, K-1\}$, $f^t_{i_k}(\bar{R}) = p^{x_{i_k}}$. By (11-1-iv)-(iv*), for each $k \in \{1, \ldots, K-2\}$, $z'_{i_k} = f_{i_{k+1}}(\bar{R})\, \bar{I}_{i_k}\, f_{i_k}(\bar{R})$.

Note that

$$z'_{i_{K-1}}\, \bar{P}_{i_{K-1}}\, (\overline{x}_{i_K}, p^{\overline{x}_{i_K}})\, \bar{I}_{i_{K-1}}\, (\overline{x}_{i_{K-1}}, p^{\overline{x}_{i_{K-1}}})\, \bar{I}_{i_{K-1}}\, f_{i_{K-1}}(\bar{R}),$$

where the strict preference relation follows from $i_K \in S$, $z'_{i_{K-1}} = (\overline{x}_{i_K}, f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})))$, (11-a): $f^t_{i_K}(\bar{R}) < p^{\overline{x}_{i_K}}$, and (11-1-ii): $-\overline{CV}_{i_K}(0; f_{i_K}(\bar{R})) < p^{\overline{x}_{i_K}} - f^t_{i_K}(\bar{R})$, the first indifference relation from (11-1-iv)-(iv*): $\{\overline{x}_{i_{K-1}}, \overline{x}_{i_K}\} \subseteq D(\bar{R}_{i_{K-1}}, p)$, and the second from $f^t_{i_{K-1}}(\bar{R}) = p^{x_{i_{K-1}}}$.

For each $i \in N \setminus \{i_k\}_{k=1}^K$, by $z'_i = f_i(\bar{R})$, $z'_i\, \bar{I}_i\, f_i(\bar{R})$.

Note that

$$\begin{aligned}
\sum_{i \in N} t'_i &= \overline{CV}_{i_K}(0; f_{i_k}(\bar{R})) + f^t_{i_K}(\bar{R}) - \overline{CV}_{i_K}(0; f_{i_K}(\bar{R})) + \sum_{k=1}^{K-2} f^t_{i_{k+1}}(\bar{R}) + \sum_{i \in N \setminus (\{i_k\}_{k=1}^K)} f^t_i(\bar{R}) \\
&= f^t_{i_K}(\bar{R}) + \sum_{k=2}^{K-1} f^t_{i_k}(\bar{R}) + \sum_{i \in N \setminus (\{i_k\}_{k=1}^K)} f^t_i(\bar{R}) \\
&= \sum_{i \in N} f^t_i(\bar{R}),
\end{aligned}$$

where the last equality follows from (11-1-iv)-(ii*): $f^t_{i_1}(\bar{R}) = 0$. Thus, z' Pareto-dominates $f(\bar{R})$ at $\bar{R}$, which contradicts *efficiency*. □

Step 2. *We derive a contradiction to conclude that $f^t_1(\hat{R}) \geq p^x$.*

Since $f^t_1(\hat{R}) < p^x$, there is $\bar{R}_1 \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{NCV}(f_1(\hat{R}))$ such that

(11-1-a) for each $z_1 \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_1(0; z_1) < p^y - t$.

Then, by $\bar{R}_1 \in \mathcal{R}_{NCV}(f_1(\hat{R}))$ and Lemma 5.4, $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$. Thus,

(11-1-b) $f^x_1(\bar{R}_1, \hat{R}_{-1}) = x \in M$ and $f^t_1(\bar{R}_1, \hat{R}_{-1}) < p^x$.

Note that $\{1\} \subseteq N'$. Suppose that $\{1\} = N'$. Since $f_1(\bar{R}_1, \hat{R}_{-1}) = f_1(\hat{R})$ and $f^x_1(\hat{R}) = x \neq 0$, $f^x_1(\bar{R}_1, \hat{R}_{-1}) = x \neq 0$. Then, by (11-i) of Lemma 5.11, it follows from (9-b) of Lemma 5.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m+1$, (ii) $f^x_{i_1}(\bar{R}_1, \hat{R}_{-1}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K-1\}$, $\{f^x_{i_k}(\bar{R}_1, \hat{R}_{-1}), f^x_{i_{k+1}}(\bar{R}_1, \hat{R}_{-1})\} \subseteq D(\hat{R}_{i_k}, p)$. Then, by Step 1 (11-b), $i_K \notin \{1\}$. Since $\{1\} = N'$, $i_K \notin N'$, which contradicts (iii): $i_K \in N'$. Thus, if $\{1\} = N'$, we obtain a contradiction.

Therefore, we assume that

(11-1-c) $\{1\} \subsetneq N'$.

Induction argument:

Let $s \geq 1$ and $N(1) \equiv \{1\}$. As induction hypothesis, we assume that there exist a set $N(s) \supseteq N(1)$ of s distinct agents and $\bar{R}_{N(s)} \in \mathcal{R}^I(z^*)^s$ such that

(11-s-a) for each $i \in N(s)$ and each $z_i \equiv (y,t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$,

(11-s-b) for some $j \in N(s)$, $f_j^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \equiv x' \in M$ and $f_j^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x'}$, and

(11-s-c) $N(s) \subsetneq N'$.

Note that (11-s-a), (11-s-b), and (11-s-c) follow from (11-1-a), (11-1-b), and (11-1-c) if $s = 1$.

We show that there exist a set $N(s+1) \supsetneq N(s)$ of $s+1$ distinct agents and $\bar{R}_{N(s+1)} \in \mathcal{R}^I(z^*)^{s+1}$ such that

(11-$(s+1)$-a) for each $i \in N(s+1)$ and each $z_i \equiv (y,t) \in M \times \mathbb{R}$ with $t < p^y$,

$$-\overline{CV}_i(0; z_i) < p^y - t, \text{ and}$$

(11-$(s+1)$-b) for some $j' \in N(s+1)$,

$$f_{j'}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \equiv x'' \in M \text{ and } f_{j'}^t(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) < p^{x''}.$$

First, we show (11-$(s+1)$-a). Since $(\bar{R}_{N(s)}, \hat{R}_{-N' \setminus N(s)}) \in \mathcal{R}^I(z^*)^{\#N'}$, (11-$s$-b) and Lemma 5.10 imply that

(B-1) for each $i \in N'$, $f_i^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \neq 0$.

Then, by (11-i) of Lemma 5.11, it follows from (9-b) of Lemma 5.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m+1$, (ii) $f_{i_1}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) = 0$, (iii) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1, \ldots, K-1\}$, $\{f_{i_k}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}), f_{i_{k+1}}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)})\} \subseteq D(\hat{R}_{i_k}, p)$. Let $x_{i_K} \equiv f_{i_K}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)})$.
Then, by Step 1-(11-a),

(B-2) $f_{i_K}^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x_{i_K}}$.

Also, by Step 1-(11-b),

(B-3) $i_K \in N' \setminus N(s)$.

Next, let $j' \equiv i_K$ and $N(s+1) \equiv N(s) \cup \{j'\}$. Then, by (B-3), $N(s+1) \supsetneq N(s)$. Also, (B-1) and (B-2) imply that $f_{i_K}^x(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) \neq 0$ and $f_{j'}^t(\bar{R}_{N(s)}, \hat{R}_{-N(s)}) < p^{x_{j'}}$. Thus, there is $\bar{R}_{j'} \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{NCV}(f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)}))$ such that

$$\text{for each } z_{j'} \equiv (y,t) \in M \times \mathbb{R} \text{ with } t < p^y,\ -\overline{CV}_{j'}(0; z_{j'}) < p^y - t,$$

Thus, (11-$(s+1)$-a) follows from (11-s-a).

Next, we show (11-$(s+1)$-b). By $\bar{R}_{j'} \in \mathcal{R}_{NCV}(f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)}))$ and Lemma 5.4, $f_{j'}(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) = f_{j'}(\bar{R}_{N(s)}, \hat{R}_{-N(s)})$. Then, by (B-1), $f_{j'}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \neq 0$. By (B-2), $f_{j'}^t(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) < p^{x_{j'}}$. Thus, (11-$(s+1)$-b) holds.

Since $N(s) \subsetneq N'$ and $j' \in N'$, $N(s+1) \subseteq N'$. Suppose that $N(s+1) = N'$. Since $(\bar{R}_{N(s+1)}, \hat{R}_{-N' \setminus N(s+1)}) \in \mathcal{R}^I(z^*)^{\#N'}$, (11-$(s+1)$-b) and Lemma 5.10 imply that

(B-4) for each $i \in N'$, $f_i^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) \neq 0$.

Then, by (11-i) of Lemma 5.11, it follows from (9-b) of Lemma 5.9 that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K \leq m+1$, (ii) $f_{i_1}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}) = 0$, (iii) for each $k \in \{1,\ldots,K-1\}$, $i_k \in N''$, and $i_K \in N'$, and (iv) for each $k \in \{1,\ldots,K-1\}$, $\{f_{i_k}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)}), f_{i_{k+1}}^x(\bar{R}_{N(s+1)}, \hat{R}_{-N(s+1)})\} \subseteq D(\hat{R}_{i_k}, p)$.

Then, by Step 1-(11-b), $i_K \notin N(s+1)$. Since $N(s+1) = N'$, $i_K \notin N'$, which contradicts (iii): $i_K \in N'$. Thus, if $N(s+1) = N'$, we obtain a contradiction.

If $N(s+1) \subsetneq N'$, we obtain a contradiction by repeating the induction argument $(\#N' - \#N(s+1))$ times. □

Part 4: Proof of Theorem 5.1.

Proof of Theorem 5.1. Let $R \in \mathcal{R}^n$, $z^* \in W_{\min}(R)$, and p be the price vector associated with z^*. By Lemma 5.5, for each $\bar{R} \in \mathcal{R}^I(z^*)^n$, each $i \in N$, and each $x \in M$, if $f_i^x(\bar{R}) = x$, then, $f_i^t(\bar{R}) \geq p^x$. Next, we prove the following claim.

Claim 5.3. *Let $k \in \{1,\ldots,n\}$ and $N_k \subseteq N$ be such that $\#N_k = k$. Then, for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\#N\setminus N_k}$, each $i \in N$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$.*

Proof of Claim 5.3. We prove Claim 5.3 by induction on k. Let $k = 1$. Let $N_1 \subseteq N$ with $\#N_1 = 1$. Let $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\#N\setminus N_1}$, $i \in N_1$, and $x \in M$ be such that $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$. Suppose that $f_i^t(R_{N_1}, \bar{R}_{-N_1}) < p^x$. Let $\bar{R}_i \in \mathcal{R}^I(z^*)$ and $\hat{x} \equiv f_i^x(\bar{R})$. Then, since $f_i^t(\bar{R}) \geq p^{\hat{x}}$, $f_i(R_{N_1}, \bar{R}_{-N_1})\, \bar{P}_i\, f_i(\bar{R})$, which contradicts *strategy-proofness*. Thus, for each $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\#N\setminus N_1}$, each $i \in N_1$, and each $x \in M$, if $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$, then, $f_i^t(R_{N_1}, \bar{R}_{-N_1}) \geq p^x$. Then, it follows from Lemma 5.11 that for each $\bar{R}_{-N_1} \in \mathcal{R}^I(z^*)^{\#N\setminus N_1}$, each $i \in N \setminus N_1$, and each $x \in M$, if $f_i^x(R_{N_1}, \bar{R}_{-N_1}) = x$, then, $f_i^t(R_{N_1}, \bar{R}_{-N_1}) \geq p^x$.

Let $k \in \{2,\ldots,n\}$. As induction hypothesis, we assume that

C: *for each $N_{k-1} \subseteq N$ with $\#N_{k-1} = k-1$, each $\bar{R}_{-N_{k-1}} \in \mathcal{R}^I(z^*)^{\#N\setminus N_{k-1}}$, each $i \in N$, and each $x \in M$, if $f_i^x(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) = x$, then, $f_i^t(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) \geq p^x$.*

Let $N_k \subseteq N$ be such that $\#N_k = k$. Let $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\#N\setminus N_k}$, $i \in N_k$ and $x \in M$ be such that $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$. Suppose that $f_i^t(R_{N_k}, \bar{R}_{-N_k}) < p^x$. Let $N_{k-1} \equiv N_k \setminus \{i\}$. Let $\hat{x} \equiv f_i^x(R_{N_{k-1}}, \bar{R}_{-N_{k-1}})$. Then, by induction hypothesis (C), $f_i^t(R_{N_{k-1}}, \bar{R}_{-N_{k-1}}) \geq p^{\hat{x}}$. Thus, $f_i(R_{N_k}, \bar{R}_{-N_k})\, \bar{P}_i\, f_i(R_{N_{k-1}}, \bar{R}_{-N_{k-1}})$, which contradicts *strategy-proofness*. Thus, for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\#N\setminus N_k}$, each $i \in N_k$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$. Then, it follows from Lemma 5.11 that for each $\bar{R}_{-N_k} \in \mathcal{R}^I(z^*)^{\#N\setminus N_k}$, each $i \in N \setminus N_k$, and each $x \in M$, if $f_i^x(R_{N_k}, \bar{R}_{-N_k}) = x$, then, $f_i^t(R_{N_k}, \bar{R}_{-N_k}) \geq p^x$. □

By Claim 5.3, for each $i \in N$ and each $x \in M$, if $f_i^x(R) = x$, then, $f_i^t(R) \geq p^x$. By Proposition 5.1, for each $i \in N$, $f_i(R)\, R_i\, z_i$. Thus, for each $i \in N$ and each $x \in M$, if $CV_i(x; z_i) < p^x$, $f_i^x(R) \neq x$. Therefore, for each $i \in N$, $f_i(R) \in B(p)$ and $f_i^x(R) \in D(R_i, p)$. Thus, $f(R)$ satisfies (WE-i) in Definition 4.1. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $x \in M$, $p^x > 0$. By Lemma 5.3, for each $x \in M$, there is $i \in N$ such that $f_i^x(R) = x$. Thus, $f(R)$ also satisfies (WE-ii) in Definition 4.1. Since p is the minimum Walrasian equilibrium price for R, we conclude that $f(R) \in W_{\min}(R)$. □

A.3 Proofs for Section 6 (Proposition 6.1)

Proof of Proposition 6.1. Let $\mathcal{R} \subseteq \mathcal{R}^E$ and $R \in \mathcal{R}^n$. Consider a simultaneous ascending (SA) auction defined in Section 6. By the definition of the SA auction, the price path $p(t)$

generated by the SA auction is nondecreasing with respect to time t. Next, for each $x \in M$, let $\hat{p}^x > C^1(R, x)$. Then, each agent demands only the null object at the price vector $\hat{p}$, that is, no overdemanded set exists at $\hat{p}$. Thus, the price path $p(\cdot)$ is bounded above, that is, for each $t \in \mathbb{R}_+$, $p(t) \leq \hat{p}$. Note that the prices are raised at a speed at least $d > 0$. Thus, there is a price vector p^* such that the price path $p(\cdot)$ generated by the SA auction converges to the price vector p^* in a finite time.

Let T be the final time of the SA auction. We show that the final price $p(T)$ is a minimum Walrasian equilibrium price for R. By the definition of SA auctions, no overdemanded set exists at the price $p(T)$. If no weakly underdemanded set exists at $p(T)$, then the desired conclusion follows from Theorem 4.1. Thus, we show that no weakly underdemanded set exists at $p(T)$. The proof consists of the following two steps.

Step 1. *Let $t' \in (0, T]$. Assume that there is a set M' of objects that is weakly underdemanded at $p(t')$. Let $N' \equiv \{i \in N : D(R_i, p(t')) \cap M' \neq \emptyset\}$. Then, (6-a) $\#N' \geq 2$, and (6-b) there exist $t'' \in (0, t')$ and $M'' \subsetneq M'$ such that $N'' \equiv \{i \in N : D(R_i, p(t'')) \cap M'' \neq \emptyset\} \subsetneq N'$ and M'' is underdemanded at $p(t'')$.*

Proof of Step 1. Since M' is weakly underdemanded at $p(t')$, for each $x \in M'$, $p^x(t') > 0$ and $\#N' \leq \#M'$. For each $i \in N$, let $z'_i \equiv (x'_i, t'_i) \in D(R_i, p(t'))$. Note that for each $i \in N \setminus N'$ and each $x \in M'$, $CV_i(x; z'_i) < p^x(t')$. For each $x \in M'$, let $q^x \equiv \max\{\{CV_j(x; z'_j) : j \in N \setminus N'\} \cup \{0\}\}$. Let $e > 0$ be such that for each $x \in M'$, $q^x < p^x(t') - e \equiv p^x$. Let $t'' \equiv \max\{t \in \mathbb{R}_+ : \text{for some } x \in M', p^x(t) \leq p^x\}$. Then, there is $x' \in M'$ such that $dp^{x'}(t'')/dt > 0$ and $p^{x'}(t'') = p^{x'}$. Since $dp^{x'}(t'')/dt > 0$, there is a minimal overdemanded set $\hat{M}$ at $p(t'')$ including x'. See Figure A.7 for an illustration.

[Figure A.7 about here]

Let $\hat{M}' \equiv \hat{M} \cap M'$. Since $x' \in M'$, $\hat{M}' \neq \emptyset$. Let

$$\hat{N}' \equiv \{i \in N' : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset \text{ and } D(R_i, p(t'')) \subseteq \hat{M}\}.$$

We show that $\#\hat{N}' > \#\hat{M}'$. If $\hat{M} \subseteq M'$, then $\hat{M}' = \hat{M}$ and for each $i \in \hat{N}'$, $D(R_i, p(t'')) \subseteq \hat{M}'$. Since $\hat{M}$ is an overdemanded set at $p(t'')$, the desired conclusion holds. Thus, we assume that $\hat{M} \not\subseteq M'$. Let $\hat{M}'' \equiv \hat{M} \setminus M'$ and $\hat{N}'' \equiv \{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\}$. Then,

$$\begin{aligned}
&\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\} \\
=&\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}''\} \cup \{i \in N : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset \text{ and } D(R_i, p(t'')) \subseteq \hat{M}\} \\
=&\hat{N}'' \cup \hat{N}',
\end{aligned}$$

where the first equality follows from $\hat{M}'' \cup \hat{M}' = \hat{M}$ and $\hat{M}'' \cap \hat{M}' = \emptyset$, and the second from $\{i \in N : D(R_i, p(t'')) \cap \hat{M}' \neq \emptyset\} \subseteq N'$. Note that for each $x \in M'$, $q^x < p^x \leq p^x(t'')$. Thus, for each $i \in N \setminus N'$ and each $x \in M'$,

$$(x'_i, p^{x'_i}(t'')) \, R_i \, (x'_i, p^{x'_i}(t')) \, R_i \, (x, q^x) \, P_i \, (x, p^x(t'')).$$

Since $\hat{M}' \subseteq M'$, for each $i \in N \setminus N'$, $D(R_i, p(t'')) \cap \hat{M}' = \emptyset$. Thus, $\hat{N}'' \cap \hat{N}' = \emptyset$. Then,

$$\begin{aligned}
\#\hat{N}'' + \#\hat{N}' &= \#\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\} \\
&> \#\hat{M} \quad (\hat{M} \text{ is an overdemanded set at } p(t'')) \\
&= \#\hat{M}'' + \#\hat{M}'.
\end{aligned}$$

Note that $\hat{M}'' \subsetneq \hat{M}$. Since $\hat{M}$ is a minimal overdemanded set at $p(t'')$, $\hat{M}''$ is not overdemanded at $p(t'')$, and so, $\#\hat{N}'' \leq \#\hat{M}''$. This implies that $\#\hat{N}' > \#\hat{M}'$.

We show (6-a). Since $\hat{M}' \neq \emptyset$, $1 \leq \#\hat{M}'$. By $\#\hat{N}' > \#\hat{M}'$ and $\hat{N}' \subseteq N'$, we have $1 \leq \#\hat{M}' < \#\hat{N}' \leq \#N'$, and thus, $\#N' \geq 2$.

Next, we show (6-b). Let $M'' \equiv M' \setminus \hat{M}'$. Since $\hat{M}' \subsetneq M'$,[30] $M'' \neq \emptyset$. By $\hat{M}' \neq \emptyset$, $M'' \subsetneq M'$. First, we show that $N'' \subseteq N' \setminus \hat{N}'$, that is, for each $i \in N''$, $i \in N'$ and $i \notin \hat{N}'$. Let $i \in N''$. Then, $D(R_i, p(t'')) \cap M'' \neq \emptyset$. Since for each $x \in M'$, $q^x < p^x(t'')$ and $M'' \subseteq M'$, for each $j \in N \setminus N'$, $D(R_j, p(t'')) \cap M' = \emptyset$. This implies $i \in N'$. Since $\hat{M}' = M' \cap \hat{M}$ implies $M'' = M' \setminus \hat{M}$, $D(R_i, p(t'')) \cap M'' \neq \emptyset$ implies $D(R_i, p(t'')) \setminus \hat{M} \neq \emptyset$. Since $\hat{N}' \subseteq \{j \in N : D(R_j, p(t'')) \subseteq \hat{M}\}$, this implies $i \notin \hat{N}'$. Thus, $N'' \subseteq N' \setminus \hat{N}'$.

Since $\#\hat{N}' > \#\hat{M}' \geq 1$, $\#\hat{N}' \geq 2$, and so, $N'' \subsetneq N'$. Finally, it follows from the inequalities below that M'' is underdemanded at $p(t'')$.

$$\begin{aligned} \#N'' &\leq \#N' - \#\hat{N}' && \text{by } \hat{N}' \subseteq N' \\ &< \#N' - \#\hat{M}' && \text{by } \#\hat{N}' > \#\hat{M}' \\ &\leq \#M' - \#\hat{M}' && \text{by } \#N' \leq \#M' \\ &= \#M''. \end{aligned}$$

□

Step 2. *There is no weakly underdemanded set at $p(T)$.*

Proof of Step 2. By contradiction, suppose that there is a set M_1 of objects that is weakly underdemanded at $p(T)$. Let $N_1 \equiv \{i \in N : D(R_i, p(T)) \cap M_1 \neq \emptyset\}$. Then, by Step 1, $\#N_1 \geq 2$, and there exist $t_1 < T$ and $M_2 \subsetneq M_1$ such that $N_2 \equiv \{i \in N : D(R_i, p(t_1)) \cap M_2 \neq \emptyset\} \subsetneq N_1$ and M_2 is underdemanded at $p(t_1)$. Since M_2 is underdemanded at $p(t_1)$, Step 1 also implies that $\#N_2 \geq 2$, and there exist $t_2 < t_1$ and $M_3 \subsetneq M_2$ such that $N_3 \equiv \{i \in N : D(R_i, p(t_2)) \cap M_3 \neq \emptyset\} \subsetneq N_2$ and M_3 is underdemanded at $p(t_2)$. Repeating this argument inductively, there is a sequence $\{N_k\} \subsetneq N_1$ such that for each $k \geq 2$, $\#N_k < \#N_{k-1}$ and $\#N_k \geq 2$. However, since N_1 is finite and for each $k \geq 2$, $N_k \subsetneq N_1$, this is a contradiction. □

References

Alcalde, J. and Barberà, S. (1994) Top dominance and the possibility of strategy-proof stable solutions to matching problems. *Economic Theory* 4: 417–435

Alkan, A. and Gale, D. (1990) The core of the matching game. *Games and Economic Behavior* 2: 203–212

Ausubel, L.M. (2004) An efficient ascending-bid auction for multiple objects. *American Economic Review* 94: 1452–1475

[30] To see this, suppose that $\hat{M}' = M'$. Since M' is weakly underdemanded at $p(t')$, $\#N' \leq \#M'$. By $\hat{M}' = M'$ and $\#\hat{N}' > \#\hat{M}'$, $\#N' \leq \#M' = \#\hat{M}' < \#\hat{N}' \leq \#N'$, which is a contradiction.

Ausubel, L.M. (2006) An efficient dynamic auction for heterogeneous commodities. *American Economic Review* 96: 602–629

Ausubel, L.M. and Milgrom, P.R. (2002) Ascending auctions with package bidding. *Frontiers of Theoretical Economics* 1: 1–42

Chew, S.H. and Serizawa, S. (2007) Characterizing the Vickrey combinatorial auction by induction. *Economic Theory* 33: 393–406

Clarke, E. (1971) Multipart pricing of public goods. *Public Choice* 8: 19–33

Demange, G. and Gale, D. (1985) The strategy structure of two-sided matching markets. *Econometrica* 53: 873–888

Demange, G., Gale, D., and Sotomayor, M. (1986) Multi-item auctions. *Journal of Political Economy* 94: 863–872

de Vries, S., Schummer, J., and Vohra, R.V. (2007) On ascending Vickrey auctions for heterogeneous objects. *Journal of Economic Theory* 132: 95–118

Groves, T. (1973) Incentives in teams. *Econometrica* 41: 617–631

Gul, F. and Stacchetti, E. (2000) The English auction with differentiated commodities. *Journal of Economic Theory* 92: 66–95

Holmström, B. (1979) Groves' scheme on restricted domains. *Econometrica* 47: 1137–1144

Kojima, F. and Manea, M. (2010) Axioms for deferred acceptance. *Econometrica* 78: 633–653

McAfee, R.P. and McMillan, J. (1996) Analyzing the airwaves auction. *Journal of Economic Perspectives* 10: 159–175

Mishra, D. and Parkes, D.C. (2007) Ascending price Vickrey auctions for general valuations *Journal of Economic Theory* 132: 335–366

Mishra, D. and Talman, D. (2010) Characterization of the Walrasian equilibria of the assignment model. *Journal of Mathematical Economics* 46: 6–20

Miyake, M. (1998) On the incentive properties of multi-item auctions. *International Journal of Game Theory* 27: 1–19

Roth, A. and Sotomayor, M. (1990) *Two-sided matching: a study in game-theoretic modeling and analysis.* Econometric Society Monograph Series. Cambridge University Press, Cambridge

Saitoh, H. and Serizawa, S. (2008) Vickrey allocation rule with income effect. *Economic Theory* 35: 391–401

Sakai, T. (2008) Second price auctions on general preference domains: two characterizations. *Economic Theory* 37: 347–356

Tadenuma, K. and Thomson, W. (1991) No-envy and consistency in economies with indivisible goods. *Econometrica* 59: 1755–1767

Vickrey, W. (1961) Counterspeculation, auctions, and competitive sealed tenders. *Journal of Finance* 16: 8–37

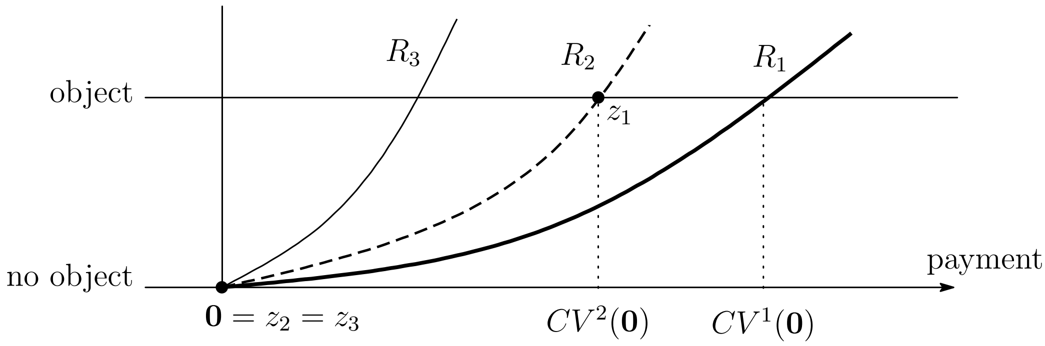

Figure 1.

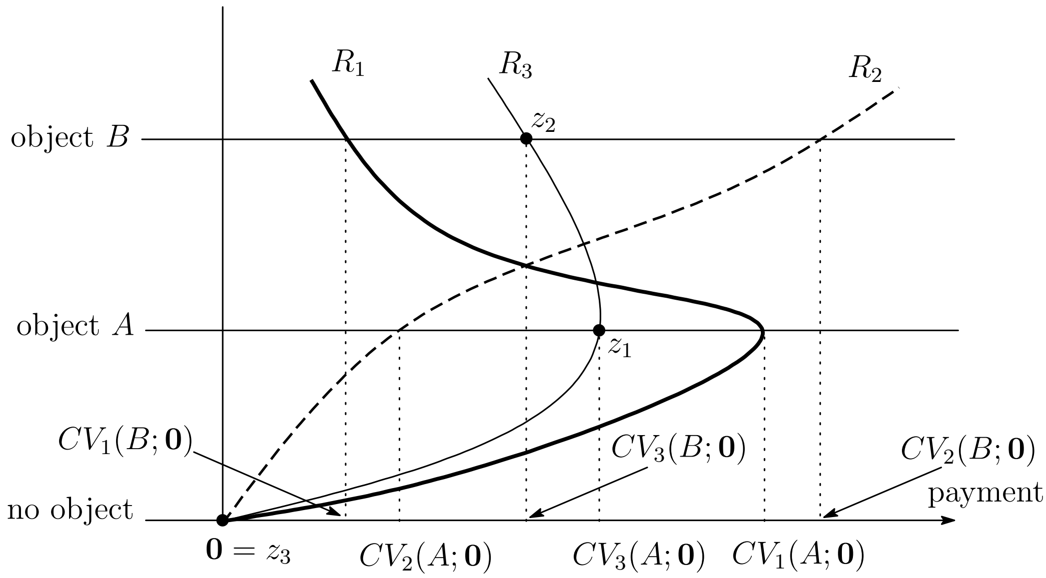

Figure 2.

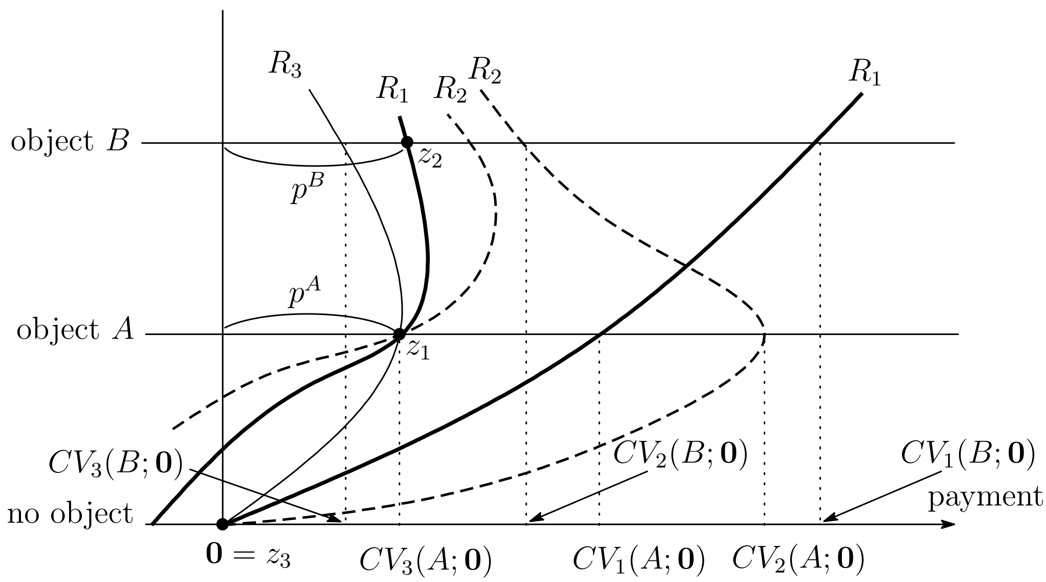

Figure 3.

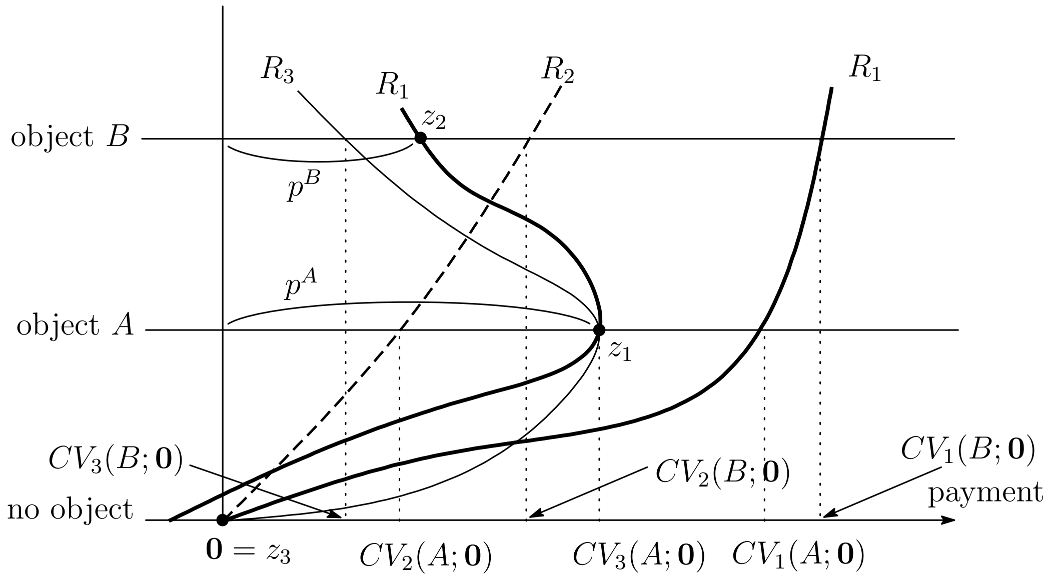

Figure 4.

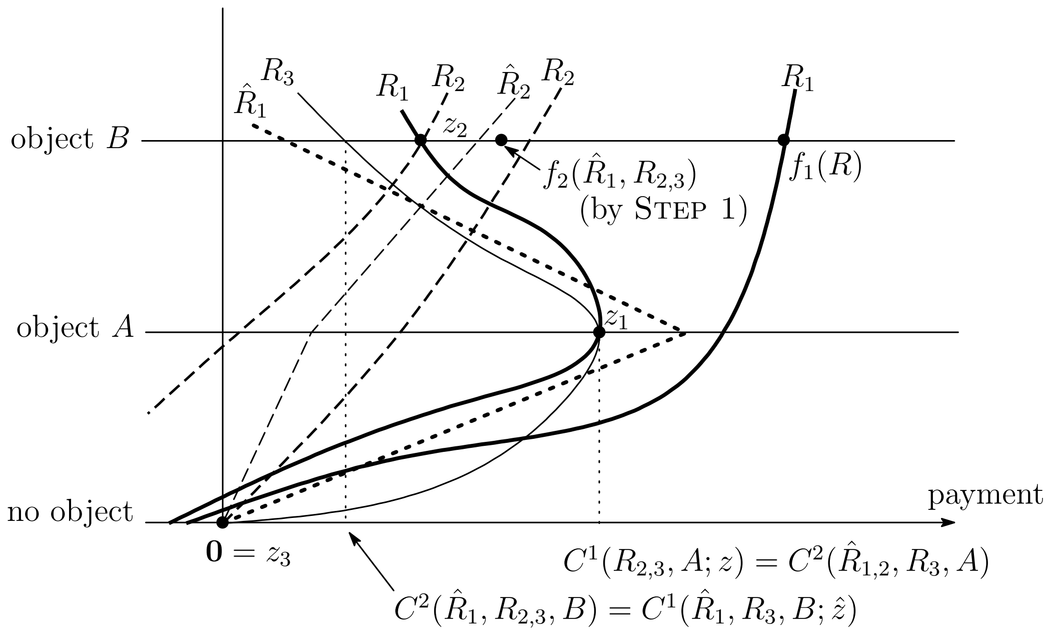

Figure 5. Illustration of proof of Proposition 5.1 for Case V in Section 2.

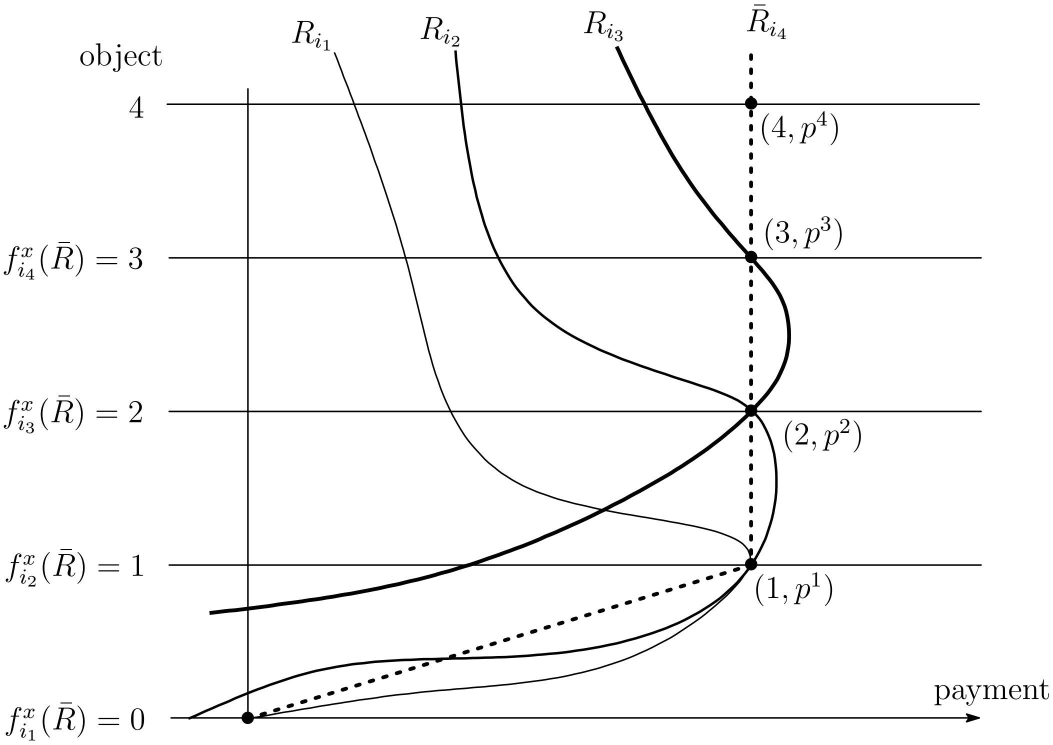

Figure 6. Illustration of (9-b) of Lemma 5.9 for $K = 4$.

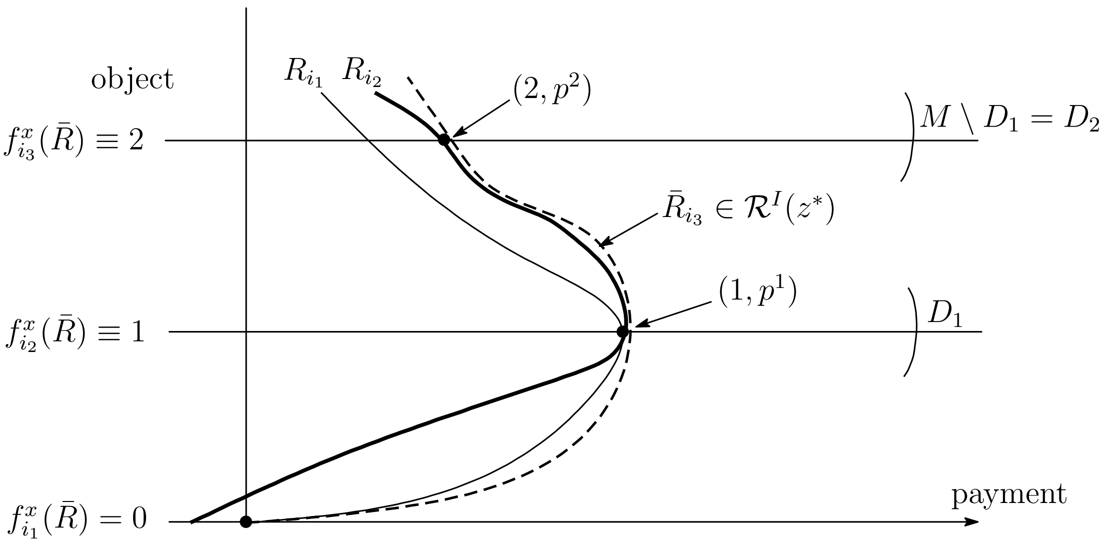

Figure 7. Illustration of proof of (9-b) in Lemma 5.9 for two objects case.

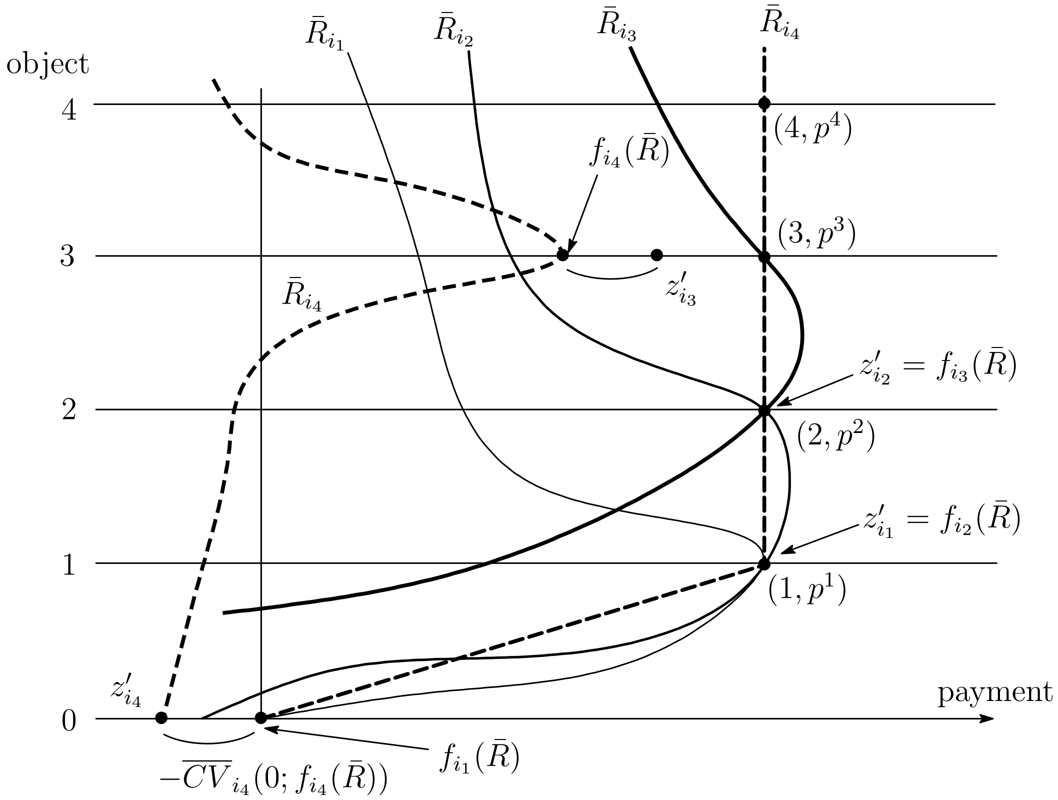

Figure 8. Illustration of the assignments of the agents in $\{i_k\}_{k=1}^{K}$ for $K = 4$.

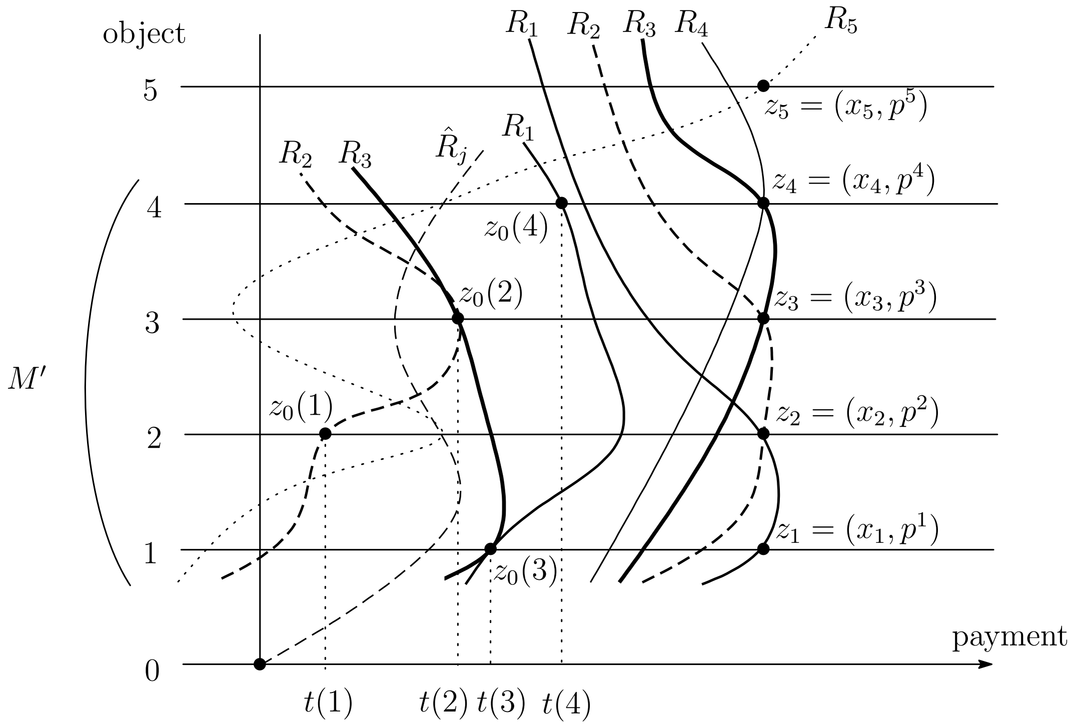

Figure A.1. Illustration of assignment sequence for the case of $m' = 4$, $x(1) = x_2$, $x(2) = x_3$, $x(3) = x_1$, and $x(4) = x_4$.

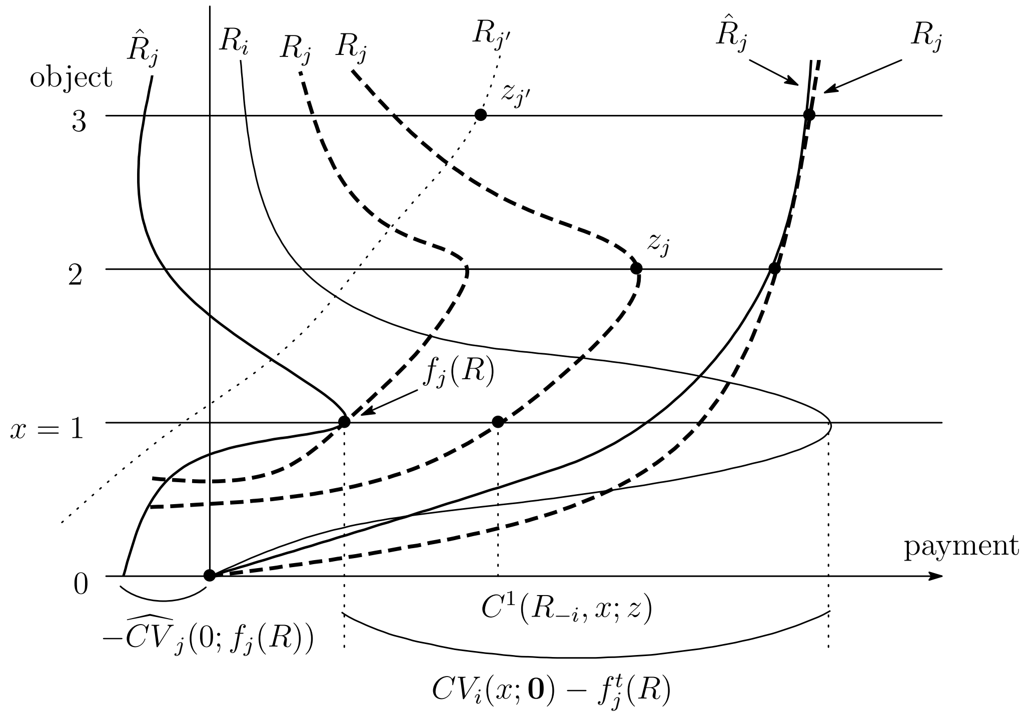

Figure A.2. Illustration of proof of Lemma 5.7.

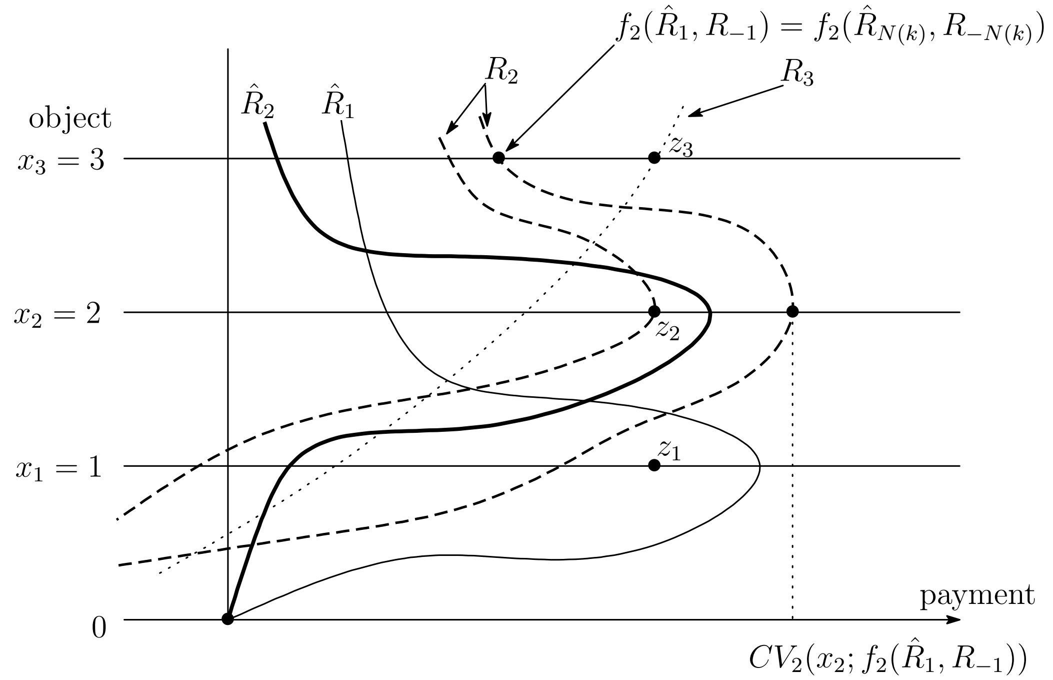

Figure A.3. Illustration of (i-$(k+1)$) and (ii-$(k+1)$) in the proof of Proposition 5.1 for $k = 1$.

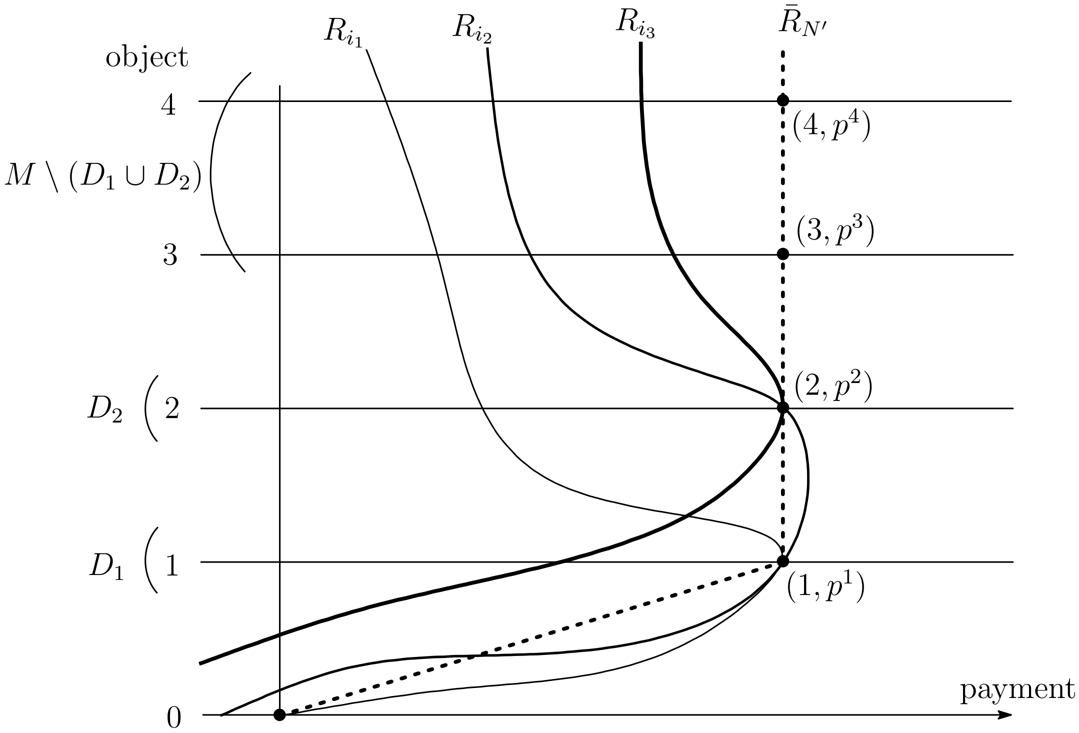

Figure A.4. Illustration of (9-k-d) of Lemma 5.9 for the case of $k = 3$, $m = 4$, $n = 5$, $N''_1 \equiv \{i_1\}$, $N''_2 \equiv \{i_2\}$, $N''_3 \equiv \{i_3\}$, and $N' \equiv N \setminus \{i_1, i_2, i_3\}$. In this case, $D_1 = \{1\}$, $D_2 = \{2\}$, and $\{j \in N : D(\bar{R}_j, p) \cap [M \setminus (D_1 \cup D_2)] \neq \emptyset\} = N'$.

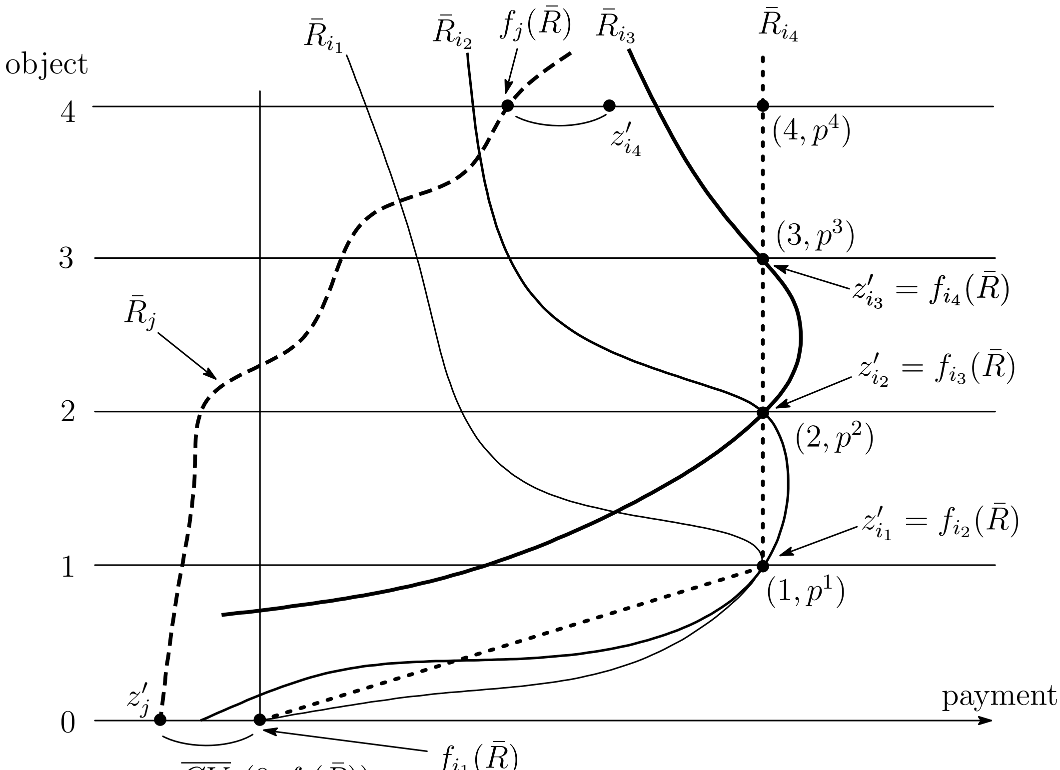

Figure A.5. Illustration of z' in (11-a) of Lemma 5.11 for $K = 4$.

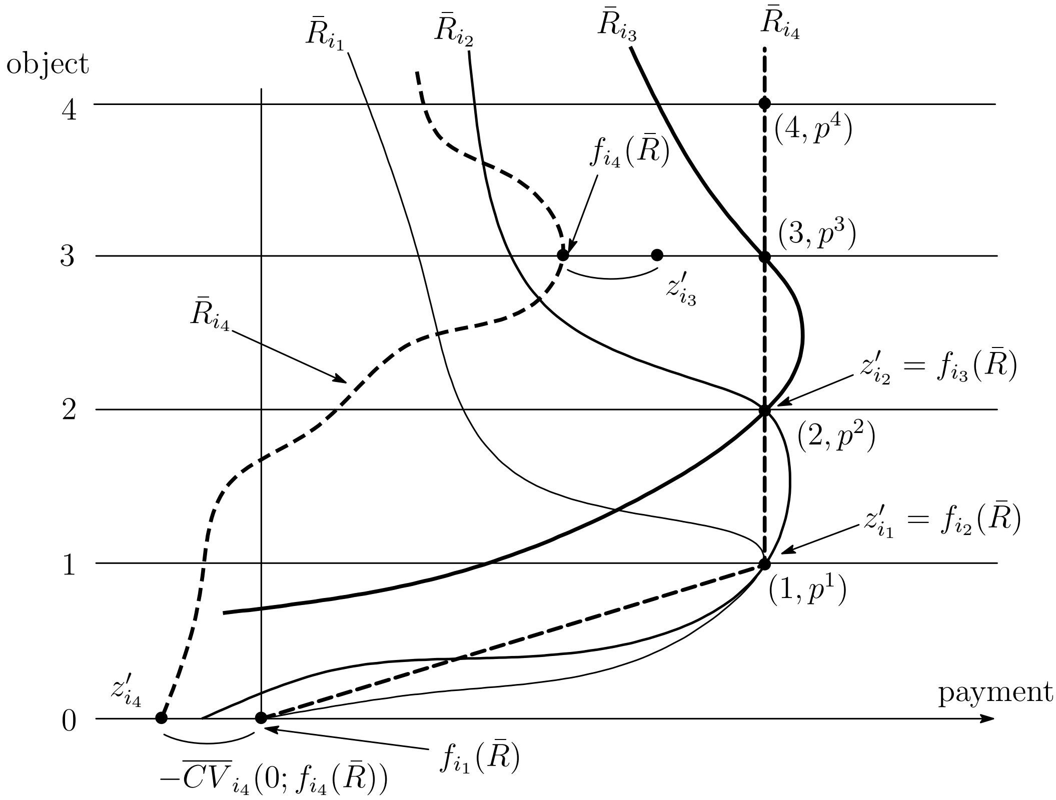

Figure A.6. Illustration of z' in (11-b) of Lemma 5.11 for $K = 4$.

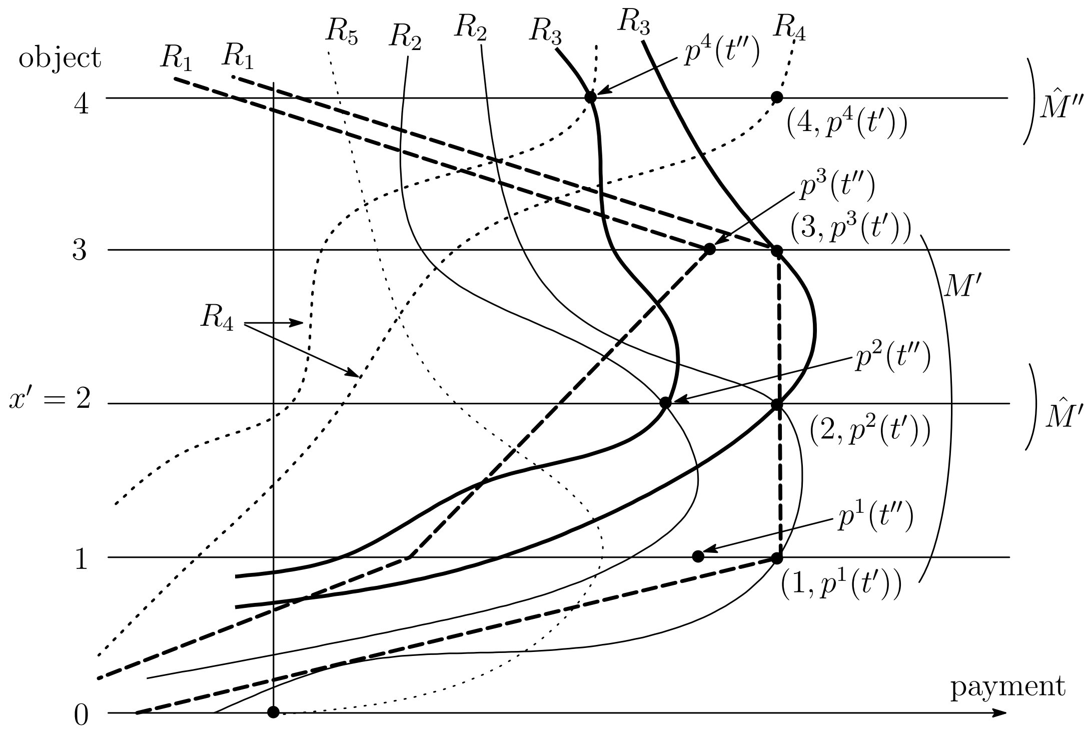

Figure A.7. Illustration of proof of Step 1 of Proposition 6.1 for the case of $m = 4$, $M' \equiv \{1, 2, 3\}$, $N' \equiv \{1, 2, 3\}$, $x' \equiv 2$, and $\hat{M}' \equiv \{2, 4\}$. In this case, $\hat{M}' = \{2\}$, $\hat{N}' = \{2, 3\}$, $\hat{M}'' = \{4\}$, $\hat{N}'' = \{4\}$, $M'' = \{1, 3\}$, and $N'' = \{1\}$.