Supplementary Material for "Strategy-proofness and Efficiency with Nonquasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule"

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In this supplement, we provide the proofs that we have omitted in "Strategy-proofness and Efficiency with Nonquasi-linear Preferences: A Characterization of Minimum Price Walrasian Rule". In Part I, we prove Fact 4.4 presented in Section 4. In Part II, we show Fact 4.5 introduced in the proof of Theorem 4.1 (Appendix, Subsection A.1).

Part I Proof of Fact 4.4.

The following theorem is useful to prove Fact 4.4.

Hall's Theorem. *Let $N \equiv \{1, \ldots, n\}$ and $M \equiv \{1, \ldots, m\}$. For each $i \in N$, let $D_i \subseteq M$. Then, (i) there is a one to one mapping $\hat{x}$ from N to M such that for each $i \in N$, $\hat{x}(i) \in D_i$ if and only if (ii) for each $N' \subseteq N$, $\# \bigcup_{i \in N'} D_i \geq \#N'$.*

Fact 4.4 (Mishra and Talman, 2010). *Let $\mathcal{R} \subseteq \mathcal{R}^E$ and $R \in \mathcal{R}^n$. A price vector p is a Walrasian equilibrium price for R if and only if no set of objects is overdemanded and no set of objects is underdemanded at p for R.*

Proof of Fact 4.4. First, we prove *only if* part of Fact 4.4. Then, we show *if* part.
Proof of "ONLY IF" part. Let p be a Walrasian equilibrium price for R. Then, there is an allocation $z = (x_i, t_i)_{i \in N}$ satisfying conditions (WE-i) and (WE-ii) in Definition 4.1. Let $M' \subseteq M$.

We show that M' is not overdemanded at p for R. Let $N' \equiv \{i \in N : D(R_i, p) \subseteq M'\}$. Since for each $i \in N'$, $x_i \in D(R_i, p) \subseteq M'$, and each indivisible object is consumed at most one agent, $\#N' = \#\{x_i : i \in N'\}$. Since $\{x_i : i \in N'\} \subseteq M'$, $\#\{x_i : i \in N'\} \leq \#M'$. Thus, $\#N' \leq \#M'$.

We show that M' is not underdemanded at p for R. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \emptyset\}$. Suppose that for each $x \in M'$, $p^x > 0$ and $\#N' < \#M'$. Note that $\#N' < \#M'$ implies that there is $x \in M'$ such that for all $i \in N$, $x_i \neq x$. Then, condition (WE-ii) implies that $p^x = 0$. This is a contradiction. Thus, $\#N' \geq \#M'$.
Proof of "IF" part. Assume that no set of objects is overdemanded and no set of objects is underdemanded at p for R.

Let $Z^* \equiv \{z = (x_i, t_i)_{i \in N} : \forall i \in N, x_i \in D(R_i, p) \text{ and } t_i = p^{x_i}\}$. First, we show $Z^* \neq \emptyset$. Suppose that there is $N' \subseteq N$ such that for each $i \in N'$, $0 \notin D(R_i, p)$ and $\#\{\bigcup_{i \in N'} D(R_i, p)\} < \#N'$. Then $\{\bigcup_{i \in N'} D(R_i, p)\}$ is overdemanded at p for R. Thus, for each $N' \subseteq N$, if for each $i \in N'$, $0 \notin D(R_i, p)$, then $\#\{\cup_{i \in N'} D(R_i, p)\} \geq \#N'$. Then, by Hall's Theorem, there is $\bar{z} \in Z$ such that for each $i \in N$, if $0 \notin D(R_i, p)$, then $\bar{x}_i \in D(R_i, p)$ and $\bar{t}_i = p^{\bar{x}_i}$. Thus, $Z^* \neq \emptyset$.

By definition, for each $z \in Z^*$, (z, p) satisfies (WE-i). We show that there is $z \in Z^*$ such that (z, p) satisfies (WE-ii). Let $M^+(p) \equiv \{x \in M : p^x > 0\}$. Let

$$z \in \arg\max_{z' \in Z^*} \#\{y \in M^+(p) : \exists i \in N \text{ s.t. } x'_i = y\}, \tag{1}$$

that is, z maximizes over Z^* the number of objects in $M^+(p)$ that are assigned to some agents. Then, by the definition of Z^*, (z, p) satisfies (WE-i).

Let $M^0 \equiv \{y \in M^+(p) : \forall i \in N, x_i \neq y\}$. Note that, if $M^0 = \emptyset$, (z, p) also satisfies (WE-ii). Thus, we show that $M^0 = \emptyset$. By contradiction, suppose that $M^0 \neq \emptyset$.

Let $N^0 \equiv \{i \in N : D(R_i, p) \cap M^0 \neq \emptyset\}$. For each $k = 1, 2, \ldots$, let $M^k \equiv \{y \in M : \exists\, i \in N^{k-1} \text{ s.t. } x_i = y\}$ and $N^k \equiv \{i \in N : D(R_i, p) \cap M^k \neq \emptyset\} \setminus \{\bigcup_{k'=0}^{k-1} N^{k'}\}$. We claim by induction that for each $k \geq 0$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

Induction argument:

Step 1. By the definition of M^0, $M^0 \subseteq M^+(p)$. Since M^0 is not underdemanded at p for R, $\#N^0 \geq \#M^0$. Thus, $M^0 \neq \emptyset$ implies that $N^0 \neq \emptyset$.

Step 2. Let $K \geq 1$. As induction hypothesis, assume that for each $k \leq K-1$, $M^k \subseteq M^+(p)$ and $N^k \neq \emptyset$.

First, we show that $M^K \subseteq M^+(p)$. Suppose that there is $x \in M^K \setminus M^+(p)$. Then, $x = 0$ or $p^x = 0$. By the induction hypothesis, there is a sequence $\{x(s), i(s)\}_{s=1}^K$ such that

$$\begin{array}{cc}
x(1) = x, & x_{i(1)} = x(1), \\
x(2) \in D(R_{i(1)}, p) \cap M^{K-1}, & x_{i(2)} = x(2), \\
x(3) \in D(R_{i(2)}, p) \cap M^{K-2}, & x_{i(3)} = x(3), \\
\vdots & \vdots \\
x(K) \in D(R_{i_{(K-1)}}, p) \cap M^1, & x_{i(K)} = x(K).
\end{array}$$

Let $x(K+1) \in D(R_{i(K)}, p) \cap M^0$. For each $s \in \{1, 2, \ldots, K\}$, let $\hat{z}_{i(s)} \equiv (x_{i(s+1)}, p^{x_{i(s+1)}})$, and for each $j \in N \setminus \{i(s)\}_{s=1}^K$, let $\hat{z}_j \equiv z_j$. Then, $\hat{z} \in Z^*$, and

$$\#\{y \in M^+(p) : \exists i \in N \text{ s.t. } \hat{x}_i = y\} = \#\{y \in M^+(p) : \exists i \in N \text{ s.t. } x_i = y\} + 1.$$

This is a contradiction to (1). Thus, $M^K \subseteq M^+(p)$.

Next, we show that $N^K \neq \emptyset$. By $M^K \subseteq M^+(p)$ and the induction hypothesis, $\bigcup_{k=1}^K M^k \subseteq M^+(p)$. Thus, since $\bigcup_{k=1}^K M^k$ is not underdemanded at p for R,

$$\# \bigcup_{k=0}^K N^k \geq \# \bigcup_{k=0}^K M^k. \tag{2}$$

By the definitions of M^k and N^k, for each $k, k' \in \{0, 1, \ldots, K\}$ with $k \neq k'$, $N^k \cap N^{k'} = \emptyset$, which also implies that $M^k \cap M^{k'} = \emptyset$. Thus,

$$\# \bigcup_{k=0}^{K} N^k = \sum_{k=0}^{K} \# N^k, \text{ and } \# \bigcup_{k=0}^{K} M^k = \sum_{k=0}^{K} \# M^k.$$

Then, by (2),

$$\sum_{k=0}^{K-1} \# N^k + \# N^K = \sum_{k=0}^{K} \# N^k \geq \sum_{k=0}^{K} \# M^k = \sum_{k=1}^{K} \# M^k + \# M^0. \tag{3}$$

For each $k \geq 1$, by $M^k \subseteq M^+(p)$, $\# M^k = \# N^{k-1}$. Thus, $\sum_{k=0}^{K-1} \# N^k = \sum_{k=1}^{K} \# M^k$. Then, by (3),

$$\# N^K \geq \# M^0.$$

Thus, by $M^0 \neq \emptyset$, $\# N^K \geq 1$, and so $N^K \neq \emptyset$.

Since $M^+(p)$ is finite, by the above induction argument, for large K, $\# \bigcup_{k=0}^{K} M^k = \sum_{k=0}^{K} \# M^k > \# M^+(p)$. Since $\bigcup_{k=0}^{K} M^k \subseteq M^+(p)$, this is a contradiction. $\square$

Part II Proof of Fact 4.5.

Let $\mathcal{R} \subseteq \mathcal{R}^E$.

Lemma A.1. *Let $i \in N$ and $R_i \in \mathcal{R}$. Let $p, q \in \mathbb{R}^m_+$ and $x, y \in L$ be such that $x \in D(R_i, p)$ and $(y, q^y) \, P_i \, (x, p^x)$. Then, $y \in M$ and $q^y < p^y$.*

Proof of Lemma A.1. Since $(y, q^y) \, P_i \, (x, p^x)$ and $x \in D(R_i, p)$, we have $(y, q^y) \, P_i \, (x, p^x) \, R_i \, \mathbf{0}$. Thus, $y \in M$. Also, by $x \in D(R_i, p)$, $(y, q^y) \, P_i \, (x, p^x) \, R_i \, (y, p^y)$. Thus, $(y, q^y) \, P_i \, (y, p^y)$ implies that $q^y < p^y$. $\square$

Let $R, \hat{R} \in \mathcal{R}^n$, and let (z, p) and $(\hat{z}, \hat{p})$ be Walrasian equilibria associated with R and $\hat{R}$, respectively. Define

$$N^1 \equiv \{i \in N : \hat{z}_i \, P_i \, z_i\}, \; M^2 \equiv \{x \in M : p^x > \hat{p}^x\},$$
$$X^1 \equiv \{x \in L : \exists \, i \in N^1 \text{ s.t. } x_i = x\}, \text{ and } \widehat{X}^1 \equiv \{x \in L : \exists \, i \in N^1 \text{ s.t. } \hat{x}_i = x\}.$$

Lemma A.2: *Decomposition* (Demange and Gale, 1985). *Let $R \in \mathcal{R}^n$ and (z, p) be a Walrasian equilibrium for R. Let $\hat{R} \in \mathcal{R}^n$ be the d-truncation of R such that for each $i \in N$, $d_i \leq -CV_i(0; z_i)$, and let $(\hat{z}, \hat{p})$ be a Walrasian equilibrium for $\hat{R}$. Then, $X^1 = \widehat{X}^1 = M^2$,*

Proof of Lemma A.2. First, we show that $\widehat{X}^1 \subseteq M^2$. Let $x \in \widehat{X}^1$. Then, there is $i \in N^1$ such that $\hat{x}_i = x$. By $i \in N^1$, $(\hat{x}_i, \hat{p}^{\hat{x}_i}) \, P_i \, (x_i, p^{x_i})$. Thus, by $x_i \in D(R_i, p)$, Lemma A.1 implies that $\hat{x}_i \in M$ and $\hat{p}^{\hat{x}_i} < p^{\hat{x}_i}$, and so $x = \hat{x}_i \in M^2$. Thus, $\widehat{X}^1 \subseteq M^2$.

Next, we show that $M^2 \subseteq X^1$. Let $x \in M^2$. Then, $x \in M$ and $0 \leq \hat{p}^x < p^x$. Thus, by (WE-ii), there is $i \in N$ such that $x_i = x$. Since $d_i \leq -CV_i(0; z_i)$, Lemma 4.2-(ii) implies that $(\hat{x}_i, \hat{p}^{\hat{x}_i}) \, P_i \, (x_i, p^{x_i})$. Thus, $i \in N^1$, and so $x = x_i \in X^1$. Thus, $M^2 \subseteq X^1$.

Note that by the definition of X^1 and $\widehat{X}^1$, $\# X^1 \leq \# N^1$ and $\# \widehat{X}^1 \leq \# N^1$. Since $\widehat{X}^1 \subseteq M^2 \subseteq M$, each agent in N^1 receives a different object, and so $\# \widehat{X}^1 = \# N^1 \geq \# X^1$.

Since $\widehat{X}^1 \subseteq M^2 \subseteq X^1$, $\#\widehat{X}^1 \leq \#M^2 \leq \#X^1$. Thus, $\#\widehat{X}^1 = \#M^2 = \#X^1$. By $\#\widehat{X}^1 = \#M^2$ and $\widehat{X}^1 \subseteq M^2$, $\widehat{X}^1 = M^2$. By $\#M^2 = \#X^1$ and $M^2 \subseteq X^1$, $M^2 = X^1$. $\square$

Lemma A.3: *Lattice Structure* (Demange and Gale, 1985). *Let $R \in \mathcal{R}^n$ and (z, p) be a Walrasian equilibrium for R. Let $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \leq -CV_i(0; z_i)$, and let $(\hat{z}, \hat{p})$ be a Walrasian equilibrium for $\hat{R}$. Then,*
(i) $p^{(-)} \equiv p \wedge \hat{p}$ is a Walrasian equilibrium price for R, and
(ii) $p^{(+)} \equiv p \vee \hat{p}$ is a Walrasian equilibrium price for $\hat{R}$.[1]

Proof of Lemma A.3. Let $N^1 \equiv \{i \in N : \hat{z}_i \, P_i \, z_i\}$ and $M^2 \equiv \{x \in M : p^x > \hat{p}^x\}$.
Proof of (i). Let $z^{(-)}$ be an allocation such that for each $i \in N^1$, $z_i^{(-)} \equiv \hat{z}_i$, and for each $i \in N \setminus N^1$, $z_i^{(-)} \equiv z_i$. We show that $(z^{(-)}, p^{(-)})$ is a Walrasian equilibrium for R.

Step 1. $(z^{(-)}, p^{(-)})$ *satisfies (WE-i).*

Let $i \in N$ and $x \in L$. In the following two cases, we show that $(x_i^{(-)}, p^{(-)x_i^{(-)}}) \, R_i \, (x, p^{(-)x})$, which implies $x_i^{(-)} \in D(R_i, p^{(-)})$.
Case 1. $i \in N^1$.
Since $x_i^{(-)} = \hat{x}_i$, by Lemma A.2, $x_i^{(-)} \in M^2$, and so $x_i^{(-)} \in M$ and $\hat{p}^{x_i^{(-)}} < p^{x_i^{(-)}}$. Thus, $p^{(-)x_i^{(-)}} = \hat{p}^{x_i^{(-)}}$.
First, we assume that $x \in M^2$. Then, by $p^{(-)x} = \hat{p}^x$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = \hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) = (x, p^{(-)x}),$$

where the preference relation follows from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$. Since $\hat{R}_i$ is the d_i-truncation of R_i, $x_i^{(-)} \neq 0$, and $x \neq 0$, Remark 4.1 implies that $(x_i^{(-)}, p^{(-)x_i^{(-)}}) \, R_i \, (x, p^{(-)x})$.
Next, we assume that $x \notin M^2$. Then, by $p^{(-)x} = p^x$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = \hat{z}_i \, P_i \, z_i \, R_i \, (x, p^x) = (x, p^{(-)x}).$$

where the strict preference relation follows from $i \in N^1$, and the second preference relation from $x_i \in D(R_i, p)$.
Case 2. $i \notin N^1$.
Since $x_i^{(-)} = x_i$, by Lemma A.2, $x_i^{(-)} \notin M^2$. Thus, $p^{x_i^{(-)}} \leq \hat{p}^{x_i^{(-)}}$ or $x_i^{(-)} = 0$. First, we assume that $x \in M^2$. Then, $p^{(-)x} = \hat{p}^x$. Note that $i \notin N^1$ implies $(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, \hat{z}_i$.
Case 2-1. $\hat{x}_i \neq 0$.
By $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, $\hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) = (x, p^{(-)x})$. Since $\hat{R}_i$ is the d_i-truncation of R_i, $\hat{x}_i \neq 0$, and $x \neq 0$, Remark 4.1 implies that $\hat{z}_i \, R_i \, (x, \hat{p}^x)$. Thus,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, \hat{z}_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Case 2-2. $\hat{x}_i = 0$.
Then, $\hat{z}_i = \mathbf{0}$. Since $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, $\widehat{CV}_i(x; \mathbf{0}) \leq \hat{p}^x$. Thus, if $CV_i(x; \mathbf{0}) \leq \widehat{CV}_i(x; \mathbf{0})$, then, $\hat{z}_i \, R_i \, (x, \hat{p}^x)$, which implies that,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, \hat{z}_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Next, assume that $CV_i(x; \mathbf{0}) > \widehat{CV}_i(x; \mathbf{0})$. Then, since $\hat{R}_i$ is the d_i-truncation of R_i, $d_i > 0$, which implies that $x_i \neq 0$.[2] Then, by $d_i \leq -CV_i(0; z_i)$, $CV_i(x; z_i) \leq \widehat{CV}_i(x; \mathbf{0}) \leq \hat{p}^x$,

[1] Denote $p \wedge \hat{p} \equiv (\min\{p^x, \hat{p}^x\})_{x \in M}$ and $p \vee \hat{p} \equiv (\max\{p^x, \hat{p}^x\})_{x \in M}$.
[2] To see this, suppose that $x_i = 0$. Then, $d_i \leq -CV_i(0; z_i) = 0$, which contradicts $d_i > 0$.

which implies that $z_i \, R_i \, (x, \hat{p}^x)$. Thus,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, (x, \hat{p}^x) = (x, p^{(-)x}).$$

Next, we assume that $x \notin M^2$. Then, $p^{(-)x} = p^x$. Since $x_i^{(-)} = x_i \in D(R_i, p)$,

$$(x_i^{(-)}, p^{(-)x_i^{(-)}}) = z_i \, R_i \, (x, p^x) = (x, p^{(-)x}).$$

Step 2. *$(z^{(-)}, p^{(-)})$ satisfies (WE-ii).*

Let $x \in M$ be such that $p^{(-)x} > 0$. We show that there is $i \in N$ such that $x_i^{(-)} = x$. Since $p^{(-)} = p \wedge \hat{p}$, $p^{(-)x} > 0$ implies that $p^x > 0$ and $\hat{p}^x > 0$.

Case 1. $x \in M^2$.

By Lemma A.2, there is $i \in N^1$ such that $\hat{x}_i = x$. Since $i \in N^1$, $x_i^{(-)} = \hat{x}_i$. Thus, $x_i^{(-)} = x$.

Case 2. $x \notin M^2$.

Since $p^x > 0$, there is $i \in N$ such that $x_i = x$. By Lemma A.2, $i \notin N^1$. This implies that $x_i^{(-)} = x_i$. Thus, $x_i^{(-)} = x$. □

Proof of (ii). Let $z^{(+)}$ be an allocation such that for each $i \in N^1$, $z_i^{(+)} \equiv z_i$, and for each $i \in N \setminus N^1$, $z_i^{(+)} \equiv \hat{z}_i$. We show that $(z^{(+)}, p^{(+)})$ is a Walrasian equilibrium for $\hat{R}$.

Step 1. *$(z^{(+)}, p^{(+)})$ satisfies (WE-i).*

Let $i \in N$ and $x \in L$. In the following two cases, we show that $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$, which implies $x_i^{(+)} \in D(\hat{R}_i, p^{(+)})$.

Case 1. $i \in N^1$.

Since $x_i^{(+)} = x_i$, by Lemma A.2, $x_i^{(+)} \in M^2$, and so $x_i^{(+)} \in M$ and $\hat{p}^{x_i^{(+)}} < p^{x_i^{(+)}}$. Thus, $p^{(+)x_i^{(+)}} = p^{x_i^{(+)}}$. First, we assume that $x \in M^2$. Since $x_i^{(+)} = x_i \in D(R_i, p)$ and $p^{(+)x} = p^x$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, R_i \, (x, p^x) = (x, p^{(+)x}).$$

Since $\hat{R}_i$ is the d_i-truncation of R_i, $x_i^{(+)} \neq 0$, and $x \neq 0$, Remark 4.1 implies that $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$.

Next, we assume that $x \notin M^2$. Then, $p^x \leq \hat{p}^x$ or $x = 0$.

Case 1-1. $x \neq 0$.

Since $x_i^{(+)} = x_i \in D(R_i, p)$ and $p^{(+)x} = \hat{p}^x \geq p^x$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, R_i \, (x, p^x) \, R_i \, (x, p^{(+)x}).$$

Since $\hat{R}_i$ is the d_i-truncation of R_i and $x_i^{(+)} \neq 0$, $(x_i^{(+)}, p^{(+)x_i^{(+)}}) \, \hat{R}_i \, (x, p^{(+)x})$.

Case 1-2. $x = 0$.

Since $\hat{R}_i$ is the d_i-truncation of R_i and $d_i \leq -CV_i(0; z_i)$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = z_i \, \hat{R}_i \, \mathbf{0} = (x, p^{(+)x}).$$

Case 2. $i \notin N^1$.

Since $x_i^{(+)} = \hat{x}_i$, by Lemma A.2, $x_i^{(+)} \notin M^2$. Thus, $p^{x_i^{(+)}} \leq \hat{p}^{x_i^{(+)}}$ or $x_i^{(+)} = 0$. If $x_i^{(+)} = 0$,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = \mathbf{0} = \hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) \, \hat{R}_i \, (x, p^{(+)x}),$$

where the first preference relation follows from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, and the second from $p^{x^{(+)}} = \max\{p^x, \hat{p}^x\}$.

Thus, we assume that $x_i^{(+)} \neq 0$. Then,

$$(x_i^{(+)}, p^{(+)x_i^{(+)}}) = \hat{z}_i \, \hat{R}_i \, (x, \hat{p}^x) \, \hat{R}_i \, (x, p^{(+)x}),$$

where the first equality follows from $p^{x_i^{(+)}} \leq \hat{p}^{x_i^{(+)}} = p^{(+)x_i^{(+)}}$, the first preference relation from $\hat{x}_i \in D(\hat{R}_i, \hat{p})$, and the second preference relation from $p^{(+)x} = \max\{p^x, \hat{p}^x\}$.

Step 2. $(x_i^{(+)}, p^{(+)})$ *satisfies (WE-ii).*

Let $x \in M$ be such that $p^{(+)x} > 0$. We show that there is $i \in N$ such that $x_i^{(+)} = x$. Since $p^{(+)} = p \vee \hat{p}$, $p^{(+)x} > 0$ implies that $p^x > 0$ or $\hat{p}^x > 0$.

Case 1. $x \in M^2$.

By Lemma A.2, there is $i \in N^1$ such that $x_i = x$. Since $i \in N^1$, $x_i^{(+)} = x_i$. Thus, $x_i^{(+)} = x$.

Case 2. $x \notin M^2$.

If $\hat{p}^x = 0$, then $\hat{p}^x = 0 < p^x$. Thus, $x \in M^2$, which is a contradiction. Thus, $\hat{p}^x > 0$. Then, there is $i \in N$ such that $\hat{x}_i = x$. By Lemma A.2, $i \notin N^1$, which implies that $x_i^{(+)} = \hat{x}_i$. Thus, $x_i^{(+)} = x$. $\square$

The following is obtained as a corollary of Lemma A.3.

Corollary A.1. *Let $R \in \mathcal{R}^n$. Let p and $\hat{p}$ be Walrasian equilibrium prices for R. Then, $p \wedge \hat{p}$ and $p \vee \hat{p}$ are also Walrasian equilibrium prices for R.*

We now proceed to prove Fact 4.5.

Fact 4.5 (Roth and Sotomayor, 1990). *Let $R \in \mathcal{R}^n$ and let $\hat{R}$ be the d-truncation of R such that for each $i \in N$, $d_i \geq 0$. Then, $p_{\min}(\hat{R}) \leq p_{\min}(R)$.*

Proof of Fact 4.5. Let $(\hat{z}, \hat{p})$ be a Walrasian equilibrium for $\hat{R}$. Then, for each $i \in N$, since $\widehat{CV}_i(0; \hat{z}_i) \leq 0$ and $d_i \geq 0$, $-d_i \leq 0 \leq -\widehat{CV}_i(0; \hat{z}_i)$. Since R is the $(-d)$-truncation of $\hat{R}$, Lemma A.3 implies that $p^{(-)} \equiv \hat{p} \wedge p_{\min}(R)$ is a Walrasian equilibrium price for $\hat{R}$. Thus, since $p_{\min}(\hat{R}) \leq p^{(-)}$, $p_{\min}(\hat{R}) \leq p_{\min}(R)$. $\square$

References

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