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TIME PREFERENCE AND DYNAMIC STABILITY IN AN N-COUNTRY WORLD ECONOMY

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Abstract

We examine stability of competitive equilibrium in an N-country world economy with capital accumulation, where each country can have either increasing marginal impatience (IMI) or decreasing marginal impatience (DMI). The necessary and sufficient condition for stability is shown as positive definiteness of a simple matrix. The condition requires that any positive perturbation in one country's wealth, adjusted for international spill-over effects on other country's savings, reduces the country's wealth accumulation. In the presence of a DMI country, the number of countries should be sufficiently small for stability. Particularly, the existence of two or more than two DMI countries implies instablility.

Keywords: Stability, decreasing marginal impatience, N-country economy.
JEL classification: F41, F32, E00.

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1 Introduction

It is important to work out how endogenous time preference formation affects dynamic properties such as stability of competitive equilibrium. It is because economists needs to know what kinds of consumer preferences are compatible with dynamically well behaved models.

The purpose of this paper is to show the local stability conditions of competitive equilibrium in an N-country world economy with endogenous time preferences, and thereby elucidate its economic implications.

When the degree of impatience, measured by the pure rate of time preference, is marginally decreasing in wealth, the wealthier are more patient and, ceteris paribus, become even wealthier over time, implying that decreasing marginal impatience (hereafter DMI) is destabilizing in itself. To ensure stability, therefore, increasing marginal impatience (hereafter IMI) has been assumed (e.g., Epstein, 1987a, b; Lucas and Stokey, 1984; Obstfeld, 1990; Ikeda, 2006). However, in general equilibrium, DMI is compatible with dynamic stability. Das (2003) and Hirose and Ikeda (2008) verify that a neo-classical production economy can be stable under even DMI. But, the analyses are limited to the closed economy setting. Our main interest is to elucidate the stability condition of a global economy with capital accumulation under country heterogeneity in terms of time preferences.

We show the necessary and sufficient condition for stability in the form of positive definiteness of a simple matrix. The resulting principal minor conditions require that any positive perturbation in one country's wealth, adjusted for international spill-over effects on other country's savings, leads to a decrease in the country's saving and hence its wealth accumulation. The result implies that, in the presence of a DMI country, the number of countries should be small enough for the interdependent world economy to be stable.

2 The model

Suppose that the world economy is composed of N countries $1, \cdots, N$, each of which is populated with the same number of infinitely-lived identical households. Without loss of generality, the population is assumed to be one. A single type of goods, which can be either consumed or invested and is tradeable across countries, is produced by competitive firms in both countries using constant-returns-to-scale technologies with capital and labor. Equities representing claims to the capital stock are traded in a perfect international financial market. In each country, one unit of labor is supplied inelastically by the representative household.

2.1 Households

The budget constraints for the representative households in country i ($i = 1, \cdots, N$) are given by

$$\dot{a}^i = ra^i + w^i - c^i, \tag{1}$$

where a^i denote the asset holdings, c^i consumption, w^i the labor wage, and r the interest rate; and a dot represents the time derivative.

We specify the preferences by assuming variable time preferences as

$$\max \int_0^\infty u^i\left(c^i\left(t\right)\right)\exp(-\Delta^i\left(t\right))\mathrm{d}t, \tag{2}$$

where $u^i\left(c^i\right)$ ($i = 1, \cdots, N$) represent the instantaneous utility functions; and Δ^i denote cumulative discount rates with instantaneous discount rates $\delta^i\left(c^i\right)$: $\Delta^i\left(t\right) = \int_0^t \delta^i\left(c^i\left(\tau\right)\right)\mathrm{d}\tau$, or

$$\dot{\Delta}^i\left(t\right) = \delta^i\left(c^i\left(t\right)\right), \Delta^i\left(0\right) = 0. \tag{3}$$

For the intertemporal preferences to be well-defined, we follow the literature (e.g., Epstein, 1987a; Obstfeld 1990; Hirose and Ikeda, 2008) in assuming that the following standard regularity conditions are valid: (C1) $u^i < 0$ ($i = 1, 2$); (C2) u^i are strictly increasing and strictly concave in c^i; (C3) $-u^i$ are log-convex in c^i; and (C4) δ^i are concave in c^i. It is known, and will be shown later, that the degree of impatience, measured by the rate of time preference, is marginally increasing or decreasing in wealth as the discount rate δ^i is increasing or decreasing in c^i. Since the regularity conditions, (C1)-(C4), are not related to the signs of the first derivatives of δ^i (hereafter $d\delta^i/dc^i$ is expressed as δ^i_c, and so on), the degree of impatience can be either marginally increasing or decreasing under the conditions.

The first-order conditions to maximize lifetime utility function (2) are summarized as[1]

$$\dot{c}^i = \sigma^i\left(c^i, \phi^i\right)\left(r - \rho^i\left(c^i, \phi^i\right)\right), \tag{4}$$

[1] For details, see Hirose and Ikeda (2008, 2012a, 2012b).

$$\dot{\phi}^i = -\xi^i\left(c^i, \phi^i\right), \tag{5}$$

where $\phi^i(t)$ represents the lifetime utility obtained from the optimal consumption stream after time t, the "generating functions" ξ^i are defined by

$$\xi^i\left(c^i, \phi^i\right) = u^i\left(c^i\right) - \phi^i \delta^i\left(c^i\right), \tag{6}$$

the rates of time preference ρ^i are computed as

$$\rho^i\left(c^i, \phi^i\right) = \delta^i\left(c^i\right) - \frac{\xi^i\left(c^i, \phi^i\right)}{\xi^i_c\left(c^i, \phi^i\right)}\delta^i_c\left(c^i\right), \tag{7}$$

and $\sigma^i = -\xi^i_c/\xi^i_{cc} > 0$ when the marginal utility of c^i (ξ^i_c) is assumed to be positive.[2]

From (7), $\rho^i_\phi = \delta^i_c(\delta^i/\xi^i_c)$ around the steady state point in which $\xi^i = 0$ (see (5)). This implies that the degree of impatience, measured by ρ^i, is marginally increasing or decreasing in the utility index ϕ^i as δ^i_c is positive or negative. We refer to consumer preferences as *increasing marginal impatience* (IMI) when δ^i_c and hence ρ^i_ϕ are positive, and as *decreasing marginal impatience* (DMI) when δ^i_c and hence ρ^i_ϕ are negative.

2.2 Firms

Let k^i and y^i ($i = 1, \cdots, N$) denote the capital stock and output in country i, respectively. For simplicity, we assume that all countries have the same per capita production function:

$$y^i = f(k^i), \tag{8}$$

satisfying $f_k > 0$, $f_{kk} < 0$. As the results of profit maximization by firms, supposing that there is no capital depreciation and no adjustment cost of investment, we obtain

$$f_k(k^i) = r \text{ and } f(k^i) - f_k(k^i)k^i = w^i. \tag{9}$$

[2] In the following analyses, we assume away a possibility of satiation ($\xi^i_c \leqq 0$) which may occur under decreasing marginal impatience ($\delta^i_c < 0$). See Hirose and Ikeda (2008) for the implications of the satiated utility of decreasing marginal impatience consumers.

By denoting the world total capital stock $K \equiv \sum_{i=1}^{N} k^i$, $k^i = \dfrac{K}{N}$ $(i = 1, \cdots, N)$ and the interest rate r can be expressed as a function of K:

$$r = \gamma(K), \text{ where } \gamma_K(K) = \frac{f_{kk}(\frac{K}{N})}{N} < 0. \tag{10}$$

2.3 The market equilibrium

The market-clearing conditions are given by

$$\sum_{i=1}^{N} c^i + \dot{K} = \sum_{i=1}^{N} y^i \text{ and } \sum_{i=1}^{N} a^i = K. \tag{11}$$

From (1), (4), (5), (9), (10), and (11), we can derive the reduced dynamic system as follows:

$$\dot{c}^i = \sigma^i\left(c^i, \phi^i\right)\left(\gamma(K) - \rho^i\left(c^i, \phi^i\right)\right) \ (i = 1, \cdots, N), \tag{12}$$

$$\dot{\phi}^i = \delta^i\left(c^i\right)\phi^i - u^i\left(c^i\right) \ (i = 1, \cdots, N), \tag{13}$$

$$\dot{K} = Nf\left(\frac{K}{N}\right) - \sum_{i=1}^{N} c^i, \tag{14}$$

$$\dot{a}^i = f\left(\frac{K}{N}\right) + \gamma(K)\left(a^i - \frac{K}{N}\right) - c^i \ (i = 1, \cdots, N-1). \tag{15}$$

This dynamic system has N pre-determined state variables, K and a^i $(i = 1, \cdots, N-1)$.

2.4 The steady-state equilibrium

From (12) through (15), the steady-state equilibrium is determined by the following equations:

$$\delta^i\left(c^{i*}\right) = \gamma(K^*) = r^* \ (i = 1, \cdots, N), \tag{16}$$

$$\sum_{i=1}^{N} c^{i*} = Nf\left(\frac{K}{N}\right), \tag{17}$$

$$\phi^{i*} = u^i\left(c^{i*}\right)/r^* \ (i = 1, \cdots, N), \tag{18}$$

$$a^{i*} = \frac{K}{N} + \frac{c^{i*} - f\left(\frac{K}{N}\right)}{r^*} \ (i = 1, \cdots, N), \tag{19}$$

where an asterisk denotes the steady-state value. The values of c^{i*} ($i = 1, \cdots, N$), K^* and r^* are jointly determined from (16) and (17). Then, ϕ^{i*} are determined from (18) and a^{i*} from (19).

3 Time Preference and Stability

By linearizing the dynamic system composed of (12) to (15) around the steady state, we obtain

$$\begin{pmatrix} \dot{c}^1 \\ \vdots \\ \dot{c}^N \\ \dot{\phi}^1 \\ \vdots \\ \dot{\phi}^N \\ \dot{K} \\ \dot{a}^1 \\ \vdots \\ \dot{a}^{N-1} \end{pmatrix} = M \begin{pmatrix} c^1 - c^{1*} \\ \vdots \\ c^N - c^{N*} \\ \phi^1 - \phi^{1*} \\ \vdots \\ \phi^N - \phi^{N*} \\ K - K^* \\ a^1 - a^{1*} \\ \vdots \\ a^{N-1} - a^{N-1*} \end{pmatrix}, \tag{20}$$

where the coefficient matrix is evaluated at the steady state; and M is defined by

$$M = \begin{pmatrix} O_{N\times N} & M_1 & O_{N\times(N-1)} \\ M_2 & rI_{N+1} & O_{(N+1)\times(N-1)} \\ M_3 & M_4 & rI_{N-1} \end{pmatrix},$$

with M_i ($i = 1, 2, 3, 4$) being given by

$$M_1 = \begin{pmatrix} -r\delta_c^1\sigma^1/\xi_c^1 & & 0 & \gamma_K\sigma^1 \\ & \ddots & & \vdots \\ 0 & & -r\delta_c^N\sigma^N/\xi_c^N & \gamma_K\sigma^N \end{pmatrix},$$

$$M_2 = \begin{pmatrix} -\xi_c^1 & & 0 \\ & \ddots & \\ 0 & & -\xi_c^N \\ -1 & \cdots & -1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} -I_{N-1} & O_{(N-1)\times 1} \end{pmatrix}, \text{ and } M_4 = \begin{pmatrix} & \gamma_K(a^1 - \frac{K}{N}) \\ O_{(N-1)\times N} & \vdots \\ & \gamma_K(a^{N-1} - \frac{K}{N}) \end{pmatrix}.$$

Since this dynamic system has N pre-determined state variables, the steady-state equilibrium is locally saddle-point stable if and only if coefficient matrix M has N negative eigenvalues. As proved in Appendix A, we thus obtain the following property.[3]

Proposition 1: The steady-state equilibrium is locally saddle-point stable if and only if

$$Q = \begin{pmatrix} r\delta_c^1 - \gamma_K & -\gamma_K & \cdots & \cdots & -\gamma_K \\ -\gamma_K & r\delta_c^2 - \gamma_K & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \ddots & & r\delta_c^{N-1} - \gamma_K & -\gamma_K \\ -\gamma_K & \cdots & \cdots & -\gamma_K & r\delta_c^N - \gamma_K \end{pmatrix}$$

is positive definite, i.e., the principal minors of Q are all positive:

- the 1st principal minor condition

$r\delta_c^i - \gamma_K > 0 \ \forall \ i$

- the 2nd principal minor condition

$\begin{vmatrix} r\delta_c^i - \gamma_K & -\gamma_K \\ -\gamma_K & r\delta_c^j - \gamma_K \end{vmatrix} > 0$ for $\forall (i, j)$, $i \neq j$

$\vdots$

- the Nth principal minor condition

[3] These stability conditions are consistent with the Appendix of Epstein (1987a).

$|Q| > 0$

Interpretation of Proposition 1

The first principal condition requires that a positive perturbation in arbitrary country i's wealth leads to a decrease in the country's savings and hence wealth accumulation. The condition is the same as the saddle-point condition that is obtained using closed economy models (e.g., Das (2003), Hirose and Ikeda (2008)).

The second principal minor condition for countries i and j, $(r\delta_c^i - \gamma_K)(r\delta_c^j - \gamma_K) - (\gamma_K)^2 > 0$, is a stability condition in which spill-over effects of a perturbation is incorporated. Consider a positive perturbation Δa^i with its effect on country j's savings being nullified by adding cancelling perturbation on a^j satisfying $-\gamma_K \Delta a^i + (r\delta_c^j - \gamma_K)\Delta a^j = 0$. The perturbation Δa^i which is augmented in this way by the consideration of the spill-over effects affects country i's savings directly by $(r\delta_c^i - \gamma_K)$ and indirectly by $-\gamma_K$ through the induced perturbation $\Delta a^j = \{\gamma_K/(r\delta_c^j - \gamma_K)\}\Delta a^i$. The principal minor condition requires that the sum of the two effects be negative.

Similarly, the nth $(n = 3, \cdots, N)$ principal minor condition requires that the augmented perturbation Δa^i, with its effects on the other countries' savings being all nullified, decreases country i's savings and hence its wealth accumulation.

The second principal minor condition in proposition 1 implies that the number of the DMI countries is required to be smaller than two for stability:

Proposition 2: If two or more than two countries have DMI, the steady-state equilibrium is unstable.

The existence of two or more than two DMI countries contradicts stability, because the second principal minor condition is violated due to destabilizing spill-over effects. To explain this, consider two DMI countries i and j which satisfy the first principal minor conditions $r\delta_c^i - \gamma_K > 0$ and $r\delta_c^j - \gamma_K > 0$. Under the conditions, the second principal minor condition is violated: an augmented positive perturbation on country i's wealth Δa^i with its effects on the country j's saving being nullified by Δa^j necessarily increases the country i's saving and its wealth accumulation $\dot{a}^i$. Therefore, The existence of two or more than two DMI countries implies instability. For the interdependent world economy to be stable, the number of the DMI countries should be less than two.

As shown by the literature (e.g., Das, 2003; Hirose and Ikeda, 2008), a closed economy populated with a DMI representative consumer can be stable. Proposition 2 implies that when it is populated with heterogenous DMI consumers, it necessarily destabilizes the economy. This is because wealth-distribution dynamics among heterogenous DMI consumers cannot be stable. For example, suppose that the economy is initially in steady-state equilibrium and consider a wealth transfer from DMI consumer (or country) j to DMI consumer (or country) i, where $\Delta a^i > 0, \Delta a^j < 0, \Delta a^n = 0$ $(n \neq i, j)$, and $\Delta K = 0$. By construction, it lowers country i's time preference (i.e., $\Delta \rho^i < 0$) and raises country j's $(\Delta \rho^j > 0)$ with leaving r unchanged. This leads to unstable process of further accumulation of a^i and decumulation of a^j.

Note that, even when there is only one DMI country in the world economy, the stability condition is necessarily violated when the number of IMI countries is large enough, because an infinitely large N makes $\gamma_K (= f_{kk}/N)$ zero and thereby violates the first principal minor condition for the DMI country.

Proposition 3: When the world economy contains a DMI country, the number of countries should be small enough for stability.

To be rough, the larger the number of countries is, the more restrictive is the stability condition, because the stability of an $(N + 1)$-country system apparently requires that each of the constituent N-country systems be stable. In this sense, an increase in the number of countries tends to destabilize the world economy.

To show this property, let us assume that δ_c^i $(i = 1, \cdots, N+1)$ and f_{kk} are constant and compare the stability conditions between an N-country system and an $(N + 1)$-country system composed of the original N countries plus a new $(N+1)$th country. Then, the stability of the augmented $(N+1)$-country system requires stronger conditions than the original N-country system for the following reasons. First, an increase in the number of countries reduces γ_K, the principal minor conditions in Proposition 1 for countries $1, \cdots, N$ are more restrictive in the augmented $(N+1)$-country system than in the original N-country system. Second, the principal minor conditions in Proposition 1 for country $(N + 1)$ are added as extra burdens to the stability conditions in the augmented $(N + 1)$-country system.

These discussions can be summarized as follows:

Proposition 4: Suppose that the world economy contains a DMI country, and that δ_c^i $(i = 1, \cdots, N+1)$ and f_{kk} are constant. Then, an increase in the number of countries, either IMI or DMI, destabilizes the world economy.

4 Conclusion

This paper examines stability of competitive equilibrium in an N-country world economy with capital accumulation, where each country's representative consumer can have either IMI or DMI. The necessary and sufficient condition for local saddle-path stability of the steady-state equilibrium is obtained in terms of positive definiteness of a simple matrix. The resulting principal minor conditions require that any positive perturbation in one country's wealth, adjusted for international spill-over effects on other country's savings, leads to a decrease in the country's saving and hence its wealth accumulation. An important corollary is that, in the presence of a DMI country, the number of interdependent countries should be sufficiently small for stability. Particularly, the existence of two or more than two DMI countries implies instability. As an implication, for a free-trade area to be stable, the number of constituent countries should not be overly large.

Appendix A: The proof of Proposition 1

Let

$$M_0 = \begin{pmatrix} O_{N\times N} & M_1 \\ M_2 & rI_{N+1} \end{pmatrix},$$

and

$$P = M_0(rI_{2N+1} - M_0) = \begin{pmatrix} P_1 & O_{N\times(N+1)} \\ O_{(N+1)\times N} & P_2 \end{pmatrix},$$

where

$$\begin{aligned} P_1 &= -M_1M_2 \\ &= \begin{pmatrix} (\gamma_K - r\delta_c^1)\sigma^1 & \gamma_K\sigma^1 & \cdots & \cdots & \gamma_K\sigma^1 \\ \gamma_K\sigma^2 & (\gamma_K - r\delta_c^2)\sigma^2 & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \ddots & & (\gamma_K - r\delta_c^{N-1})\sigma^{N-1} & \gamma_K\sigma^{N-1} \\ \gamma_K\sigma^N & \cdots & \cdots & \gamma_K\sigma^N & (\gamma_K - r\delta_c^N)\sigma^N \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} P_2 &= -M_2M_1 \\ &= \begin{pmatrix} -r\delta_c^1\sigma^1 & & O & \xi_c^1\gamma_K\sigma^1 \\ & \ddots & & \vdots \\ O & & -r\delta_c^N\sigma^N & \xi_c^N\gamma_K\sigma^N \\ -r\delta_c^1\sigma^1/\xi_c^1 & \cdots & -r\delta_c^N\sigma^N/\xi_c^N & \gamma_K\Sigma_{i=1}^N\sigma^i \end{pmatrix}. \end{aligned}$$

Lemma A.1: r is an eigenvalue of M_0.

Proof: $|rI_{2N+1} - M_0| = \begin{vmatrix} rI_N & -M_1 \\ -M_2 & O_{(N+1)\times(N+1)} \end{vmatrix}$
$= |rI_N| \left|O_{(N+1)\times(N+1)} - (-M_2)(rI_N)^{-1}(-M_1)\right| = |rI_N| \left|\frac{1}{r}(-A_NA_N)\right|$
$= \frac{1}{r}|P_2| = 0$, since P_2 is singular.

Lemma A.2: If λ is an eigenvalue of M_0, $\lambda(r-\lambda)$ is an eigenvalue of P. Therefore, if χ is an eigenvalue of P, at least either $\frac{1}{2}(r+\sqrt{r^2-4\chi})$ or $\frac{1}{2}(r-\sqrt{r^2-4\chi})$ is an eigenvalue of M_0.

Proof: Let $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$, $V = (v_1, \cdots, v_N)$, where λ_i $(i = 1, \cdots, N)$ denote the eigenvalues of M_0, and v_i $(i = 1, \cdots, N)$ the eigenvector corresponding to λ_i, that is, $M_0 V = V\Lambda \Leftrightarrow V^{-1}M_0 V = \Lambda \Leftrightarrow M_0 = V\Lambda V^{-1}$.

Then, $P = M_0(rI_{2N+1} - M_0) = V\Lambda V^{-1}(rI_{2N+1} - V\Lambda V^{-1}) = V(r\Lambda)V^{-1} - V\Lambda^2 V^{-1} = V\{\Lambda(rI - \Lambda)\}V^{-1}$, and hence $V^{-1}PV = \Lambda(rI_{2N+1} - \Lambda) = diag(\lambda_1(r-\lambda_1), \cdots, \lambda_N(r-\lambda_N))$.

Lemma A.3: The eigenvalues of P_1 and 0 are the eigenvalues of P_2.

Proof: Let $\Xi_1 = \begin{pmatrix} 1/\xi_c^1 & & & O \\ & \ddots & & \\ & & 1/\xi_c^n & \\ O & & & 1 \end{pmatrix}$, $\Xi_2 = \begin{pmatrix} \xi_c^1 & & & O \\ & \ddots & & \\ & & \xi_c^n & \\ O & & & 1 \end{pmatrix}$,

$$\Xi_3 = \begin{pmatrix} 1 & & O & 0 \\ & \ddots & & \vdots \\ O & & 1 & 0 \\ -1 & \cdots & -1 & 1 \end{pmatrix}, \ \Xi_4 = \begin{pmatrix} 1 & & O & 0 \\ & \ddots & & \vdots \\ O & & 1 & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}. \text{ Then,}$$

$$|\chi I_{N+1} - P_2| = \begin{vmatrix} \chi + r\delta_c^1\sigma^1 & & O & -\xi_c^1\gamma_K\sigma^1 \\ & \ddots & & \vdots \\ O & & \chi + r\delta_c^n\sigma^n & -\xi_c^n\gamma_K\sigma^n \\ r\delta_c^1\sigma^1/\xi_c^1 & \cdots & r\delta_c^n\sigma^n/\xi_c^n & \chi - \gamma_K\Sigma_{i=1}^N\sigma^i \end{vmatrix}$$

$$= \left| \Xi_1 \begin{pmatrix} \chi + r\delta_c^1\sigma^1 & & O & -\xi_c^1\gamma_K\sigma^1 \\ & \ddots & & \vdots \\ O & & \chi + r\delta_c^n\sigma^n & -\xi_c^n\gamma_K\sigma^n \\ r\delta_c^1\sigma^1/\xi_c^1 & \cdots & r\delta_c^n\sigma^n/\xi_c^n & \chi - \gamma_K\Sigma_{i=1}^N\sigma^i \end{pmatrix} \Xi_2 \right|$$

$(\because |\Xi_1|\,|\Xi_2| = 1)$

$$= \begin{vmatrix} \chi + r\delta_c^1\sigma^1 & & O & -\gamma_K\sigma^1 \\ & \ddots & & \vdots \\ O & & \chi + r\delta_c^n\sigma^n & -\gamma_K\sigma^n \\ r\delta_c^1\sigma^1 & \cdots & r\delta_c^n\sigma^n & \chi - \gamma_K\Sigma_{i=1}^N\sigma^i \end{vmatrix}$$

$$= \left| \Xi_3 \begin{pmatrix} \chi + r\delta_c^1\sigma^1 & & O & -\gamma_K\sigma^1 \\ & \ddots & & \vdots \\ O & & \chi + r\delta_c^n\sigma^n & -\gamma_K\sigma^n \\ r\delta_c^1\sigma^1 & \cdots & r\delta_c^n\sigma^n & \chi - \gamma_K\Sigma_{i=1}^N\sigma^i \end{pmatrix} \Xi_4 \right|$$

$(\because |\Xi_3| = |\Xi_4| = 1)$

$$
= \begin{vmatrix}
(r\delta_c^1-\gamma_K)\sigma^1 & -\gamma_K\sigma^1 & \cdots & \cdots & -\gamma_K\sigma^1 & -\gamma_K\sigma^1 \\
-\gamma_K\sigma^2 & (r\delta_c^2-\gamma_K)\sigma^2 & & \ddots & \vdots & \vdots \\
\vdots & & \ddots & & \vdots & \vdots \\
\vdots & \ddots & & (r\delta_c^{n-1}-\gamma_K)\sigma^{n-1} & -\gamma_K\sigma^{n-1} & \vdots \\
-\gamma_K\sigma^n & \cdots & \cdots & -\gamma_K\sigma^n & (r\delta_c^n-\gamma_K)\sigma^n & -\gamma_K\sigma^n \\
0 & \cdots & \cdots & \cdots & 0 & \chi
\end{vmatrix}
$$

$$
= \chi \begin{vmatrix}
(r\delta_c^1-\gamma_K)\sigma^1 & -\gamma_K\sigma^1 & \cdots & \cdots & -\gamma_K\sigma^1 \\
-\gamma_K\sigma^2 & (r\delta_c^2-\gamma_K)\sigma^2 & & \ddots & \vdots \\
\vdots & & \ddots & & \vdots \\
\vdots & \ddots & & (r\delta_c^{n-1}-\gamma_K)\sigma^{n-1} & -\gamma_K\sigma^{n-1} \\
-\gamma_K\sigma^n & \cdots & \cdots & -\gamma_K\sigma^n & (r\delta_c^n-\gamma_K)\sigma^n
\end{vmatrix}
$$

$$
= \chi \left|\chi I_N - M_1\right|.
$$

Lemma A.4: If λ is an eigenvalue of M_0 satisfying that $\lambda(r-\lambda)$ is an eigenvalue of P_1, $r-\lambda$ is also an eigenvalue of M_0.

Proof: Let $\begin{pmatrix} \upsilon_1 \\ \vdots \\ \upsilon_N \end{pmatrix}$ is the eigenvector of P_1 corresponding to the $\lambda(r-\lambda)$.

Then, the eigenvector of M_0 corresponding to λ is given by

$$
\begin{pmatrix} (r-\lambda)\upsilon_1 \\ \vdots \\ (r-\lambda)\upsilon_N \\ \xi_c^1\upsilon_1 \\ \vdots \\ \xi_c^n\upsilon_N \\ \Sigma_{i=1}^N \upsilon_i \end{pmatrix},
$$

that is

$$
A \begin{pmatrix} (r-\lambda)\upsilon_1 \\ \vdots \\ (r-\lambda)\upsilon_N \\ \xi_c^1\upsilon_1 \\ \vdots \\ \xi_c^n\upsilon_N \\ \Sigma_{i=1}^N \upsilon_i \end{pmatrix} = \lambda \begin{pmatrix} (r-\lambda)\upsilon_1 \\ \vdots \\ (r-\lambda)\upsilon_N \\ \xi_c^1\upsilon_1 \\ \vdots \\ \xi_c^n\upsilon_N \\ \Sigma_{i=1}^N \upsilon_i \end{pmatrix}.
$$

Therefore,

$$A\begin{pmatrix} \lambda\upsilon_1 \\ \vdots \\ \lambda\upsilon_N \\ \xi_c^1\upsilon_1 \\ \vdots \\ \xi_c^n\upsilon_N \\ \Sigma_{i=1}^N\upsilon_i \end{pmatrix} = (r-\lambda)\begin{pmatrix} \lambda\upsilon_1 \\ \vdots \\ \lambda\upsilon_N \\ \xi_c^1\upsilon_1 \\ \vdots \\ \xi_c^n\upsilon_N \\ \Sigma_{i=1}^N\upsilon_i \end{pmatrix},$$

implying that $r-\lambda$ is also an eigenvalue of M_0.

From Lemmas A.1-A.4, we obtain lemma A.5.

Lemma A.5: Letting χ_i $(i = 1, \cdots, N)$ be the eigenvector of P_1, the eigenvalues of M_0 are $\frac{1}{2}(r + \sqrt{r^2 - 4\chi_i})(> 0)$, $\frac{1}{2}(r - \sqrt{r^2 - 4\chi_i})$ $(i = 1, \cdots, N)$, and r.

Noting that $\frac{1}{2}(r - \sqrt{r^2 - 4\chi_i}) < 0(\geqq 0)$ for $\chi_i < 0(\geqq 0)$, we obtain lemma A.6.

Lemma A.6: M_0 (and hence M) have N negative eigenvalues if and only if eigenvalues of P_1 are all negative.

Let $D = diag(\sigma^1, \cdots, \sigma^n)$ and

$$Q = \begin{pmatrix} r\delta_c^1 - \gamma_K & -\gamma_K & \cdots & \cdots & -\gamma_K \\ -\gamma_K & r\delta_c^2 - \gamma_K & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \ddots & & r\delta_c^{N-1} - \gamma_K & -\gamma_K \\ -\gamma_K & \cdots & \cdots & -\gamma_K & r\delta_c^N - \gamma_K \end{pmatrix}.$$

Then, $P_1 = -DQ$, and we obtain lemma A.7.

Lemma A.7: All eigenvalues of P_1 are negative if and only if Q is positive definite.

Proof: All eigenvalues of $P_1(= -DQ)$ are negative:
$U^{-1}(-DQ)U = diag(\chi_1, \cdots, \chi_N)$ where $\chi_i < 0\ \forall i$

$\Leftrightarrow$ All eigenvalues of $-D^{1/2}QD^{1/2}$ are negative:
$(D^{-1/2}U)^{-1}(-D^{1/2}QD^{1/2})(D^{-1/2}U) = diag(\chi_1, \cdots, \chi_N)$
$\Leftrightarrow -D^{1/2}QD^{1/2}$ is negative definite: $z'(-D^{1/2}QD^{1/2})z < 0 \ \forall z \neq 0$
$\Leftrightarrow Q$ is positive definite: $x'Qx > 0 \ \forall x \neq 0$ (by letting $x = D^{1/2}z$)

From Lemmas A.6 and A.7, we obtain Proposition 1.

Appendix B: The interpretation of Proposition 1

For simplicity of notation, we consider the interpretation of the leading principal minor conditions in this appendix. Let $\tilde{Q}_{(n)}$ $(n = 1, \cdots, N)$ denote the nth order leading principal submatrix of Q.

A perturbation $(\Delta a^1, \cdots, \Delta a^n)$ changes ρ^i $(i = 1, \cdots, N)$ and r approximately as

$$\Delta\rho^i = \frac{r\delta_c^i}{\xi_c^i}\Delta\phi^i = \frac{r\delta_c^i}{\xi_c^i}\left(\frac{\xi_c^i}{r}\Delta c^i\right) = \delta_c^i(r\Delta a^i) = r\delta_c^i\Delta a^i,$$
$$\Delta r = \gamma_K \Delta K = \gamma_K(\Sigma_{i=1}^N \Delta a^i).$$

When $\Delta\rho^i - \Delta r > 0$ (< 0), it leads to a decrease (an increase) in the country i's savings.

- The interpretation of the first order leading principal minor condition: $\left|\tilde{Q}_{(1)}\right| = r\delta_c^1 - \gamma_K > 0$

For a perturbation $(\Delta a^1, 0, \cdots, 0)$ with $\Delta a^1 > 0$,
$\Delta\rho^1 - \Delta r = (r\delta_c^1 - \gamma_K)\Delta a^1 = \left|\tilde{Q}_{(1)}\right| \Delta a^1 > 0$

This implies that the perturbation leads to a decrease in the country 1's savings.

- The interpretation of the nth $(n = 2, \cdots, N)$ order leading principal minor condition: $\left|\tilde{Q}_{(n)}\right| > 0$

From $\tilde{Q}_{(n)} = \begin{pmatrix} \tilde{Q}_{(n-1)} & \Gamma_{(n-1)} \\ \Gamma'_{(n-1)} & r\delta_c^n - \gamma_K \end{pmatrix}$, where $\Gamma_{(n-1)} = \begin{pmatrix} -\gamma_K \\ \vdots \\ -\gamma_K \end{pmatrix}$,

$$\left|\tilde{Q}_{(n)}\right| = \left|\tilde{Q}_{(n-1)}\right| \{(r\delta_c^n - \gamma_K) - \Gamma'_{(n-1)}\tilde{Q}^{-1}_{(n-1)}\Gamma_{(n-1)}\}.$$

Supposing that the $(n-1)$th order leading principal minor condition $\left|\tilde{Q}_{(n-1)}\right| > 0$ is valid,

nth order leading principal minor condition: $\left|\tilde{Q}_{(n)}\right| > 0 \Leftrightarrow (r\delta_c^n - \gamma_K) - \Gamma'_{(n-1)}\tilde{Q}^{-1}_{(n-1)}\Gamma_{(n-1)} > 0$

Then, for a perturbation $(\Delta a^1, \cdots, \Delta a^n, 0, \cdots, 0)$ with $\Delta a^n > 0$ whose effect on savings of countries $1, \cdots, n-1$ being nullified as

$$\begin{pmatrix} \Delta\rho^1 - \Delta r \\ \vdots \\ \Delta\rho^{n-1} - \Delta r \end{pmatrix} = \tilde{Q}_{(n-1)} \begin{pmatrix} \Delta a^1 \\ \vdots \\ \Delta a^{n-1} \end{pmatrix} + \Gamma_{(n-1)}\Delta a^n = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e., $$\begin{pmatrix} \Delta a^1 \\ \vdots \\ \Delta a^{n-1} \end{pmatrix} = -\tilde{Q}^{-1}_{(n-1)}\Gamma_{(n-1)}\Delta a^n,$$

$$\Delta\rho^n - \Delta r = \begin{pmatrix} \Gamma'_{(n-1)} & r\delta_c^n - \gamma_K & \Gamma'_{(N-n)} \end{pmatrix} \begin{pmatrix} \Delta a^1 \\ \vdots \\ \Delta a^{n-1} \\ \Delta a^n \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$= \Gamma' \begin{pmatrix} \Delta a^1 \\ \vdots \\ \Delta a^{n-1} \end{pmatrix} + (r\delta_c^n - \gamma_K)\Delta a^n$$

$$= \{(r\delta_c^n - \gamma_K) - \Gamma'_{(n-1)}\tilde{Q}^{-1}_{(n-1)}\Gamma_{(n-1)}\}\Delta a^n > 0.$$

This implies that the perturbation leads to a decrease in the country n's savings.

- The interpretation of positive definiteness

Q is positive definite $\Leftrightarrow x'Qx > 0, \forall x \neq 0$

By letting $x = (\Delta a^1, \cdots, \Delta a^N)'$,

$$x'Qx = (\Delta a^1, \cdots, \Delta a^N) \begin{pmatrix} r\delta_c^1\Delta a^1 - \gamma_K\Delta K \\ \vdots \\ r\delta_c^N\Delta a^N - \gamma_K\Delta K \end{pmatrix} = (\Delta a^1, \cdots, \Delta a^N) \begin{pmatrix} \Delta\rho^1 - \Delta r \\ \vdots \\ \Delta\rho^N - \Delta r \end{pmatrix}.$$

Thus, Q is positive definite $\Leftrightarrow \sum_{i=1}^{N} \Delta a^i(\Delta\rho^i - \Delta r) > 0$ for any perturbation $(\Delta a^1, \cdots, \Delta a^N) \neq 0$

Appendix C: The proof of Proposition 3

(i) The stability condition in the original N-country system:

$$Q_N = \begin{pmatrix} r\delta_c^1 - \frac{f_{kk}}{N} & -\frac{f_{kk}}{N} & \cdots & \cdots & -\frac{f_{kk}}{N} \\ -\frac{f_{kk}}{N} & r\delta_c^2 - \frac{f_{kk}}{N} & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \ddots & & r\delta_c^{N-1} - \frac{f_{kk}}{N} & -\frac{f_{kk}}{N} \\ -\frac{f_{kk}}{N} & \cdots & \cdots & -\frac{f_{kk}}{N} & r\delta_c^N - \frac{f_{kk}}{N} \end{pmatrix}$$

is positive definite

(ii) The stability condition in the augmented $(N+1)$-country system:

$$Q_{N+1} = \begin{pmatrix} r\delta_c^1 - \frac{f_{kk}}{N+1} & -\frac{f_{kk}}{N+1} & \cdots & \cdots & \cdots & -\frac{f_{kk}}{N+1} \\ -\frac{f_{kk}}{N+1} & r\delta_c^2 - \frac{f_{kk}}{N+1} & & & & \vdots \\ \vdots & & \ddots & & \ddots & \vdots \\ \vdots & \ddots & & r\delta_c^{N-1} - \frac{f_{kk}}{N+1} & & \vdots \\ \vdots & & & & r\delta_c^N - \frac{f_{kk}}{N+1} & -\frac{f_{kk}}{N+1} \\ -\frac{f_{kk}}{N+1} & \cdots & \cdots & \cdots & -\frac{f_{kk}}{N+1} & r\delta_c^{N+1} - \frac{f_{kk}}{N+1} \end{pmatrix}$$

is positive definite.

(iii) The principal minor conditions for countries $1, \cdots, N$ in the augmented $(N+1)$-country system:

$$\tilde{Q}_{N+1(N)} = \begin{pmatrix} r\delta_c^1 - \frac{f_{kk}}{N+1} & -\frac{f_{kk}}{N+1} & \cdots & \cdots & -\frac{f_{kk}}{N+1} \\ -\frac{f_{kk}}{N+1} & r\delta_c^2 - \frac{f_{kk}}{N+1} & & \ddots & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & \ddots & & r\delta_c^{N-1} - \frac{f_{kk}}{N+1} & -\frac{f_{kk}}{N+1} \\ -\frac{f_{kk}}{N+1} & \cdots & \cdots & -\frac{f_{kk}}{N+1} & r\delta_c^N - \frac{f_{kk}}{N+1} \end{pmatrix}$$

is positive definite.

Lemma C.1: (iii)$\rightarrow$(i) is valid, but the reverse is not valid.

Proof: $\tilde{Q}_{N+1(N)} = Q_N - f_{kk}\left(\frac{1}{N} - \frac{1}{N+1}\right)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$, and hence

$x'\tilde{Q}_{N+1(N)}x = x'Q_N x + \left\{-f_{kk}\left(\frac{1}{N} - \frac{1}{N+1}\right)\right\}\left(\Sigma_{i=1}^{N} x_i\right)^2$ (where $x = (x_1, \cdots, x_N)'$).

Noting that $-f_{kk}\left(\frac{1}{N} - \frac{1}{N+1}\right) > 0$, $x'\tilde{Q}_{N+1(N)}x > 0 \ \forall x \neq 0 \rightarrow x'Q_N x > 0 \ \forall x \neq 0$.

Lemma C.2: (ii)$\rightarrow$(iii) is valid, but the reverse is not valid.

(This lemma is apparent, because $\tilde{Q}_{N+1(N)}$ is the principal submatrix of Q_N.)

From Lemmas C.1 and C.2, (ii)$\rightarrow$(i) is valid, but the reverse is not valid.

References

[1] Das, M, 2003. Optimal growth with decreasing marginal impatience. Journal of Economic Dynamics & Control 27, 1881-1898.

[2] Devereux, M., Shi, S., 1991. Capital accumulation and the current account in a two-country model. Journal of International Economics 30, 1-25.

[3] Epstein, L.G., 1987a. A simple dynamic general equilibrium model. Journal of Economic Theory 41, 68-95.

[4] Epstein, L.G., 1987b. The global stability of efficient intertemporal allocations. Econometrica 55, 329-355.

[5] Epstein, L.G., Hynes, J.A., 1983. The rate of time preference and dynamic economic analysis. Journal of Political Economy 91, 611-625.

[6] Harrison, G.W., Lau, M.I., Williams, M.B., 2002. Estimating individual discount rates in Denmark: A field experiment. American Economic Review 92, 1606–1617.

[7] Hirose, K., Ikeda, S., 2008. On decreasing marginal impatience. Japanese Economic Review 59, 259-274.

[8] Hirose, K., Ikeda, S., 2012a. Decreasing marginal impatience in a two-country economy. Journal of Economics 105, 247-262.

[9] Hirose, K., Ikeda, S., 2012b. Decreasing and Increasing Marginal Impatience and the Terms of Trade in an Interdependent World Economy. Journal of Economic Dynamics & Control 36, 1551-1565.

[10] Ikeda, S., 2001. Weakly non-separable preferences and the Harberger-Laursen-Metzler effect. Canadian Journal of Economics 34, 290-307.

[11] Ikeda, S., 2006. Luxury and wealth. International Economic Review 47, 495-526.

[12] Ikeda, S., Ohtake, F., Tsutsui, Y., 2006. Time discounting, hyperbolic discounting, and intertemporal choices: Evidences from experiments and questionnaires. mimeo.

[13] Jafarey, S., Park, H., 1998. The dynamics of optimal wealth distributions with recursive utility. Economic Letters 61, 149-158.

[14] Lawrance, E.C., 1991. Poverty and the rate of time preference, Evidence from panel data. Journal of Political Economy 99, 54-77.

[15] Lucas, R., Stokey, N., 1984. Optimal growth with many consumers. Journal of Economic Theory 32, 139-171.

[16] Obstfeld, M., 1982. Aggregate spending and the terms of trade: Is there a Laursen-Metzler effect?. Quarterly Journal of Economics 97, 251-270.

[17] Obstfeld, M., 1990. Intertemporal dependence, impatience, and dynamics. Journal of Monetary Economics 26, 45-76.

[18] Samwick, A.A., 1998. Discount rate heterogeneity and social security reform. Journal of Development Economics 57, 117–146.

[19] Svensson, L.E.O., Razin, A., 1983. The terms of trade and the current account: The Harberger-Laursen-Metzler effect. Journal of Political Economy 91, 97-125.

[20] Uzawa, H., 1968. Time Preference, the consumption function and optimum asset holdings, in: Wolfe J.N. (Ed.), Value Capital and Growth: Papers in Honour of Sir John Hicks. Aldine, Chicago, pp.485-504.