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EFFICIENCY AND STRATEGY-PROOFNESS IN OBJECT ASSIGNMENT PROBLEMS WITH MULTI DEMAND PREFERENCES

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Efficiency and strategy-proofness in object assignment problems with multi demand preferences*

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Abstract

Consider the problem of allocating objects to agents and how much they should pay. Each agent has a preference relation over pairs of a set of objects and a payment. Preferences are not necessarily quasi-linear. Non-quasi-linear preferences describe environments where payments influence agents' abilities to utilize objects. This paper is to investigate the possibility of designing *efficient* and *strategy-proof* rules in such environments. A preference relation is *single demand* if an agent wishes to receive at most one object; it is *multi demand* if whenever an agent receives one object, an additional object makes him better off. We show that if a domain contains all the single demand preferences and at least one multi demand preference relation, and there are more agents than objects, then no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers* on the domain.

Keywords: strategy-proofness, efficiency, multi demand preferences, single demand preferences, non-quasi-linear preferences, minimum price Walrasian rule
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1 Introduction

Consider an object assignment model with money. Each agent receives a set of objects, pays money, and has a preference relation over a set of objects and a payment. An allocation specifies how the objects are allocated to the agents and how much they pay. An (*allocation*) *rule* is a mapping from the class of admissible preference profiles, which we call "domain," to the set of allocations. An allocation is *efficient* if without reducing the total payment, no other allocation makes all agents at least as well off and at least one agent better off. A rule is *efficient* if it always selects an efficient allocation. *Strategy-proofness* is a condition of incentive compatibility. It requires that each agent should have an incentive to report his true preferences. This paper is to investigate the possibility of designing *efficient* and *strategy-proofness* rules.

Our model can be treated as one of the multi-object auction models. Much literature on auction theory makes an assumption on preferences, "quasi-linearity." It states that valuations over objects are not affected by payment level. On the quasi-linear domain, *i.e.*, the class of quasi-linear preferences, rules so-called "VCG rules" (Vickrey, 1961; Clarke, 1971; Groves, 1973) satisfy *efficiency* and *strategy-proofness*, and they are only rules satisfying those properties (Holmstöm, 1979).

As Marshall (1920) demonstrates, preferences are approximately quasi-linear if payments for goods are sufficiently low. However, in important applications of auction theory such as spectrum license allocations, house allocations, etc., prices are often equal to or exceed agents' annual revenues. Excessive payments for the objects may damage agents' budgets to purchase complements for effective uses of the objects, and thus may influence the benefits from the objects. Or agents may need to obtain loans to pay high amounts, and typically financial costs are nonlinear in borrowings. This factor also makes agents' preferences non-quasi-linear.[1] In such important applications, quasi-linearity is not a suitable assumption.[2]

Some authors studying object assignment problems do not assume quasi-linearity but make a different assumption on preferences, "single demand" property.[3] It states that an agent wishes to receive at most one object. On the single demand domain, *i.e.*, the class of single demand preferences, it is known that the minimum price Walrasian rules are well-defined.[4] The minimum price Walrasian (MPW) rules are rules that assign an allocation associated with the minimum price Walrasian equilibria for each preference profile. Demange and Gale (1985) show that the MPW rules are *strategy-proof* on the single demand domain. It is straightforward that in addition to *efficiency*, the MPW rules satisfy two properties on this domain: *individual rationality*; *no subsidy for losers*. *Individual rationality* states that the bundle assigned to an agent is at least as good as getting no object and paying zero. *No subsidy for losers* states that the payment of an agent who receives no object is nonnegative. Morimoto and Serizawa (2015) show that on the single demand domain, the MPW rules are only rules satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

[1] See Saitoh and Serizawa (2008) for numerical examples.

[2] Ausubel and Milgrom (2002) also discuss the importance of the analysis under non-quasi-linear preferences. Also see Sakai (2008) and Baisa (2013) for more examples of non-quasi-linear preferences.

[3] For example, see Andersson and Svensson (2014), Andersson et al. (2015), and Tierney (2015).

[4] Precisely, if agents have unit demand preferences, minimum price Walrasian equilibria exist. See Quinzi (1984), Gale (1984), and Alkan and Gale (1990).

Although the assumption of the single demand property is suitable for some important cases such as house allocation, etc., the number of such applications is limited. In many cases, there are agents who wish to receive more than one object, and indeed, many authors analyze such cases.[5]

Now, one natural question arises. Is it possible to design *efficient* and *strategy-proof* rules on a domain which is not the quasi-linear domain or the single demand domain? This is the question we address in this paper. To state our result precisely, we define a property, which we call the "multi demand"property. A preference relation satisfies the *multi demand* property if when an agent receives an object, an additional object makes him better off. We start from the single demand domain, and expand the domain by adding multi demand preferences. We show that on any domain that includes the single demand domain and contains at least one multi demand preference relation, no rule satisfies *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

This article is organized as follows. In Section 2, we introduce the model and basic definitions. In Section 3, we define the minimum price Walrasian rule. In Section 4, we state our result and show the proof. Section 5 concludes.

2 The model and definitions

Consider an economy where there are $n \geq 2$ agents and $m \geq 2$ indivisible objects. We denote the set of agents by $N \equiv \{1, \ldots, n\}$ and the set of objects by $M \equiv \{1, \ldots, m\}$. Let $\mathcal{M}$ be the power set of M. For each $a \in M$, with abuse of notation, we sometimes write a to denote $\{a\}$. Each agent receives a subset of M and pays some amount of money. Thus, the **consumption set** is $\mathcal{M} \times \mathbb{R}$ and a generic **(consumption) bundle** of agent i is denoted by $z_i = (A_i, t_i) \in \mathcal{M} \times \mathbb{R}$. Let $\mathbf{0} \equiv (\emptyset, 0)$.

Each agent i has a complete and transitive preference relation R_i over $\mathcal{M} \times \mathbb{R}$. Let P_i and I_i be the strict and indifferent relations associated with R_i. A set of preferences is called a **domain** and a generic domain is denoted by $\mathcal{R}$.[6] The following are basic properties of preferences.

Money monotonicity: For each $A_i \in \mathcal{M}$ and each $t_i, t'_i \in \mathbb{R}$ with $t_i < t'_i$, $(A_i, t_i)\ P_i\ (A_i, t'_i)$.

Possibility of compensation: For each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ and each $A'_i \in \mathcal{M}$, there are $t'_i, t''_i \in \mathbb{R}$ such that $(A_i, t_i)\ R_i\ (A'_i, t'_i)$ and $(A'_i, t''_i)\ R_i\ (A_i, t_i)$.

Continuity: For each $z_i \in \mathcal{M} \times \mathbb{R}$, the **upper contour set** at z_i, $UC_i(z_i) \equiv \{z'_i \in \mathcal{M} \times \mathbb{R} : z'_i\ R_i\ z_i\}$, and the **lower contour set** at z_i, $LC_i(z_i) \equiv \{z'_i \in \mathcal{M} \times \mathbb{R} : z_i\ R_i\ z'_i\}$, are both closed.

Desirability of object: (i) For each $(a, t_i) \in \mathcal{M} \times \mathbb{R}$, $(a, t_i)\ P_i\ (\emptyset, t_i)$, and (ii) for each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ and each $A'_i \in \mathcal{M}$ with $A'_i \subseteq A_i$, $(A_i, t_i)\ R_i\ (A'_i, t_i)$.[7]

[5] For example, see Gul and Stacchetti (1999, 2000), Bikhchandani and Ostroy (2002), Papai (2003), Ausubel (2004, 2006), Mishra and Parkes (2007), de Vries et al (2007), and Sun and Yang (2006, 2009, 2014).

[6] Formally, the domain of an allocation rule is a set of preference profiles. In this article, however, we simply call a set of preferences a domain, because we define rules on a Cartesian product of the same set of preferences.

[7] Condition (ii) requires free disposal. Morimoto and Serizawa (2015) also define *desirability of object* but require only condition (i), because in their model it is assumed that each agent can receive at most one object.

Definition 1 A preference relation is **classical** if it satisfies money monotonicity, possibility of compensation, continuity, and desirability of object.[8]

Let $\mathcal{R}^C$ be the class of classical preferences, and we call it the **classical domain**. Throughout this paper, we assume that preferences are classical.

Note that by money monotonicity, possibility of compensation and continuity, for each $R_i \in \mathcal{R}^C$, each $z_i \in \mathcal{M} \times \mathbb{R}$, and each $A_i \in \mathcal{M}$, there is a unique payment, $CV_i(A_i; z_i) \in \mathbb{R}$, such that $(A_i, CV_i(A_i; z_i))\ I_i\ z_i$. We call this payment the **compensated valuation** of A_i from z_i for R_i. Note that by money monotonicity, for each $(A_i, t_i), (A'_i, t'_i) \in \mathcal{M} \times \mathbb{R}$, $(A_i, t_i)\ R_i\ (A'_i, t'_i)$ if and only if $CV_i(A'_i; (A_i, t_i)) \leq t'_i$.

Definition 2 A preference relation $R_i \in \mathcal{R}^C$ is **quasi-linear** if for each $(A_i, t_i), (A'_i, t'_i) \in \mathcal{M} \times \mathbb{R}$ and each $t''_i \in \mathbb{R}$, $(A_i, t_i)\ I_i\ (A'_i, t'_i)$ implies $(A_i, t_i + t''_i)\ I_i\ (A'_i, t'_i + t''_i)$.

Let $\mathcal{R}^Q$ be the class of quasi-linear preferences, and we call it the **quasi-linear domain**. Obviously, $\mathcal{R}^Q \subsetneq \mathcal{R}^C$.

Remark 1 Let $R_i \in \mathcal{R}^Q$. Then,
(i) there is a **valuation function** $v_i : \mathcal{M} \to \mathbb{R}_+$ such that $v_i(\emptyset) = 0$, and for each $(A_i, t_i), (A'_i, t'_i) \in \mathcal{M} \times \mathbb{R}$, $(A_i, t_i)\ R_i\ (A'_i, t'_i)$ if and only if $v_i(A'_i) - t'_i \leq v_i(A_i) - t_i$, and
(ii) for each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ and each $A'_i \in \mathcal{M}$, $CV_i(A'_i; (A_i, t_i)) - t_i = v_i(A'_i) - v_i(A_i)$.

Now we define important classes of preferences. The following definition captures preferences of agents who desire to consume only one object.

Definition 3 A preference relation $R_i \in \mathcal{R}^C$ satisfies the **single demand** property if
(i) for each $(a, t_i) \in \mathcal{M} \times \mathbb{R}$, $(a, t_i)\ P_i\ (\emptyset, t_i)$, and
(ii) for each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ with $|A_i| > 1$, there is $a \in A_i$ such that $(A_i, t_i)\ I_i\ (a, t_i)$.[9]

Condition (i) says that an agent prefers any object to no object. Condition (ii) says that if an agent receives a set consisting of several objects, there is an object in the set that makes him indifferent to the set. Let $\mathcal{R}^U$ be the class of single demand preferences and we call it the **single demand domain**. Obviously, $\mathcal{R}^U \subsetneq \mathcal{R}^C$.

Note that the single demand property is equivalent to the following:
(i′) for each $a \in M$ and each $t_i \in \mathbb{R}$, $CV_i(a; (\emptyset, t_i)) > t_i$, and
(ii′) for each $A_i \in \mathcal{M}$ and each $t_i \in \mathbb{R}$, $CV_i(A_i; (\emptyset, t_i)) = \max_{a \in A_i} CV_i(a; (\emptyset, t_i))$.[10]

Figure 1 illustrates a single demand preference relation.

***** FIGURE 1 (Single demand preference relation) ENTERS HERE *****

We also consider preferences of agents who desire to consume more than one object. The following definition captures preferences of agents who desire to consume until a fixed number of objects.

[8] Morimoto and Serizawa (2015) also define classical preferences as a preferences satisfying the same properties that we impose. However, because of condition (ii) of desirability of object, their definition is slightly different from ours.

[9] Given a set X, $|X|$ denotes the cardinality of X.

[10] Gul and Stacchetti (1999) define the single demand property for quasi-linear preferences. However they do not require condition (i′). In their model, a preference relation $R_i \in \mathcal{R}^Q$ satisfies the single demand property if for each $A_i \in \mathcal{M} \setminus \{\emptyset\}$, $v_i(A_i) = \max_{a \in A_i} v_i(a)$. This condition corresponds to condition (ii′).

Definition 4 Let $k \in \{2 \ldots, m\}$. A preference relation $R_i \in \mathcal{R}^C$ satisfies the $k-$**objects demand** property if
(i) for each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ with $|A_i| < k$, and each $a \in \mathcal{M} \setminus A_i$, $(A_i \cup \{a\}, t_i)\ P_i\ (A_i, t_i)$, and
(ii) for each $(A_i, t_i) \in \mathcal{M} \times \mathbb{R}$ with $|A_i| \geq k$, there is $A'_i \subseteq A_i$ with $|A'_i| \leq k$ such that $(A_i, t_i)\ I_i\ (A'_i, t_i)$.[11]

Condition (i) says that an agent prefers an additional object until he gets k objects. Condition (ii) says that if an agent receives a set consisting of at least k objects, there is a subset consisting of at most k objects that makes him indifferent to the original set.

The $k-$objects demand property is equivalent to the following:
(i′) for each $A_i \in \mathcal{M}$ with $|A_i| < k$, each $a \in \mathcal{M} \setminus A_i$, and each $t_i \in \mathbb{R}$, $CV_i(A_i \cup \{a\}; (\emptyset, t_i)) > CV_i(A_i; (\emptyset, t_i))$, and
(ii′) for each $A_i \in \mathcal{M}$ with $|A_i| \geq k$ and each $t_i \in \mathbb{R}$, $CV_i(A_i; (\emptyset, t_i)) = \max_{A'_i \subseteq A_i, |A'_i| \leq k} CV_i(A'_i; (\emptyset, t_i))$.

Figure 2 illustrates a $k-$objects demand preference relation, when $k = 2$.

***** FIGURE 2 (2−objects demand preference relation) ENTERS HERE *****

The following class of preferences is larger than the classes of k-objects demand preferences. The definition requires only that when an agent receives an object, an additional one make him better off.

Definition 5 A preference relation $R_i \in \mathcal{R}^C$ satisfies the **multi demand** property if for each $(a, t_i) \in M \times \mathbb{R}$ and each $A_i \in \mathcal{M}$ with $A_i \supsetneq \{a\}$, $(A_i, t_i)\ P_i\ (a, t_i)\ P_i\ (\emptyset, t_i)$.

Let $\mathcal{R}^M$ be the class of preferences satisfying the multi demand property and call it the **multi demand domain**. Note that for each $k \in \{2, \ldots, m\}$, preferences satisfying the $k-$objects demand property satisfy the multi demand property. But there are multi demand preferences that do not satisfy the $k-$objects demand property for any $k \in \{2, \ldots, m\}$ (See Figure 3). Note also that no multi demand preference relation satisfies the single demand property, *i.e.*, $\mathcal{R}^M \cap \mathcal{R}^U = \emptyset$.

The multi demand property is equivalent to the following: for each $a \in M$, each $A_i \in \mathcal{M}$ with $A_i \supsetneq \{a\}$, and each $t_i \in \mathbb{R}$, $CV_i(A_i; (\emptyset, t_i)) > CV_i(a; (\emptyset, t_i)) > t_i$. Figure 3 illustrates a multi demand preference relation. Note that this preference relation does not satisfy the $k-$objects demand property for any $k \in \{2, \ldots, m\}$.

***** FIGURE 3 (Multi demand but not $k-$demand preference relation) ENTERS HERE *****

An **object allocation** is an n-tuple $A \equiv (A_1, \cdots, A_n) \in \mathcal{M}^n$ such that $A_i \cap A_j = \emptyset$ for each $i, j \in N$ with $i \neq j$. We denote the set of object allocations by $\mathcal{A}$. A **(feasible) allocation** is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((A_1, t_1), \ldots, (A_n, t_n)) \in (\mathcal{M} \times \mathbb{R})^n$ such that $(A_1, \ldots, A_n) \in \mathcal{A}$. We denote the set of feasible allocations by $\mathcal{Z}$. Given $z \in \mathcal{Z}$, we denote the object allocation and the agents' payments at z by $A \equiv (A_1, \ldots, A_n)$ and $t \equiv (t_1 \ldots, t_n)$, respectively.

A **preference profile** is an n-tuple $R \equiv (R_1, \cdots R_n) \in \mathcal{R}^n$. Given $R \in \mathcal{R}^n$ and $i \in N$, let $R_{-i} \equiv (R_j)_{j \neq i}$.

[11] In Gul and Stacchetti (1999), this notion is called $k-$*satiation*.

An allocation rule, or simply a **rule** on $\mathcal{R}^n$ is a function $f : \mathcal{R}^n \to \mathcal{Z}$. Given a rule f and $R \in \mathcal{R}^n$, we denote the bundle assigned to agent i by $f_i(R)$ and we write $f_i(R) = (A_i(R), t_i(R))$.

Now, we introduce standard properties of rules. The first property states that for each preference profile, a rule chooses an efficient allocation. An allocation $z \equiv ((A_i, t_i))_{i\in N} \in \mathcal{Z}$ is **(Pareto-)efficient** for $R \in \mathcal{R}^n$ if there is no feasible allocation $z' \equiv ((A'_i, t'_i))_{i\in N} \in \mathcal{Z}$ such that

$$\text{(i) for each } i \in N,\ z'_i\ R_i\ z_i, \text{ (ii) for some } j \in N, z'_j\ P_i\ z_j, \text{ and (iii) } \textstyle\sum_{i\in N} t'_i \geq \sum_{i\in N} t_i.$$

Remark 2 By money monotonicity and continuity, efficiency is equivalent to the condition that there is no feasible allocation $z' \equiv ((A'_i, t'_i))_{i\in N} \in \mathcal{Z}$ such that

$$\text{(i') for each } i \in N,\ z'_i\ I_i\ z_i, \text{ and (ii') } \textstyle\sum_{i\in N} t'_i > \sum_{i\in N} t_i.$$

Efficiency: For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

The second property states that no agent benefits from misrepresenting his preferences.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R)\ R_i\ f_i(R'_i, R_{-i})$.

The third property states that an agent is never assigned a bundle that makes him worse off than he would be if he had received no object and paid nothing.

Individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ \mathbf{0}$.

The fourth property states that the payment of each agent is always nonnegative.

No subsidy: For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.

The final property is a weaker variant of the fourth: if an agent receives no object, his payment is nonnegative.

No subsidy for losers: For each $R \in \mathcal{R}^n$ and each $i \in N$, if $A_i(R) = \emptyset$, $t_i(R) \geq 0$.

3 Minimum price Walrasian rule

In this section we define the minimum price Walrasian rules and state several facts related to them.

Let $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$ be a price vector. The **budget set** at p is defined as $B(p) \equiv \{(A_i, t_i) \in \mathcal{M} \times \mathbb{R} : t_i = \sum_{a\in A_i} p^a\}$. Given $R_i \in \mathcal{R}$, the **demand set** at p for R_i is defined as $D(R_i, p) \equiv \{z_i \in B(p) : \text{for each } z'_i \in B(p),\ z_i\ R_i\ z'_i\}$.

Remark 3 For each $R_i \in \mathcal{R}^U$, each $p \in \mathbb{R}^m_{++}$, and each $(A_i, t_i) \in D(R_i, p)$, $|A_i| \leq 1$.

Definition 6 Let $R \in \mathcal{R}^n$. A pair $((A, t), p) \in \mathcal{Z} \times \mathbb{R}^m_+$ is a **Walrasian equilibrium** for R if

$$\begin{aligned}&\text{(WE-i) for each } i \in N,\ (A_i, t_i) \in D(R_i, p), \text{ and}\\ &\text{(WE-ii) for each } y \in M, \text{ if } a \notin A_i \text{ for each } i \in N, \text{ then, } p^a = 0.\end{aligned}$$

Condition (WE-i) says that each agent receives a bundle that he demands. Condition (WE-ii) says that an object's price is zero if it is not assigned to anyone. Given $R \in \mathcal{R}^n$, let $W(R)$ and $P(R)$ be the sets of Walrasian equilibria and prices for R, respectively.

Remark 4 Let $R \in (\mathcal{R}^U)^n$ and $p \in P(R)$. If $n > m$, then $p^a > 0$ for each $a \in M$.

The facts below are results for models in which each agent can receive at most one object. On the other hand, each agent can receive several objects in our model. By condition (ii) of the single demand property, however, the same results hold for single demand preferences.

Fact 1 *For each $R \in (\mathcal{R}^U)^n$, a Walrasian equilibrium for R exists.*

Fact 1 is shown by several authors.[12] Fact 2 below states that for each preference profile, there is a unique minimum Walrasian equilibrium price vector.

Fact 2 (Demange and Gale, 1985) *For each $R \in (\mathcal{R}^U)^n$, there is a unique $p \in P(R)$ such that for each $p' \in P(R)$, $p \leq p'$.*[13]

Let $p_{\min}(R)$ denote this price vector for R. A **minimum price Walrasian equilibrium (MPWE)** is a Walrasian equilibrium associated with the minimum price. Although there might be several minimum price Walrasian equilibria, they are indifferent for each agent, *i.e.,* for each $R \in \mathcal{R}^n$, each pair $(z, p_{\min}(R)), (z', p_{\min}(R)) \in W(R)$, and each $i \in N$, $z_i \; I_i \; z'_i$.

Definition 7 A rule f on $\mathcal{R}^n$ is a **minimum price Walrasian (MPW) rule** if for each $R \in \mathcal{R}^n$, $(f(R), p_{\min}(R)) \in W(R)$.

The fact below states that on the single demand domain, the minimum price Walrasian rules satisfy the properties stated in the fact, and that if there are more agents than objects, they are the unique rules satisfying them.

Fact 3 (Demange and Gale, 1985; Morimoto and Serizawa, 2015) *(i) The minimum price Walrasian rules on $(\mathcal{R}^U)^n$ satisfy efficiency, strategy-proofness, individual rationality and no subsidy. (ii) Let $n > m$. Then, the minimum price Walrasian rules are the only rules on $(\mathcal{R}^U)^n$ satisfying efficiency, strategy-proofness, individual rationality and no subsidy for losers.*

4 Main result

We extend domains from the single demand domain by adding multi demand preferences and investigate whether *efficient* and *strategy-proof* rules still exist on such domains. In marked contrast to Fact 3 in Section 3, the results on expanded domains are negative. Namely, if there are more agents than objects, and the domain includes the single demand domain and contains at least one multi demand preference relation, then there exits no rule satisfying *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy for losers.*

Theorem *Let $n > m$. Let $R_0 \in \mathcal{R}^M$ and $\mathcal{R}$ be such that $\mathcal{R} \supseteq \mathcal{R}^U \cup \{R_0\}$. Then, no rule on $\mathcal{R}^n$ satisfies efficiency, strategy-proofness, individual rationality and no subsidy for losers.*

[12] See, for example, Quinzi (1984), Gale (1984), and Alkan and Gale (1990).
[13] For each $p, p' \in \mathbb{R}^m$, $p \leq p'$ if and only if for each $i \in \{1, \ldots, m\}$, $p^i \leq p'^i$.

Remark 5 In this paper, we assume that all the agents have the common domain $\mathcal{R}$. If each agent $i \in N$ has his own domain $\mathcal{R}_i$, Theorem can be strengthen as follows. Suppose $\mathcal{R}_i \supseteq \mathcal{R}^U$ for each $i \in N$, and there is $j \in N$ and $R_j \in \mathcal{R}^M$ such that $R_j \in \mathcal{R}_j$. Then, there is no rule on $\prod_{i \in N} \mathcal{R}_i$ satisfying *efficiency*, *strategy-proofness*, *individual rationality* and *no subsidy for losers*.

Proof: Suppose by contradiction that there is a rule f on $\mathcal{R}^n$ satisfying the four properties.

PART I. We state six lemmas that are used in the proof. Lemma 1 below states that if an agent receives no object, then his payment is zero. This is immediate from *individual rationality* and *no subsidy for losers*. Thus we omit the proof.

Lemma 1 (Zero payment for losers) *Let $R \in \mathcal{R}^n$ and $i \in N$. If $A_i(R) = \emptyset$, $t_i(R) = 0$.*

Lemma 2 below states that all objects are always assigned. This follows from *efficiency*, $n > m$, and desirability of objects. We omit the proof.

Lemma 2 (Full object assignment) *For each $R \in \mathcal{R}^n$ and each $a \in M$, there is $i \in N$ such that $a \in A_i(R)$.*

Lemma 3 below states that if an agent has a single demand preference relation, then he does not receive more than one object. This follows from *efficiency*, the single demand property, and $n > m$. We omit the proof.

Lemma 3 (Single object assignment) *Let $R \in \mathcal{R}^n$ and $i \in N$. If $R_i \in \mathcal{R}^U$, $|A_i(R)| \leq 1$.*

Lemma 4 below is a necessary condition for *efficiency*.

Lemma 4 (Necessary condition for *efficiency*) *Let $R \in \mathcal{R}^n$ and $i, j \in N$ with $i \neq j$. Let $A_i, A_j \in \mathcal{A}$ be such that $A_i \cap A_j = \emptyset$ and $A_i \cup A_j \subseteq A_i(R) \cup A_j(R)$. Then, $CV_i(A_i; f_i(R)) + CV_j(A_j; f_j(R)) \leq t_i(R) + t_j(R)$.*

Proof: Suppose by contradiction that $CV_i(A_i; f_i(R)) + CV_j(A_j; f_j(R)) > t_i(R) + t_j(R)$. Let $z' \in \mathcal{Z}$ be such that $z'_i = (A_i, CV_i(A_i; f_i(R)))$, $z'_j = (A_j, CV_j(A_j; f_j(R)))$, and for each $k \in N \setminus \{i, j\}$, $z'_k = f_k(R)$. Then $z'_k \ I_k \ f_k(R)$ for each $k \in N$. Moreover, $CV_i(A_i; f_i(R)) + CV_j(A_j; f_j(R)) + \sum_{k \neq i,j} t_k(R) > \sum_{k \in N} t_k(R)$. By Remark 2, this is a contradiction to *efficiency*. □

By Lemma 1, Lemma 4 and $n > m$, we can show that f satisfies *no subsidy*.

Lemma 5 (*No subsidy*) *For each $R_i \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq 0$.*

Proof: (Figure 4.) Suppose by contradiction that $t_i(R) < 0$ for some $R \in \mathcal{R}^n$ and $i \in N$. Let $R'_i \in \mathcal{R}^U \cap \mathcal{R}^Q$ be such that for each $a \in M$, $v'_i(a) < \min_{j \in N} CV_j(a; \mathbf{0})$. Note that by $R'_i \in \mathcal{R}^U$ and desirability of object, for each $A_i \in \mathcal{M} \setminus \{\emptyset\}$, $v'_i(A_i) < \min_{j \in N} CV_j(A_i; \mathbf{0})$.

First, suppose $A_i(R'_i, R_{-i}) = \emptyset$. By Lemma 1, $t_i(R'_i, R_{-i}) = 0$. By $v'_i(A_i(R)) > 0 > t_i(R)$,

$$f_i(R) = (A_i(R), t_i(R)) \ P'_i \ (A_i(R), v'_i(A_i(R))) \ I'_i \ \mathbf{0} = f_i(R'_i, R_{-i}).$$

This is a contradiction to *strategy-proofness*. Hence, $A_i(R'_i, R_{-i}) \neq \emptyset$.

By $n > m$ and $A_i(R'_i, R_{-i}) \neq \emptyset$, there is $j \in N \setminus \{i\}$ such that $A_j(R'_i, R_{-i}) = \emptyset$. By Lemma 1, $f_j(R'_i, R_{-i}) = \mathbf{0}$. Let $A_i \equiv \emptyset$ and $A_j \equiv A_i(R'_i, R_{-i})$. Then, $A_i \cap A_j = \emptyset$ and $A_i \cup A_j \subseteq A_i(R'_i, R_{-i}) \cup A_j(R'_i, R_{-i})$. Moreover,

$$\begin{aligned}
&CV'_i(A_i; f_i(R'_i, R_{-i})) + CV_j(A_j; f_j(R'_i, R_{-i})) \\
&\quad = CV'_i(\emptyset; f_i(R'_i, R_{-i})) + CV_j(A_i(R'_i, R_{-i}); \mathbf{0}) && \text{(by } f_j(R'_i, R_{-i}) = \mathbf{0}\text{)} \\
&\quad > CV'_i(\emptyset; f_i(R'_i, R_{-i})) + v'_i(A_i(R'_i, R_{-i}) && \text{(by the def. of } R'_i\text{)} \\
&\quad = CV'_i(\emptyset; f_i(R'_i, R_{-i})) + t_i(R'_i, R_{-i}) - CV'_i(\emptyset; f_i(R'_i, R_{-i})) && \text{(by Remark 1 (ii))} \\
&\quad = t_i(R'_i, R_{-i}) \\
&\quad = t_i(R'_i, R_{-i}) + t_j(R'_i, R_{-i}). && \text{(by } t_j(R'_i, R_{-i}) = 0\text{)}
\end{aligned}$$

This is a contradiction to Lemma 4. □

***** FIGURE 4 (Illustration of proof of Lemma 5)) ENTERS HERE *****

Lemma 6 below states that f coincides with an MPW rule on $(\mathcal{R}^U)^n$. This is immediate from Fact 3 (ii). Thus we omit the proof.

Lemma 6 *For each $R \in (\mathcal{R}^U)^n$, $(f(R), p_{\min}(R)) \in W(R)$.*

Part II. The proof of Theorem has five steps.

Step 1: Constructing a preference profile.

Let $R_1 \equiv R_0$, and

$$\begin{aligned}
\bar{t}_1 &\equiv \max\{t_1 - CV_1(\emptyset; (A_1, t_1)) : (A_1, t_1) \in \mathcal{M} \times \mathbb{R}, (A_1, t_1)\ R_1\ \mathbf{0}, t_1 \geq 0\}, \text{ and} \\
\underline{t}_1 &\equiv \min\{CV_1(A_1; (a, t_1)) - t_1 : (a, t_1) \in \mathcal{M} \times \mathbb{R}, (a, t_1)\ R_1\ \mathbf{0}, t_1 \geq 0, A_1 \supsetneq \{a\}\}.
\end{aligned}$$

Note that $\bar{t}_1$ and $\underline{t}_1$ are well-defined and $\bar{t}_1 > \underline{t}_1$.[14,15] Since R_1 satisfies the multi demand property, we have $\underline{t}_1 > 0$. Let $a^* \in M$ be such that for each $a \in M$, $(a^*, \underline{t}_1)\ R_1\ (a, \underline{t}_1)$. By desirability of object and money monotonicity, there is $t_1^* \in (0, \underline{t}_1)$ such that $(a^*, t_1^*)\ P_1\ \mathbf{0}$. By money monotonicity, for each $a \in M \setminus \{a^*\}$, $CV_1(a; (a^*, t_1^*)) < CV_1(a; (a^*, \underline{t}_1)) \leq \underline{t}_1$. We may assume $a^* = 1$ since the other cases can be treated in the same way. Let $p \in \mathbb{R}^m_{++}$ be such that $p^1 = t_1^*$, and for each $a \in M \setminus \{1\}$, $\max\{CV_1(a; (1, p^1)), 0\} < p^a < \underline{t}_1$. Figure 5 illustrates R_1 and p.

14 By money monotonicity and possibility of compensation, the sets $\{t_1 - CV_1(\emptyset; (A_1, t_1)) : (A_1, t_1) \in \mathcal{M} \times \mathbb{R}, (A_1, t_1)\ R_1\ \mathbf{0}, t_1 \geq 0\}$ and $\{CV_1(A_1; (a, t_1)) - t_1 : (a, t_1) \in \mathcal{M} \times \mathbb{R}, (a, t_1)\ R_1\ \mathbf{0}, t_1 \geq 0, A_1 \supsetneq \{a\}\}$ are bounded. By continuity of R_1, compensated valuation is a continuous function, and thus, we can show that the two sets are closed.

15 Let $(a, t_1) \in \mathcal{M} \times \mathbb{R}$ be such that $(a, t_1)\ R_1\ \mathbf{0}$ and $t_1 \geq 0$. Let $A_1 \supsetneq \{a\}$ and $t'_1 \equiv CV_1(A_1; (a, t_1))$. Then, $(A_1, t'_1)\ I_1\ (a, t_1)\ R_1\ \mathbf{0}$. By desirability of object, $t'_1 = CV_1(A_1; (a, t_1)) \geq t_1 \geq 0$. Thus, $(A_1, t'_1) \in \{(A''_1, t''_1) \in \mathcal{M} \times \mathbb{R} : (A''_1, t''_1)\ R_1\ \mathbf{0}, t''_1 \geq 0\}$. Therefore, $t'_1 - CV_1(\emptyset; (A_1, t_1)) \leq \bar{t}_1$.

Note that by desirability of object, $CV_1(\emptyset; (A_1, t'_1)) = CV_1(\emptyset; (a, t_1)) < t_1$. Thus,

$$\underline{t}_1 \leq CV_1(A_1; (a, t_1)) - t_1 = t'_1 - t_1 < t'_1 - CV_1(\emptyset; (A_1, t'_1)) \leq \bar{t}_1.$$

***** FIGURE 5 (R_1 and p) ENTERS HERE ******

Let $R_2 \in \mathcal{R}^U$ satisfy the following conditions:

$$
\begin{aligned}
&(2-1) \quad CV_2(a;\mathbf{0}) \begin{cases} > \bar{t}_1 & \text{if } a = 1, \\ \in (p^2, \underline{t}_1) & \text{if } a = 2, \\ < p^a & \text{otherwise,} \end{cases} \\
&(2-2) \quad \text{for each } a \in M \setminus \{2\}, CV_2(a;(2,p^2)) < 0, \text{ and} \\
&(2-3) \quad CV_2(\emptyset;(2,0)) > \max_{a \in M \setminus \{1\}} CV_2(a;\mathbf{0}) - \underline{t}_1.
\end{aligned}
$$

For each $i \in \{3,\ldots,m\}$, let $R_i \in \mathcal{R}^U$ satisfy the following two conditions:

$$
\begin{aligned}
&(i-1) \quad CV_i(a;\mathbf{0}) \begin{cases} < p^1 & \text{if } a = 1, \\ > \bar{t}_1 & \text{otherwise,} \end{cases} \\
&(i-2) \quad \text{for each } a \in M \setminus \{i\}, CV_i(a;(i,p^i)) < 0, \text{ and} \\
&(i-3) \quad CV_i(\emptyset;(1,0)) > p^1 - \bar{t}_1.
\end{aligned}
$$

Figure 6 illustrates R_2 and R_i for $i \in \{3,\ldots,m\}$.

***** FIGURE 6 (R_2, and R_i ($i \in \{3,\ldots,m\}$)) ENTERS HERE *****

By $n > m$, there are at least $m+1$ agents. Let $R_{m+1} \in \mathcal{R}^U \cap \mathcal{R}^Q$ be such that for each $a \in M$, $v(a) = p^a$. If there are more than $m+1$ agents, then for each $i \in \{m+2,\ldots,n\}$, let $R_i \in \mathcal{R}^U \cap \mathcal{R}^Q$ be such that for each $a \in M$, $v_i(a) < \min_{b \in M, j \in \{1,\ldots,m+1\}} CV_j(b;\mathbf{0})$. Denote $R \equiv (R_1,\ldots,R_n)$. □

Step 2: For each $i \in \{m+2,\ldots,n\}$, $A_i(R) = \emptyset$.

Suppose by contradiction that for some $i \in \{m+2,\ldots,n\}$, $A_i(R) \neq \emptyset$. By $R_i \in \mathcal{R}^U$ and Lemma 3, there is $a \in M$ such that $A_i(R) = a$. Since there are only m objects, there is $j \in \{1,\ldots,m+1\}$ such that $A_j(R) = \emptyset$. By Lemma 1, $t_j(R) = 0$.

Let $A_i \equiv \emptyset$ and $A_j \equiv a$. Then $A_i \cap A_j = \emptyset$ and $A_i \cup A_j \subseteq A_i(R) \cup A_j(R)$. Moreover,

$$
\begin{aligned}
&CV_i(A_i; f_i(R)) + CV_j(A_j; f_j(R)) \\
&\quad = CV_i(\emptyset; f_i(R)) + CV_j(a;\mathbf{0}) && (\text{by } f_j(R) = \mathbf{0}) \\
&\quad = t_i(R) - v_i(a) + CV_j(a;\mathbf{0}) && (\text{by } R_i \in \mathcal{R}^Q \text{ and Remark 1(ii)}) \\
&\quad > t_i(R) - CV_j(a;\mathbf{0}) + CV_j(a;\mathbf{0}) && (\text{by } v_i(a) < CV_j(a;\mathbf{0})) \\
&\quad = t_i(R) \\
&\quad = t_i(R) + t_j(R). && (\text{by } t_j(R) = 0)
\end{aligned}
$$

This is a contradiction to Lemma 4. □

Given $i \in N$ and $R_{-i} \in \mathcal{R}^{n-1}$, we define the **option set** of agent i for R_{-i} by

$$o_i(R_{-i}) \equiv \{z_i \in \mathcal{M} \times \mathbb{R} : \exists R_i \in \mathcal{R} \text{ s.t. } f_i(R_i, R_{-i}) = z_i\}.$$

Step 3: Let $a \in M$. If $a = 1$, then $(a, t^a) \in o_1(R_{-1})$ for some $t^a \leq p^1$. If $a \neq 1$, then $(a, t^a) \in o_1(R_{-1})$ for some $t^a \geq p^a$.

Let $R'_1 \in \mathcal{R}^U$ be such that

$$
\begin{aligned}
&(1'-1) \quad CV'_1(a;\mathbf{0}) > \max_{i \in N} CV_i(a;\mathbf{0}),\\
&(1'-2) \quad \text{for each } b \in M \setminus \{a\}, CV'_1(b;\mathbf{0}) < \min_{i \in N} CV_i(b;\mathbf{0}), \text{ and}\\
&(1'-3) \quad \text{for each } b \in M \setminus \{a\}, CV'_1(b;(a,p^a)) < 0.
\end{aligned}
$$

Figure 7 illustrates R'_1.

***** FIGURE 7 (R'_1) ENTERS HERE *****

By $(R'_1, R_{-1}) \in (\mathcal{R}^U)^n$ and Lemma 6, $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}). Let $\hat{p} \equiv p_{\min}(R'_1, R_{-1})$. By $(R'_1, R_{-1}) \in (\mathcal{R}^U)^n$ and Lemma 3, $|A_i(R'_1, R_{-1})| \leq 1$ for each $i \in N$. In the following two paragraphs, we show $(a, \hat{p}^a) \in o_1(R_{-1})$.

First, suppose $A_1(R'_1, R_{-1}) = \emptyset$. Then $t_1(R'_1, R_{-1}) = \mathbf{0}$ by Lemma 1. By $A_1(R'_1, R_{-1}) = \emptyset$, there is $i \in N \setminus \{1\}$ such that $A_i(R'_1, R_{-1}) = a$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $t_i(R'_1, R_{-1}) = \hat{p}^a$. By *individual rationality*, $f_i(R'_1, R_{-1})\ R_i\ \mathbf{0}$, and therefore $\hat{p}^a = t_i(R) \leq CV_i(a;\mathbf{0})$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $f_1(R'_1, R_{-1}) \in D(R'_1, \hat{p})$. Thus, $f_1(R'_1, R_{-1})\ R'_1\ f_i(R'_1, R_{-1}) = (a, \hat{p}^a)$. Therefore,

$$CV'_1(a;\mathbf{0}) = CV'_1(a; f_1(R'_1, R_{-1})) \leq \hat{p}^a \leq CV_i(a;\mathbf{0}).$$

This is a contradiction to $(1'-1)$. Hence, $A_1(R'_1, R_{-1}) \neq \emptyset$.

Next, suppose $A_1(R'_1, R_{-1}) = b$ for some $b \in M \setminus \{a\}$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $t_1(R'_1, R_{-1}) = \hat{p}^b$. By *individual rationality*, $(b, \hat{p}^b) = f_1(R'_1, R_{-1})\ R'_1\ \mathbf{0}$, and thus, $\hat{p}^b \leq CV'_1(b;\mathbf{0})$. By $n > m$, there is $i \in N \setminus \{1\}$ such that $A_i(R'_1, R_{-1}) = \emptyset$. By Lemma 1, $f_i(R'_1, R_{-1}) = \mathbf{0}$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $f_i(R'_1, R_{-1}) \in D(R_i, \hat{p})$. Thus, $f_i(R'_1, R_{-1})\ R_i\ f_1(R'_1, R_{-1}) = (b, \hat{p}^b)$. Therefore,

$$CV_i(b;\mathbf{0}) = CV_i(b; f_i(R'_1, R_{-1}) \leq \hat{p}^b \leq CV'_1(b;\mathbf{0}).$$

This is a contradiction to $(1'-2)$. Thus, $A_1(R'_1, R_{-1}) \neq b$ for each $b \in M \setminus \{a\}$. By $A_1(R'_1, R_{-1}) \neq \emptyset$ and $|A_1(R'_1, R_{-1})| \leq 1$, we have $A_1(R'_1, R_{-1}) = a$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $t_1(R'_1, R_{-1}) = \hat{p}^a$. Hence, $(a, \hat{p}^a) \in o_1(R_{-1})$.

Next, we show that $\hat{p}^a \leq p^a$ if $a = 1$, and $\hat{p}^a \geq p^a$ otherwise.

Case 1: $a = 1$. Let $z \in \mathcal{Z}$ be such that for each $i \in \{1, \ldots, m\}$, $z_i = (i, p^i)$, and for each $i \in \{m+1, \ldots, n\}$, $z_i = \mathbf{0}$. We show that for each $i \in N$, if $i = 1$, $z_1 \in D(R'_1, p)$, and if $i \neq 1$, $z_i \in D(R_i, p)$. Then, we conclude that (z, p) is a Walrasian equilibrium for (R'_1, R_{-1}), and therefore, by $\hat{p} = p_{\min}(R'_1, R_{-1})$, $\hat{p}^1 \leq p^1$.

Note that by $p \in \mathbb{R}^m_{++}$ and Remark 3, if $i = 1$, then $|A_1| \leq 1$ for each $(A_1, t_1) \in D(R'_1, p)$, and if $i \neq 1$, then $|A_i| \leq 1$ for each $(A_i, t_i) \in D(R_i, p)$.

Subcase 1-1: $i = 1$. By $(1'-3)$ and $p \in \mathbb{R}^m_+$, $CV'_1(b;(1,p^1)) < 0 \leq p^b$ for each $b \in M \setminus \{1\}$. Thus, $(1, p^1)\ P'_1\ (b, p^b)$ for each $b \in M \setminus \{1\}$. Also by $(1'-3)$ and desirability of object, $CV'_1(\emptyset;(1,p^1)) < 0$, and this implies $(1,p^1)\ P'_1\ \mathbf{0}$. Thus, $z_1 = (1, p^1) \in D(R'_1, p)$.

Subcase 1-2: $i \in \{2, \ldots, m\}$. By $(i-2)$ and $p \in \mathbb{R}^m_+$, $CV_i(b; (i, p^i)) < 0 \leq p^b$ for each $b \in M \setminus \{i\}$. Thus, $(i, p^i)\ P_i\ (b, p^b)$ for each $b \in M \setminus \{i\}$. Also by $(i-2)$ and desirability of object, $CV_i(\emptyset; (i, p^i)) < 0$, and this implies $(i, p^i)\ P_i\ \mathbf{0}$. Thus, $z_i = (i, p^i) \in D(R_i, p)$.

Subcase 1-3: $i = m+1$. By the def. of R_{m+1}, for each $b \in M$, $CV_{m+1}(b; \mathbf{0}) = v_{m+1}(b) = p^b$, and this implies $\mathbf{0}\ I_{m+1}\ (b, p^b)$. Thus, $z_{m+1} = \mathbf{0} \in D(R_{m+1}, p)$.

Subcase 1-4: $i \in \{m+2, \ldots, n\}$. For each $b \in M$, $CV_i(b; \mathbf{0}) = v_i(b) < v_{m+1}(b) < p^b$, and this implies $\mathbf{0}\ P_i\ (b, p^b)$. Thus, $z_i = \mathbf{0} \in D(R_i; \mathbf{0})$.

Case 2: $a \in \{2 \ldots, m\}$. Let $i = a$. Suppose by contradiction that $\hat{p}^a < p^a$. By $(i-2)$ and $\hat{p} \in \mathbb{R}^m_+$, $CV_i(b; (a, \hat{p}^a)) < CV_i(b; (a, p^a)) < 0 \leq \hat{p}^b$ for each $b \in M \setminus \{a\}$. Thus $(a, \hat{p}^a)\ P_i\ (b, \hat{p}^b)$ for each $b \in M \setminus \{a\}$. Also by $(i-2)$ and desirability of object, $CV_i(\emptyset; (a, \hat{p}^a)) < CV_i(\emptyset; (a, p^a)) < 0$, and this implies $(a, \hat{p}^a)\ P_i\ \mathbf{0}$. Note that by $n < m$ and Remark 4 (i), $\hat{p} \in \mathbb{R}^m_{++}$. Thus by Remark 3, $D(R_i, \hat{p}) = \{(a, \hat{p}^a)\}$. Since $f(R'_1, R_{-1})$ is an MPWE allocation for (R'_1, R_{-1}), $A_i(R'_1, R_{-1}) = a$. This is a contradiction to $A_1(R'_1, R_{-1}) = a$. □

Step 4: $|A_1(R)| > 1$.

Suppose by contradiction that $|A_1(R)| \leq 1$. Remember that by Lemma 3, $|A_i(R)| \leq 1$ for each $i \in \{2, \ldots, m+1\}$, and that by Step 2, $A_i(R) = \emptyset$ for each $i \in \{m+2, \ldots, n\}$. Thus, since there are m objects and they are always assigned by Lemma 2, there is $i \in \{2, m+1\}$ such that $|A_i(R)| = 1$.

By Step 3, there is $t^1 \leq p^1$ such that $(1, t^1) \in o_1(R_{-1})$, and for each $a \in M \setminus \{1\}$, there is $p^a \leq t^a$ such that $(a, t^a) \in o_1(R_{-1})$. By the def. of p, for each $a \in M \setminus \{1\}$, $CV_1(a; (1, t^1)) \leq CV_1(a; (1, p^1)) < p^a \leq t^a$, and thus, $(1, t^1)\ P_1\ (a, t^a)$. Therefore, by $|A_1(R)| \leq 1$ and *strategy-proofness*, $A_1(R) = 1$.

By $A_1(R) = 1$, there is $a \in M \setminus \{1\}$ such that $A_i(R) = a$. Let $A_1 \equiv \{1, a\}$ and $A_i \equiv \emptyset$. Then, $A_1 \cap A_i = \emptyset$ and $A_1 \cup A_i \subseteq A_1(R) \cup A_i(R)$. By *individual rationality*, $f_1(R)\ R_1\ \mathbf{0}$, and by Lemma 5, $t_1(R) \geq 0$. Thus, by $A_1(R) = 1$ and the def. of $\underline{t}_1$, $CV_1(\{1, a\}; f_1(R)) - t_1(R) \geq \underline{t}_1$. Therefore, by $A_1 = \{1, a\}$ and $A_i = \emptyset$,

$$CV_1(A_1; f_1(R)) + CV_i(A_i; f_i(R)) \geq \underline{t}_1 + t_1(R) + CV_i(\emptyset; f_i(R)). \quad (1)$$

We derive a contradiction in each of the following cases since $i = 2$ or $i = m+1$.

Case 1: $i = 2$. By *individual rationality*, $f_2(R)\ R_2\ \mathbf{0}$, and thus, $t_2(R) \leq CV_2(a; \mathbf{0})$. By $(2-2)$ and Lemma 5, $CV_2(a; (2, 0)) < 0 \leq t_2(R)$ and thus, $CV_2(\emptyset; f_2(R)) > CV_2(\emptyset; (2, 0))$. Therefore,

$$\begin{aligned}
&CV_2(\emptyset; f_2(R)) \\
&> \max_{b \in M \setminus \{1\}} CV_2(b; \mathbf{0}) - \underline{t}_1 && \text{(by } CV_2(\emptyset; f_2(R)) > CV_2(\emptyset; (2,0)) \text{ and } (2-3)) \\
&\geq t_2(R) - CV_2(a; \mathbf{0}) + \max_{b \in M \setminus \{1\}} CV_2(b; \mathbf{0}) - \underline{t}_1 && \text{(by } t_2(R) \leq CV_2(a; \mathbf{0})) \\
&\geq t_2(R) - \underline{t}_1. && \text{(by } a \neq 1) \quad (2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&CV_1(A_1; f_1(R)) + CV_2(A_2; f_2(R)) \\
&\quad \geq \underline{t}_1 + t_1(R) + CV_2(\emptyset; f_2(R)) && \text{(by (1))} \\
&\quad > \underline{t}_1 + t_1(R) + t_2(R) - \underline{t}_1 && \text{(by (2))} \\
&\quad = t_1(R) + t_2(R).
\end{aligned}$$

This is a contradiction to Lemma 4.

Case 2: $i = m+1$. Note that

$$\begin{aligned}
&CV_1(A_1; f_1(R)) + CV_{m+1}(A_{m+1}; f_{m+1}(R)) \\
&\quad = \underline{t}_1 + t_1(R) + CV_{m+1}(\emptyset; f_{m+1}(R)) && \text{(by (1))} \\
&\quad > \underline{t}_1 + t_1(R) + t_{m+1}(R) - v_{m+1}(a) && \text{(by } R_{m+1} \in \mathcal{R}^Q \text{ and Remark 1 (ii))} \\
&\quad = \underline{t}_1 + t_1(R) + t_{m+1}(R) - p^a && \text{(by the def. of } R_{m+1}) \\
&\quad > t_1(R) + t_{m+1}(R). && \text{(by } p^a < \underline{t}_1)
\end{aligned}$$

This is a contradiction to Lemma 4. □

Step 5: Completing the proof.

By *individual rationality*, $f_1(R)\ R_1\ \mathbf{0}$, and by Lemma 5, $t_1(R) \geq 0$. Therefore, by the def. of $\bar{t}_1$, $t_1(R) - CV_1(\emptyset; f_1(R)) \leq \bar{t}_1$. Thus, by desirability of object, for each $a \in A_1(R)$,

$$t_1(R) - CV_1(A_1(R) \setminus \{a\}; f_1(R)) \leq t_1(R) - CV_1(\emptyset; f_1(R)) \leq \bar{t}_1. \tag{3}$$

Since $|A_1(R)| > 1$ by Step 4, we have either $A_2(R) = \emptyset$ or $A_i(R) = \emptyset$ for some $i \in \{3, \ldots, m\}$. We derive a contradiction for each case.

Case 1: $A_2(R) = \emptyset$. By Lemma 1, $t_2(R) = 0$. Remember that by Step 2, for each $i \in \{m+2, \ldots, n\}$, $A_i(R) = \emptyset$. Thus, we have three subcases: $1 \in A_1(R)$; $1 \in A_i(R)$ for some $i \in \{3 \ldots, m\}$; $1 \in A_{m+1}(R)$.

Subcase 1-1: $1 \in A_1(R)$. Let $A_1 \equiv A_1(R) \setminus \{1\}$ and $A_2 \equiv 1$. Then, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 \subseteq A_1(R) \cup A_2(R)$. Moreover,

$$\begin{aligned}
&CV_1(A_1; f_1(R)) + CV_2(A_2; f_2(R)) \\
&\quad = CV_1(A_1(R) \setminus \{1\}; f_1(R)) + CV_2(1; f_2(R)) \\
&\quad \geq t_1(R) - \bar{t}_1 + CV_2(1; \mathbf{0}) && \text{(by (3) and } f_2(R) = \mathbf{0}) \\
&\quad > t_1(R) && \text{(by } (2-1)) \\
&\quad = t_1(R) + t_2(R). && \text{(by } t_2(R) = 0)
\end{aligned}$$

This is a contradiction to Lemma 4.

Subcase 1-2: $1 \in A_i(R)$ for some $i \in \{3, \ldots, m\}$. By $R_i \in \mathcal{R}^U$ and Lemma 3, $A_i(R) = 1$. Let $A_2 \equiv 1$ and $A_i \equiv \emptyset$. Then, $A_2 \cap A_i = \emptyset$ and $A_2 \cup A_i \subseteq A_2(R) \cup A_i(R)$. By Lemma 5, $t_i(R) \geq 0$. Thus by $A_i(R) = 1$ and $(i-3)$, $CV_i(\emptyset; f_i(R)) \geq CV_i(\emptyset; (1,0)) > p^1 - \bar{t}_1$. By *individual rationality*, $f_i(R)\ R_i\ \mathbf{0}$, and thus, by $(i-1)$, $t_i(R) \leq CV_i(1;\mathbf{0}) < p^1$. Therefore,

$$
\begin{aligned}
&CV_2(A_2; f_2(R)) + CV_i(A_i; f_i(R)) \\
&\quad = CV_2(1;\mathbf{0}) + CV_i(\emptyset; f_i(R)) && \text{(by } f_2(R) = \mathbf{0}\text{)} \\
&\quad > \bar{t}_1 + p^1 - \bar{t}_1 && \text{(by } (2-1) \text{ and } CV_i(\emptyset; f_i(R)) > p^1 - \bar{t}_1\text{)} \\
&\quad = p^1 \\
&\quad > t_i(R) && \text{(by } p^1 > t_i(R)\text{)} \\
&\quad = t_2(R) + t_i(R). && \text{(by } t_2(R) = 0\text{)}
\end{aligned}
$$

This is a contradiction to Lemma 4.

Subcase 1-3: $1 \in A_{m+1}(R)$. By $R_{m+1} \in \mathcal{R}^U$ and Lemma 3, $A_{m+1}(R) = 1$. Let $A_2 \equiv 1$ and $A_{m+1} \equiv \emptyset$. Then, $A_2 \cap A_{m+1} = \emptyset$ and $A_2 \cup A_{m+1} \subsetneq A_2(R) \cup A_{m+1}(R)$. Moreover,

$$
\begin{aligned}
&CV_2(A_2; f_2(R)) + CV_{m+1}(A_{m+1}; f_{m+1}(R)) \\
&\quad = CV_2(1;\mathbf{0}) + CV_{m+1}(\emptyset; f_{m+1}(R)) && \text{(by } f_2(R) = \mathbf{0}\text{)} \\
&\quad > \bar{t}_1 + t_{m+1}(R) - v_{m+1}(1) && \text{(by } (2-1),\ R_{m+1} \in \mathcal{R}^Q\text{, and Remark 1 (ii))} \\
&\quad = \bar{t}_1 + t_{m+1}(R) - p^1 && \text{(by } v_{m+1}(1) = p^1\text{)} \\
&\quad > \bar{t}_1 + t_{m+1}(R) - \underline{t}_1 && \text{(by } p^1 < \underline{t}_1\text{)} \\
&\quad > t_{m+1}(R) && \text{(by } \underline{t}_1 < \bar{t}_1\text{)} \\
&\quad = t_2(R) + t_{m+1}(R). && \text{(by } t_2(R) = 0\text{)}
\end{aligned}
$$

This is a contradiction to Lemma 4.

Case 2: $A_i(R) = \emptyset$ for some $i \in \{3, \ldots, m\}$. By $|A_1(R)| > 1$, there is $a \in M \setminus \{1\}$ such that $a \in A_1(R)$. Let $A_1 \equiv A_1(R) \setminus \{a\}$ and $A_i = a$. Then, $A_1 \cap A_i = \emptyset$ and $A_1 \cup A_i \subseteq A_1(R) \cup A_i(R)$. Moreover,

$$
\begin{aligned}
&CV_1(A_1; f_1(R)) + CV_i(A_i; f_i(R)) \\
&\quad = CV_1(A_1(R) \setminus \{a\}; f_1(R)) + CV_i(a;\mathbf{0}) && \text{(by } f_i(R) = \mathbf{0}\text{)} \\
&\quad > t_1(R) - \bar{t}_1 + \bar{t}_1 && \text{(by (3) and } (i-1)\text{)} \\
&\quad = t_1(R) \\
&\quad = t_1(R) + t_i(R). && \text{(by } t_i(R) = 0\text{)}
\end{aligned}
$$

This is a contradiction to Lemma 4. □

5 Concluding remarks

In this article, we considered an object assignment problem with money where each agent can receive more than one object. We focused on domains that include the single demand domain and contain some multi demand preferences. We studied allocation rules satisfying *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*, and showed that if the domain includes the single demand domain and contains at least one multi demand preference relation, and there are more agents than objects, then no rule on the domain satisfies the four properties. As we discussed in Section 1, for the applicability to various important cases, we investigated the possibility of designing *efficient* and *strategy-proof* rules on a domain which is not the quasi-linear domain or the single demand domain. Our result suggests the difficulty of designing *efficient* and *strategy-proof* rules on such a domain. We state two remarks on our result.

Maximal domain. Some literature on *strategy-proofness* addresses maximal domains on which there are rules satisfying desirable properties.[16] A domain $\mathcal{R}$ is a *maximal domain* for a list of property on rules if there is a rule on $\mathcal{R}^n$ satisfying the properties, and for each $\mathcal{R}' \supsetneq \mathcal{R}$, no rule on $(\mathcal{R}')^n$ satisfies the properties. Our theorem almost implies that when the number of agents is greater than that of objects, the single demand domain is a maximal domain for *efficiency*, *strategy-proofness*, *individual rationality*, and *no subsidy for losers*.

However, our theorem does not imply such a maximal domain result in that only multi demand preferences can be added to the single demand domain to derive the non-existence of rules satisfying the above properties. For example, consider a preference relation such that there is $k \in \{2, \ldots, m\}$ such that (i) for each $(A_i, t_i) \in \mathcal{M}$ with $|A_i| \leq k$, there is $a \in A_i$ such that $(A_i, t_i)\, I_i\, (a, t_i)$, and (ii) for each $(A_i, t_i) \in \mathcal{M}$ with $|A_i| \geq k$, and each $a \in M \setminus A_i$, $(A_i \cup \{a\})\, P_i\, (A_i, t_i)$. This preference relation does not satisfy the single demand property nor the $k'-$object demand property for any $k' \in \{2, \ldots, m\}$. When such a preference relation is added to the single demand domain, our theorem does not exclude the possibility that some rules satisfy the four properties. Hence, it is an open question whether there exist rules satisfying the four properties when a non-multi demand preference is added to the single demand domain.

Identical objects. Some literature on object assignment problems also study the case in which the objects are identical.[17] In this paper, we assume that the objects are not identical. And this assumption plays an important role in our proof. Therefore, our theorem does not exclude the possibility that when objects are identical, multi demand preferences can be added to the single demand domain while keeping the existence of rules satisfy the four properties.

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[16] For example, see Ching and Serizawa (1998), Berga and Serizawa (2000), Massó and Neme (2001), Ehlers (2002), etc.

[17] For example, see Saitoh and Serizawa (2008), Ashlagi and Serizawa (2012), Adachi (2014), etc.

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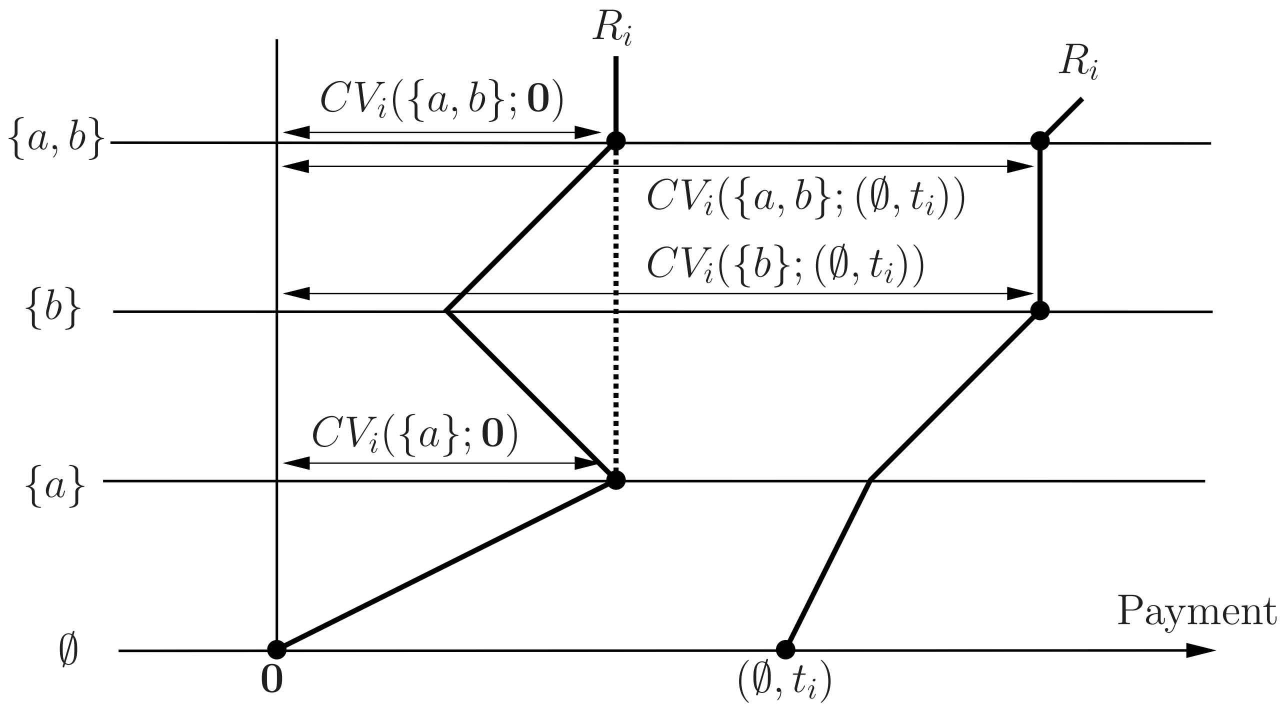

Figure 1: Single demand preference relation.

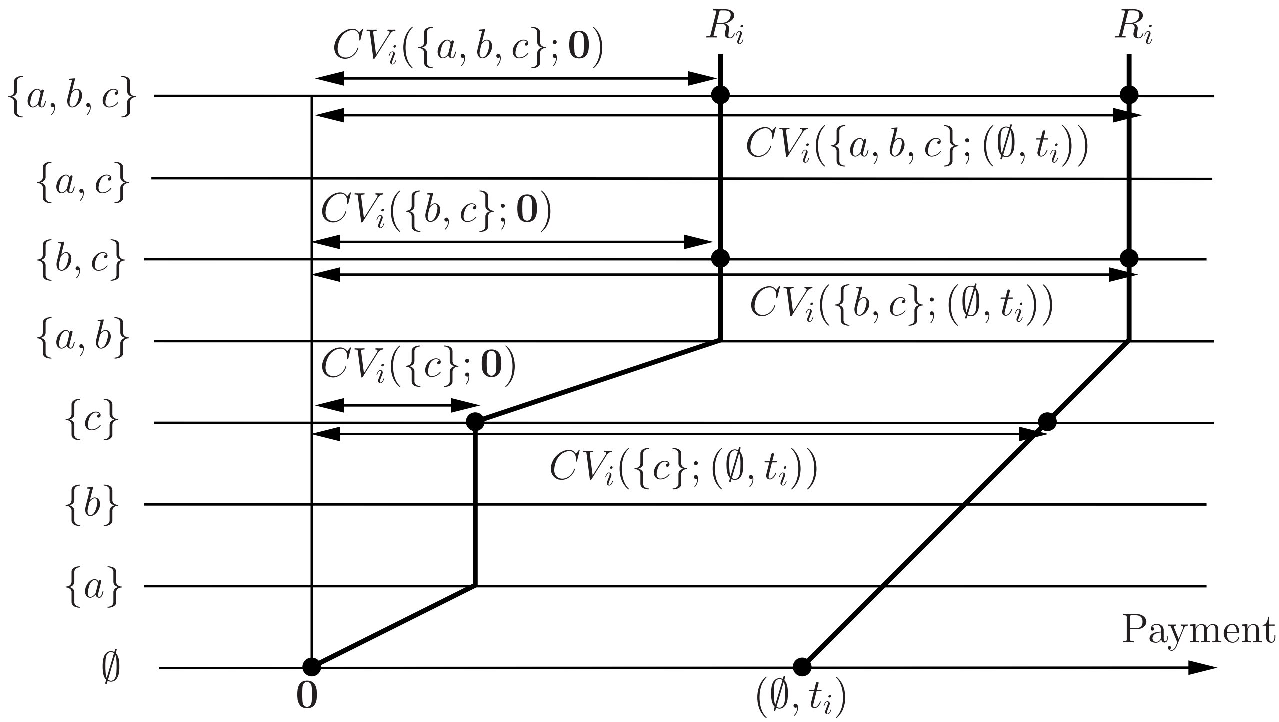

Figure 2: 2−objects demand preference relation.

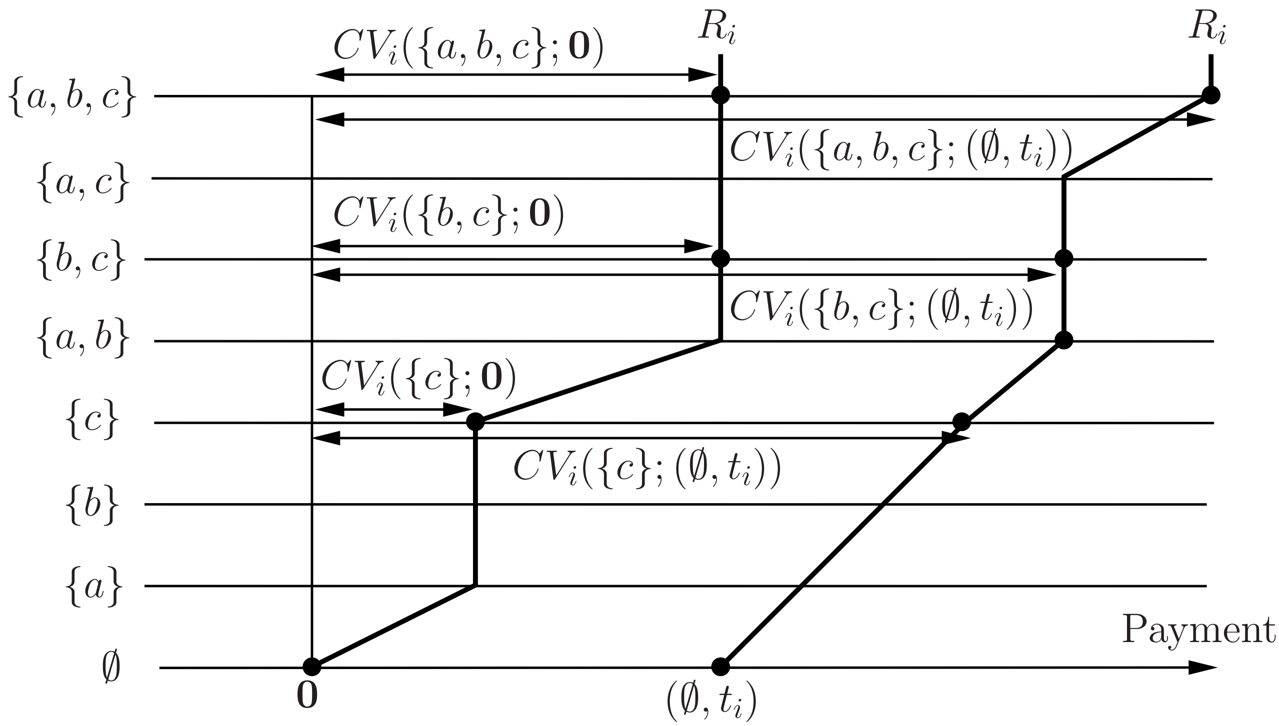

Figure 3: Multi demand but not $k-$demand preference relation.

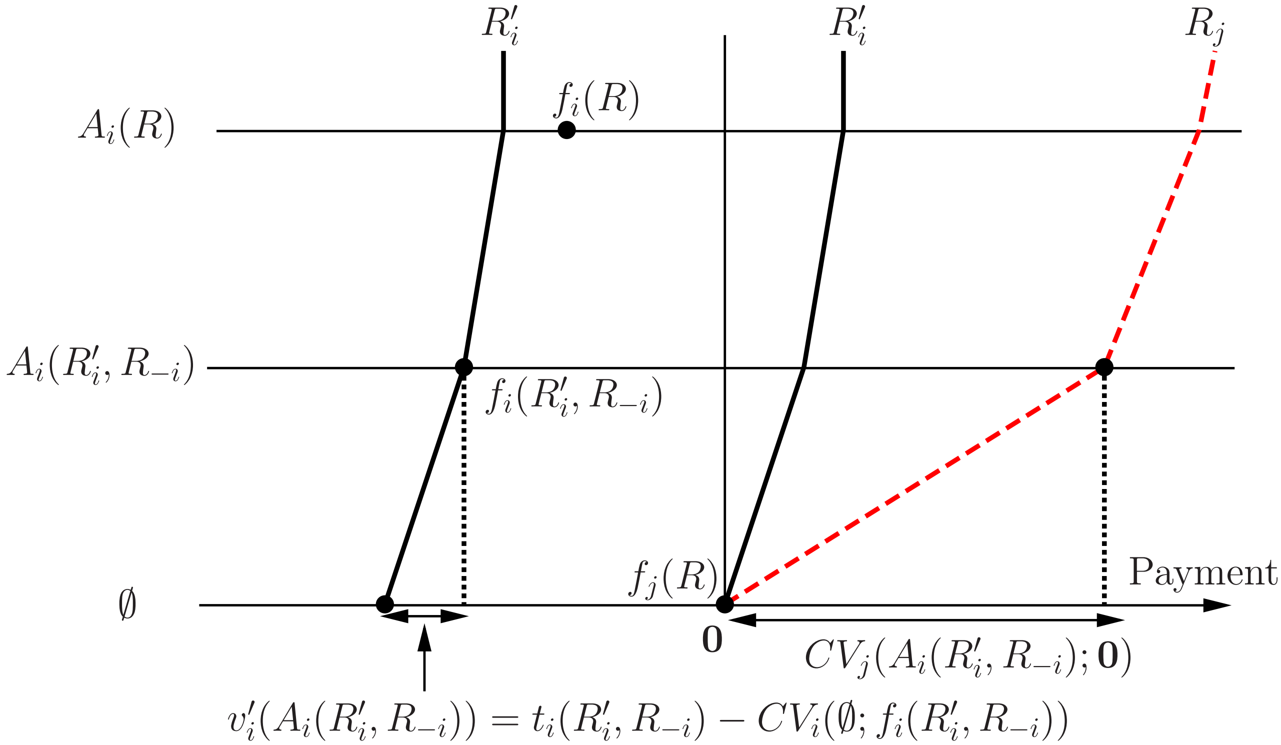

Figure 4: Illustration of proof of Lemma 5.

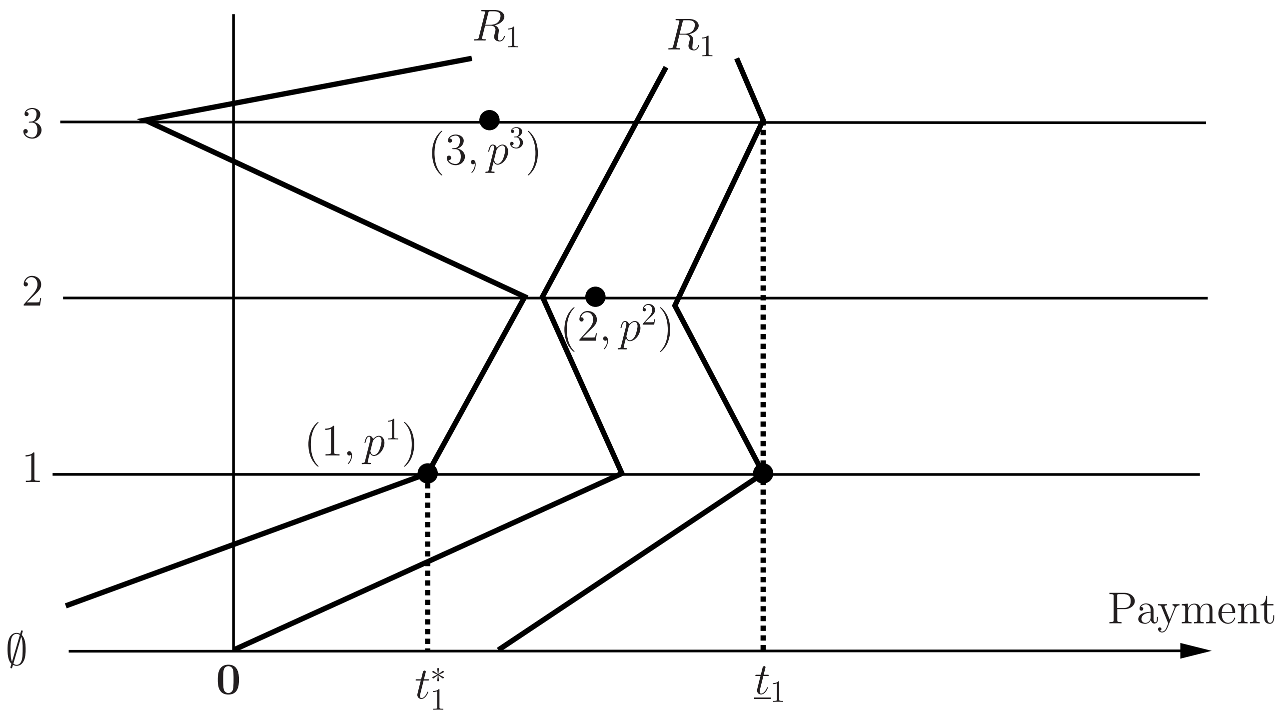

Figure 5: R_1 and p.

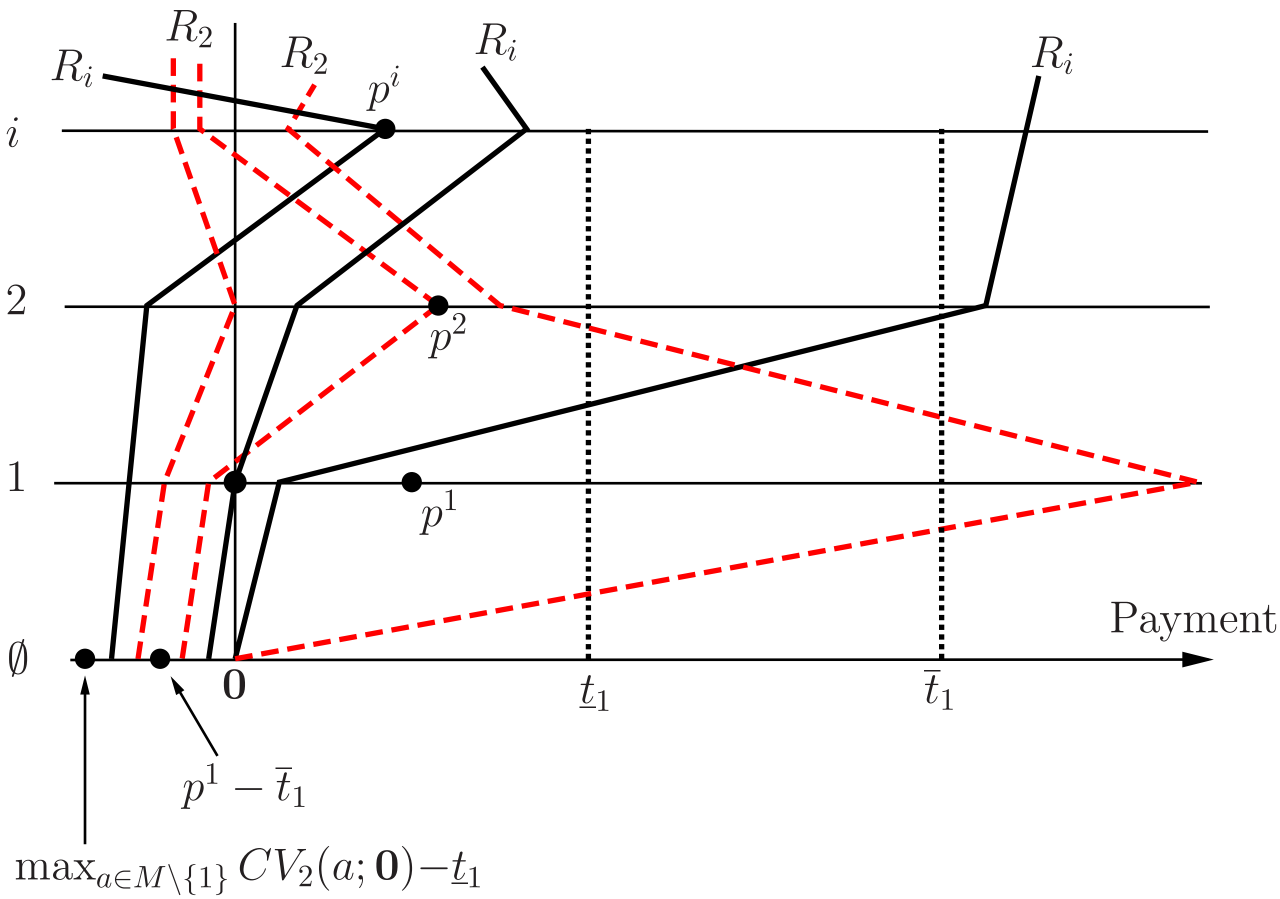

Figure 6: R_2 and R_i ($i \in \{3, \ldots, m\}$).

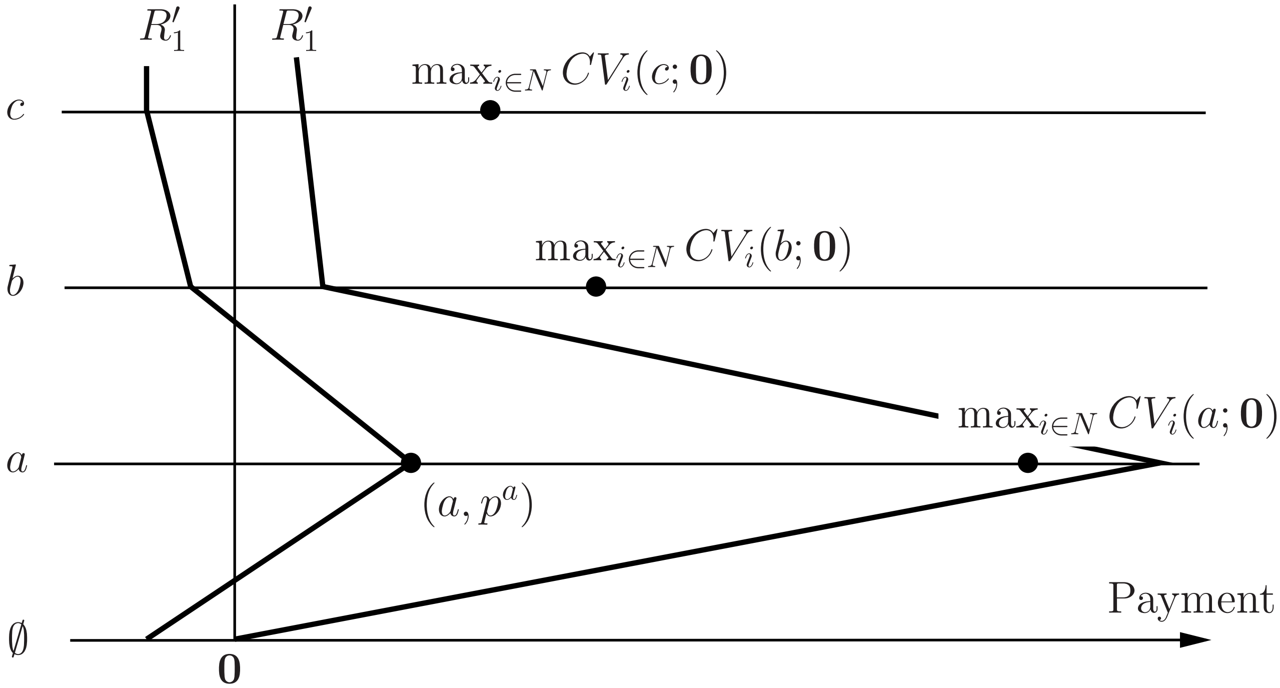
R'_1
R'_1
$\max_{i \in N} CV_i(c; \mathbf{0})$
c
$\max_{i \in N} CV_i(b; \mathbf{0})$
b
$\max_{i \in N} CV_i(a; \mathbf{0})$
a
(a, p^a)
Payment
$\emptyset$
$\mathbf{0}$

Figure 7: R'_1.