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Abstract. There is a divisible commodity and money. Each agent has an endowment of the two goods and continuous, monotone, convex preferences over bundles. Agents may benefit from trade. An exchange rule is a mapping that, for each profile of preferences, calculates for each agent a trade that he finds acceptable, given his preferences. It is known that no strategy-proof exchange rule always yields Pareto efficient outcomes. Strategy-proofness, however, is quite strong. We may instead ask: if we insist upon Pareto efficiency, how frequently will the exchange rule be manipulable? We identify a large subdomain, $\mathcal{D}$, of quasilinear economies on which any efficient exchange rule will be densely manipulable. Moreover, we show the set of manipulable economies is non-meagre. For generic economies outside of $\mathcal{D}$, there exist rules that are locally *non*-manipulable.

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It is well known that there is no efficient trading rule (mechanism) that makes true preference revelation a dominant strategy. Thus, in a world with private information, realizing all gains from trade is problematic. Obviously, we should explore rules with less attractive incentive properties, possibly by relying on higher order rationality (as in Nash equilibrium), by relying on knowledge of each trader's beliefs (as in Bayes-Nash equilibrium), or by considering other, novel weakenings. We join a recent strand of the literature that seeks to *quantify the opportunities* to gain from deceiving a rule with desirable properties (Maus et al., 2006, 2007; Andersson et al., 2014). The benefit of this approach is that it remains agnostic as to the actual behavior the agents. The cost, of course, is difficulty. In particular, we consider a classical model of trading in divisible goods, so the space of preferences (agent types) is necessarily infinite dimensional. As there is no translation invariant, non-trivial measure, and no satisfactory way to extend the notion of "Lebesgue measure zero" to an infinite dimensional space (Stinchcombe, 2001), we instead work with purely topological notions: denseness and Baire category. Denseness is familiar to economists, and Baire category is a simple application of elementary topology. We shall review these notions later, but to preview our results, we find that the set of economies at which some agent can manipulate an efficient trading rule is dense in the set of economically meaningful economies, and is "significant" in the Baire sense. Ours is the first paper to derive the denseness of manipulability in the pure exchange setting, and the first (to our knowledge) to study Baire category in any model. We emphasize the importance of the latter approach: dense sets may nonetheless be quite small.

Hurwicz (1972) first showed that, for two agents, there is no *strategy-proof*, *efficient*, and *individually rational* rule in the classical setting. Later work on the two-agent case dropped the *voluntary participation* requirement and found that each *strategy-proof* and *efficient* rule is dictatorial (Zhou, 1991; Schummer, 1996; Hashimoto, 2008). For more than two agents, Kato and Ohseto (2002) found non-dictatorial rules, but they also conjectured, with strong justification from other results below, that any *strategy-proof* and *efficient* rule must, at each economy, have one agent consuming at the origin. Characterizing the general structure of what is possible with an arbitrary population remains a difficult open question.

For the exchange setting, it is important that agents' endowments be respected as outside options. With this extra constraint, inroads have been made to the multiple-agent problem. Serizawa (2002), working with homothetic preferences, showed that no *strategy-proof* and *efficient* rule respects endowments as welfare lower bounds. Serizawa and Weymark (2003) extended this to show that, in fact, for any positive lower bound, each *strategy-proof* and *efficient* rule will violate this bound for at least one economy. Goswami et al. (2014) showed the corresponding result for quasilinear economies.

I argue that *strategy-proofness* is too strong a requirement. It is unnecessarily robust: we do not really believe that agents will manipulate at every chance they get. This is not because agents are inherently honest but rather because manipulation is costly; it requires information acquisition and strategizing. The cost of this robustness is a great loss of freedom, in terms of designing rules. *Strategy-proofness* governs how the rule changes as the economy changes, each choice constraining the next. The resulting "contamination effect" is most clearly seen in the following case: in a two agent model, if there is a single economy for which a *strategy-proof* rule awards agent i the entire social endowment, then in fact the rule must *always* award agent i the entire social endowment. *Efficiency*, in contrast, is a punctual property; an allocation can be judged efficient for an economy without reference to any other hypothetical economies. By studying the point-wise violation of incentive constraints, we shut-off the contamination effect. Unlike the models studied in Carroll (2012), local incentive compatibility *does not* imply global incentive compatibility here.

In terms of methods, this paper is most closely related to Goswami et al. (2014). Both papers take advantage of the special structure of the efficient set in quasilinear economies, and both apply results from auction theory. Aside from the result already mentioned, their contribution is a dictatorship result for arbitrary numbers of agents. However, to arrive at this, they strengthen the contamination effect of *strategy-proofness* by further imposing *non-bossiness* and a continuity condition. As already mentioned, we completely shut off this effect, which forces us to find a more general characterization of the set of *efficient* allocations. Furthermore, we must strengthen the classical Green-Laffont-Holmström Theorem (Green and Laffont, 1977; Holmström, 1977) to show the topological significance of manipulable economies.

It is debatable what makes a set in a topological space "significant." However, if we honor the topology, we should say a set that contains no open sets, even after taking its closure, is negligible. In a complete metric space, the class of countable unions of negligible sets is a σ-ideal, whose elements are called the "meagre" sets. We show that the set of manipulable economies is not in this class: it is non-meagre.

Denote by $\mathcal{D}^\circ$ those economies for which the Walrasian allocation of money is, for all agents, positive. We show that *any minimally stable* rule will densely be manipulable on $\mathcal{D}^\circ$. In fact, the manipulable economies form a non-meagre subset of $\mathcal{D}^\circ$. Moreover, for every economy in the complement of $\mathcal{D}^\circ$, if the economy is replicated sufficiently many times, then there is a rule that is non-manipulable in a neighborhood of the replicated economy.

The negativity of my findings is proportional to the significance of the economies in $\mathcal{D}^\circ$. I study the "partial equilibrium" case: there is a divisible good and money, and preferences are quasilinear in money. The classical motivation for studying the partial equilibrium model is that the commodity in question makes up a small portion of each agent's expenditure. Given such a motivation, $\mathcal{D}^\circ$ is in fact *all* of the economies of interest and thus my result is completely negative. That said, in much of the mechanism design literature, quasilinearity is taken as a primitive, without the classical motivation.

Results similar to what we present here have been obtained for economies with both public and private goods. Hurwicz and Walker (1990), addressed the question that is symmetric to ours: insisting on *strategy-proofness*, how often is the resulting rule inefficient? The answer is "densely." Studying the same question as we study, Beviá and Corchón (1995), also found that *min-stable* rules are densely manipulable. This paper, in addition to improving upon previous results, further highlights the difference between economies with and without public goods. In our case, unlike the model of Beviá and Corchón (1995), dense manipulability no longer extends to the entire domain. Thus, we provide further confirmation that public goods impose more stringent incentive constraints.

1. Model

1.1. **Environment and Preferences.**
There are two divisible goods: a commodity, indexed as X, and money, indexed as M. There is a finite set N of agents. Each agent $i \in N$ has a commonly known endowment $\omega_i := (\omega_{iX}, \omega_{iM}) \in \mathbb{R}^2_+$ of the two goods, with $\omega_{iM} > 0$. The point $\Omega := \sum \omega_i \geq 0$ is the total supply of goods in the economy.[1]

Let $\mathcal{U}$ be the set of increasing and concave functions $u : [0, \Omega_X] \to \mathbb{R}$ with the property that $u(0) = 0$. Endow $\mathcal{U}$ with the topology of uniform convergence. For each $i \in N$, there is $u_i \in \mathcal{U}$ such that i's preferences can be represented by the function

$$U_i(x_i, m_i) := u_i(x_i) + m_i.$$

We therefore identify the space of preferences with the space $\mathcal{U}$.

The endowments are to be reallocated such that each agent i receives a bundle $(x_i, m_i) \in \mathbb{R}^2_+$. An **allocation** is therefore a list of bundles $((x_i, m_i))_{i \in N} \in \left(\mathbb{R}^2_+\right)^N$. An allocation is **feasible** if $\sum_{i \in N}(x_i, m_i) \leq \Omega$. The set of feasible allocations is denoted Z, with typical element denoted z.

Endowment remains fixed throughout; therefore, an economy is identified by its preference profile, which is in turn identified by an element $u := (u_i)_{i \in N} \in \mathcal{U}^N$. A **social choice rule**, hereafter simply called a **rule**, is a function $\varphi : \mathcal{U}^N \to Z$.

1.2. **Min-Stability.**
We are primarily interested in the following two properties.

Voluntary Participation: An agent $i \in N$ boycotts bundle (x_i, m_i) if $u_i(x_i) + m_i < u_i(\omega_{iX}) + \omega_{iM}$. A rule φ satisfies *voluntary participation* if, for each economy $u \in \mathcal{U}^N$, there are no agents who boycott $\varphi(u)$.

Efficiency: A feasible allocation (x, m) is *efficient* for economy u if there exists no feasible allocation (x', m') such that, for each $i \in N$, $u_i(x'_i) + m'_i \geq u_i(x_i) + m_i$, and for at least one agent $j \in N$ the inequality is strict. A rule φ is *efficient* if, for each $\mathcal{U}^N$, $\varphi(u)$ is *efficient* for u.

If a rule fails either *voluntary participation* or *efficiency*, it may be undermined in practice. Even if an agent believed that, in expectation, a rule would improve his

[1] For $\{x, y\} \subseteq \mathbb{R}^K$, we write $x > y$ only if, for each coordinate $k \in K$, $x_k > y_k$.

welfare, he might refuse *ex post* to execute a trade that would make him worse off. Or, if a rule is not *efficient*, agents would seek further trades after it is executed. The expectation of further trades would change the entire model, perhaps undermining the very goals of the rule. Thus, we define

Minimal Stability: A rule is *min-stable* if it is *efficient* and satisfies *voluntary participation*.

A necessary condition for efficiency is material balance: $\sum_{i \in N} x_i = \Omega_X$ and $\sum_{i \in N} m_i = \Omega_M$. Furthermore, each *efficient* allocation is supported by at least one hyperplane (line). We refer to this line by its normal, which is in turn identified by its first coordinate; we assume without loss of generality that the second coordinate is 1. Thus a typical line of support is denoted $p \in \mathbb{R}$, which identifies the normal vector $(p, 1)$.

1.3. **Manipulability.** Once we have found a desirable *min-stable* rule, we would hope implement it. To do so, we would need some form of incentive compatibility. We study dominant strategy incentive compatibility, but not in the traditional sense. Rather than insisting on incentive compatibility for the entire domain of economies, we seek to measure the set of economies at which incentive compatibility fails.

Manipulability of φ at $\boldsymbol{u}$: There are an agent $i \in N$ and a preference relation $\hat{u}_i \in \mathcal{U}$ such that

$$U_i\left(\varphi_i\left(\hat{u}_i, u_{-i}\right)\right) > U_i\left(\varphi_i\left(u\right)\right).$$

Collect such profiles in set M^{φ}.

1.4. **Baire Category.** Economists are familiar with the notion of *denseness*. A subset A of a topological space is dense in the subset B if each open set $U \subseteq B$ has $A \cap U \neq \emptyset$. Intuitively, we may approximate any point of B by points in A. While the denseness of A implies that it is pervasive, it does *not* imply it is large. Note for example that the rational numbers are dense in the reals, and yet $\mathbb{R} \setminus \mathbb{Q}$ is still "most" of the real numbers.

A set is **meagre** if it is a countable union of sets, each one closed and nowhere dense. As the countable union of singletons, the rational numbers are meagre. The

Cantor set is meagre. Somewhat surprisingly, the functions that are differentiable on at least one point are a meagre subset of continuous functions.

The importance of meagre sets is highlighted by the Baire Category theorem, which states that any complete metric space must be built from *at least uncountably many* meagre sets.

Theorem (The Baire Category Theorem)**.** *Let X be a complete metric space and let $\{A_i\}_{i\in\mathbb{N}}$ be a countable family such that each A_i is meagre in X. Then $X \supsetneq \cup_{i\in\mathbb{N}} A_i$.*

The set $\overline{\mathcal{U}}$, the closure of $\mathcal{U}$, is a complete metric space. Clearly, $\mathcal{U}$ contains the interior of $\overline{\mathcal{U}}$. Thus, the theorem applies, and it follows that if the set of manipulable economies is meagre, then we can safely ignore it and just implement min-stable rules.

2. Results

The theorems presented apply to the large class of economies in which there is a Walrasian allocation with all agents consuming a positive quantity of money. Note that for a fixed profile of preferences, augmenting the endowment of money for the relevant agents is guaranteed to produce such an outcome. Since we maintain a fixed endowment profile, we instead state the condition as a function of preferences. Denote the Walrasian correspondence W. Let $\mathcal{D}^\circ := \{u \in \mathcal{U}^N : \exists z \in W(u) \forall i \in N,\ z_{iM}(u) > 0\}$, the set of economies for which, under the Walrasian correspondence, each agent consumes a positive quantity of money. Let $\mathcal{D} := \overline{\mathcal{D}^\circ}$.

Theorem 1. *Assume φ is a* min-stable *rule. Then M^φ is dense in $\mathcal{D}$.*

Thus, the manipulable economies are pervasive, yet this doesn't prevent them from being small. The following result does.

Theorem 2. *Assume φ is a* min-stable *rule. Then M^φ is nowhere meagre in $\mathcal{D}$*

We may wonder if the results above take advantage of the large size of $\mathcal{U}$. In particular, economists typically are interested in the domain of differentiable and strictly concave preferences, which we shall denote $\mathcal{U}^*$. The proof of the following theorem is just a simplification of the proof of Theorem 1, so it is omitted.

Theorem 1'. *Assume φ is a min-stable rule defined on $(\mathcal{U}^*)^N$. Then $\mathcal{M}^\varphi$ is dense in $\mathcal{D} \cap (\mathcal{U}^*)^N$.*

The following proposition provides a sense in which $\mathcal{D}$ is the maximal domain on which a result like Theorem 1' can be shown. Before stating the proposition, we must first recall the notion of *replica economy*. An economy may be replicated a natural number $\nu \in \mathbb{N}$ times. This creates a new set $\nu * N$ of agents such that there is a function $\mu : \nu * N \to N$ with the property that, for each $i \in N$, $|\mu^{-1}(i)| = \nu$. The agents in $\mu^{-1}(i)$ are "copies" of agent i. For each $i' \in \nu * N$, his endowment is $\omega_{i'} := \omega_{\mu(i')}$, and these endowments define the new feasible set $\nu * Z$ in the natural way. We denote the resulting economy $\nu * u$. The set of all ν-replica economies is thus $\nu * \mathcal{U}^N$.

Proposition 1. *Let $u \in (\mathcal{U}^*)^N \setminus \mathcal{D}$. Assume that, for each $i \in N$, $W_i(u) \neq \omega_i$. Then there exists a natural number $\nu \in \mathbb{N}$, a rule $\varphi^u : \nu * (\mathcal{U}^*)^N \to \nu * Z$, and a neighborhood $V \ni \nu * u$ such that φ^u is non-manipulable in V.*

3. Conclusion

We have shown that *efficiency* and *voluntary participation* bring with them plentiful opportunities for agents to manipulate. It is worth noting as well that the manipulations available to agents are, in a sense which we will not make formal, simple. Two types of manipulations suffice: to declare an almost-linear preference relation that prefers one's endowment to one's current allocation, or to make an arbitrarily small deviation.

It is also worth noting, however, that our result makes full use of the large preference domain, and the fact that Vickrey-Clarke-Groves (VCG) mechanisms are not budget-balanced in general. There may be sub-domains on which budget-balanced, *voluntary* VCG rules exist. On such a domain, our result would not hold.

Appendix A. Proof of Theorem 1

We begin with a characterization of the efficient set, which is denoted, for each $u \in \mathcal{U}$, by $\mathcal{E}(u)$. To this end, consider the allocations available when the feasibility constraint for money is ignored. For each economy $u \in \mathcal{U}^N$ we study the program

$$\begin{aligned} \mathbf{V}_N(u) \coloneqq \quad & \max_{\tilde{x}} \sum_{i \in N} u_i(\tilde{x}_i) \\ s.t. \quad & \Omega_X - \sum_{i \in N} \tilde{x}_i \geq 0, \\ \forall i \in N, \quad & x_i \geq 0 \end{aligned} \tag{A.1}$$

Since the objective is continuous and the constraint set compact, the maximum is attained. Since Slater's constraint qualification is obviously satisfied, it is both necessary and sufficient to study the saddle points of the Lagrangian,

$$L^0(x, p, \mu) \coloneqq \sum_{i \in N} u_i(x_i) + p(\Omega_X - \sum_{i \in N} x_i) + \sum_{i \in N} \mu_i x_i. \tag{A.2}$$

For economy u, denote by $\mathcal{S}(u)$ the saddle points of expression A.2. Denote by $X^*(u)$ the projection of $\mathcal{S}(u)$ on the $x \in \mathbb{R}^N$ variable and $P^*(u)$ the projection on the p variable.

Since each u_i is concave, at each $x_i \in \mathbb{R}_+$, the set $Du_i(x_i)$ of subderivatives is well-defined, and is an interval. In particular, for $x_i > 0$, denote by $\underleftarrow{u}_i(x_i)$ and $\underrightarrow{u}_i(x_i)$ the left and right hand derivatives, respectively, and note that $Du_i(x_i) = [\underrightarrow{u}_i(x_i), \underleftarrow{u}_i(x_i)]$. Set $\underleftarrow{u}_i(0) \coloneqq \infty$ and $\underrightarrow{u}_i(\Omega_X) \coloneqq 0$.

Lemma 1. *Let $p \in P^*(u)$ and $x \in X^*(u)$. Then for each $i \in N$, $p \in Du_i(x_i)$.*

Proof. Let $p^1 \in P^*(u)$ and $x^0 \in X^*(u)$. There are (x^1, p^1, μ^1) and $(x^0, p^0, \mu^0) \in \mathcal{S}(u)$. By definition, for each $x' \in \mathbb{R}^N$ and $(p', \mu') \in \mathbb{R} \times \mathbb{R}^N$,

$$L^0(x^1, p', \mu') \geq L^0(x^1, p^1, \mu^1) \geq L^0(x', p^1, \mu^1).$$

Since x^0 and x^1 are both solutions to the problem, $\sum_{i \in N} u_i(x_i^1) = \sum_{i \in N} u_i(x_i^0)$. Since preferences are increasing, $\sum_{i \in N} x_i^1 = \Omega_X = \sum_{i \in N} x_i^0$. Thus, expanding the

Lagrangians and making replacements, we write

$$\begin{aligned}\sum_{i\in N} u_i(x_i^0) + p'\left(\Omega_X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \mu_i' x_i^1 \\ \geq \sum_{i\in N} u_i(x_i^0) + p^1\left(\Omega_X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \mu_i^1 x_i^1 \\ \geq \sum_{i\in N} u_i(x_i') + p^1\left(\Omega_X - \sum_{i\in N} x_i'\right) + \sum_{i\in N} \mu_i^1 x_i'\end{aligned}$$

Since the inequalities hold for arbitrary x', we may replace x' with x^0. Noting that $\sum_{i\in N} \mu_i^1 x_i^1 = 0$, the second inequality then yields $\sum_{i\in N} \mu_i^1 x_i^0 \leq 0$. Since $x^0 \geq 0$ and $\mu^1 \geq 0$, $\sum_{i\in N} \mu_i^1 x_i^0 = 0$. Thus,

$$\begin{aligned}\sum_{i\in N} u_i(x_i^0) + p'\left(\Omega_X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \mu_i' x_i^0 \\ \geq \sum_{i\in N} u_i(x_i^0) + p^1\left(\Omega_X - \sum_{i\in N} x_i^0\right) + \sum_{i\in N} \mu_i^1 x_i^0 \\ \geq \sum_{i\in N} u_i(x_i') + p^1\left(\Omega_X - \sum_{i\in N} x_i'\right) + \sum_{i\in N} \mu_i^1 x_i',\end{aligned}$$

where the first inequality comes from the fact that $L^0(x^0, p', \mu') \geq L^0(x^0, p^0, \mu^0) = \sum_{i\in N} u_i(x_i^0)$. We have deduced that

$$L^0(x^0, p', \mu') \geq L^0(x^0, p^1, \mu^1) \geq L^0(x', p^1, \mu^1).$$

Since x' and (p', μ') were arbitrary, we conclude that $(x^0, p^1, \mu^1) \in \mathcal{S}(u)$.

Let $(x, p, \mu) \in \mathcal{S}(u)$. As a saddle point, for each $x' \in \mathbb{R}^N$, $L^0(x, p, \mu) \geq L^0(x', p, \mu)$. In particular, let $x' := (x_i + \varepsilon, x_{-i})$. Then we have

$$\frac{u_i(x_i + \varepsilon) - u_i(x_i)}{\varepsilon} \leq (p - \mu_i).$$

Assuming $x_i > 0$, we find the inequality corresponding to $x - \varepsilon$, take limits, and deduce

$$\underleftarrow{u}_i(x_i) \leq (p - \mu_i) \leq \underrightarrow{u}_i(x_i).$$

If $x_i > 0$ then complementary slackness implies $\mu_i = 0$ and the lemma is shown. If $x_i = 0$, then $Du_i(x_i)$ is unbounded above, and since $\mu_i > 0$, we have $\underleftarrow{u}_i(x_i) \leq p$. ■

Corollary 1. *Either* $|P^*(u)| = 1$ *or* $|X^*(u)| = 1$.

Proof. Let $x \in X^*(u)$. There is $i \in N$ with $x_i > 0$. Assume that $|P^*(u)| > 1$. Since Lemma 1 implies $P^*(u) \subseteq Du_i(x_i)$, deduce that $\underleftarrow{u}_i(x_i) < \underrightarrow{u}_i(x_i)$. Since u_i is concave, for each $x'_i < x_i$, $\underleftarrow{u}_i(x'_i) \geq \underrightarrow{u}_i(x_i) > \underleftarrow{u}_i(x_i)$ and so $\underleftarrow{u}_i(x_i) \notin Du_i(x'_i)$. It follows that $Du_i(x'_i) \cap Du_i(x_i)$ is either empty or contains the single point $\underleftarrow{u}_i(x'_i) = \underrightarrow{u}_i(x_i)$. Since $|P^*(u)| > 1$, it is not possible for $P^*(u) \subseteq Du_i(x'_i)$. Lemma 1 thus implies no element of $X^*(u)$ has $x'_i < x_i$. A symmetric argument shows that no element of $X^*(u)$ has $x'_i > x_i$. Since i is arbitrary, $X^*(u)$ is a singleton. ■

We partition the Pareto set in two subsets, $Z^*(u) := \{z \in Z : z_X \in X^*(u)\}$ and its complement. The Second Welfare Theorem allows us to study the Pareto set via each individual's optimal choice from a Walrasian budget. For each $z \in \mathcal{E}(u)$, there is a price p and a list $(w_i)_{i \in N} \in \mathbb{R}^N_+$ such that each $z_i := (x_i, m_i)$ solves the program

$$\begin{aligned} \max \quad & u_i(x_i) + m_i \\ s.t. \quad & w_i - px_i - m_i \geq 0, \\ & x_i \geq 0, \; m_i \geq 0. \end{aligned} \tag{A.3}$$

If $z_i = 0$ then $w_i = 0$ and the problem is trivial. Otherwise, $w_i > 0$ and Slater's constraint qualification is satisfied. We again use the saddle-point method. Let β_i denote the Lagrange multiplier on the budget constraint. Let λ_{iX} denote the multiplier for the non-negativity constraint of the commodity and let λ_{iM} denote the corresponding multiplier for money. Denote the Lagrangian $L(x_i, m_i, \beta_i, \lambda_i; p, w_i)$, where the price and wealth parameters are suppressed when context makes them clear.

Assume $(x_i, m_i, \beta_i, \lambda_i)$ is a saddle point for the problem with price p and wealth w_i. By studying the expression $L(x_i \pm \varepsilon, m_i, \beta_i, \lambda_i) - L(x_i, m_i, \beta_i, \lambda_i) \leq 0$, we find

$$p\beta_i + \lambda_{iX} \in Du_i(x_i). \tag{A.4}$$

The expression $L(x_i, m_i \pm \varepsilon, \beta_i, \lambda_i) - L(x_i, m_i, \beta_i, \lambda_i) \leq 0$ in turn yields that

$$\beta_i = 1 + \lambda_{iM}. \tag{A.5}$$

Lemma 2. *Assume $z := (x, m) \in \mathcal{E}(u)$ and p supports z for u. Then $p \leq \max P^*(u)$.*

Proof. There is $i \in N$ with $x_i > 0$ and $x_i \geq \min X_i^*(u)$. Then $\lambda_{iX} = 0$ and by line A.4, we find $p\beta_i \in Du_i(x_i)$. Let $x_i^* \in X_i^*(u)$ satisfy $x_i^* = \min X_i^*(u)$. By concavity, $Du_i(x_i) \leq Du_i(x_i^*)$.[2] By Lemma 1, $P^*(u) \subseteq Du_i(x_i^*)$. Therefore, we use line A.5 and the fact that $\lambda_M \geq 0$ to find

$$\max P^*(u) \geq p\beta_i = (1 + \lambda_{iM})p \geq p. \tag{A.6}$$

■

Lemma 3. *Let $z := (x, m) \in \mathcal{E}(u)$ be supported by price p. Assume that the individual optimization problem (A.3) for each $i \in N$ has a saddle point of the form $(x_i, m_i, 1, (\lambda_{iX}, 0))$. Then $z \in Z^*(u)$.*

Proof. Since, for each $x_i' \in [0, \Omega_X]$, $L(x_i, m_i, 1, (\lambda_{iX}, 0)) - L(x_i', m_i, 1, (\lambda_{iX}, 0)) \geq 0$, we deduce that

$$u(x_i) - px_i + \lambda_{iX}x_i \geq u(x_i') - px_i' + \lambda_{iX}x_i'.$$

Sum over agents and add $p\Omega_X$ to each side to arrive at

$$\sum_{i \in N} u_i(x_i) + p(\Omega_X - \sum_{i \in N} x_i) + \sum_{i \in N} \lambda_{iX}x_i \geq \sum_{i \in N} u_i(x_i') + p(\Omega_X - \sum_{i \in N} x_i') + \sum_{i \in N} \lambda_{iX}x_i'. \tag{A.7}$$

Note that this is precisely $L^0(x, p, \lambda_X) \geq L^0(x', p, \lambda_X)$. Since $z \in \mathcal{E}(u)$ and preferences are increasing, $\Omega_X - \sum_{i \in N} x_i = 0$. Since for each $i \in N$, $\lambda_{iX}x_i = 0$, we deduce that for $p' \geq 0$ and $\lambda_X' \in \mathbb{R}_+^N$, $L^0(x, p, \lambda_X) \leq L^0(x, p', \lambda_X')$. Thus, (x, p, λ) is a saddle point of L^0 and therefore $x \in X^*(u)$. ■

Lemma 4. *Let $z := (x, m) \in \mathcal{E}(u) \setminus Z^*(u)$. For each $x^* \in X^*(u)$ there is $j \in N$ such that $m_j = 0$ and $x_j < x_j^*$.*

Proof. Let p support z for u. Let $x^* \in X^*(u)$. Suppose that for each $i \in N$ with $x_i < x_i^*$, $m_i > 0$.

[2]For sets A and $B \subseteq \mathbb{R}$, write $A \geq B$ when, $\inf A \geq \sup B$.

Case 1. $p \in P^*(u)$

Consider $i \in N$ with $x_i < x_i^*$. By construction, there is $m^* \in \mathbb{R}$, possibly negative, such that $z_i^* \coloneqq (x_i^*, m_i^*)$ is an optimal solution to the relaxed individual's problem where the non-negativity constraint $m_i \geq 0$ is ignored. Since $m_i > 0$, z_i is *also* an optimal solution to this relaxed problem.

Consider $j \in N$ with $x_j > x_j^*$. Since $x_j > x_j^*$, there is $m_j^* \leq m_j$ such that $z_j^* \coloneqq (x_j^*, m_j^*)$ is affordable at this budget. Since $p \in P^*(u)$, z_j^* is also optimal.

By construction, $\sum_{i \in N} z_{iX}^* = \Omega_X$. By adding $\min\{0, (\sum_{i \in N} m_i^*) - \Omega_M\}$ units of money to the economy, z^* becomes feasible, in which case $z^* \in Z^*(u)$ would hold. However, since each $i \in N$ is indifferent between z_i^* and z_i, in this hypothetical economy, z solves each agent's individual problem with, for each $i \in N$, $\lambda_{iM} = 0$. Thus, by Lemma 3, $x \in X^*(u)$, a contradiction.

Case 2. $p < \min P^*(u)$

Let $i \in N$ have $x_i < x_i^*$. Again, if the agent could consume negative money, x_i^* would be an optimal choice at prices $p' \in P^*(u)$. Thus, since $p \leq p'$, there is $x^{**} \geq x^*$ optimal if the agent could consume negative money. Since in fact $m_i > 0$ and $x_i < x_i^*$, it follows by convexity of preferences that there is $x_i' \in \,]x_i, x_i^*[\, \subseteq \,]x_i, x_i^{**}[$ optimal and affordable for i at his actual individual problem. Let the associated consumption of money be m_i'.

We examine the saddle-point condition $L(x_i, m_i, \beta_i, \lambda_i; p, w_i) - L(x_i', m_i', \beta_i, \lambda_i; p, w_i) \geq 0$. Since (x_i, m_i) and (x_i', m_i') are both optimal choices to the individual's problem A.3, $u(x_i) + m_i = u(x_i') + m_i'$, so these terms cancel. The remaining terms are

$$\beta_i(w_i - w_i + p(x_i' - x_i) + m_i' - m_i) + \lambda_{iX}(x_i - x_i') + \lambda_{iM}(m_i - m_i') \geq 0.$$

Since both points are on the budget line, $p(x_i' - x_i) + m_i' - m_i = 0$. Since $m_i > 0$, $\lambda_{iM} = 0$. Thus we deduce that $\lambda_{iX}(x_i - x_i') \geq 0$. Since $\lambda_{iX} \geq 0$ and $x_i - x_i' < 0$, we have $\lambda_{iX} = 0$.

By concavity, $\underline{u}_i(x_i) \geq \underline{u}_i(x_i^*)$. By Lemma 1, $\underline{u}_i(x_i^*) \geq \min P^*(u)$. By line A.4, $p\beta_i + \lambda_{iX} \geq \underline{u}_i(x_i)$. Since $\lambda_{iM} = 0$, by line A.5, $\beta_i = 1$. Since $\lambda_{iX} = 0$, we conclude $p \geq \min P^*(u) > p$.

■

Lemma 5. *Let $u \in \mathcal{D}^\circ$ and $\varphi(u) \in \mathcal{E}(u) \setminus X^*(u)$. Then $u \in M^\varphi$.*

Proof. Let $(x,m) = \varphi(u) \in \mathcal{E}(u) \setminus Z^*(u)$ and $z^W := (x^W, m^W) \in W(u)$ be such that, for each $i \in N$, $m_i > 0$. Let p^W be a Walrasian price for z^W. Lemma 3 thus implies that $z^W \in Z^*(u)$ and $p^W \in P^*(u)$. If for each $i \in N$, $x_i \geq x_i^W$, then by feasibility $x = x^W \in W(u) \subseteq X^*(u)$, a contradiction. Therefore, there is $i \in N$ with $x_i < x_i^W$ and $m_i = 0$. Thus, $z_i < z_i^W$, and it follows that i is consuming strictly within his budget set given price p^W.

At profile u, agent i has a profitable deviation from truth-telling: Let i declare a differentiable $\hat{u}_i$ with the properties:

$$\begin{aligned} \hat{u}_i'\left(x_i^W\right) &= p^W \\ \hat{U}_i(\omega_i) &> \hat{U}_i\left(\varphi_i(u)\right). \end{aligned}$$

To see why the inequality is feasible, consider the function given for each $\tilde{z} = (\tilde{x}, \tilde{m})$ by $\tilde{U}(\tilde{x}, \tilde{m}) = p^W \cdot \tilde{x} + \tilde{m}$. Clearly, $\tilde{U}(\omega_j) > \tilde{U}(\varphi_j(u))$. Let $\hat{u}$ be a smooth, concave function that is sufficiently close to the linear function $\tilde{x} \mapsto p^W \cdot \tilde{x}$.

Denote $\hat{u} := (\hat{u}^j, u^{-j})$. Clearly, $x^W \in X^*(\hat{u})$. Since $x_i^W > x_i \geq 0$, and since $\hat{u}_i$ is differentiable, in fact $X^*(\hat{u}) = \{x^W\}$ and $P^*(\hat{u}) = \{p^W\}$. *Voluntary participation* requires $\varphi_i(\hat{u}) \neq \varphi_i(u)$. Since φ is *efficient*, it is supported by $p' \leq p^W$. Thus if $\varphi_{iM}(\hat{u}) > 0$, then since $\hat{u}_i$ is smooth and strictly concave, $\varphi_{iX} \geq x^W$. If $\varphi_{iM}(\hat{u}) = 0$, then by *voluntary participation*, $\varphi_{iX}(\hat{u}) > \varphi_{iX}(u)$. In either case, $\varphi_i(\hat{u}) \gneq \varphi_i(u)$. Since preferences are increasing, $U_i(\varphi_i(\hat{u})) > U_i(\varphi_i(u))$.

■

Lemma 6. *Let $u \in \mathcal{U}^N$ and $x \in X^*(u)$. Let $\varepsilon > 0$. Then there are $u^\varepsilon \in \mathcal{U}^N$, a list $(\alpha_i, \beta_i, \gamma_i)_{i \in N}$, and a neighborhood $\Pi_{i \in N} U_i \ni x$ such that:*

- *for each $i \in N$, each $\tilde{x}_i \in U_i$, $u_i^\varepsilon(\tilde{x}_i) = \alpha_i \log(\tilde{x}_i + \beta_i) + \gamma_i$, and $|u_i^\varepsilon(\tilde{x}_i) - u_i(\tilde{x})| < \varepsilon$,*
- *for each $i \in N$, each $\tilde{x}_i \notin U_i$, $u_i^\varepsilon(\tilde{x}_i) = u_i(\tilde{x}_i)$,*
- *$X^*(u^\varepsilon) = \{x\}$.*

Proof. The proof is constructive. Let $p \in P^*(u)$ and $i \in N$. Since u_i is concave and increasing, for each $\delta > 0$ there is $x_i' \in \mathbb{R}$ such that $|x_i' - x_i| < \delta$ and u_i is twice differentiable at x_i'. We first calibrate α_i and β_i so that

$$\begin{aligned} \frac{d}{dx}\left[\alpha_i \log\left(x_i+\beta_i\right)\right] &= p \\ \text{and} \quad \frac{d^2}{dx^2}\left[\alpha_i \log\left(x'+\beta_i\right)\right] &> \frac{d^2}{dx^2} u_i(x'). \end{aligned}$$

The first condition ensures that $\{x\} = X^*(u^\varepsilon)$ while the second ensures that u_i uniformly dominates u_i^ε. Let $\bar{\gamma}_i := u_i(x) - \alpha_i \log\left(x'+\beta_i\right)$. For each $\gamma \in]0, \bar{\gamma}_i[$, there is a neighborhood $]\underline{x}(\gamma), \bar{x}(\gamma)[$ containing x such that for each $\tilde{x} \in]\underline{x}(\gamma), \bar{x}(\gamma)[$, $\alpha_i \log(\tilde{x}+\beta_i)+\gamma < u_i(\tilde{x})$ and for each $y \in \{\underline{x}(\gamma), \overline{x}(\gamma)\}$, $\alpha_i \log(y+\beta_i)+\gamma = u_i(y)$. For each $\gamma \in \mathbb{R}$, define u_i^γ such that

$$u_i^\gamma(\tilde{x}) := \begin{cases} \alpha_i \log(\tilde{x}+\beta_i)+\gamma & \tilde{x} \in]\underline{x}(\gamma), \bar{x}(\gamma)[\\ u_i(\tilde{x}) & \textit{otherwise.} \end{cases}$$

For $\gamma_i^* \in]0, \bar{\gamma}_i[$ with $|\bar{\gamma}_i - \gamma_i^*|$ sufficiently small, $\left(u_i^{\gamma_i^*}\right)_{i\in N}$ satisfies the requirements of the Lemma.

It is simple to verify that the calibration of α_i and β_i exists as required. ■

We can now give

Proof of Theorem 1. From lemma 5, we know that $u \in \mathcal{D} \setminus M^\varphi$ implies $\varphi(u) \in Z^*(u)$. Our proof is by contradiction: assume M^φ is *not* dense in $\mathcal{D}$. Then $\mathcal{D} \setminus M^\varphi$ contains an open set V. Without loss of generality, assume $V = \Pi_{i\in N} V_i$. Therefore, $\varphi\big|_V$ is a *strategy-proof* rule that implements a selection from the correspondence Z^*. Since $\mathcal{U}$ is smoothly path connected, we may apply the Green-Laffont-Holmström Theorem (see Holmström (1977)): $\varphi\big|_V$ is a VCG mechanism. For each $u \in V$, write $\varphi\big|_V(u) = (x^*(u), t(u))$. We then have from Holmström (1977) that $\sum_{i\in N} t_i(u) = \Omega_m$ for each $u \in V$ if and only if there is a list of functions $(f_i)_{i\in N}$, with $f_i : V_{-i} \to \mathbb{R}$, such that for each $u \in V$,

$$\mathbf{V}_N(u) = \sum_{i\in N} f_i\left(u_{-i}\right). \tag{A.8}$$

Given $\varepsilon > 0$, let u^ε approximate u such that each u_i^ε is a logarithm in a neighborhood of $x_i^*(u)$, as in Lemma 6. Consider a one-dimensional subdomain $\mathcal{A} \subset \mathcal{U}$, containing u^ε, such that for each $\tilde{u}_i \in \mathcal{A}$, there is $\tilde{\alpha}_i \in \mathbb{R}$ with $\tilde{u}_i(\cdot)|_{V_i} = \tilde{\alpha}_i \log(\cdot+\beta_i)+\gamma_i$.

Identify a profile of preferences $\tilde{u} \in \mathcal{A}^N$ by its list $\tilde{\alpha} := (\tilde{\alpha}_i)_{i \in N}$ of parameters. Let $\tilde{\Omega}_X := \Omega_X + \sum_{i \in N} \beta_i$. Given this specification, x^* has a closed form near x. Letting $\tilde{\Omega}_X := \Omega_X + \sum_{i \in N} \beta_i$, it is easy to verify that, for each $\alpha \in \mathcal{A}^N$,

$$x_i^*(\alpha) = \frac{\alpha_i \tilde{\Omega}_X}{\sum_{i \in N} \alpha_i} - \beta_i.$$

The envelope theorem and further calculation then yields the formula

$$\frac{\partial^k}{\partial \alpha_1 \partial \alpha_2 \cdots \partial \alpha_k} \mathbf{V}_N(\alpha) = (k-2)! \left(\frac{-1}{\sum_{j \in N} \alpha_j} \right)^{k-1}.$$

However, equation A.8 implies that for each $\alpha \in \mathcal{A}^N$,

$$\text{(A.9)} \qquad \frac{\partial^n \mathbf{V}_N(u)}{\partial \alpha_1 \partial \alpha_2 \cdots \partial \alpha_n} = 0.$$

Therefore, $\mathbf{V}_N(\alpha)$ does not satisfy A.8. By choosing $\varepsilon > 0$ sufficiently small, we can guarantee that $\mathcal{A}^N \cap V \neq \emptyset$, a contradiction. ■

Appendix B. Proof of Theorem 2

B.1. **Some results on Baire category in product sets.** Before proving Theorem 2, we need a few supporting results. For this subsection, we put aside the economic environment and study the subsets of a general product set $\Phi := \Pi_{i=1}^K \Phi_k$. Assume that each Φ_i is second-countable.

Write $A \sqsubseteq B$ to denote the relation "set A is a dense subset of set B." Write $A \,\underline{\underline{\circledcirc}}\, B$ to denote the relation "$A \subseteq B$ and $int(A)$ is dense in B." A set $A \subseteq B$ is **generic** in B if $B \setminus A$ is meagre. We denote this relation by $A \trianglelefteq B$. We leave it to the reader to verify that these relations are transitive.

For each $A \subseteq \Phi$, each $\bar{k} \in \{1, \ldots, K\}$, denote by $A_{\bar{k}}$ the projection of A onto $\Phi_{\bar{k}}$. Denote by $A_{-\bar{k}}$ the projection of A onto $\Pi_{k \neq \bar{k}} \Phi_k$.

Separate-denseness[3]: Set $A \subseteq \Phi$ is separately dense in $B \subseteq \Phi$, written $A \sqsubseteq^+ B$, if for each $\bar{k} \in \{1, \ldots, K\}$, there is a set $A^*_{-\bar{k}} \sqsubseteq B_{-\bar{k}}$ such that, for each $a_{-\bar{k}} \in A^*_{-\bar{k}}$,

$$\{a_{\bar{k}} : (a_{\bar{k}}, a_{-\bar{k}}) \in A\} \sqsubseteq B_{\bar{k}}.$$

[3]A related notion in descriptive set theory is the complement of a *slim* set, defined in Gruenhage et al. (2007).

Separate-genericity: Set $A \subseteq \Phi$ is an separately generic in $B \subseteq \Phi$, written $A \trianglelefteq^{+} B$, if for each $\bar{k} \in \{1, \ldots, K\}$, there is a set $A^*_{-\bar{k}} \trianglelefteq B_{-\bar{k}}$ such that, for each $a_{-\bar{k}} \in A^*_{-\bar{k}}$,

$$\{a_{\bar{k}} : \ (a_{\bar{k}}, a_{-\bar{k}}) \in A\} \trianglelefteq B_{\bar{k}}.$$

Lemma 7. *An open and dense set is separately generic.*

Proof. Let $A \subseteq \Phi$ be an open set. Clearly, A_{-k} is open in Φ_{-k}. Let $a_{-k} \in A_{-k}$. The set $A_k(a_{-k}) \coloneqq \{a_k \in A_k : (a_k, a_{-k}) \in A\}$ is open in A_k. Thus, if it is not dense in Φ_k, then there is a basic open set $U_k \subseteq \Phi_k$ such that $A^c_k\left(a_{-k}\right) \coloneqq \Phi_k \setminus A_k\left(a_{-k}\right) \supseteq U_k$. Let $V_{-k}(U_k) \coloneqq \{a_{-k} \in \Phi_{-k} : A^c_k\left(a_{-k}\right) \supseteq U_k\}$.

We shall show that if $V_{-k}(U_k)$ is somewhere dense in Φ_{-k}, then A fails to be dense in Φ. Suppose there is an open set $U_{-k} \subseteq \overline{V_{-k}(U_k)} \subseteq \Phi_{-k}$. Let $U \coloneqq U_k \times U_{-k}$. Assume there is $a \in A \cap U$. Since A and U are open, there is an open set U' with $a \in U' \subseteq A \cap U$. Then for each $a' \in U'$ we have $A_k(a'_{-k})$ open and $\emptyset \neq A_k\left(a'_{-k}\right) \cap U' \subseteq U_k$, a contradiction. Therefore, $A \cap U = \emptyset$ and it follows that A is not dense.

Thus, if $A \sqsubseteq \Phi$, then for each basic open $U_k \subseteq \Phi_k$, $V_{-k}(U_k)$ is nowhere dense in Φ_{-k}. Denote by $\mathcal{B}$ the basic open sets of Φ_k. Let $V^*_{-k} \coloneqq \bigcup_{U'_k \in \mathcal{B}} V_{-k}(U'_k)$. Since $\mathcal{B}$ is countable, V^*_{-k} is meagre in Φ_{-k}. Therefore,

$$\Phi_{-k} \setminus V^*_{-k} = \{a_{-k} : A_k\left(a_{-k}\right) \underline{\boxdot} \Phi_k\}$$

is generic in Φ_{-k}. ∎

Lemma 8. *A generic set is separately generic.*

Proof. Let A be a generic set and assume $A = \cap_{n \in \mathbb{N}} A_n$, where each A_n is open and dense. By Lemma lemma 7, each A_n is separately generic.

Let $k \in \{1, \ldots, K\}$. For each $n \in \mathbb{N}$, there is a set $A^*_{-k,n} \trianglelefteq \Phi_{-k}$ such that, for each $a_{-k,n} \in A^*_{-k,n}$, $A(a_{-k,n}) \coloneqq \{a_k : (a_k, a_{-k,n}) \in A\} \trianglelefteq A_{k,n}$. Let $A^*_{-k} \coloneqq \cap_{n \in \mathbb{N}} A^*_{-k,n}$. Clearly, $A^*_{-k} \trianglelefteq \Phi_{-k}$. Let $a_{-k} \in A^*_{-k}$. Then, for each $n \in \mathbb{N}$, there is $a_{-k,n} \in A^*_{-k,n}$ such that $a_{-k} = a_{-k,n}$. It follows that

$$\{a_k : (a_k, a_{-k}) \in A\} = \bigcap_{n \in \mathbb{N}} A(a_{-k}) = \bigcap_{n \in \mathbb{N}} A(a_{-k,n}) \trianglelefteq \bigcap_{n \in \mathbb{N}} A_{k,n} \trianglelefteq A_k,$$

so transitivity of the $\trianglelefteq$ relation yields the result. ∎

Corollary 2. *A generic set is separately dense.*

B.2. **The non-meagreness of the manipulable set.**

Lemma 9. *Let $V \subseteq \mathcal{R}^N$ be open, and let $U^* \sqsubseteq^+ V$. Let $\psi : U^* \to Z$ be an individually rational, budget balanced VCG rule. Then there is $\psi^* : V \to Z$, also an individually rational, budget balanced VCG rule, with $\psi^*|_{U^*} = \psi$.*

Proof. Since ψ is a VCG rule, there is a list of functions $(h_i)_{i\in N}$, each $h_i : U^*_{-i} \to \mathbb{R}$, such that for each $u \in U^*$ and each $i \in N$, $\psi_i(u) = (x^*(u), p^*(u) + h_i(u_{-i}))$, where $p^*(u)$ is the pivot rule payment. Let $u \in V \setminus U^*$. Thus, u is in the closure of U^*, so there is a sequence $(u^n)^{n\in\mathbb{N}} \subseteq U^*$ such that $u^n \to u$. Since ψ is individually rational, for each $i \in N$, the sequence $\left(h_i(u^n_{-i})\right)^{n\in\mathbb{N}}$ is bounded below. Since ψ is budget balanced, $\sum_{i\in N} h_i(u^n_{-i}) \equiv 0$, so each sequence $\left(h_i(u^n_{-i})\right)^{n\in\mathbb{N}}$ is also bounded above. Therefore, $\left(h_1(u^n_{-1}), h_2(u^n_{-2}), \ldots, h_{|N|}\left(u^n_{-|N|}\right)\right)^{n\in\mathbb{N}}$ has a convergent subsequence, σ. It follows that $u^{\sigma(n)} \to u$ and we may define, for each $i \in N$, $h^*_i(u_{-i}) = \lim_{n\to\infty} h_i\left(u^{\sigma(n)}_{-i}\right)$. Let ψ^* be the VCG rule with functions $(h^*_i)_{i\in N}$ as parameters. Individual rationality follows from continuity of preferences. Budget balance follows from the result in Holmström (1977), since now, for each $u \in V$, $\sum_{i\in N} h_i(u_{-i}) = 0$. ■

Proposition 2. *$\mathcal{D} \setminus \mathcal{M}^\varphi$ is nowhere separately dense in $\mathcal{D}$.*

Proof of Theorem 2. If $\mathcal{M}^\varphi \cap \mathcal{D}$ were meagre, $\mathcal{D} \setminus \mathcal{M}^\varphi$ would be a generic subset of $\mathcal{D}$. By Corollary 2, this is not the case. ■

Proof of Proposition 2. Suppose by way of contradiction that there is an open set $V \subseteq \mathcal{D}$ and a set $U \subseteq \mathcal{D} \setminus \mathcal{M}^\varphi$ such that $U \sqsubseteq^+ V$. By Lemma 5, $\varphi|_U$ implements $Z^*|_U$. Let $i \in N$ and let U^*_{-i} be the set designated in the definition of separate-denseness. Let $u_{-i} \in U^*_{-i}$ and, following our earlier convention,

$$U_i(u_{-i}) \coloneqq \{u_i \in \mathcal{U} : (u_i, u_{-i}) \in U\}.$$

Recall that our hypothesis implies $U_i(u_{-i}) \sqsubseteq V_i$ and $U^*_{-i} \sqsubseteq V_{-i}$.

Since $U \subseteq \mathcal{D} \setminus \mathcal{M}^\varphi$, for each pair $\{u_i, u'_i\} \subseteq U_i(u_{-i})$,

$$u_i\left(\varphi_{iX}(u_i, u_{-i})\right) + \varphi_{iM}(u_i, u_{-i}) \geq u_i\left(\varphi_{iX}(u'_i, u_{-i})\right) + \varphi_{iM}(u'_i, u_{-i}). \tag{B.1}$$

Let $h_i(u_{-i}) \coloneqq \varphi_{iM}(u_i, u_{-i}) - \left(\mathbf{V}_{N\setminus i}(u) - \mathbf{V}_N(u)\right)$.

Claim. For each $u_i' \in U_i$, $\varphi_{iM}(u_i', u_{-i}) = \mathbf{V}_{N\setminus i}(u_i', u_{-i}) - \mathbf{V}_N(u_i', u_{-i}) + h_i(u_{-i})$.

Proof. We construct a *strategy-proof* function $\psi : V \to \mathbb{R}^{2\times N}$ satisfying the claim. We do not require ψ to be feasible. Let $O \coloneqq \varphi_i(U_i(u_{-i}), u_{-i})$. Line B.1 implies that, for each $u_i' \in U_i(u_{-i})$, $\varphi_i(u_i', u_{-i}) \in \arg\max_{(x,m)\in O} u_i'(x) + m$. Let $\psi_i|_{U_i(u_{-i})} = \varphi_i|_{U_i(u_i-i)}$. For each $u_i' \in V_i \setminus U_i(u_{-i})$, let $\psi_i(u_i', u_{-i}) \in \arg\max_{(x,m)\in\bar{O}} u_i'(x) + m$. Since X^* is upper hemi-continuous and $U_i(u_{-i}) \sqsubseteq V_i$, $\psi_{iX} \in X^*|_{V_i\times\{u_{-i}\}}$. By the Green-Laffont-Holmström Theorem, ψ_i has the required form. The claim follows. □

We have found a list of functions $(h_i)_{i\in N}$ such that φ has the VCG form whenever $u \in U$. Thus $\varphi|_U$ is an individually rational, budget balanced VCG rule. Lemma 9 then implies that it can be extended to an individually rational, budget balanced VCG rule on V, contradicting Theorem 1. ■

Appendix C. Proof of Proposition 1

Proposition 1 is an obvious corollary of the following proposition, which we show in these appendices.

Proposition 3. *Let $u \in \mathcal{U}^N$, $z := (x, m) := W(u)$, and let $p \in \mathbb{R}_+$ be the (unique) Walrasian price of the commodity, in terms of money, at profile u. Assume*

(1) For each $i \in N$, $(x_i, m_i) \neq \omega_i$;

(2) There is a non-empty set of agents $N' \subset N$ such that, for each $i \in N'$,

(a) $m_i = 0$,

(b) $p < \frac{du_i}{dx}(x_i + \frac{x_i}{|N'|-1})$, and

(c) $u_i\left(x_i - \frac{x_i}{|N'|}\right) > u_i(\omega_X)$.

Then there is a rule φ and a neighborhood $V \ni u$ such that for each $u' \in V$, φ is not *manipulable at u'.*

Assume $u \in \mathcal{U}^N$ satisfies conditions 1 and 2a of Proposition 3. We first construct a *min-stable* rule φ^p and an open set $V \ni u$ such that the restriction $\varphi^p|_V$ is *strategy-proof*. It is without loss of generality to assume V is rectangular; that is, there are open sets $(V_i)_{i\in N}$ such that $V = \Pi_{i\in N} V_i$.

It will be convenient to have a "dummy" agent 0 with the following formal properties: $0 \notin N$ and for any allocation (x, m), feasible or otherwise, $(x_0, m_0) = (0, 0)$. This second point is not to be superseded by any definitions that follow. Thus if $i^* : \mathcal{U}^N \to N$ is a function designating an agent for each economy, and if for some function $f : \mathcal{U}^N \to \mathbb{R}^2$ we set $\varphi_{i^*(u)}(u) := f(u)$, then for the case $i^*(u) = 0$, $\varphi_{i^*(u)}(u) := (0, 0)$, regardless of $f(u)$.

Given preference $u_i \in \mathcal{U}$, price $p' \in \mathbb{R}_{++}$, and endowment $\omega' \in \mathbb{R}^2$, denote by $D(u_i, p', \omega') \in \mathbb{R}^2$ agent i's demanded consumption. If $\omega' := \omega$, the notation for endowment is suppressed. For each $i \in N$ and each $p' \in \mathbb{R}_{++}$, we define an adjusted endowment function $\omega_i(\cdot; p')$. First, define functions

$$f(i, \hat{u}; p') := \frac{\sum_{j \in N \setminus (N' \cup \{i\})} D_M(u_j, p) - D_M(\hat{u}_j, p')}{|N \setminus N'| - 1}.$$

and

$$g(i, \hat{u}; p') := \frac{\sum_{j \in N \setminus N'} D_X(u_j, p) - D_X(\hat{u}_j, p')}{|N'|} + \frac{\sum_{j \in N' \setminus i} D_X(u_j, p)}{|N'| - 1}.$$

Finally, for each $i \in N$ and each $p' \in \mathbb{R}_{++}$, let

$$\omega_i(\hat{u}; p') := \begin{cases} (\omega_{iX}, f(i, \hat{u}; p')) & i \in N \setminus N' \\ (g(i, \hat{u}, p'), 0) & i \in N'. \end{cases}$$

For each $p' \in \mathbb{R}_{++}$, each $\hat{u} \in V$, and each $i \in N$, $\varphi_i(\hat{u}; p') := D(\hat{u}_i, p', \omega_i(\hat{u}; p))$. By construction, agents cannot influence their own adjusted endowment. It follows that the function $\varphi(\cdot; p)|_V$ is *strategy-proof*. If V is sufficiently small, it is also the case that $\varphi(\cdot; p)|_V$ is *efficient* and *voluntary*. *Efficiency* results when V is small enough that, for each $\hat{u} \in V$, at $\varphi(\hat{u}; p)$, it remains the case that the $N \setminus N'$ agents are consuming on the interior. Similarly, *voluntary participation* holds when $\omega_i(\hat{u}, p)$ and ω_i do not differ by much. Thus, the restricted rule $\varphi(\cdot; p)|_V$ is *min-stable* and *strategy-proof*.

Assume now that $u \in \mathcal{U}^N$ satisfies all the conditions of Proposition 3. We extend $\varphi(\cdot; p)|_V$ to a rule $\Phi : \mathcal{U}^N \to Z$ such that for each $u' \in V$, $\Phi(u')$ is not manipulable at u'. It may be necessary to make stronger assumptions on V.

Let $V^* := \left\{\hat{u} \in \mathcal{U}^N : \exists i \in N,\ \tilde{u}_i \in \mathcal{U},\ s.t.\ (\tilde{u}_i, \hat{u}_{-i}) \in V\right\}$. For each $\hat{u} \in V^* \setminus V$ there is an agent $i^*(\hat{u})$ such that there is $\tilde{u}_{i^*(\hat{u})} \in \mathcal{U}$ making $\left(\tilde{u}_{i^*(\hat{u})}, \hat{u}_{-i^*(\hat{u})}\right) \in V$. Let $\underline{p} := \min\left\{p^W(\tilde{u}_i, u_{-i}) :\ i \in N,\ \tilde{u}_i \in \mathcal{U}\right\}$ and $\overline{p} := \max\left\{p^W(\tilde{u}_i, u_{-i}) :\ i \in N,\ \tilde{u}_i \in \mathcal{U}\right\}$. Let $A := \left\{y \in \mathbb{R}^2 : \left(\underline{p}, 1\right)(y-\omega) \leq 0,\ (\bar{p}, 1)(y-\omega) \leq 0\right\}$. Let $\hat{u} \in V^* \setminus V$, $i^* := i^*(\hat{u})$ and set

$$\Phi_{i^*}(\hat{u}) := \arg \max_{(x,m) \in A \cup \{D(\hat{u}_{i^*}, p, \omega_{i^*}(\hat{u};p))\}} \hat{u}_{i^*}(x) + m.$$

Note that for each $\hat{u} \in V^* \setminus V$, $\pi(\hat{u}) := \frac{d\hat{u}_{i^*(\hat{u})}}{dx}(\Phi_{i^*X}(\hat{u}))$ is well-defined. Let $\Delta(\hat{u}) := D(\hat{u}_{i^*}, \pi(\hat{u}), \omega_{i^*}(\hat{u}; \pi(\hat{u}))) - \Phi_{i^*}(\hat{u})$ and define

$$\bar{\Delta}_i(\hat{u}) := \begin{cases} \left|N \setminus (N' \cup \{i^*\})\right|^{-1} (0, \Delta_M(\hat{u})) & i \in N \setminus N' \\ \left|N' \setminus \{i^*\}\right|^{-1} (\Delta_X(\hat{u}), 0) & i \in N' \\ (0,0) & \text{otherwise.} \end{cases}$$

Thus we define, for each $i \neq i^*(\hat{u})$, $\varphi_i(\hat{u}; p') := D(\hat{u}_i, p', \omega_i(\hat{u}; p')) + \bar{\Delta}_i(\hat{u})$.

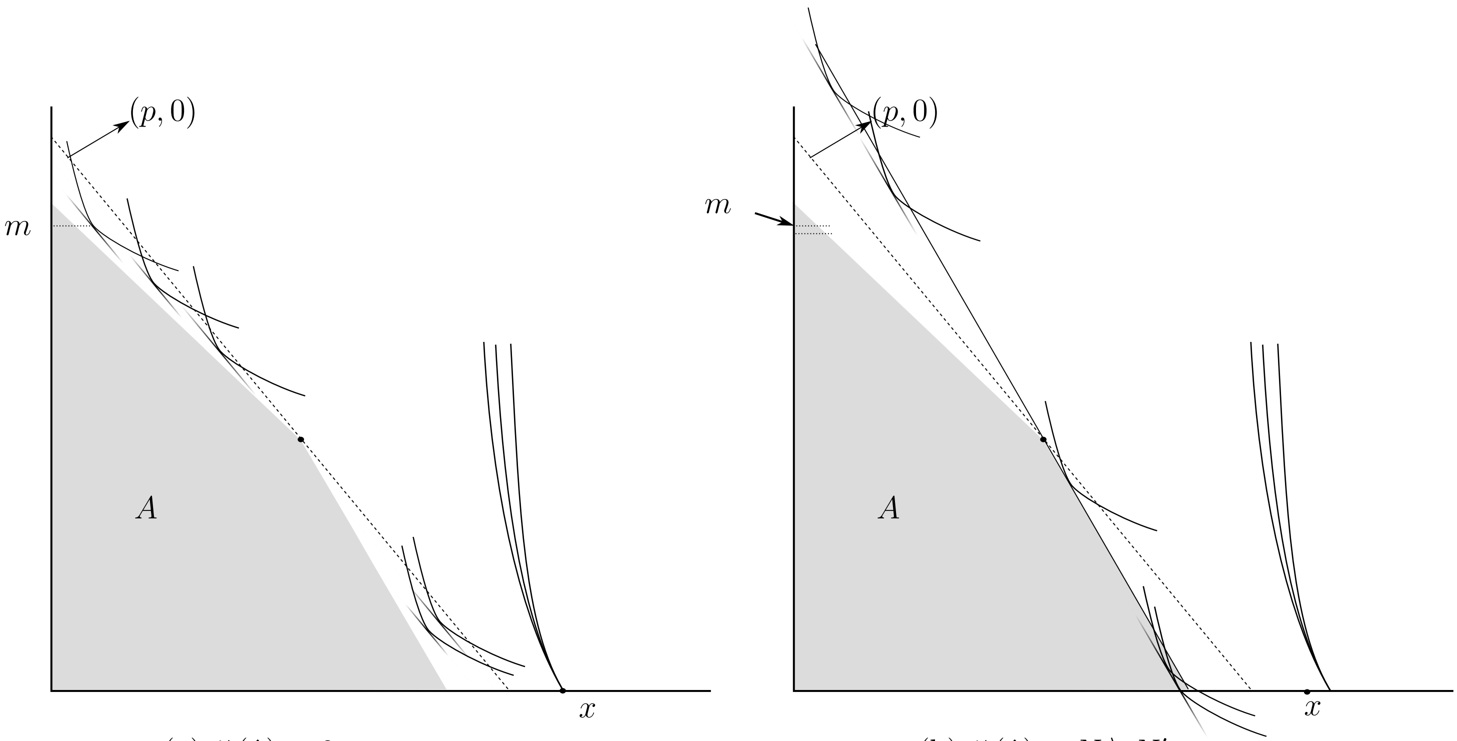
In fact π could also be defined on V, in which case $\pi|_V$ is identically p. Therefore we define, for each $u \in \mathcal{U}^N$, each $i \in N$

$$\Phi_i(u) := \begin{cases} \varphi_i^p(u) & u \in V \\ \varphi_i^{\pi(u)}(u) & u \in V^* \setminus V,\ i \neq i^*(u) \\ \Phi_{i^*}(u) & u \in V^* \setminus V,\ i = i^*(u) \\ W(u) & u \notin V^*. \end{cases}$$

Figure C.1 illustrates the idea of the rule Φ. Panel (a) shows a case where $\hat{u} \in V$. Note that all the interior agents have $D_X(\hat{u}_i, p) < D_X(u_i, p)$ and consequently $D_M(\hat{u}_i, p) > D_M(u_i, p)$. Thus, their endowments are all adjusted downward. The excess of commodity X is absorbed by the N' agents. In panel (b) an agent $i \in N$ has departed from V_i. Note that in the picture, $\Delta(\hat{u}) < 0$.

C.1. **Feasibility of Φ .**

Proposition 4. $\sum_{i \in N} \Phi_i \equiv \Omega$.



(a) $i^*(\hat{u}) = 0$ (b) $i^*(\hat{u}) \in N \setminus N'$

Figure C.1. Rule Phi

Proof. Assume $\hat{u} \in V^* \setminus V$. The case $\hat{u} \in V$ is simpler and is in fact nested in this proof. Since for each $i \neq i^*(\hat{u})$, $\Phi_i(\hat{u}) := \varphi_i^{\pi(\hat{u})}(\hat{u})$, we have

$$\sum_{i \in N} \Phi_i(\hat{u}) = \Phi_{i^*(\hat{u})}(\hat{u}) + \sum_{i \in N \setminus \{i^*(\hat{u})\}} \left(D\left(\hat{u}_i, \pi(\hat{u}), \omega_i(\hat{u}; \pi(\hat{u}))\right) + \bar{\Delta}_i(\hat{u}) \right). \tag{C.1}$$

For simpler notation, let $i^* := i^*(\hat{u})$ and $\pi := \pi(\hat{u})$.

$$\begin{aligned}
\sum_{i \in N \setminus \{i^*\}} \bar{\Delta}_i(\hat{u}) &= \sum_{i \in N' \setminus \{i^*\}} \bar{\Delta}_i(\hat{u}) \\
&= \sum_{i \in N' \setminus \{i^*\}} \frac{\Delta_i(\hat{u})}{|N \setminus \{i^*\}|} \\
&= \sum_{i \in N' \setminus \{i^*\}} \frac{\Delta_i(\hat{u})}{|N \setminus \{i^*\}|} \\
&= \sum_{i \in N' \setminus \{i^*\}} \frac{D\left(\hat{u}_{i^*}, \pi(\hat{u}), \omega_{i^*}\left(\hat{u}; \pi(\hat{u})\right)\right) - \Phi_{i^*}\left(\hat{u}\right)}{|N \setminus \{i^*\}|} \\
&= D\left(\hat{u}_{i^*}, \pi(\hat{u}), \omega_{i^*}\left(\hat{u}; \pi(\hat{u})\right)\right) - \Phi_{i^*}\left(\hat{u}\right).
\end{aligned}$$

Thus equation C.1 simplifies to

(C.2)

$$\sum_{i \in N} \Phi_i\left(\hat{u}\right) = D\left(\hat{u}_{i^*}, \pi, \omega_{i^*}\left(\hat{u}; \pi(\hat{u})\right)\right) + \sum_{i \in N \setminus \{i^*\}} D\left(\hat{u}_i, \pi, \omega_i(\hat{u}; \pi)\right) = \sum_{i \in N} D\left(\hat{u}_i, \pi, \omega_i(\hat{u}; \pi)\right).$$

By assuming $|N|$ is large, $\bar{p} - \underline{p}$ can be made arbitrarily small. Note that $\pi \in [\underline{p}, \bar{p}]$. Thus if V is sufficiently small, the demand of each $i \in N \setminus N'$ given price π and adjusted endowment $\omega_i\left(\hat{u}; \pi\right)$ is interior. Thus, by quasilinearity $D\left(\hat{u}_i, \pi, \omega_i(u_{i^*}, \hat{u}_{-i^*}; \pi(\hat{u}))\right) = D\left(\hat{u}_i, \pi\right) + \left(0, \omega_i(u_{i^*}, \hat{u}_{-i^*}; \pi(\hat{u}))\right)$. Similarly, each $i \in N'$ continues to demand a boundary bundle and therefore $D\left(\hat{u}_i, \pi, \omega_i(u_{i^*}, \hat{u}_{-i^*}; \pi)\right) = \omega_i(\hat{u}; \pi) = (\omega_{iX}(\hat{u}; \pi), 0)$. Recall that

$$\omega_{iX}(\hat{u}; \pi) = \frac{\sum_{j \in N \setminus N'} D_X\left(u_j, p\right) - D_X\left(\hat{u}_j, \pi\right)}{|N'|} + \frac{\sum_{j \in N' \setminus i} D_X\left(u_j, p\right)}{|N'| - 1}.$$

Therefore,

$$\begin{aligned}
\sum_{i \in N'} \omega_{iX}(\hat{u}; \pi) &= \sum_{i \in N'} \left(\frac{\sum_{j \in N \setminus N'} D_X\left(u_j, p\right) - D_X\left(\hat{u}_j, \pi\right)}{|N'|} + \frac{\sum_{j \in N' \setminus i} D_X\left(u_j, p\right)}{|N'| - 1} \right) \\
&= \sum_{j \in N \setminus N'} D_X\left(u_j, p\right) - D_X\left(\hat{u}_j, \pi\right) + \sum_{j \in N'} D_X\left(u_j, p\right) \\
&= \sum_{j \in N} D_X\left(u_j, p\right) - \sum_{j \in N \setminus N'} D_X\left(\hat{u}_j, \pi\right).
\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i\in N} D_X\left(\hat{u}_i,\pi,\omega_i(\hat{u};\pi)\right) &= \sum_{i\in N\setminus N'} D_X\left(\hat{u}_i,\pi\right)+\sum_{i\in N'}\omega_{iX}(\hat{u};\pi)\\ &= \sum_{i\in N\setminus N'} D_X\left(\hat{u}_i,\pi\right)+\sum_{i\in N} D_X\left(u_i,p\right)-\sum_{i\in N\setminus N'} D_X\left(\hat{u}_i,\pi\right)\\ &= \sum_{i\in N} D_X\left(u_i,p\right)=\Omega_X.\end{aligned}$$

Symmetrically,

$$\begin{aligned}\sum_{i\in N\setminus N'} D_M\left(\hat{u}_i,\pi,\omega_i(u_{i^*},\hat{u}_{-i^*};\pi(\hat{u}))\right) &= \sum_{i\in N\setminus N'}\left[D_M\left(\hat{u}_i,\pi\right)+\omega_{iM}(\hat{u};\pi)\right]\\ &= \sum_{i\in N\setminus N'} D_M\left(\hat{u}_i,\pi\right)\\ &\quad+\sum_{i\in N\setminus N'}\left[\frac{\sum_{j\in N\setminus(N'\cup\{i\})} D_M\left(u_j,p\right)-D_M(\hat{u}_j,\pi)}{|N\setminus N'|-1}\right]\\ &= \sum_{i\in N\setminus N'} D_M\left(\hat{u}_i,\pi\right)+\sum_{i\in N\setminus N'} D_M\left(u_j,p\right)-D_M(\hat{u}_j,\pi)\\ &= \sum_{i\in N\setminus N'} D_M\left(u_j,p\right)=\Omega_M.\end{aligned}$$

■

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