ISSN (Print) 0473-453X
Discussion Paper No. 1001
ISSN (Online) 2435-0982

STRATEGY-PROOF MULTI-OBJECT MECHANISM DESIGN: EX-POST REVENUE MAXIMIZATION WITH NON-QUASILINEAR PREFERENCES

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Revised January 2020
May 2017

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Strategy-proof multi-object mechanism design: Ex-post revenue maximization with non-quasilinear preferences *

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January 7, 2020

Abstract

A seller is selling multiple objects to a set of agents, who can buy at most one object. Each agent's preference over (object, payment) pairs need not be quasilinear. The seller considers the following desiderata for her mechanism, which she terms *desirable*: (1) *strategy-proofness*, (2) *ex-post individual rationality*, (3) *equal treatment of equals*, (4) *no wastage* (every object is allocated to some agent). The minimum Walrasian equilibrium price (MWEP) mechanism is desirable. We show that at each preference profile, the MWEP mechanism generates more revenue for the seller than any desirable mechanism satisfying no subsidy. Our result works for the quasilinear domain, where the MWEP mechanism is the VCG mechanism, and for various non-quasilinear domains, some of which incorporate positive income effect of agents. We can relax no subsidy to *no bankruptcy* in our result for certain domains with positive income effect.

Keywords. multi-object allocation; strategy-proofness; ex-post revenue maximization; minimum Walrasian equilibrium price mechanism; non-quasilinear preferences; no wastage; equal treatment of equals.

JEL Code. D82, D47, D71, D63.

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*We have benefited from insightful comments of three referees and an associate editor. We are also grateful to Brian Baisa, Elizabeth Baldwin, Inés Moreno de Barreda, Dirk Bergemann, Yeon-koo Che, Vincent Crawford, Bhaskar Dutta, Albin Erlanson, Johannes Horner, Paul Klemperer, Hideo Konishi, Nicolas S. Lambert, Andrew Mackenzie, Komal Malik, Paul Milgrom, Noam Nisan, Andrew Postlewaite, Larry Samuelson, James Schummer, Ilya Segal, Alex Teytelboym, Ryan Tierney, Rakesh Vohra, Jörgen Weibull, and various seminar participants for their comments. We gratefully acknowledge financial support from the Joint Usage/Research Center at ISER, Osaka University and the Japan Society for the Promotion of Science (Kazumura, 14J05972; Serizawa, 15J01287, 15H03328, 15H05728).

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1 Introduction

One of the most challenging problems in microeconomic theory is the design of a revenue maximizing mechanism in the multi-object allocation problems. Ever since the seminal work of Myerson (1981) for solving the revenue maximizing mechanism in the single object environment, advances in the mechanism design literature have convinced researchers that it is difficult to precisely describe a revenue maximizing mechanism in multi-object environments.

In the literature on revenue maximizing mechanism design, authors conventionally impose only incentive compatibility and individual rationality conditions, and try to find a mechanism that maximizes the (expected) revenue among mechanisms satisfying those conventional conditions. When allocating public assets, governments are supposed to pursue several goals, such as fairness and efficiency, besides revenue maximization.[1] Since our main focus is on revenue maximization, we impose only moderate desiderata for other goals on mechanisms. In other words, we define several conditions embodying other goals, and maximize revenue in the class of mechanisms satisfying those new conditions along with the conventional incentive and participation constraints.

We study the problem of allocating m indivisible heterogenous objects to $n > m$ agents, each of whom can be assigned at most one object (unit demand agents) – such unit demand settings are common in allocating houses in public housing schemes (Andersson and Svensson, 2014), selling team franchises in professional sports leagues, and even in selling a small number of spectrum licenses (Binmore and Klemperer, 2002).[2][3] Agents in our model can have non-quasilinear preferences over consumption bundles - (object, payment) pairs.

We briefly describe the additional axioms that we impose for our revenue maximization exercise. *Equal treatment of equals* is a desideratum for fairness, and requires that two agents

[1]For example, Klemperer (2002) discusses the list of goals pursued in UK 3G auction conducted in 2000.

[2]When a professional cricket league, called the *Indian Premier League (IPL)* was started in India in 2007, professional teams were sold to interested owners (bidders) by an auction. Since it does not make sense for an owner to have two teams, the unit demand assumption is satisfied in this problem. See the Wiki entry of IPL for details:

https://en.wikipedia.org/wiki/Indian_Premier_League and a news article here: http://content-usa.cricinfo.com/ipl/content/current/story/333193.html

[3]Although modern spectrum auctions involve sale of of *bundles* of spectrum licenses, Binmore and Klemperer (2002) report that one of the biggest spectrum auctions in the UK involved selling a fixed number of licenses to bidders, each of whom can be assigned at most one license. The unit demand setting is also one of the few *computationally* tractable model of combinatorial auction studied in the literature (Blumrosen and Nisan, 2007).

having identical preferences be assigned consumption bundles (i.e., (object, payment) pairs) to which they are indifferent. *No wastage* is a desideratum for a mild form of efficiency, and requires that every object be allocated to some agent. We term a mechanism *desirable* if it satisfies strategy-proofness, ex-post individual rationality, equal treatment of equals, and no wastage.

The mechanism we identify in this paper is based on a market clearing idea. A price vector on objects is called a *Walrasian equilibrium price (WEP) vector* if there is an allocation of objects such that each agent gets an object from his demand set. Demange and Gale (1985) showed that the set of WEP vectors is always a non-empty compact lattice in our model. This means that there is a unique minimum WEP vector.[4] The *minimum Walrasian equilibrium price (MWEP)* mechanism selects the minimum WEP vector at every profile of preferences and uses a corresponding equilibrium allocation. The MWEP mechanism is desirable (Demange and Gale, 1985) and satisfies *no subsidy*. No subsidy requires that payment of each agent be non-negative. In the quasilinear domain of preferences, the MWEP mechanism coincides with the Vickrey-Clarke-Groves (VCG) mechanism (Leonard, 1983). However, we emphasize that outside the quasilinear domain, a naive generalization of the VCG mechanism to non-quasilinear preferences is not strategy-proof (Morimoto and Serizawa, 2015).[5] This also means that for an arbitrary domain of classical preferences, the MWEP mechanism is very different from a generalization of the VCG mechanism.

We show that on a variety of domains (the set of admissible preferences), the *MWEP* mechanism is *ex-post revenue optimal* among all desirable mechanisms satisfying *no subsidy*, i.e., for each preference profile, the MWEP mechanism generates more revenue for the seller than any desirable mechanism satisfying no subsidy (Theorem 1). Further, we show that if the domain includes all positive income effect preferences, then the MWEP mechanism is ex-post revenue optimal in the class of all desirable and *no bankruptcy* mechanisms (Theorem 2). No bankruptcy is a weaker condition than no subsidy and requires the sum of payments of all agents across all profiles be bounded below.

Our results are robust in the following sense. First, the MWEP mechanism maximizes ex-post revenue. Hence, we can recommend the MWEP mechanism without resorting to any prior-based maximization. Notice that ex-post revenue optimality is much stronger than expected (ex-ante) revenue optimality, and mechanisms satisfying ex-post revenue optimality

[4]Results of this kind were earlier known for quasilinear preferences (Shapley and Shubik, 1971; Leonard, 1983).

[5]See Section 6.2 in Morimoto and Serizawa (2015).

rarely exist. Second, our results hold on a variety of domains. This is in contrast to many papers in the literature on mechanism design in which results are established only on quasi-linear domain. Our main result (Theorem 1) holds on every domain satisfying a *richness* condition. The richness condition requires the domain to include enough variety preferences. However it is weak enough to be satisfied by various well known domains, such as the quasi-linear domain, the classical domain, the domain of positive income effect preferences, and any domain including one of those domains.

Ours is the first paper to study revenue maximization in a multi-object allocation problem when preferences of agents are not quasilinear. While quasilinearity is standard and popular in the literature, its practical relevance is debatable in many settings. There are at least two obvious reasons why quasilinearity may fail in practice. First, bidders in auctions usually invest in various supporting products and processes to realize the full value of the object. For instance, cellular companies invest in communication infrastructure development, a sports team owner invests in marketing, and so on. Such ex-post investments cannot be assumed to be independent of the payments in auctions. This hints at an explicit effect of payments in the auction mechanism on the values of objects in these problems. Another source of non-quasilinearity is borrowing costs. Usually, bidders in large auctions (like spectrum auctions, housing auctions, etc.) borrow to pay for objects. The higher interest rates imposed on the larger amount of borrowings make preferences non-quasilinear.[6]

Finding optimal mechanism (expected revenue maximizing mechanism subject to Bayesian incentive compatibility and individual rationality) in multi-object auction environment is a difficult problem – see Section 8 for an extensive literature review. The main difficulty is that the traditional Myersonian approach works by figuring out the binding incentive constraints, which is difficult to characterize in multi-object auction models. This is a long recognized problem (Armstrong, 2000). The literature is developing new toolkits to solve these problems – see Carroll (2016), who imposes an additional informational robustness condition to figure out the binding incentive constraints; and Daskalakis et al. (2017), who use optimal transportation theory to find the optimal mechanism in a single agent problem. While we certainly do not introduce any new method to solve the multidimensional mechanism design problems, our results show that one can circumvent some of the difficulties in these problems by imposing additional axioms.

We briefly discuss the practical relevance of two of our axioms: equal treatment of equals and no wastage. Later, we elaborate the kind of mechanisms we rule out by imposing these

[6]We will discuss the effect of borrowing cost on preferences in Subsection 4.1.

axioms. Equal treatment of equals is arguably the weakest fairness axiom in the literature – as Aristotle (1995) writes, justice is considered to mean "equality for those who are equal, and not for all".[7] Sometimes, there are practical implications of violating fairness – for instance, Deb and Pai (2016) cite many legal implications of violating *symmetry* in mechanisms, which is a stronger property than equal treatment of equals.

Efficiency is an important goal for governments. Although Pareto efficiency is a standard efficiency desideratum in the literature, since we focus on revenue maximization, we impose no wastage, a much weaker desideratum. Unlike Pareto efficiency, no wastage is an easily detectable axiom (detecting violation of Pareto efficiency requires the knowledge of preferences). Violation of no wastage in government auctions creates a lot of controversies in the public, and often, the unsold objects are resold.[8] In such environments, governments cannot commit to reserve prices even though expected revenue maximization may require them. Indeed, McAfee and McMillan (1987); Ashenfelter and Graddy (2003); Jehiel and Lamy (2015); Hu et al. (2017) report that many real-life auctions have zero reserve price – McAfee and McMillan (1996), Jehiel and Lamy (2015) and Hu et al. (2017) build theoretical models to explain it as an equilibrium phenomenon. While our results do not provide a theory for why the seller should not keep a reserve price, we show that if the seller uses a mechanism satisfying no wastage and other desirable properties, then the MWEP mechanism is ex-post revenue optimal.

Finally, the MWEP mechanism is Pareto efficient and can be implemented as a simultaneous ascending auction (SAA) - for quasilinear domains, see Demange et al. (1986), and for non-quasilinear domains, see Morimoto and Serizawa (2015).[9] SAAs have distinct advantages of practical implementation and are often used in practice to allocate multiple objects. The efficiency foundations for SAAs have been well-established. Because of their practical importance, it is worth providing alternate foundations for SAAs. Our results provide a *revenue maximization* foundation for SAAs. This differentiates our results from previous research on

[7]The quote is from Aristotle's Book III titled "The Theory of Citizenship and Constitutions". It can be found in Part C of the book, titled "The Principle of Oligarchy and Democracy and the Nature of Distributive Justice", in Chapter 9 and paragraph 1280a7.

[8]As an example, the Indian spectrum auctions reported a large number of unsold spectrum blocks in 2016, and all of them are supposed to be re-auctioned. See the following news article: http://www.livemint.com/Industry/xt5r4Zs5RmzjdwuLUdwJMI/Spectrum-auction-ends-after-lukewarm-response-from-telcos.html

[9]To be precise, the MWEP mechanism can be implemented as a simple ascending price auction with a sufficiently small price increment.

the MWEP mechanism and SAA, most of which focus on efficiency properties (Ausubel and Milgrom, 2002).

2 Preliminaries

A seller has m objects to sell, denoted by $M := \{1, \ldots, m\}$. There are $n > m$ agents (buyers), denoted by $N := \{1, \ldots, n\}$. Each agent can receive at most one object (unit demand preference). Let $L := M \cup \{0\}$, where 0 is the null object, which is assigned to any agent who does not receive any object in M – thus, the null object can be assigned to more than one agent. Note that the unit demand restriction can either be a restriction on preferences or an institutional constraint. For instance, objects may be substitutable when houses are being allocated in a public housing scheme (Andersson and Svensson, 2014). The unit demand restriction can also be institutional as was the case in the spectrum license auction in UK in 2000 (Binmore and Klemperer, 2002) or in the Indian Premier League auction. As long as the mechanism designer restricts messages in the mechanisms to *only* use information on preferences over individual objects, our results apply.

The consumption set of every agent is the set $L \times \mathbb{R}$, where a typical (consumption) bundle $z \equiv (a, t)$ corresponds to object $a \in L$ and payment $t \in \mathbb{R}$. Notice that t denotes the amount *paid* by an agent to the designer. Now, we formally introduce preferences of agents and the notion of a desirable mechanism.

2.1 The preferences

A preference ordering R_i (of agent i) over $L \times \mathbb{R}$, with strict part P_i and indifference part I_i, is **classical** if it satisfies the following assumptions:

1. **Money monotonicity.** for every $t, t' \in \mathbb{R}$ with $t > t'$ and for every $a \in L$, we have $(a, t')\ P_i\ (a, t)$.

2. **Desirability of objects.** for every $t \in \mathbb{R}$ and for every $a \in M$, $(a, t)\ P_i\ (0, t)$.

3. **Continuity.** for every $z \in L \times \mathbb{R}$, the sets $\{z' \in L \times \mathbb{R} : z'\ R_i\ z\}$ and $\{z' \in L \times \mathbb{R} : z\ R_i\ z'\}$ are closed.

4. **Possibility of compensation.** for every $z \in L \times \mathbb{R}$ and for every $a \in L$, there exists a pair $t, t' \in \mathbb{R}$ such that $z\ R_i\ (a, t)$ and $(a, t')\ R_i\ z$.

A classical preference R_i is *quasilinear* if there exists $v \in \mathbb{R}^{|L|}$ such that for every $a, b \in L$ and $t, t' \in \mathbb{R}$, $(a, t)\ R_i\ (b, t')$ if and only if $v_a - t \geq v_b - t'$. We refer to v as the valuation of the agent, and we normalize v_0 to 0. The idea of valuation may be generalized as follows for non-quasilinear preferences.

Definition 1 *The* **valuation** *at a classical preference R_i for object $a \in L$ with respect to bundle $z \in L \times \mathbb{R}$ is defined as $V^{R_i}(a, z)$, which uniquely solves $(a, V^{R_i}(a, z))\ I_i\ z$.*

Hence, $V^{R_i}(a, z)$ is the amount t agent i is willing to pay so that he is indifferent between (a, t) and z. A straightforward consequence of our assumptions is that for every $a \in L$, for every $z \in L \times \mathbb{R}$, and for every classical preference R_i, the valuation $V^{R_i}(a, z)$ exists. For any R and for any $z \in L \times \mathbb{R}$, the valuations at R with respect to z is a vector in $\mathbb{R}^{|L|}$.

An illustration of the valuation is shown in Figure 1. In the figure, the horizontal

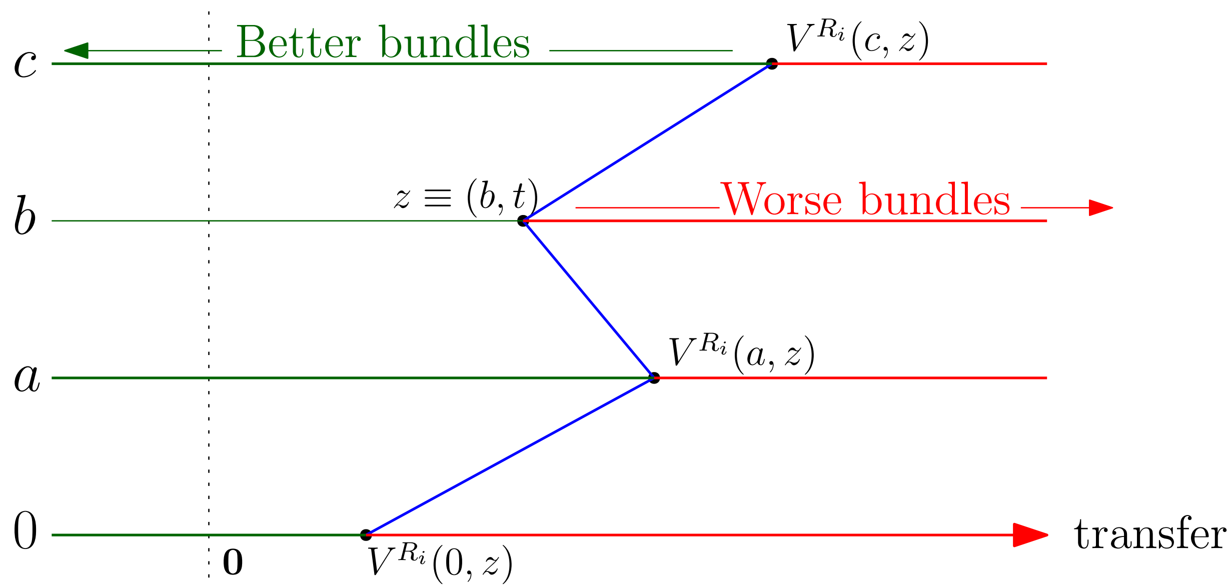

Figure 1: Valuation at a preference

lines correspond to objects: $L = \{0, a, b, c\}$. The horizontal lines indicate payment levels. Hence, the consumption set consists of the four lines. For example, z denotes the bundle consisting of object b and the payment equal to the distance of z from the vertical dotted line. A preference R_i can be described by drawing (non-intersecting) indifference vectors through these consumption bundles (lines). One such indifference vector passing through z is shown in Figure 1. This indifference vector actually consists of four points: $(0, V^{R_i}(0, z)), (a, V^{R_i}(a, z)), z \equiv (b, t)$, and $(c, V^{R_i}(c, z))$ as shown. Parts of the indifference line in Figure 1 which lie between the consumption bundle lines is useless and has no meaning, and it is only displayed for convenience. As we go to the right along the horizontal lines starting from any bundle, we get worse bundles (due to money monotonicity). Similarly, bundles to the left of a particular bundle are better than that bundle. This is shown in Figure 1 with respect to the indifference vector.

Our modeling of preferences captures income effects even though we do not model income explicitly. We explain this point when we introduce positive income effect in Section 4.1.

2.2 Desirable mechanisms

Let $\mathcal{R}^C$ denote the set of all classical preferences and $\mathcal{R}^Q$ denote the set of all quasilinear preferences. We will consider an arbitrary subset of classical preferences $\mathcal{R} \subseteq \mathcal{R}^C$ - we will put specific restrictions on $\mathcal{R}$ later. A preference of agent i is denoted by $R_i \in \mathcal{R}$. A preference profile is a list of preferences $R \equiv (R_1, \ldots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N'}$, respectively.

An *object allocation* is an n-tuple $(a_1, \ldots, a_n) \in L^n$ such that no real (non-null) object is assigned to two agents, i.e., $a_i \neq a_j$ for all $i, j \in N$ with $a_i, a_j \neq 0$. The set of all object allocations is denoted by A. A (feasible) allocation is an n-tuple $((a_1, t_1), \ldots, (a_n, t_n)) \in (L \times \mathbb{R})^n$ such that $(a_1, \ldots, a_n) \in A$, where (a_i, t_i) is the bundle of agent i. Let Z denote the set of all feasible allocations. For every allocation $(z_1, \ldots, z_n) \in Z$, we will denote by z_i the bundle of agent i.

An **mechanism** is a map $f : \mathcal{R}^n \to Z$. By definition, we restrict ourseleves to **deterministic** mechanisms. Allowing for randomization will entail considering preferences over lotteries of allocations. This brings substantial difficulty in modeling and analysis. We do not know how our results will extend if we allow for randomization.

At a preference profile $R \in \mathcal{R}^n$, we denote the bundle of agent i in mechanism f as $f_i(R) \equiv (a_i(R), t_i(R))$, where $a_i(R)$ and $t_i(R)$ are respectively the object allocated to agent i and i's payment at preference profile R. We call $a(\cdot) \equiv (a_1(\cdot), \ldots, a_n(\cdot))$ and $t(\cdot) \equiv (t_1(\cdot), \ldots, t_n(\cdot))$ the object allocation mechanism and the payment mechanism, respectively of f.

DEFINITION **2** *A mechanism $f : \mathcal{R}^n \to Z$ is* **desirable** *if it satisfies the following properties:*

1. **Strategy-proofness.** *for every $i \in N$, for every $R_{-i} \in \mathcal{R}^{n-1}$, and for every $R_i, R'_i \in \mathcal{R}$, we have*

$$f_i(R_i, R_{-i}) \; R_i \; f_i(R'_i, R_{-i}).$$

2. **(Ex-post) individual rationality (IR).** *for every* $i \in N$*, for every* $R \in \mathcal{R}^n$*, we have* $f_i(R)\ R_i\ (0,0)$.

3. **Equal treatment of equals (ETE).** *for every* $i, j \in N$*, for every* $R \in \mathcal{R}^n$ *with* $R_i = R_j$*, we have* $f_i(R)\ I_i\ f_j(R)$.

4. **No wastage (NW).** *for every* $R \in \mathcal{R}^n$ *and for every* $a \in M$*, there exists some* $i \in N$ *such that* $a_i(R) = a$.

Besides desirability, for some of our results, we will require some form of restrictions on payments.

Definition 3 *A mechanism* $f : \mathcal{R}^n \to Z$ *satisfies* **no subsidy** *if for every* $R \in \mathcal{R}^n$ *and for every* $i \in N$*, we have* $t_i(R) \geq 0$.

No subsidy can be considered desirable to exclude "fake" agents, who participate in mechanisms just to take away available subsidy. It is an axiom satisfied by most standard mechanisms in practice. It is also motivated by the fact that in many settings, the seller may not have any means to finance any agent.

3 The minimum Walrasian equilibrium price mechanism

In this section, we define the notion of a Walrasian equilibrium, and use it to define a desirable mechanism. A price vector $p \in \mathbb{R}_+^{|L|}$ defines a price for every object with $p_0 = 0$. At any price vector $p \in \mathbb{R}_+^{|L|}$, let $D(R_i, p) := \{a \in L : (a, p_a)\ R_i\ (b, p_b)\ \forall\ b \in L\}$ denote the demand set of agent i with preference R_i at price vector p.

Definition 4 *An object allocation* $(a_1, \ldots, a_n) \in A$ *and a price vector* $p \in \mathbb{R}_+^{|L|}$ *is a* **Walrasian equilibrium** *at a preference profile* $R \in \mathcal{R}^n$ *if*

1. $a_i \in D(R_i, p)$ *for all* $i \in N$ *and*

2. $p_a = 0$ *for all* $a \in M \setminus \{a_1, \ldots, a_n\}$.

We refer to p *and* $((a_1, p_{a_1}), \ldots, (a_n, p_{a_n}))$ *defined above as a* **Walrasian equilibrium price vector** *and a* **Walrasian equilibrium allocation** *at* R *respectively.*

Since we assume $n > m$ and preferences satisfy desirability of objects, the conditions of Walrasian equilibrium imply that for all $a \in M$, we have $a_i = a$ for some $i \in N$.[10]

A Walrasian equilibrium price vector p is a **minimum Walrasian equilibrium price vector** at preference profile R if for every Walrasian equilibrium price vector p' at R, we have $p_a \leq p'_a$ for all $a \in L$. At every $R \in (\mathcal{R}^C)^n$, a Walrasian equilibrium exists (Alkan and Gale, 1990), the set of Walrasian equilibrium price vectors forms a lattice with a unique minimum and a unique maximum Walrasian equilibrium price vector (Demange and Gale, 1985). We denote the minimum Walrasian equilibrium price vector at $R \in (\mathcal{R}^C)^n$ as $p^{min}(R)$. Notice that by desirability of objects, if $n > m$, then for every $a \in M$, we have $p^{min}_a(R) > 0$.[11]

We give an example to illustrate the notion of minimum Walrasian equilibrium price vector. Suppose $N = \{1, 2, 3\}$ and $M = \{a, b\}$. Figure 2 shows some indifference vectors of a preference profile $R \equiv (R_1, R_2, R_3)$ and the corresponding minimum Walrasian equilibrium price vector $p^{min}(R) \equiv p^{min} \equiv (p^{min}_0 = 0, p^{min}_a, p^{min}_b)$.

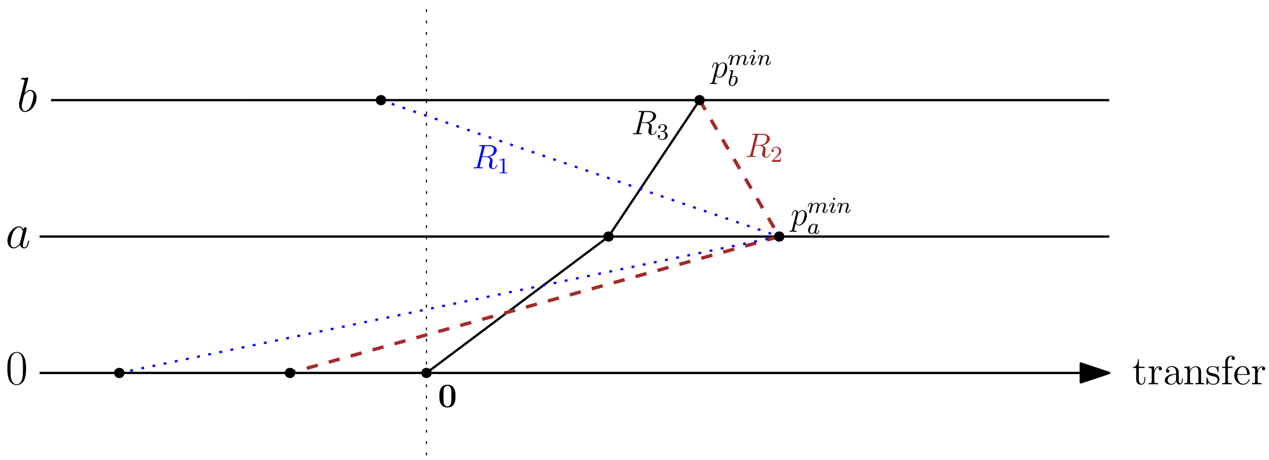

Figure 2: The minimum Walrasian equilibrium price vector

First, note that

$$D(R_1, p^{min}) = \{a\}, D(R_2, p^{min}) = \{a, b\}, D(R_3, p^{min}) = \{0, b\}.$$

[10] To see this, suppose that there is $a \in M$ such that $a_i \neq a$ for each $i \in N$. Then, by the second condition of Walrasian equilibrium, $p_a = 0$. By $n > m$, $a_i = 0$ for some $i \in N$. By desirability of objects, $(a, 0)\ P_i\ (a_i, 0)$, contradicting the first condition of Walrasian equilibrium.

[11] To see this, suppose $p^{min}_a(R) = 0$ for some $a \in M$. Then any agent $i \in N$ who is not assigned in the Walrasian equilibrium will prefer $(a, 0)$ to $(0, 0)$ contradicting the fact that he is assigned a bundle from his demand set. Indeed, this argument holds for any Walrasian equilibrium price vector.

Hence, a Walrasian equilibrium is the allocation where agent 1 gets object a, agent 2 gets object b, and agent 3 gets the null object at the price vector p^{min}. Also, p^{min} is the minimum Walrasian equilibrium price vector. To see this, let p be any other Walrasian equilibrium price vector. If $p_a < p_a^{min}$ and $p_b < p_b^{min}$, then no agent demands the null object, contradicting Walrasian equilibrium. Thus, $p_a \geq p_a^{min}$ or $p_b \geq p_b^{min}$. If $p_b < p_b^{min}$, then by $p_a \geq p_a^{min}$, both agents 2 and 3 will demand only object b, contradicting Walrasian equilibrium. Thus, $p_b \geq p_b^{min}$. But, if $p_a < p_a^{min}$, both agents 1 and 2 will demand only object a, a contradiction to Walrasian equilibrium. Hence, $p \geq p^{min}$.

We now describe a desirable mechanism satisfying no subsidy. The mechanism picks a minimum Walrasian equilibrium allocation at every profile of preferences. Although the minimum Walrasian equilibrium price vector is unique at every preference profile, there may be multiple supporting object allocations – all these object allocations must be indifferent to all the agents. To handle this multiplicity problem, we introduce some notation. Let $Z^{min}(R)$ denote the set of all allocations at a minimum Walrasian equilibrium at preference profile R. Note that if $n > m$ and $((a_1, p_{a_1}), \ldots, (a_n, p_{a_n})) \in Z^{min}(R)$ then $p \equiv (p_a)_{a \in L} = p^{min}(R)$.

Definition 5 *A mechanism* $f^{min} : \mathcal{R}^n \to Z$ *is a* **minimum Walrasian equilibrium price (MWEP)** *mechanism if* $f^{min}(R) \in Z^{min}(R) \; \forall \; R \in \mathcal{R}^n$.

As discussed earlier, at any preference profile R, $Z^{min}(R)$ may contain multiple allocations but each agent is indifferent between its allocations in this set. Hence, we refer to f^{min} as *the* MWEP mechanism, even though there can be more than one MWEP mechanism (depending on which allocation in $Z^{min}(R)$ is picked at every R).

Demange and Gale (1985) showed that the MWEP mechanism is strategy-proof. Clearly, it also satisfies IR, ETE, NW, and no subsidy. We document this fact below.

Fact 1 *The MWEP mechanism is desirable and satisfies no subsidy.*

In fact, Demange and Gale (1985) show that the MWEP mechanism satisfies a stronger incentive property called *(weak) group-strategy-proofness*, which means that for each $R \in \mathcal{R}^n$, there are no coalition $N' \subseteq N$, of agents and $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N'$, $f_i(R'_{N'}, R_{-N'}) \; P_i \; f_i(R)$. Further, the MWEP mechanism satisfies stronger fairness properties - it is *anonymous* (permuting preferences of agents does not change the outcome) and *envy-free.*

It is worth comparing the MWEP mechanism with the VCG mechanism for quasilinear preferences. Indeed, there is a naive way to generalize the VCG mechanism to any classical preference domain. Consider a preference profile R. For every agent $i \in N$ with preference R_i, let $v_i^a := V^{R_i}(a, (0, 0))$ for all $a \in M$. Let $v_i^0 = 0$ for all $i \in N$. Now, we compute the allocation and payments according to the VCG mechanism with respect to this profile of vectors $(v_1, \ldots, v_n)$. Such a generalized VCG mechanism coincides with the MWEP mechanism if the domain is the quasilinear domain (Leonard, 1983). Else, the generalized VCG mechanism is very different from the MWEP mechanism. Further, it is not strategy-proof if the domain is not the quasilinear domain (Morimoto and Serizawa, 2015).

4 The results

In this section, we formally state our two results. The proofs of both the results are in Section 7. Before we state the results, we explain the domain richness they use.

4.1 Rich domains

For each pair of price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$, we write $p > \hat{p}$ if $p_a > \hat{p}_a$ for all $a \in M$. The domain of preferences that we consider for our first result requires the following richness.

Definition 6 *A domain of preferences $\mathcal{R}$ is* **rich** *if for all $a \in M$ and for every $\hat{p}$ with $\hat{p}_a > 0, \hat{p}_b = 0$ for all $b \neq a$ and for every $p > \hat{p}$ there exists $R_i \in \mathcal{R}$ such that*

$$D(R_i, \hat{p}) = \{a\} \text{ and } D(R_i, p) = \{0\}.$$

Figure 3 illustrates this notion of richness with two objects a and b - two possible price vectors p and $\hat{p}$ are shown and two indifference vectors of a preference R_i are shown such that $D(R_i, p) = \{0\}$ and $D(R_i, \hat{p}) = \{a\}$.

The requirement of the richness condition is weak enough to be satisfied by many domains of interest. Obviously, if a domain of preferences is rich, then any superset of that domain is also rich. We give below some interesting examples of rich domains. Any superset of these domains are also rich.

- Any domain of preferences containing $\mathcal{R}^Q$ satisfies richness. To see this, fix an object $a \in M$ and a price vector $\hat{p}$ with $\hat{p}_b = 0$ for all $b \neq a$ and $\hat{p}_a > 0$. Consider any other price vector $p > \hat{p}$. Now, consider the quasilinear preference R_i given by the valuation vector v such that

$$v_b = \begin{cases} \hat{p}_b + 2\epsilon & \text{if } b = a, \\ \epsilon & \text{if } b \neq a, \end{cases}$$

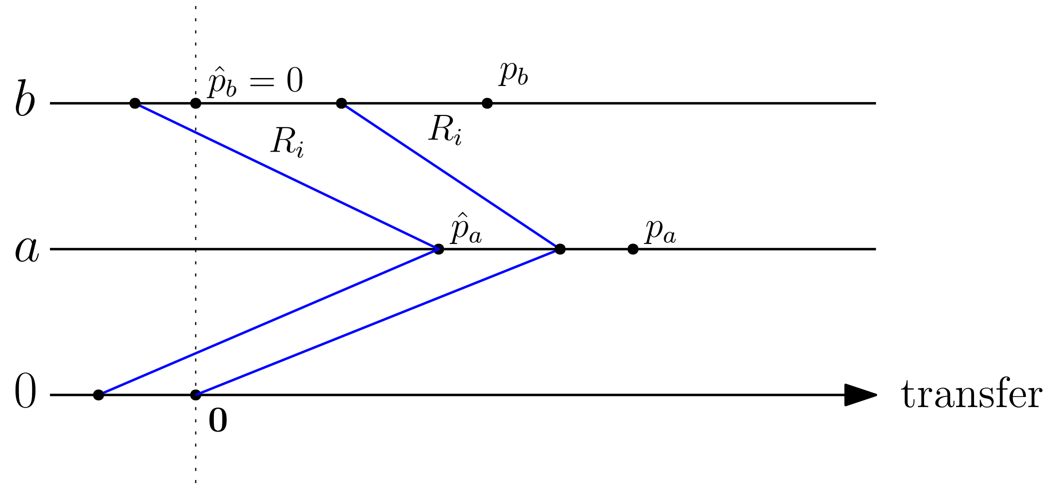

Figure 3: Illustration of richness

where $\epsilon > 0$ is small enough such that $v_a = \hat{p}_a + 2\epsilon < p_a$ and $\epsilon < p_b$ for all $b \in M \setminus \{b\}$. This means that $D(R_i, \hat{p}) = \{a\}$ but $D(R_i, p) = \{0\}$.

- The set of all positive income effects preferences and the set of all non-negative income effect preferences satisfy richness.

Definition 7 *A preference R_i satisfies* **positive income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have*

$$(b, t' - \delta)\ P_i\ (a, t - \delta) \quad \forall\ \delta > 0.$$

A preference R_i satisfies **non-negative income effect** *if for every $a, b \in L$ and for every t, t' with $t < t'$ and $(b, t')\ I_i\ (a, t)$, we have*

$$(b, t' - \delta)\ R_i\ (a, t - \delta) \quad \forall\ \delta > 0.$$

Let $\mathcal{R}^{++}$ be the set of all positive income effect preferences and $\mathcal{R}^{+}$ be the set of all non-negative income effect preferences.

A standard definition of positive income effect will say that a preferred object is more preferred as income increases. We do not model income explicitly, but the zero payment corresponds to the endowed income. Thus, in our model, when income increases by $\delta > 0$, the origin of consumption space moves to right by δ. This movement is equivalent to sliding indifference vectors to left. In other words, if the origin is fixed, the increase of income by δ is expressed as the decrease of payments of all bundles by δ. In the above definition, $(b, t')\ I_i\ (a, t)$ and $t' > t$ imply that object b is strictly preferred to object a at any common payment levels $t'' \in [t, t']$. Then, positive income effect requires that when payments are decreased by δ, b will be preferred to a, i.e., $(b, t' - \delta)\ P_i\ (a, t - \delta)$. Hence, our modeling of preferences captures income effects even though we do not model income explicitly.

Both $\mathcal{R}^+$ and $\mathcal{R}^{++}$ are rich domains. The fact that $\mathcal{R}^+$ is rich follows from the observation that $\mathcal{R}^Q \subseteq \mathcal{R}^+$ and $\mathcal{R}^Q$ is rich. Even though $\mathcal{R}^{++} \cap \mathcal{R}^Q = \emptyset$, $\mathcal{R}^{++}$ is still a rich domain.

- The set of all quasi-linear preferences with non-linear borrowing cost satisfies richness. Imagine a situation in which an agent has a quasilinear preference with valuation v, but has to borrow money from banks at interest rate $r > 0$ if his payment for an object exceeds his income $I > 0$. Then, given $t \in \mathbb{R}$, his *cost* of payment, which we denote by $c(t, I, r)$, is as follows.

$$c(t, I, r) = \begin{cases} t & \text{if } t \leq I, \\ I + (t - I)(1 + r) & \text{if } t > I. \end{cases}$$

  Thus, for each pair $(a, t), (b, t') \in L \times \mathbb{R}$, the agent weakly prefers (a, t) to (b, t') if and only if $v(a) - c(t, I, r) \geq v(b) - c(t', I, r)$. Such preferences are obviously not quasilinear. Let $\mathcal{R}^B$ be the set of all such preferences. Then, $\mathcal{R}^B$ is rich.

- The set of all single-peaked preferences satisfies richness. Imagine a condominium in which each floor has one room. Some agents prefer the highest floor because of good views, some prefer the lowest to avoid walking up stairs, and some prefer middle floors. Then, it is natural that each agent has a single-peaked preference – an ideal floor, and as we go away from the ideal floor, we go down our preference.

  Formally, there is a strict order $\succ$ over L such that for each $a \in M$, $a \succ 0$. A preference R_i is **single-peaked** if there is a unique object $\tau(R_i)$ such that for all $t \in \mathbb{R}$

  – $(\tau(R_i), t)\ P_i\ (a, t)$ for all $a \in M \setminus \{\tau(R_i)\}$ and
  – if $\tau(R_i) \succ a \succ b$ or $b \succ a \succ \tau(R_i)$, then $(a, t)\ P_i\ (b, t)$.

  In other words, an agent with preference R_i has a "peak" floor, say $\tau(R_i)$, such that when the prices of all floors are the same, he prefers $\tau(R_i)$ to other floors, and for any

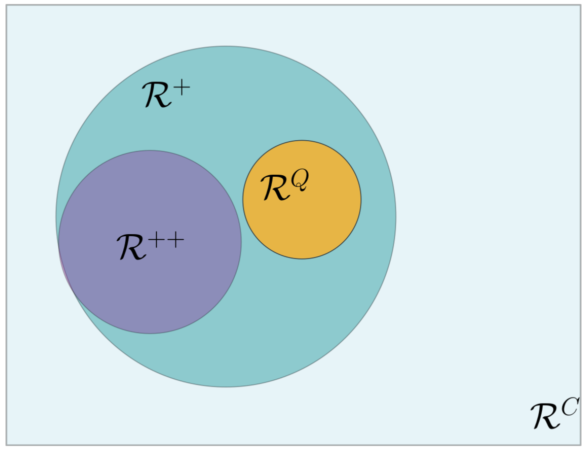

Figure 4: Illustration of relationship between rich domains: $\mathcal{R}^C, \mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}$.

two floors a and b, if $b \succ a \succ \tau(R_i)$ or $\tau(R_i) \succ a \succ b$, he preferes a to b. Let $\mathcal{R}^S$ be the set of all single-peaked preferences. Then, $\mathcal{R}^S$ is rich.

We can summarize the above discussions in this claim.

Claim 1 *The following domains are rich:* $\mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^B, \mathcal{R}^S, \mathcal{R}^C$.

We omit a formal proof for the above claim. However, the intuition for its proof is similar to the quasilinear domain proof outlined above. Given $a \in M$ and two price vectors $p, \hat{p} \in \mathbb{R}_+^{|L|}$ with $\hat{p} < p$, in those domains, we can find a preference R_i that has two indifference vectors satisfying the following: $V^{R_i}(a, (0,0)) < p_a$ and for each $b \in M \setminus \{a\}$, $V^{R_i}(b, (0,0))$ is close to zero and $V^{R_i}(b, (a, \hat{p}_a)) < 0$. Then, for preferences satisfying these conditions, the demand sets at p and $\hat{p}$ contain only 0 and a, respectively. The relationship between some of the rich domains are shown in Figure 4.

Claim 1 shows that many plausible domains satisfy our definition of richness. Of course, richness is a condition which ensures a variety of preferences in the domain. For instance, if we just take a domain containing two (or any finite) quasilinear preferences, it will not satisfy richness.

A concrete domain which violates richness appears in Zhou and Serizawa (2018). They consider a domain where objects are *commonly ranked*. For instance, suppose there are two objects, $M = \{a, b\}$, and there is a common ranking of objects given by the ordering $\succ$: $b \succ a$. Further, assume that agents have quasilinear preferences over consumption bundles. A quasilinear preference, represented by a valuation vector $v_i \in \mathbb{R}^2_{++}$, must satisfy $v_{ib} > v_{ia} > v_{i0} = 0$. This means that the set of quasilinear preferences satisfying common object ranking is a smaller subset of $\mathcal{R}^Q$.[12] Such a domain cannot satisfy richness. To see this, consider a price vector where $\hat{p}_a > 0$ and $\hat{p}_b = 0$ (as in Definition 6). By common object ranking, for any valuation vector v_i of any agent i, we must have $v_{ib} > v_{ia}$ and this means that $v_{ib} - \hat{p}_b > v_{ia} - \hat{p}_a$. This implies that agent i with this preference cannot demand object a at price vector $\hat{p}$. This means that the common object ranking domain of Zhou and Serizawa (2018) is not rich.

Another domain which violates richness is the *identical objects domain*. As the name suggests, in this domain, all the objects are identical. This means that if the prices are the same for all the objects, then the agent is indifferent between all the objects. Just as we argued about the common object ranking domain, it is not difficult to see that the identical objects domain violates richness – with identical objects $v_{ia} = v_{ib}$ and the arguments do not change in the previous paragraph. Adachi (2014) studies the domain of quasilinear preferences when objects are identical and provides an example of a desirable mechanism satisfying no subsidy, which is not the Vickrey auction (the MWEP mechanism in this case).

4.2 Ex-post revenue maximization of desirable mechanisms

We now formally state our first main result. For any mechanism $f : \mathcal{R}^n \to Z$, we define the **revenue** at preference profile $R \in \mathcal{R}^n$ as

$$\textsc{Rev}^f(R) := \sum_{i \in N} t_i(R).$$

Definition 8 *A mechanism* $f : \mathcal{R}^n \to Z$ **revenue dominates** *another mechanism* $g : \mathcal{R}^n \to Z$ *if*

$$\textsc{Rev}^f(R) \geq \textsc{Rev}^g(R) \quad \forall\ R \in \mathcal{R}^n.$$

A mechanism is **ex-post revenue optimal** *among a class of mechanisms if it belongs to this class of mechanisms and revenue dominates every mechanism in this class.*

Since revenue domination is not a complete binary relation in a typical class of mechanisms, an ex-post revenue optimal mechanism rarely exists. Our main result shows that an ex-post revenue optimal mechanism exists among the class of desirable mechanisms satisfying no subsidy, and it is the MWEP mechanism.

[12]As Zhou and Serizawa (2018) argue that the restriction of common object ranking is a natural one in many settings. For instance, objects are houses located on a street with a public facility located at one end of the street or objects are condominiums located on different floors of an apartment building.

Theorem 1 *Suppose $\mathcal{R}$ is a rich domain of preferences. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on $\mathcal{R}^n$.*

Theorem 1 clearly implies that even if we do *expected* revenue maximization with respect to *any* prior on the preferences of agents, we will only get the MWEP mechanism among the class of desirable and no subsidy mechanisms.

We use Claim 1 to spell out our result in specific domains.

Corollary 1 *Suppose $\mathcal{R} \in \{\mathcal{R}^Q, \mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^B, \mathcal{R}^S, \mathcal{R}^C\}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on $\mathcal{R}^n$.*

In the quasilinear domain, the outcome of the MWEP mechanism coincides with the Vickrey-Clarke-Groves (VCG) mechanism. Hence, the VCG mechanism is ex-post revenue optimal in the quasilinear domain among the class of desirable mechanisms satisfying no subsidy. Note that Holmstrom's celebrated theorem (Holmstrom, 1979) does not imply this result since it uses Pareto efficiency but we do not. Similarly, Krishna and Perry (1998) show that among the class of *Pareto efficient*, BIC and IIR mechanisms, the VCG mechanism maximizes expected revenue among all *Pareto efficient*, BIC, and IIR mechanisms in the quasilinear domain. This result works for multiple object auction problems even when agents can be allocated more than object. Again, this result uses Pareto efficiency but we do not.

A closer inspection of the richness reveals that if p is too small, then richness requires the existence of a preference where the valuations (with respect to $(0, 0)$) for real objects is very small. We can weaken this richness to a weaker condition which requires that valuations lie in an interval of the form (v^{min}, v^{max}), where v^{min} and v^{max} are any lower and upper bounds on the valuation of the objects such that $v^{\max} > v^{min} \geq 0$ and $v^{max} \in \mathbb{R}_+ \cup \{+\infty\}$. Theorem 1 continues to hold in such domains.

We now show how Theorem 1 can be strengthened in some specific rich domains. In particular, if the domain contains all the positive income effect preferences, then our result can be strengthened – we can replace no subsidy in Theorem 1 by the following no bankruptcy condition.

Definition 9 *A mechanism $f : \mathcal{R}^n \to Z$ satisfies* **no bankruptcy** *if there exists $\ell \leq 0$ such that for every $R \in \mathcal{R}^n$, we have $\sum_{i \in N} t_i(R) \geq \ell$.*

Obviously, no bankruptcy is a weaker property than no subsidy.[13] No bankruptcy is motivated by settings where the seller has limited means to finance the auction participants. Theorem 1 can now be strengthened in the positive income effect domain.

THEOREM **2** *Suppose $\mathcal{R} \supseteq \mathcal{R}^{++}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no bankruptcy defined on $\mathcal{R}^n$.*

Analogous to Corollary 1, the following is a corollary of Theorem 2.

COROLLARY **2** *Suppose $\mathcal{R} \in \{\mathcal{R}^+, \mathcal{R}^{++}, \mathcal{R}^C\}$. The MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no bankruptcy defined on $\mathcal{R}^n$.*

4.3 Pareto efficiency

Since no wastage is a minimal form of efficiency axiom, it is natural to explore the implications of stronger forms of efficiency. We now discuss the implications of Pareto efficiency in our problem and relate it to our results. Before we formally define it, we must reiterate that no wastage is a much weaker but more testable axiom in practice than Pareto efficiency.

DEFINITION **10** *A mechanism $f : \mathcal{R}^n \to Z$ is* **Pareto efficient** *if at every preference profile $R \in \mathcal{R}^n$, there exists no allocation $((\hat{a}_1, \hat{t}_1), \ldots, (\hat{a}_n, \hat{t}_n)) \in Z$ such that*

$$(\hat{a}_i, \hat{t}_i) \ R_i \ f_i(R) \qquad \forall \ i \in N$$
$$\sum_{i \in N} \hat{t}_i \geq \textsc{Rev}^f(R),$$

with either the second inequality holding strictly or some agent i strictly preferring $(\hat{a}_i, \hat{t}_i)$ to $f_i(R)$.

The above definition is the appropriate notion of Pareto efficiency in this setting. Notice that by distributing some money among all the agents, we can always make each agent better off than the allocation in any mechanism. Hence, the above definition requires that there

[13] In the literature, the no-deficit condition is sometimes imposed instead of no subsidy. A mechanism $f : \mathcal{R}^n \to Z$ satisfies *no deficit* if for each $R \in \mathcal{R}^n$, $\sum_{i \in N} t_i(R) \geq 0$. It is clear that no bankruptcy is weaker than no deficit.

should not exist another allocation where the auctioneer's revenue is not less and every agent is weakly better off.

The MWEP mechanism is Pareto efficient (Morimoto and Serizawa, 2015). Our results establish that even if a seller maximizes her revenue with this weak form of efficiency, it will be forced to use a Pareto efficient mechanism. We state this as corollaries below.

Corollary **3** *Let $\mathcal{R}$ be rich and $f : \mathcal{R}^n \to Z$ be ex-post revenue optimal among desirable mechanisms satisfying no subsidy. Then, f is efficient.*

Corollary **4** *Let $\mathcal{R} \supseteq \mathcal{R}^{++}$ and $f : \mathcal{R}^n \to Z$ be ex-post revenue optimal among desirable mechanisms satisfying no bankruptcy. Then, f is efficient.*

In other words, even if the seller maximizes her revenue among the set of all desirable mechanisms satisfying no subsidy (or no bankruptcy in the positive income effect domain), it will be forced to use a Pareto efficient mechanism. Hence, we get Pareto efficiency as a corollary without imposing it explicitly.

5 Desirable mechanisms satisfying no subsidy

An important question is whether the MWEP mechanism is the *unique* desirable mechanism satisfying no subsidy in rich domains. The answer to this question will depend on the domain of the mechanism. We can answer this question for two important rich domains. For the non-negative income effect domain, we provide below a family of desirable mechanisms satisfying no subsidy. In the Supplementary Appendix B.1, we include an example, due to Tierney (2019), of a desirable mechanism satisfying no subsidy for the quasilinear domain. These mechanisms are different from the MWEP mechanism. Hence, at least in these two rich domains, we can conclude that the MWEP mechanism is not the unique desirable mechanism satisfying no subsidy and our ex-post revenue maximization requirement is necessary for Theorem 1.

Now, we describe a family of desirable mechanisms satisfying no subsidy for the non-negative income effect domain $\mathcal{R}^+$. No mechanism in this family is the MWEP mechanism. To describe the idea of the new mechanisms, recall that in a slot machine, a player wins money if numbers or symbols in the slot constitute a "winning combination". Our mechanism is a variant of the MWEP mechanism. The variation resembles a slot machine in the sense that if for an agent $i \in N$, the preference profile of the other agents are aligned in a special way and constitutes a "discounting combination", then agent i can get discounts from her payment in the MWEP mechanism. To retain the properties of the MWEP mechanism, we need to give such discounts carefully. The discounting combination and discounts are cleverly constructed such that the new mechanism remains desriable and satisfies no subsidy.

Though the family of mechanisms can be defined very generally, we define it for the simple case when $M = \{a, b\}$ and $N = \{1, 2, 3, 4\}$. First, we formalize the idea of a discounting combination in this case.

DEFINITION **11** *A* **discounting combination** *is a collection of three distinct preferences $T \equiv \{R^\alpha, R^\beta, R^\gamma\} \subset \mathcal{R}^+$ such that there are two price vectors $p^T, \overline{p}^T \in \mathbb{R}^3_+$ with the following properties: $0 < p^T_a = p^T_b < \overline{p}^T_a = \overline{p}^T_b$ and for each $R_i \in \{R^\alpha, R^\beta, R^\gamma\}$,*

$$(a, \overline{p}^T_a) \ I_i \ (b, \overline{p}^T_b) \ I_i \ (0, 0) \text{ and } (a, p^T_a) \ I_i \ (b, p^T_b).$$

Hence, a discounting combination requires three preferences such that two of its indifference vectors satisfy some condition: (1) the indifference vector through $(a, \overline{p}^T_a)$ also passes through $(b, \overline{p}^T_b)$ and $(0, 0)$; (2) the indifference vector through (a, p^T_a) passes through (b, p^T_b). If a preference is quasilinear, this is impossible to achieve since these two indifference vectors are parallel and there can only be one preference which can have these two indifference vectors. On the other hand, a discounting combination requires three unique preferences. Hence, a discounting combination (as defined in Definition 11) cannot be defined in quasilinear domain.

We denote by $\mathcal{T}$ a set of discounting combinations. We say a set of discounting combinations $\mathcal{T}$ is **disjoint** if for each $T, T' \in \mathcal{T}$ with $T \neq T'$, $T \cap T' = \emptyset$. Given a set of discounting combinations $\mathcal{T}$, a profile of preferences $R \in (\mathcal{R}^+)^4$ is a **discounting combination** for agent i if $\{R_j : j \neq i\} \in \mathcal{T}$.

Our family of mechanisms will be defined using a disjoint set of discounting combinations. In particular, for every disjoint set of discounting combinations, we will define a desirable mechanism satisfying no subsidy which is different from the MWEP mechanism.

For the two claims below, we fix a disjoint set of discounting combinations $\mathcal{T}$. Hence, for each $T \in \mathcal{T}$, there exists two price vectors p^T and $\overline{p}^T$ as defined in Definition 11. The claims below relate the minimum Walrasian equilibrium allocation to these price vectors at a discounting combination.

CLAIM **2** *If $R \equiv (R_1, R_2, R_3, R_4) \in (\mathcal{R}^+)^4$ is a discounting combination for agent i and $\{R_j : j \neq i\} = T \in \mathcal{T}$, then $p^{min}(R) = \overline{p}^T$.*

Proof: Let $R \equiv (R_1, R_2, R_3, R_4) \in (\mathcal{R}^+)^4$ be a discounting combination for agent i and $\{R_j : j \neq i\} = T$. Then by the definition of discounting combination, $D(R_j, \overline{p}^T) = L$ for each $j \neq i$. Hence, $\overline{p}^T$ is a Walrasian equilibrium price vector.

Let $p' \leq \overline{p}^T$ be such that $p'_a < p^T_a$ or $p'_b < p^T_b$. Then by the definition of discounting combination, $0 \notin D(R_j, p')$ for every $j \neq i$. Hence, only agent i may demand 0. Thus, by $n = 4$ and $m = 2$, p' cannot be a Walrasian equilibrium. ■

Note that since $\mathcal{T}$ is disjoint and each set in $\mathcal{T}$ contains only distinct preferences, at any preference profile R, there can be a maximum of two agents for whom R is a discounting combination. Further, if R is a discounting combination for i and j, then $\{R_k : k \neq i\} = \{R_k : k \neq j\}$. Hence, Claim 2 is consistent with such preference profiles.

The next claim establishes an important property involving discounting combinations – we will use this property crucially to define our mechanism. The proof of this claim is given in Appendix A.

Claim **3** *For each* $R \in (\mathcal{R}^+)^4$*, there exists an object allocation* $(a_1, \ldots, a_4)$ *such that* $\{a_1, a_2, a_3, a_4\} = \{0, a, b\}$ *and for each* $i \in N$*,*

1. *if* R *is a discounting combination for agent* i *with* $\{R_j : j \neq i\} = T \in \mathcal{T}$*, then* $a_i \in D(R_i, p^T)$*,*
2. *if* R *is not a discounting combination for agent* i*, then there exists a minimum Walrasian equilibrium price allocation* $((b_1, p^{min}_{b_1}(R)), \ldots, (b_4, p^{min}_{b_4}(R))) \in Z^{min}(R)$ *such that* $b_i = a_i$*.*

Claim 3 shows that for every disjoint set of discounting combinations $\mathcal{T}$, at every preference profile R, we can identify an object allocation satisfying (1) and (2) of Claim 3. Though there may be more than one such object allocation, we identify one such object allocation at every R and use it to formally define our mechanism.

Definition **12** *Given a disjoint set of discounting combinations* $\mathcal{T}$*, the* **MWEP mechanism with discounting combinations** $\mathcal{T}$*, denoted by* $f^{\mathcal{T}}$*, is defined as follows: for every* $R \in (\mathcal{R}^+)^4$*,* $(a^{\mathcal{T}}_1(R), \ldots, a^{\mathcal{T}}_4(R))$ *is an object allocation satisfying Claim 3, and for every* $i \in N$

$$t^{\mathcal{T}}_i(R) = \begin{cases} p^T_{a^{\mathcal{T}}_i(R)} & \text{if } R \text{ is a discounting combination for } i, \\ p^{min}_{a^{\mathcal{T}}_i(R)}(R) & \text{otherwise,} \end{cases}$$

where p^T *is the price vector defined in Definition 11 for every* $T \in \mathcal{T}$*.*

It is clear that $f^{\mathcal{T}}$ satisfies individual rationality, equal treatment of equals, no wastage, and no subsidy. We show that it is also strategy-proof. As shown below in the proof, Claim 3 plays an important role in showing strategy-proofness.

PROPOSITION 1 *For every disjoint set of discounting combinations $\mathcal{T}$, the mechanism $f^{\mathcal{T}}$ is strategy-proof.*

Proof: Fix $R \in (\mathcal{R}^+)^4$, and $i \in N$. If R is not a discounting combination for i, then by changing his preference to R'_i, (R'_i, R_{-i}) is not a discounting combination for i. By Claim 3, in both the preference profiles, we can pick the respective minimum Walrasian equilibrium allocation, and by Demange and Gale (1985), i cannot manipulate to R'_i.

If R is a discounting combination for i with respect to discounting combination T, then by changing his preference to R'_i, (R'_i, R_{-i}) is also a discounting combination for i with respect to discounting combination T. As a result, we get that $a_i^{\mathcal{T}}(R) \in D(R_i, p^T)$ and $a_i^{\mathcal{T}}(R'_i, R_{-i}) \in D(R'_i, p^T)$. Clearly, agent i cannot manipulate to R'_i. ■

Note that if R is a discounting combination for i, then she pays according to p^T which is lower than $\overline{p}^T = p^{min}(R)$ (Claim 2). Hence, $f^{\mathcal{T}}$ is different from the MWEP mechanism. By varying the choice of $\mathcal{T}$, we generate a family of such mechanisms. Note that even though these mechanisms are defined for $\mathcal{R}^+$, $f^{\mathcal{T}}$ can be defined on a smaller domain $\mathcal{D} \subseteq \mathcal{R}^+$ as long as we can define $\mathcal{T}$. However, as discussed earlier, a discounting combination T containing only quasilinear preferences is not possible. Hence, there is no set of discounting combinations in $\mathcal{R}^Q$ (the domain of quasilinear preferences). But we show in Supplementary Appendix B.1 that the MWEP mechanism is not the unique desirable mechanism satisfying no subsidy on $\mathcal{R}^Q$.

The above family of mechanisms clarify that we cannot afford to drop ex-post revenue maximization in Theorem 1. In that sense, our result is *not* entirely an axiomatic exercise, and revenue maximization is an essential part of our result. At the same time, ex-post revenue maximization is about picking a desirable mechanism satisfying no subsidy which revenue dominates every such mechanism. Here, if this property is treated as an axiom, then our result can be viewed as providing an axiomatic foundation for the MWEP mechanism.

6 Discussion

In this section, we discuss various extensions and interpretations of our results.

6.1 Extending a quasilinear domain result

In this section, we interpret Theorem 1 as a generalization of a revenue maximization result in the single object quasilinear domain setting. In particular, we impose no wastage in Myerson's analysis of optimal single object auction. We observe that if agents' values are independently and identically distributed (IID, hereafter) random variables, then the optimal mechanism is the Vickrey auction. We then interpret our Theorem 1 as an extension of this result.

Consider the single object auction setting and the quasilinear domain. So, the valuation v_i of each agent is drawn from $(0, \infty)$ using independent and identical cumulative distribution function Γ (with pdf γ). Define the *virtual value* function of each agent i as $w(v_i) = v_i - \frac{1-\Gamma(v_i)}{\gamma(v_i)}$ for each $v_i \in \mathbb{R}_{++}$. We assume that the virtual valuation function is increasing (*regularity*).

Myerson (1981) optimizes expected revenue over the set of all *Bayesian incentive compatible* and *interim individually rational* mechanisms. He also allows for randomization. Our first observation here points out that we get the Vickrey auction to be optimal if we restrict the set of mechanisms by imposing no wastage. As will be clear, the IID assumption of distribution plays a crucial role for this result.

To see this, consider a Bayesian incentive compatible (BIC) and interim individual rational (IIR) mechanism $f \equiv (a, t)$, where f can be a random mechanism.[14] Myerson (1981) shows that the expected revenue from such a mechanism can be written as:

$$\mathbb{E}_v\big[\sum_{i \in N} w(v_i) a_i(v)\big],$$

where $\mathbb{E}_v$ denotes the expectation across all valuation profiles using the IID distribution Γ. Since $a_i(v)$ is the object allocation probability of agent i at valuation profile v, the term $w(v_i)a_i(v)$ is the expected virtual value of agent i at valuation profile v. Hence, to maximize expected revenue, it is sufficient to maximize expected virtual value at every valuation profile.

Now, suppose we want to maximize expected revenue over all BIC, IIR, and *no wastage* mechanisms. Since at every valuation profile v, we have to allocate the object and we need to maximize $\sum_{i \in N} w(v_i)a_i(v)$, we can do so by setting $a_k(v) = 1$ if $k \in \arg\max_{i \in N} w(v_i)$. In other words, we allocate the object to the agent with the *highest virtual value*.[15] Since the distribution is regular, *revenue equivalence* pins down the payment such that the resulting mechanism is BIC and IIR. Of course, it satisfies no wastage by construction. In fact, this

[14] We do not formally define BIC and IIR as they are standard in the literature.

[15] If no wastage is not imposed, then as Myerson (1981) shows, the expected revenue maximizing mechanism should not sell the object when virtual values of all the agents are negative.

mechanism is the Vickrey auction since the agent with the highest virtual value is also the agent with the highest value – this is not the case if the IID is assumption is dropped. We list this as a fact below, and note that no analogue of this fact is known in the literature in our model (multiple heterogenous objects with unit demand agents).

Fact **2** (Myerson (1981)) *Suppose there is a single object ($m = 1$) and the domain of preferences is the quasilinear domain with valuations of agents identically and independently distributed and the distribution satisfies regularity. Then, the Vickrey auction maximizes* **expected revenue** *over the set of all* **BIC and IIR** *mechanisms satisfying no wastage.*

	Fact 2	Theorem 1
Number of objects	$m = 1$	$m < n$
Domain of preferences	Quasilinear	Any rich domain (Corollary 1)
Distributional assumptions	IID values with regularity	None
Revenue maximization	Ex-ante (expected)	Ex-post (at every profile)
Incentive compatibility	BIC	Strategy-proofness
Axioms	BIC, IIR, and no wastage	Strategy-proofness, Ex-post IR, no wastage, ETE and no subsidy
Random mechanisms	Considered	Not considered

Table 1: Comparison of Fact 2 and Theorem 1

Clearly, there are many ways in which Theorem 1 generalizes Fact 2: (i) it considers more than one object; (ii) it allows for domains of preferences which can contain non-quasilinear preferences; (iii) it shows ex-post revenue maximizing mechanism; (iv) it does not assume anything about the priors (i.e., completely prior-free prediction)[16]. But these generalizations come with some further assumptions. Theorem 1 only considers strategy-proof mechanisms (as compared to BIC mechanisms in Fact 2). Theorem 1 does not consider random mechanisms while Fact 2 optimizes over random mechanisms. Finally, Theorem 1 imposes no subsidy and equal treatment of equals but Fact 2 does not impose these axioms. These differences are summarized in Table 1.

[16]For instance, Fact 2 is no longer true if values of agents are drawn from different distributions. In that case, the object goes to the agent with the highest *virtual value*, and such an agent need not have the highest value (the Vickrey auction gives the object to the agent with the highest value).

6.2 A lower bound on expected revenue

In this section, we show the consequence of *dropping* the following axioms from Theorem 1: no wastage, equal treatment of equals, and no subsidy. Hence, we will consider a larger class of mechanisms than in Theorem 1 - mechanisms which are strategy-proof and (ex-post) individually rational. We will also be interested in maximizing *expected* revenue.

We explicitly focus on quasilinear domain of preferences, where the MWEP mechanism is the VCG mechanism. A preference R_i of agent i can be represented by a valuation vector $v_i \equiv \{v_{ia}\}_{a \in L}$, where the valuation for the null object is normalized to zero: $v_{i0} = 0$. The discussion in this section will require assumptions on prior distribution of values of agents. We will assume a symmetric prior: every agent i draws valuation for object $j \in M$ using the same distribution Γ from $(0, \infty)$, which is strictly increasing, differentiable, and admits a density function γ. Hence, the joint distribution of valuations is independent (across agents and objects) and identical (IID). Further, we will assume that Γ is *regular*, which means $x - \frac{1-\Gamma(x)}{\gamma(x)}$ is increasing in x.

Given a mechanism $f \equiv (a, t)$ on the quasilinear domain, we denote the **expected** revenue from f as

$$\mathbb{E}(\text{Rev}^f) = \mathbb{E}_v\big[\sum_{i \in N} t_i(v)\big],$$

where $t_i(v)$ is the payment of agent i at valuation profile v. Using this, we define the notion of the standard optimal mechanism, which we call *ex-ante revenue optimality.*

Definition 13 *A mechanism $f : (\mathcal{R}^Q)^n \to Z$ is* **ex-ante revenue optimal** *among a class of mechanisms in the quasilinear domain if for every mechanism $g : (\mathcal{R}^Q)^n \to Z$ in this class, we have*

$$\mathbb{E}(\text{Rev}^f) \geq \mathbb{E}(\text{Rev}^g).$$

If the number of agents is at least twice the number of objects, Roughgarden et al. (2012) show that the **expected** revenue of the VCG mechanism in our model is at least half the expected revenue from the ex-ante revenue optimal mechanism among the class of strategy-proof and ex-post individually rational mechanisms. Since the MWEP mechanism coincides with the VCG mechanism in the quasilinear domain, the following corollary is immediate from Theorem 1 and their result.

Corollary 5 *Suppose $n \geq 2m$. Let $f : (\mathcal{R}^Q)^n \to Z$ be an ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy in $(\mathcal{R}^Q)^n$. Let*

$g : (\mathcal{R}^Q)^n \to Z$ be an ex-ante revenue optimal mechanism among the class of mechanisms satisfying strategy-proofness and ex-post individual rationality in $(\mathcal{R}^Q)^n$. Then,

$$\mathbb{E}(\text{Rev}^f) \geq \frac{1}{2}\mathbb{E}(\text{Rev}^g)$$

The main point about Corollary 5 is that the ratio of expected revenue from f and g is bounded by a *constant*, i.e., it does not depend on m or n.[17]

Corollary 5 shows that by requiring no wastage, equal treatment of equals, and no subsidy, we do not sacrifice arbitrary proportion of expected revenue in the quasilinear domain. Further, we gain in terms of robustness since the VCG mechanism is a prior-free mechanism and is ex-post revenue optimal in the class of desirable mechanisms satisfying no subsidy.

6.3 Our axioms

As we have discussed at several places, the expected revenue maximization with multiple objects is a difficult problem. The objective of our exercise was to see the implication of additional axioms on this problem. These axioms surprisingly gave us a robust solution – ex-post revenue maximization. Futher, our result can incorporate non-quasilinear preferences. Such a robust prediction is rarely seen in the mechanism design literature.

How do our axioms help in achieving such a robust result? Below, we give some examples to illustrate the implications of our axioms on the result. In particular, we show that each of the axioms are necessary to get our result.

Notion of incentive compatibility and IR. Consider a mechanism that chooses the maximum Walrasian equilibrium allocation at every profile. Such a mechanism will satisfy no subsidy and all the properties of desirability except strategy-proofness. Similarly, the MWEP mechanism supplemented by a *participation fee* satisfies no subsidy and all the properties of desirability except ex-post IR. Both these mechanisms generate more revenue than the MWEP mechanism. Hence, strategy-proofness and ex-post IR are necessary for our results to hold.

[17] Roughgarden et al. (2012) have similar constant approximation bounds for $n < 2m$ also, and we can use them to derive analogues of Corollary 5 for $m < n < 2m$. Further, Corollary 5 does not explicitly consider randomized and Bayesian incentive compatibile mechanisms. But results in similar spirit can also be obtained by allowing for such mechanisms. This is because Chawla et al. (2015) show that similar bounds can be obtained between deterministic and randomized ex-ante revenue optimal mechanisms.

What is less clear is if we can relax the notion of incentive compatibility to Bayesian incentive compatibility in our results. In Appendix B.2, we consider the single object auction model in the quasilinear domain, and show that a modified first-price auction (modified only at a set of zero measure valuation profiles) is Bayesian incentive compatible, ex-post individually rational, and satisfies no wastage, equal treatment of equals, and no subsidy. Clearly, a first-price auction generates more revenue than the Vickrey mechanism at some valuation profiles.

At least in quasilinear domain, we know that there are many *revenue equivalent* Bayesian incentive compatible mechanisms to the Vickrey auction (for single object). Hence, our notion of incentive compatibility eliminates such mechanisms.

No wastage. We have already argued about why no wastage as an axiom makes sense. In Section 6.1, we gave an example of a result in the quasilinear domain where we explicitly point out the implication of no wastage. In short, no wastage gets rid of all mechanisms with reserve price. Hence, it is easy to see that no wastage is required for our result – in the quasilinear domain of preferences with one object, Myerson (1981) shows that the Vickrey mechanism with an *optimally* chosen reserve price maximizes expected revenue for independent and identically distributed values of agents. Such a mechanism wastes the object and generates more revenue than the Vickrey mechanism at some profiles of preferences.

No wastage is also necessary in a more indirect manner. Consider the domain of quasilinear preferences with two objects $M \equiv \{a, b\}$ and $N = \{1, 2, 3\}$. We show that the seller may increase her revenue by *not* selling all the objects. Consider a profile of valuations as follows:

$$v_1(a) = v_1(b) = 5$$

$$v_2(a) = v_2(b) = 4$$

$$v_3(a) = v_3(b) = 1.$$

The MWEP price at this profile is $p_a^{min} = p_b^{min} = 1$, which generates a revenue of 2 to the seller. On the other hand, suppose the seller conducts a Vickrey mechanism of object a only. Then, he generates a revenue of 4. Hence, the seller can increase her ex-post revenue at some profiles of valuations by withholding objects. Notice that withholding objects is a stronger violation of efficiency, and is easier to detect than misallocating the objects among agents.

Equal treatment of equals. There are various fairness notions in the mechanism design literature. A typical notion of *ex-post* fairness is envy freeness (Varian, 1974; Sprumont, 2013). A typical notion of *ex-ante* fairness is anonymity (Sprumont, 1991; Moulin and Shenker, 1992; Barbera and Jackson, 1995).[18] Equal treatment of equals is the weakest fairness notion in the sense that it is weaker than each of envy-freeness and anonymity.

There are ample examples of mechanisms violating equal treatment of equals in the mechanism design literature In the single object auction model in quasilinear domain, if values of agents are drawn from different distributions, then revenue is maximized by an *asymmetric* mechanism - see Section 6.1 for a precise statement. Hence, for some profiles of preferences, such mechanisms must generate more revenue than the Vickrey auction. Equal treatment of equals rules out such mechanisms.

Another example that shows the necessity of equal treatment equals in our result is the following. Suppose that there are one object and two agents, and preferences are quasilinear. Hence, the preference of each agent $i \in \{1, 2\}$ can be described by his *valuation* for the object v_i.

We define the following mechanism: the object is first offered to agent 1 at price $p > 0$; if agent 1 accepts the offer, then he gets the object at price p and agent 2 does not get anything and does not pay anything; else, agent 2 is given the object for free.

This mechanism generates a revenue of p whenever $v_1 > p$ (but generates zero revenue otherwise). However, note that the Vickrey mechanism generates a revenue of v_2 when $v_1 > v_2$. Hence, if $v_1 > p > v_2$, then this mechanism generates more revenue than the Vickrey mechanism. Also, this mechanism satisfies no subsidy and all the properties of desirability except equal treatment of equals.

No subsidy. It is tempting to conjecture that no subsidy can be relaxed in quasilinear domain of preferences. The following example shows that this need not be true.

Consider an example with one object and two agents in the quasilinear domain - hence, preferences of agents can be represented by their valuations v_1 and v_2. Further, assume that valuations lie in $\mathbb{R}_{++}$. Choose $k \in (0, 1)$ and define the mechanism $f \equiv (a, t)$ as follows: for every (v_1, v_2)

$$a(v_1, v_2) = \begin{cases} (1,0) & \text{if } kv_1 > v_2, \\ (0,1) & \text{otherwise,} \end{cases}$$

[18] Anonymity is sometime called *symmetry* in the literature (Manelli and Vincent, 2010; Deb and Pai, 2016). Though it is stronger than equal treatment of equals in our model, it is often used when random mechanisms are allowed.

$$t_1(v_1, v_2) = \begin{cases} -(v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 0, \\ \frac{v_2}{k} - (v_2 - kv_2) & \text{if } a_1(v_1, v_2) = 1, \end{cases}$$

$$t_2(v_1, v_2) = \begin{cases} 0 & \text{if } a_2(v_1, v_2) = 0, \\ kv_1 & \text{if } a_2(v_1, v_2) = 1. \end{cases}$$

It is straightforward to check that the mechanism is strategy-proof. It is also not difficult to see that utilities of the agents are always non-negative, and hence, individual rationality holds. Finally, if $v_1 = v_2$, we have

$$a_1(v_1, v_2) = 0, a_2(v_1, v_2) = 1, \quad t_1(v_1, v_2) = -(v_2 - kv_2), t_2(v_1, v_2) = kv_1.$$

Hence, net utility of agent 1 is $v_2 - kv_2$ and that of agent 2 is $v_1 - kv_1$, which are equal since $v_1 = v_2$. This shows that the mechanism satisfies equal treatment of equals.

However, the mechanism pays agent 1 when he does not get the object. Thus, it violates no subsidy. The revenue from this mechanism when $kv_1 > v_2$ is

$$v_2\Big(\frac{1}{k} + k - 1\Big) \geq v_2.$$

The Vickrey mechanism generates a revenue of v_2 when $kv_1 > v_2$. Hence, this mechanism generates more revenue than the Vickrey mechanism when $kv_1 > v_2$. This shows that we cannot drop no subsidy from Theorem 1. [19]

6.4 Extension to other combinatorial auction models

It is not clear whether Theorem 1 extends to other models of combinatorial auctions. First, if we just consider the quasilinear preference domain, the VCG mechanism generalizes to other combinatorial auction models. However, a crucial feature of the VCG mechanism in our model (heterogenous objects and unit demand buyers) is that it coincides with the MWEP mechanism. This plays a crucial role in all our proofs. This equivalence is lost in other models of combinatorial auctions (Gul and Stacchetti, 1999; Bikhchandani and Ostroy, 2006), and further, the Walrasian equilibrium price vector may fail to exist in other models (Bikhchandani and Ostroy, 2006). Hence, it is not clear how our result extends to

[19] Further inspection reveals that the revenue from this mechanism when $v_1 = v_2 = v$ is $kv - v(1-k) = v(2k-1)$. So, if $k < \frac{1}{2}$, this revenue approaches $-\infty$ as $v \to \infty$. Hence, this mechanism even violates no bankruptcy.

other models of combinatorial auctions (even in the quasilinear domain). We keep this as an agenda for future research.

On the other hand, when the set of preferences include all or a very rich class of non-quasilinear preferences, strategy-proofness and Pareto efficiency (along with other axioms) have been shown to be incompatible if the unit demand assumption is violated - (Kazumura and Serizawa, 2016) show this for multi-object allocation problems where agents can be allocated more than one object; (Baisa, 2019, Forthcoming) shows this for homogeneous object allocation problems where agents can be assigned any number of units. In other words, no canonical mechanism is known to exist once we relax the unit demand assumption, and it is not clear how Theorem 1 extends.

7 The proofs of Theorems 1 and 2

In this section, we present all the proofs. The proofs use the following fact very crucially: the MWEP mechanism chooses a Walrasian equilibrium outcome. Before diving into the proofs, we want to stress here that a greedy approach of proving our results would be to first prove that any desirable mechanism satisfying no subsidy and maximizing revenue must be Pareto efficient. In the quasilinear domain, using revenue equivalence will then pin down the MWEP (VCG) mechanism. This approach will fail in our setting because our results work even without quasilinearity and revenue equivalence does not hold in such domains. Further, it is not obvious even in quasilinear domain that the desirability, no subsidy, and the revenue optimality implies Pareto efficiency. Our proofs work by showing various implications of desirability and no subsidy on consumption bundles of agents. It uses richness of the domain to derive these implications. In that sense, it departs from traditional Myersonian techniques, where revenue maximization is a programming problem with object allocation mechanisms as decision variables.

It is worth discussing how our proofs are different from Morimoto and Serizawa (2015), who characterize the MWEP mechanism. Their focus is on Pareto efficiency and their proofs depend on this. Since we use only no wastage as a efficiency desideratum, which is much weaker than Pareto-efficiency, we need to develop our own proof techniques to establish our results.

We start off by showing an elementary lemma which shows that at every preference profile, if a mechanism gives every agent weakly better consumption bundles than the MWEP mechanism, then its revenue is no more than any MWEP mechanism. This lemma will be used to prove both our results.

LEMMA **1** *For every mechanism $f : \mathcal{R}^n \to Z$ and for every $R \in \mathcal{R}^n$, the following holds:*

$$\big[f_i(R)\ R_i\ f_i^{min}(R)\ \forall\ i \in N\big] \Rightarrow \big[\text{REV}^{f^{min}}(R) \geq \text{REV}^{f}(R)\big],$$

where f^{min} is the MWEP mechanism.

Proof: Fix a profile of preferences $R \in \mathcal{R}^n$ and denote $f_i^{min}(R) = (a_i, p^{min}_{a_i}(R))$ for each $i \in N$. Now, for every $i \in N$, we have $f_i(R) \equiv (a_i(R), t_i(R))\ R_i\ (a_i, p^{min}_{a_i}(R))$ and by the Walrasian equilibrium property, $(a_i, p^{min}_{a_i}(R))\ R_i\ (a_i(R), p^{min}_{a_i(R)}(R))$. This gives us $t_i(R) \leq p^{min}_{a_i(R)}(R)$ for each $i \in N$. Hence,

$$\text{REV}^f(R) = \sum_{i \in N} t_i(R) \leq \sum_{i \in N} p^{min}_{a_i(R)}(R) \leq \text{REV}^{f^{min}}(R),$$

where the last inequality follows from $p^{min}(R) \in \mathbb{R}^{|L|}_+$. ■

7.1 Proof of Theorem 1

We start with a series of Lemmas before providing the main proof. Throughout, we assume that $\mathcal{R}$ is a rich domain of preferences and f is a desirable mechanism satisfying no subsidy on $\mathcal{R}^n$. For the proofs, we need the following definition.

DEFINITION **14** *A preference R_i is (a, t)-**favoring** for $t \geq 0$ and $a \in M$ if for price vector p with $p_a = t, p_b = 0$ for all $b \neq a$, we have $D(R_i, p) = \{a\}$.*

An equivalent way to state this is that R_i is (a, t)-favoring for $t > 0$ and $a \in M$ if $V^{R_i}(b, (a, t)) < 0$ for all $b \neq a$. A slightly stronger version of (a, t)-favoring preference is the following.

DEFINITION **15** *A preference R_i is $(a, t)^{\epsilon}$-favoring for $t \geq 0$, $a \in M$, and $\epsilon > 0$ if it is (a, t)-favoring and*

$$V^{R_i}(a, (0, 0)) < t + \epsilon$$
$$V^{R_i}(b, (0, 0)) < \epsilon\ \forall\ b \in M \setminus \{a\}.$$

The following lemma shows that if $\mathcal{R}$ is rich, then $(a, t)^{\epsilon}$-favoring preferences exist for every $(a, t) \in M \times \mathbb{R}_+$ and $\epsilon > 0$.

Lemma **2** *Suppose $\mathcal{R}$ is rich. Then, for every bundle $(a,t) \in M \times \mathbb{R}_+$ and for every $\epsilon > 0$, there exists a preference $R_i \in \mathcal{R}$ such that it is $(a,t)^\epsilon$-favoring.*

Proof: Define $\hat{p}$ as follows: $\hat{p}_a = t, \;\; \hat{p}_b = 0 \;\; \forall\, b \neq a$.
Define p as follows: $p_a = t + \epsilon, \;\; p_0 = 0, \;\; p_b = \epsilon \;\; \forall\, b \in M \setminus \{a\}$.
By richness, there exists R_i such that $D(R_i, \hat{p}) = \{a\}$ and $D(R_i, p) = \{0\}$. But this implies that R_i is (a,t)-favoring. Further, $V^{R_i}(a,(0,0)) < t + \epsilon$ and $V^{R_i}(b,(0,0)) < \epsilon \; \forall\, b \in M \setminus \{a\}$. Hence, R_i is $(a,t)^\epsilon$-favoring. ■

Using this, we prove the following lemma which will be used in the proof.

Lemma **3** *For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, for every $t \in \mathbb{R}_+$, if there exists $j \neq i$ such that R_j is $(a_i(R), t)$-favoring, then $t_i(R) > t$.*

Proof: Suppose $t_i(R) \leq t$. Since R_j is $(a_i(R), t)$-favoring, $t_i(R) \leq t$ implies that R_j is also $f_i(R) \equiv (a_i(R), t_i(R))$-favoring. Consider a preference profile $R' \equiv (R'_i = R_j, R'_{-i} = R_{-i})$. By equal treatment of equals (since $R'_i = R'_j = R_j$),

$$f_i(R') \; I_j \; f_j(R'). \tag{1}$$

We argue that $f_i(R') = f_i(R)$. If $a_i(R') = a_i(R)$, then strategy-proofness implies that $t_i(R') = t_i(R)$ and we are done. Assume for contradiction that $a_i(R) = a \neq b = a_i(R')$. By strategy-proofness, $(b, t_i(R')) \; R'_i \; (a, t_i(R))$, which implies that $t_i(R') \leq V^{R'_i}(b, (a, t_i(R)))$. Since $R'_i = R_j$ is $(a, t_i(R))$-favoring, we have $V^{R'_i}(b, (a, t_i(R))) < 0$. This implies that $t_i(R') < 0$, which is a contradiction to no subsidy. Hence, we have

$$f_i(R') = f_i(R). \tag{2}$$

Combining Inequality (1) and Equation (2), we get that $f_i(R) \; I_j \; f_j(R')$. Hence, $t_j(R) = V^{R_j}(a_j(R'), f_i(R)) < 0$, where the strict inequality followed from the fact R_j is $f_i(R)$-favoring and $a_i(R) = a_i(R') \neq a_j(R')$. This is a contradiction to no subsidy. ■

We will now prove Theorem 1 using these lemmas.

Proof of Theorem 1

Proof: Fix a desirable mechanism $f : \mathcal{R}^n \to Z$ satisfying no subsidy, where $\mathcal{R}$ is a rich domain of preferences. Fix a preference profile $R \in \mathcal{R}^n$. Let $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism f^{min} at R. Let $\underline{p} \equiv \min_{a\in M} p_a^{min}(R)$. Clearly, $\underline{p} > 0$. For simplicity of notation, we will denote $z_i \equiv (a_i, p_i)$, where $p_i \equiv p_{a_i}^{min}(R)$ for all $i \in N$. We prove that $f_i(R)\ R_i\ z_i$ for all $i \in N$, and by Lemma 1, we will be done.

Assume for contradiction that there is some agent $i \in N$ such that $z_i\ P_i\ f_i(R)$. We first construct a finite sequence of *distinct* agents and preferences, without loss of generality $(1, R'_1), \ldots, (n, R'_n)$, satisfying certain properties. Let $N_0 \equiv \emptyset$, $N_k \equiv \{1, \ldots, k\}$ for each $k \geq 1$, and $(R'_{N_0}, R_{-N_0}) \equiv R$. This sequence satisfies the properties that for every $k \in \{1, \ldots, n\}$,

1. $z_k\ P_k\ f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ for each $k \geq 1$,
2. $a_k \neq 0$,
3. R'_k is $(z_k)^{\epsilon_k}$-favoring for some $\epsilon_k > 0$ with $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$.

Now, we construct this sequence inductively.

Step 1 - Constructing $(1, R'_1)$. Let $i = 1$. By our assumption, $z_1\ P_1\ f_1(R)$. This implies $p_1 - V^{R_1}(a_1, f_1(R) < 0$. Thus, there is $\epsilon_1 > 0$ such that $\epsilon_1 < \min\{V^{R_1}(a_1, f_1(R)) - p_1, \underline{p}\}$. By Lemma 2, there is a $(z_1)^{\epsilon_1}$-favoring preference R'_1. Suppose $a_1 = 0$. Then, $(0, 0) = z_1\ P_1\ f_1(R)$, which contradicts individual rationality. Hence, $a_1 \neq 0$.

Step 2 - Constructing (k, R'_k) for $k > 1$. We proceed inductively - suppose, we have already constructed $(1, R'_1), \ldots, (k-1, R'_{k-1})$ satisfying Properties 1, 2, and 3. By no wastage and the fact that $a_{k-1} \neq 0$, there is agent $j \in N$ such that $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$.

If $j = k - 1$, then individual rationality of f and $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ imply that

$$t_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_{k-1}}(a_{k-1}, (0,0)) < p_{k-1} + \epsilon_{k-1} < V^{R_{k-1}}(a_{k-1}, f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}})),$$

where the second inequality followed from the fact that R'_{k-1} is $(z_{k-1})^{\epsilon_{k-1}}$-favoring, and the last inequality followed from the definition of ϵ_{k-1}. Thus, by $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$, we have

$$f_{k-1}(R'_{N_{k-1}}, R_{-N_{k-1}})\ P_{k-1}\ f_{k-1}(R'_{N_{k-2}}, R_{-N_{k-2}}),$$

which contradicts strategy-proofness. Hence, $j \neq k - 1$.

If $j \in N_{k-2}$, then by individual rationality of f, we get

$$t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) \leq V^{R'_j}(a_{k-1}, (0,0)) < \epsilon_j < p_{k-1}, \tag{3}$$

where the second inequality followed from the fact that R'_j is $(z_j)^{\epsilon_j}$-favoring and $j \neq (k-1)$, and the last inequality followed from the definition of ϵ_j. But, notice that agent $(k-1) \neq j$ and R'_{k-1} is z_{k-1}-favoring (since it is $(z_{k-1})^{\epsilon_{k-1}}$-favoring). Further $a_j(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$. Then, Lemma 3 implies that $t_j(R'_{N_{k-1}}, R_{-N_{k-1}}) > p_{k-1}$, which contradicts Inequality (3).

Thus, we have established $j \notin N_{k-1}$, i.e., j is a distinct agent not in N_{k-1}. Hence, we denote $j \equiv k$, and note that

$$z_k \; R_k \; z_{k-1} \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}}),$$

where the first preference relation follows from the Walrasian equilibrium property and the second follows from the fact that $a_k(R'_{N_{k-1}}, R_{-N_{k-1}}) = a_{k-1}$ and $p_{k-1} < t_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ (Lemma 3). Hence Property 1 is satisfied for agent k. Next, if $a_k = 0$, then $(0,0) = z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ contradicts individual rationality. Hence, Property 2 also holds. By $z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$, $p_k - V^{R_k}(a_k, f_k(R'_{n_{k-1}}, R_{-N_{k-1}})) > 0$. Thus, there is $\epsilon_k > 0$ such that $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$. Hence, by Lemma 2, there is a $z_k^{\epsilon_k}$-favoring R'_k.

Thus, we have constructed a sequence $(1, R'_1), \ldots, (n, R'_n)$ such that $a_k \neq 0$ for all $k \in N$. This is impossible since $n > m$, giving us the required contradiction.

Finally, we show that the MWEP mechanism is the unique ex-post revenue optimal mechanism among the class of desirable mechanisms satisfying no subsidy defined on a rich domain. Suppose $\hat{f} \equiv (\hat{a}, \hat{t})$ is another (not the MWEP) desirable mechanism satisfying no subsidy that is ex-post revenue optimal among the class of desirable mechanisms satisfying no subsidy. Then, there is some preference profile R and an agent i such that the object $\hat{a}_i(R)$ assigned to agent i by the mechanism $\hat{f}$ is not in her demand set at $p^{min}(R)$. Let $(a_j, p^{min}_{a_j}(R))$ denote the consumption bundle assigned to each agent $j \in N$ at preference profile R by the MWEP mechanism f^{min}. Hence,

$$(a_i, p^{min}_{a_i}(R)) \; P_i \; (\hat{a}_i(R), p^{min}_{\hat{a}_i(R)}(R)).$$

For all $j \neq i$, by the definition of Walrasian equilibrium, we have

$$(a_j, p^{min}_{a_j}(R)) \; R_j \; (\hat{a}_j(R), p^{min}_{\hat{a}_j(R)}(R)).$$

In the first part of the proof of this theorem, we have already shown that for all $j \in N$,

$$(\hat{a}_j(R), \hat{t}_j(R)) \; R_j \; (a_j, p^{min}_{a_j}(R)).$$

Combining the above relations, for all $j \in N$, we have $(\hat{a}_j(R), \hat{t}_j(R))\ R_j\ (\hat{a}_j(R), p^{min}_{\hat{a}_j(R)}(R))$ with strict relation holding for agent i. This implies that $\hat{t}_j(R) \leq p^{min}_{\hat{a}_j(R)}(R)$ with strict inequality holding for agent i. Adding it over all the agents, we get

$$\text{Rev}^{\hat{f}}(R) = \sum_{j \in N} \hat{t}_j(R) < \sum_{j \in N} p^{min}_{\hat{a}_j(R)}(R) \leq \text{Rev}^{f^{min}}(R),$$

which is a contradiction to the ex-post revenue optimality of $\hat{f}$. ■

7.2 Proof of Theorem 2

We now fix a desirable mechanism $f : \mathcal{R}^n \to Z$, where $\mathcal{R} \supseteq \mathcal{R}^{++}$. Further, we assume that f satisfies no bankruptcy, where the corresponding bound as $\ell \leq 0$. We start by proving an analogue of Lemma 3.

Lemma 4 *For every preference profile $R \in \mathcal{R}^n$, for every $i \in N$, and every $(a, t) \in M \times \mathbb{R}_+$ with $a = a_i(R)$, if there exists $j \neq i$ such that for each $b \in L \setminus \{a\}$,*

$$V^{R_j}(b, (a, t)) < -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + \ell,$$

then $t_i(R) > t$.

Proof: Assume for contradiction $t_i(R) \leq t$. Consider $R'_i = R_j$. By strategy-proofness, $f_i(R'_i, R_{-i})\ R'_i\ f_i(R) = (a, t_i(R))$. By equal treatment of equals,

$$f_j(R'_i, R_{-i})\ I_j\ f_i(R'_i, R_{-i})\ R_j\ (a, t_i(R)).$$

Note that either $a_i(R'_i, R_{-i}) \neq a$ or $a_j(R'_i, R_{-i}) \neq a$. Without loss of generality, assume that $a_j(R'_i, R_{-i}) = b \neq a$. Then, using the fact that $(b, t_j(R'_i, R_{-i}))\ R_j\ (a, t_i(R))$ and $t_i(R) \leq t$, we get

$$\begin{aligned} t_j(R'_i, R_{-i}) &\leq V^{R_j}(b, (a, t_i(R))) \\ &\leq V^{R_j}(b, (a, t)) \\ &< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + \ell. \end{aligned}$$

By individual rationality

$$t_i(R'_i, R_{-i}) \leq V^{R'_i}(a_i(R'_i, R_{-i}), (0, 0)) \leq \max_{c \in M} V^{R'_i}(c, (0, 0)).$$

Further, individual rationality also implies that for all $k \notin \{i, j\}$,

$$t_k(R'_i, R_{-i}) \leq V^{R_k}(a_k(R'_i, R_{-i}), (0, 0)) \leq \max_{c \in M} V^{R_k}(c, (0, 0)).$$

Adding these three sets of inequalities above, we get

$$\begin{aligned}
&\sum_{k \in N} t_k(R'_i, R_{-i}) \\
&< -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + \ell + \max_{c \in M} V^{R'_i}(c, (0, 0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0, 0)) \\
&= -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + \ell + \max_{c \in M} V^{R_j}(c, (0, 0)) + \sum_{k \in N \setminus \{i,j\}} \max_{c \in M} V^{R_k}(c, (0, 0)) \\
&\leq -n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + (n-1)\big(\max_{k \in N \setminus \{i\}} \max_{c \in M} V^{R_k}(c, (0, 0))\big) + \ell \\
&\leq \ell.
\end{aligned}$$

This contradicts no bankruptcy. ■

Using Lemma 4, we can mimic the proof of Theorem 1 to complete the proof of Theorem 2. We start by defining a class of positive income effect preferences by strengthening the notion of $(a, t)^\epsilon$-favoring preference. For every $(a, t) \in M \times \mathbb{R}_+$, for each $\epsilon > 0$, and for each $\delta > 0$, let $\mathcal{R}((a, t), \epsilon, \delta)$ be the set of preferences such that for each $\hat{R}_i \in \mathcal{R}((a, t), \epsilon, \delta)$, the following holds:

1. $\hat{R}_i$ is $(a, t)^\epsilon$-favoring and
2. $V^{\hat{R}_i}(b, (a, t)) < -\delta$ for all $b \neq a$.

A graphical illustration of $\hat{R}_i$ is provided in Figure 5. Since $\delta > 0$, it is clear that a $\hat{R}_i$ can be constructed in $\mathcal{R}((a, t), \epsilon, \delta)$ such that it exhibits positive income effect. Hence, $\mathcal{R}^{++} \cap \mathcal{R}((a, t), \epsilon, \delta) \neq \emptyset$.

Proof of Theorem 2

Proof: Now, we can mimic the proof of Theorem 1. We only show parts of the proof that requires some change. As in the proof of Theorem 1, by Lemma 1, we only need to show that for every profile of preferences $R \in \mathcal{R}^n$ and for every $i \in N$, $f_i^{min}(R)\ R_i\ f(R)$, where f^{min} is the MWEP mechanism. Assume for contradiction that there is some profile of preferences

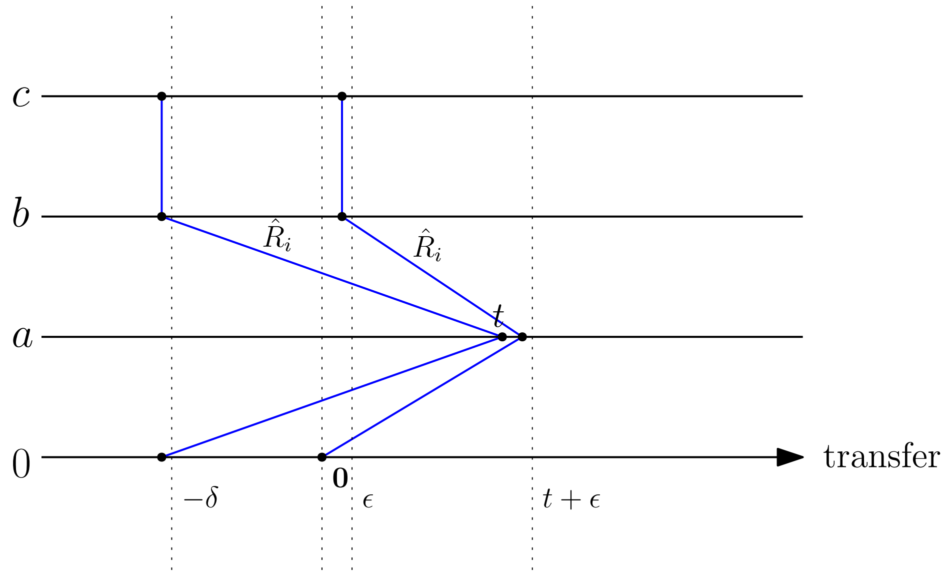

Figure 5: Illustration of $\hat{R}_i$

$R \in \mathcal{R}^n$ and some agent $i \in N$ such that $z_i \; P_i \; f_i(R)$, where $(z_1, \ldots, z_n) \equiv f^{min}(R)$ be the allocation chosen by the MWEP mechanism at R. Let $\underline{p} \equiv \min_{a \in M} p_a^{min}(R)$. For simplicity of notation, we will denote $z_j \equiv (a_j, p_j)$, where $p_j \equiv p_{a_j}^{min}(R)$, for all $j \in N$.

Define $\bar{\delta} > 0$ as follows:

$$\bar{\delta} := n\big(\max_{k \in N} \max_{c \in M} V^{R_k}(c, (0, 0))\big) - \ell.$$

We first construct a finite sequence of agents and preferences, without loss of generality $(1, R_1'), \ldots, (n, R_n')$, satisfying certain properties. Let $N_0 \equiv \emptyset$, $N_k \equiv \{1, \ldots, k\}$ for each $k \geq 1$, and $(R'_{N_0}, R_{-N_0}) \equiv R$. This sequence satisfies the properties that for every $k \in \{1, \ldots, n\}$,

1. $z_k \; P_k \; f_k(R'_{N_{k-1}}, R_{-N_{k-1}})$ for each $k \geq 1$,
2. $a_k \neq 0$,
3. $R_k' \in \mathcal{R}^+ \cap \mathcal{R}(z_k, \epsilon, \bar{\delta})$ for some $\epsilon_k > 0$ with $\epsilon_k < \min\{V^{R_k}(a_k, f_k(R'_{N_{k-1}}, R_{-N_{k-1}})) - p_k, \underline{p}\}$.

Now, we can complete the construction of this sequence inductively as in the proof of Theorem 1 (using Lemma 4 instead of Lemma 3), giving us the desired contradiction.

The uniqueness proof is identical to the proof of uniqueness given in Theorem 1. ■

8 Relation to the literature

Ever since the work of Myerson (1981), various extensions of his work to multi-object case have been attempted in quasilinear domain. Most of these extensions focus on the single agent (or, screening problem of a monopolist) with additive valuations (value for a bundle of objects is the sum of values of objects). Armstrong (1996, 2000) are early papers that show the difficulty in extending Myerson's optimal mechanisms to multiple objects case - he identifies optimal mechanisms for the cases where agents' preferences are binary, i.e, the valuations of each agent on a given object are only low and high values, but demonstrates that it is too complicated to identify optimal mechanisms for other cases.[20] Rochet and Choné (1998) show how to extend the convex analysis techniques in Myerson's work to multidimensional environment and point out various difficulties in the derivation of an optimal mechanism. These difficulties are more precisely formulated in the following line of work for the single agent additive valuation case: (1) optimal mechanism may require randomization (Thanassoulis, 2004; Manelli and Vincent, 2007); (2) simple mechanism like selling each good separately (Daskalakis et al., 2017) and selling all the goods as a grand bundle (Manelli and Vincent, 2006) are optimal for very specific distributions; (3) there is inherent revenue non-monotonicity of the optimal mechanism - if we take two distributions with one first-order stochastic-dominating the other, the optimal mechanism revenue may not increase (Hart and Reny, 2015); (4) the optimal mechanism may require an infinite menu of prices (Hart and Nisan, 2019). The complexity of revenue maximization in our model is apparent in the work of Thirumulanathan et al. (2017), who show that the analysis becomes intractable even for one buyer (with unit demand) and two objects (with uniformly distributed values).

Since these extensions are for a single agent who has *additive* valuation for bundles of objects, this may give the impression that the multi-object optimal mechanism design problem is difficult *only* when agents can be allocated more than one object. However, this impression is not true - the source of the problem is the multiple dimension of private information, which continues to exist even in the unit demand model considered in our paper. In our model, even with quasilinearity, the multiple dimensions of private information will be valuations for each object. As illustrated in Armstrong (1996, 2000), the multiple dimensions of private information implies that the incentive constraints become complicated to handle. In quasi-linear domain, the Myersonian approach to this problem would pin down payments of agents

[20] Whenever we say optimal mechanisms, we mean, like in Myerson (1981), an expected revenue maximizing mechanism under incentive and participation constraints with respect to some prior distribution.

in terms of object allocation rules using the well known revenue equivalence formula (Krishna and Maenner, 2001; Milgrom and Segal, 2002). Then, the objective function (maximizing sum of expected payments) is rewritten in terms of object allocation rule. On the constraint side, necessary and sufficient conditions are identified for the object allocation rule to be implementable (Rochet, 1987; Jehiel et al., 1999; Bikhchandani et al., 2006), and they are put as constraints. Whether agents can be allocated at most one object or multiple objects, the multidimensional nature of private information makes *both* the revenue equivalence formula and the constraints of the optimization problem become extremely difficult to handle. Vohra (2011) provides a linear programming approach to study such multidimensional mechanism design problems and points out similar difficulties.

Further, it is unclear how some of the above single agent results can be extended to the case of multiple agents. In the multiple agent problems, the set of feasible allocations starts interacting with the incentive constraints of the agents. Further, the standard Bayesian incentive compatibility constraints become challenging to handle. Note that in the single agent problem, these notions of incentive compatibility are equivalent, and for one-dimensional mechanism design problems, they are equivalent in a useful sense (Manelli and Vincent, 2010; Gershkov et al., 2013). Because we work in a model without quasilinearity, we are essentially operating in an "infinite" dimensional mechanism design problem. Hence, we should expect the problems discussed in quasilinear environment to appear in an even more complex way in our model.

To circumvent the difficulties from the multidimensional private information and multiple agents, a literature in computer science has developed approximately optimal mechanisms for our model - multiple objects and multiple agents with unit demand agents (but with quasilinearity). Contributions in this direction include Chawla et al. (2010a,b); Briest et al. (2010); Cai et al. (2012). Many of these approximate mechanisms allow for randomization. Further, these approximately optimal mechanisms involve reserve prices and violate no wastage axiom. It is unlikely that these results extend to environments without quasilinearity.

Finally, the Myersonian approach may not work if preferences are not quasilinear. In a companion paper (Kazumura et al., 2017), we investigate mechanism design without quasilinearity more abstractly and illustrate the difficulty of solving the single object optimal mechanism design problem. Hence, solving for full optimality without imposing the additional axioms that we put seems to be even more challenging in our model. In that sense, our results provide a useful resolution to this complex problem.

Our work can be connected to a result by Bulow and Klemperer (1996) and its extension by Roughgarden et al. (2015). In Bulow and Klemperer (1996), it was shown that (under standard independent and identical agent assumption with *regular* distribution) a single object optimal mechanism (with quasilinear preferences) for n agents generates less expected revenue than a single object Vickrey mechanism for $(n + 1)$ agents. Hence, if the cost of recruiting an agent is small, then the Vickrey mechanism can be recommended.[21] This result has been extended to our multi-object unit-demand agent setting with quasi-linear preferences: the expected revenue maximizing mechanism for n agents generates less expected revenue than the VCG mechanism for $(m + n)$ agents, where m is the number of objects (Roughgarden et al., 2015) - note that the MWEP mechanism is the VCG mechanism in the quasilinear domain. Our results complement these results by establishing an axiomatic revenue maximizing foundation of the MWEP mechanism (even when preferences are not quasilinear).[22]

We motivated our no wastage axiom by saying that the seller may not be able to commit to a no sale in future if the objects are not sold. If the seller can commit to a mechanism after a no-sale, then we can invoke a revelation principle and our results will follow. However, in many realistic settings, the seller is not able to commit to a future mechanism. Skreta (2015) analyzes a single object auction model in quasilinear domain and models the non-commitment of the seller explicitly. In a finite-period model, she finds that the expected revenue maximizing mechanism takes the same form as in the case of commitment. Her optimal mechanism does not satisfy no wastage, i.e., it is still optimal for the seller to not trade the object at the last period.

Ausubel and Cramton (1999) consider a model of a seller selling *identical* objects to a set of buyers who can consume at most one unit. They assume quasilinear preferences and explore the conequences of ex-post resale. They show that the Vickrey auction with reserve price stands out as the optimal (expected revenue maximizing) mechanism with resale (in a subclass of allocation rules called the *monotonic aggregate* allocation rules). They also offer other results to show that an inefficient allocation is suboptimal if there is perfect resale.

[21] Of course, one can argue that if we have $(n + 1)$ agents, then the seller must use the *expected revenue maximizing* mechanism for $(n + 1)$ agents. The main point in Bulow and Klemperer (1996) is that the Vickrey mechanism is a prior-free robust mechanism, whereas the expected revenue maximizing mechanism requires knowledge of priors.

[22] The computer science literature is interested in such prior-free bounds on optimal multidimensional mechanisms (which is hard to compute) - a recent paper by Eden et al. (2017) provide further extensions of Bulow-Klemperer results in multi-object environments where buyers can consume more than one object but have additive valuations.

While they do not consider non-quasilinear preferences and the heterogenous objects model, their results also hint that some form of revenue maximization and perfect resale leads to a restricted Pareto efficient mechanism (i.e., whenever there is sale, the object is allocated efficiently).

There is a short but important literature on object allocation problem with non-quasilinear preferences. Baisa (2016) considers the single object model and allows for randomization with non-quasilinear preferences. He introduces a novel mechanism in his setting and studies its optimality properties (in terms of revenue maximization). We do not consider randomization and our solution concept is different from his. Further, ours is a model with multiple objects.

The literature with non-quasilinear preferences and multiple objects have traditionally looked at Pareto efficient mechanisms. As discussed earlier, the closest paper is Morimoto and Serizawa (2015) who consider the same model as ours. They characterize the MWEP mechanism using Pareto efficiency, individual rationality, incentive compatibility, and no subsidy when the domain includes *all* classical preferences - see an extension of this characterization in a smaller domain in Zhou and Serizawa (2018). Similar characterizations are also available for other settings: Sakai (2008, 2013b,a) provide such characterizations in the single object auction model; Saitoh and Serizawa (2008); Ashlagi and Serizawa (2012); Adachi (2014) in the homogeneous object auction model with unit demand preferences. Pareto efficiency and the *complete* class of classical preferences play a critical role in pinning down the MWEP mechanism in these papers. As we point out in Section 5, even in the quasilinear domain of preferences, there are desirable mechanisms satisfying no subsidy which are different from the MWEP mechanism. By imposing revenue maximization as an objective instead of Pareto efficiency, we get the MWEP mechanism in our model. Pareto efficiency is obtained as an implication (Corollaries 3 and 4). Finally, our results work for not only the complete class of classical preferences, but for a large variety of domains, such as the class of all quasilinear preferences, one including all non-quasilinear preferences, one including all preferences exhibiting positive income effects, etc.

Tierney (2019) considers axioms like *no discrimination*, *welfare continuity*, and some stronger form of strategy-proofness to give various characterizations of the MWEP mechanism with reserve prices in the quasilinear domain. Using our result, he shows that in the *quasilinear domain*, the MWEP mechanism is the unique mechanism satisfying strategy-proofness, *no-discrimination*, individual rationality, no wastage, and *welfare continuity*.

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A Proof of Claim 3

Before we start the proof of Claim 3, we point out a technical property of non-negative income effect preferences. The claim below shows a form of monotonicity of demand sets with non-negative income effect preferences.

Claim **4** *Let* $p, p' \in \mathbb{R}^3_+$ *be price vectors such that* $p'_a = p'_b < p_a = p_b$*. For each* $R_i \in \mathcal{R}^+$*, if* $D(R_i, p) \cap M \neq \emptyset$*, then* $D(R_i, p') \subseteq D(R_i, p)$*.*

Proof: Let $D(R_i, p) \cap M \neq \emptyset$. Then, by $p'_a = p'_b < p_a = p_b$, $0 \notin D(R_i, p')$. Assume for contradiction that $D(R_i, p') \setminus D(R_i, p) \neq \emptyset$. Then, by $0 \notin D(R_i, p')$, without loss of generality, let $a \in D(R_i, p')$ and $a \notin D(R_i, p)$.

By $a \in D(R_i, p')$ and $p'_b < p_b$, $(a, p'_a)\ R_i\ (b, p'_b)\ P_i\ (b, p_b)$. This implies $p'_a < V^{R_i}(a, (b, p_b))$. Thus, $\delta := V^{R_i}(a, (b, p_b)) - p'_a > 0$. By $R_i \in \mathcal{R}^+$ and $\delta > 0$, $(b, p_b - \delta)\ R_i\ (a, V^{R_i}(a, (b, p_b)) - \delta)$.

By $D(R_i, p) \cap M \neq \emptyset$ and $a \notin D(R_i, p)$, we have $b \in D(R_i, p)$ and $(b, p_b)\ P_i\ (a, p_a)$. The latter implies $V^{R_i}(a, (b, p_b)) < p_a = p_b$. Thus, $p_b - \delta = p_b - (V^{R_i}(a, (b, p_b)) - p'_a) > p_b - (p_b - p'_a) = p'_a = p'_b$.

By the definition of δ, $V^{R_i}(a, (b, p_b)) - \delta = p'_a$. Thus by $p_b - \delta > p'_b$, $(b, p'_b)\ P_i\ (b, p_b - \delta)\ R_i\ (a, V^{R_i}(a, (b, p_b)) - \delta) = (a, p'_a)$. This contradicts $a \in D(R_i, p')$. ■

Proof of Claim 3

Proof: Let $R \in (\mathcal{R}^+)^4$ and $S(R) := \{i \in N : R \text{ is discounting combination for } i\}$. If $S(R)$ is empty, then the claim follows because $Z^{min}(R)$ is non-empty. As we discussed just above Claim 3, if $S(R)$ is non-empty, then $|S(R)| \leq 2$. Further, if $S(R) = \{i, j\}$, then R is a discounting combination for i and j with respect to the same discounting combination. So, we consider two cases.

Case 1. $S(R) = \{i, j\}$. Note that there is a common discounting combination $T \in \mathcal{T}$ for agents i and j. By Claim 2, we have $p^{min}(R) = \overline{p}^T$. Since R is a discounting combination for two agents, for every $k \in N = \{1, 2, 3, 4\}$, we must have $R_k \in T$ with $R_i = R_j$. Thus, by Definition 11 , for every $k \in N = \{1, 2, 3, 4\}$, $D(R_k, \overline{p}^T) = \{0, a, b\}$, and $D(R_i, p^T) = D(R_j, p^T) = \{a, b\}$.

Consider an object allocation (a_1, a_2, a_3, a_4) such that $a_i, a_j \in \{a, b\}$ and for each $k \in N \setminus \{i, j\}$, $a_k = 0$. Then, (a_1, a_2, a_3, a_4) satisfies Conditions (1) and (2) of the claim for this case.

CASE 2. $S(R) = \{i\}$. Let the discounting combination of i be $T := \{R_k : k \neq i\} \in \mathcal{T}$. By Claim 2, we have

$$p^{min}(R) = \overline{p}^T. \tag{4}$$

Also, by (b) of Definition 11, we have

$$D_k(R_k, \overline{p}^T) = \{0, a, b\} \qquad k \in N \setminus \{i\}. \tag{5}$$

Consider an object allocation (a_1, a_2, a_3, a_4) such that $a_i \in D(R_i, p^T)$ and $\{a_1, a_2, a_3, a_4\} = \{0, a, b\}$. Then, since $S(R) = \{i\}$, the object allocation (a_1, a_2, a_3, a_4) satisfies Condition (1) of the claim. To show Condition (2) of the claim, we consider two subcases.

CASE 2A. Suppose $D(R_i, \overline{p}^T) \cap M \neq \emptyset$. Then, by Claim 4 we get $a_i \in D(R_i, p^T) \subseteq D(R_i, \overline{p}^T)$. Thus, by Equations (4) and (5), we have $((a_1, \overline{p}^T_{a_1}), \ldots, (a_4, \overline{p}^T_{a_4})) \in Z^{min}(R)$. Thus, (a_1, a_2, a_3, a_4) also satisfies Condition 2.

CASE 2B. Suppose $D(R_i, \overline{p}^T) \cap M = \emptyset$, i.e., $D(R_i, \overline{p}^T) = \{0\}$. Choose any $k \notin S(R)$. Then, $k \neq i$. Consider any object allocation (b_1, b_2, b_3, b_4) satisfying $\{b_1, b_2, b_3, b_4\} = \{0, a, b\}$ such that $b_k = a_k$ and $b_i = 0$. By $D(R_i, \overline{p}^T) = \{0\}$, by Equations (4) and (5), we get $((b_1, \overline{p}^T_{b_1}), \cdots, (b_4, \overline{p}^T_{b_4})) \in Z^{min}(R)$. Thus, (a_1, a_2, a_3, a_4) also satisfies Condition 2.

This exhausts all the cases and completes the proof. ■

B Supplementary Appendix

The supplementary appendix contains two sections. In the first section, we provide a desirable mechanism satisfying no subsidy on the quasilinear domain. This mechanism is not the MWEP mechanism. In the second section, we modify the first-price auction so that it satisfies equal treatment of equals.

B.1 Non-MWEP desirable mechanisms in quasilinear domain

We reproduce a desirable mechanism satisfying no subsidy from Tierney (2019). The mechanism is defined for a simple case of three objects: $M := \{a, b, c\}$ and five agents: $N := \{1, 2, 3, 4, 5\}$. Before formally defining the mechanism, we emphasize two points: (1) the mechanism we define works similar to the family of mechanisms defined in Section 5, but a careful look will reveal that there are significant differences; (2) even though we give one desirable mechanism satisfying no subsidy in the quasilinear domain, it can be extended to a family of desriable mechanisms satisfying no subsidy (containing the mechanism of Tierney (2019) but excluding the MWEP mechanism). We elaborate on these two points at the end of this section.

To define this mechanism formally, first we pick up four quasi-linear preferences to specify a "discounting combination". This is shown in Table 2. Denote the quasilinear preference corresponding to valuation functions $v^\alpha, v^\beta, v^\gamma, v^\lambda$ as $R^\alpha, R^\beta, R^\gamma, R^\lambda$ respectively.

	a	b	c
v^α	2	3	4
v^β	2	3	ϵ
v^γ	ϵ	3	4
v^λ	2	ϵ	4

Table 2: Four quasilinear preferences - $\epsilon > 0$ but arbitrarily close to zero

Denote $T \equiv \{R^\alpha, R^\beta, R^\gamma, R^\lambda\}$. The mechanism we describe works in the class of all quasilinear preferences $\mathcal{R}^Q$. For any $i \in N$, we say a profile of preferences $R \equiv (R_1, \dots, R_5) \in (\mathcal{R}^Q)^5$ is a **discounting combination** for agent i if the preferences of agents other than agent i coincide with T, i.e., $\{R_j : j \neq i\} = T$. We say a preference profile $R \in (\mathcal{R}^Q)^5$ is **discounting** if R is a discounting combination for some agent i.

Before describing the mechanism, we make a comment about discounting preference profiles.

Claim **5** *If $R \in (\mathcal{R}^Q)^5$ is discounting, $p_a^{min}(R) = 2, p_b^{min}(R) = 3, p_c^{min} = 4$.*

Proof: Consider the suggested price vector p with $p_a = 2, p_b = 3, p_c = 4$. If R is discounting, at least four agents have the null object in their demand set at p. Further, each object is demanded by some agent at p. Hence, p is clearly a WEP. To see that it is the minimum WEP, consider a price vector $p' \leq p$. In that case, if price of at least two objects are lower at p' than p, then there are at least four agents who do not demand the null object at p'. Since the number of objects is only three, p' cannot be a WEP. Hence, price of exactly one object, say x, is lower at p' than p. But then, $|\{i \in N : \{x\} = D(R_i, p)\}| > 1$, contradicting the fact p' is a WEP. ■

We now give the analogue of Claim 3 without a proof.

Claim **6** *For each $R \in (\mathcal{R}^Q)^5$, there exists an object allocation $(a_1, \ldots, a_5)$ such that $\{a_1, \ldots, a_5\} = L$ and for each $i \in N$,*

1. *if R is a discounting combination for agent i with $\{R_j : j \neq i\} = T$, then $a_i \in D(R_i, p^T)$, where $p_a^T = 1, p_b^T = 2, p_c^T = 3$,*

2. *if R is not a discounting combination for agent i, then there exists a minimum Walrasian equilibrium price allocation $((b_1, p_{b_1}^{min}(R)), \ldots, (b_5, p_{b_5}^{min}(R))) \in Z^{min}(R)$ such that $b_i = a_i$.*

Proof: Let $R \in (\mathcal{R}^+)^5$ and $S(R) := \{i \in N : R$ is discounting combination for $i\}$. If $S(R)$ is empty, then the claim follows because $Z^{min}(R)$ is non-empty. If $S(R)$ is non-empty, then $|S(R)| \leq 2$ by the definition of discounting combination. So, we consider two cases.

Case 1. $S(R) = \{i, j\}$. Since R is a discounting combination for two agents, for every $k \in N$, we must have $R_k \in T$. Also, $R_i = R_j \in T$ and by definition of T and p^T, there must exist two objects in M, say $\{x, y\}$, such that $\{x, y\} \subseteq D(R_i, p^T) = D(R_j, p^T)$. Let $M \setminus \{x, y\} = \{z\}$ and k be an agent such that $z \in D(R_k, p^{min}(R))$ – Claim 5 and Definition of T ensures that such an agent $k \in N \setminus \{i, j\}$ exists. Consider an object allocation $(a_1, a_2, a_3, a_4, a_5)$ such that $a_i, a_j \in \{x, y\}$, $\{a_k\} = z$ and for each $h \in N \setminus \{i, j, k\}$, $a_h = 0$. Thus, $a_i \in D(R_i, p^T)$ and $a_j \in D(R_j, p^T)$. By Claim 5, $p_a^{min}(R) = 2, p_b^{min}(R) = 3, p_c^{min}(R) = 4$. Hence, $a_i \in D(R_i, p^{min}(R))$

and $a_j \in D(R_j, p^{min}(R))$. By definition, $z \in D(R_k, p^{min}(R))$. Also, for every $R_h \in T$, we have $0 \in D(R_h, p^{min}(R))$. As a result, the allocation $((a_1, p^{min}_{a_1}(R), \ldots, (a_5, p^{min}_{a_5}(R)))$ is a minimum Walrasian equilibrium price allocation for R. Hence, the object allocation $(a_1, a_2, a_3, a_4, a_5)$ satisfies the assertions of the claim.

Case 2. $S(R) = \{i\}$. Hence, $T = \{R_k : k \neq i\}$. By Claim 5, $p^{min}(R)$ is such that for every $k \in N \setminus \{i\}$, we have $D(R_k, p^{min}(R)) = \{0, x, y\}$ for some $x, y \in \{a, b, c\}$ and $\{a, b, c\} \subseteq \cup_{k \neq i} D(R_k, p^{min}(R))$. There are two subcases to consider now.

Case 2a. Suppose $D(R_i, p^{min}(R)) \cap M \neq \emptyset$. Then, by the definition of quasilinear preferences and p^T, $D(R_i, p^T) \subseteq D(R_i, p^{min}(R))$. Further, since $p^T < p^{min}(R)$, we have $0 \notin D(R_i, p^T)$. Let $x \in D(R_i, p^T) \subseteq D(R_i, p^{min}(R))$ and $\{y, z\} := M \setminus \{x\}$. By $\{a, b, c\} \subseteq \cup_{k \neq i} D(R_k, p^{min}(R))$, there are $j, k \in N \setminus \{i\}$ such that $y \in D(R_j, p^{min}(R))$ and $z \in D(R_k, p^{min}(R))$. Then, $b_i = x$, $b_j = y, b_k = z$, and $b_h = 0$ for all $h \notin \{i, j, k\}$ is an object allocation such that $((b_1, p^{min}_{b_1}(R)), \ldots, (b_n, p^{min}_{b_n}(R))) \in Z^{min}(R)$.

Case 2b. Suppose $D(R_i, p^{min}(R)) \cap M = \emptyset$, i.e., $D(R_i, p^{min}(R)) = \{0\}$. We also know that for every $k \in N \setminus \{i\}$, we have $D(R_k, p^{min}(R)) = \{0, x, y\}$ for some $x, y \in \{a, b, c\}$ and $\{a, b, c\} \subseteq \cup_{k \neq i} D(R_k, p^{min}(R))$. Since $|N \setminus \{i\}| = 4$, for every $k \in N \setminus \{i\}$ and for every $x \in D(R_k, p^{min}(R))$, there exists an allocation $((b_1, p^{min}_{b_1}(R)), \ldots, (b_n, p^{min}_{b_n}(R))) \in Z^{min}(R)$ such that $b_k = x$. Hence, we can choose any $a_i \in D(R_i, p^T)$ and allocate objects in $\{0, a, b, c\} \setminus \{a_i\}$ among agents in $N \setminus \{i\}$ in such a way that every agent in $N \setminus \{i\}$ receives an object in the demand set and satisfy the assertions of the claim.

This exhausts all the cases and completes the proof. ■

Claim 6 shows that for discounting combination T, at every preference profile $R \in (\mathcal{R}^Q)^5$, we can identify an object allocation satisfying (1) and (2) of Claim 6. Again, there may be more than one such object allocation, but we identify one such object allocation at every R and use it to formally define our mechanism.[23]

Definition 16 *Given the discounting combination T, the* **MWEP mechanism with discounting combination** T *is denoted by f^T and defined as follows: for every $R \in (\mathcal{R}^Q)^5$,*

[23]It is straightforward to extend this mechanism to a family of mechanisms by specifying a set of discounting combinations as in Section 5.

$(a_1^T(R), \ldots, a_5^T(R))$ *is an object allocation satisfying Claim 6, and for every* $i \in N$

$$t_i^T(R) = \begin{cases} p^T_{a_i^T(R)} & \text{if } R \text{ is a discounting combination for } i, \\ p^{min}_{a_i^T(R)}(R) & \text{othewise,} \end{cases}$$

where p^T *is the price vector defined as* $p_a^T = 1, p_b^T = 2, p_c^T = 3.$

It is clear that f^T satisfies individual rationality, equal treatment of equals, no wastage, and no subsidy. Just like Proposition 1, we can show that f^T is strategy-proof (we skip the similar proof).

PROPOSITION **2** *The mechanism* f^T *is strategy-proof.*

Clearly, f^T is not the MWEP mechanism since $p^T < p^{min}(R)$ whenever R is discounting. Hence, this establishes the fact that the MWEP mechanism is not the only desirable mechanism satisfying no subsidy in the quasilinear domain.

Though the mechanism f^T looks similar to the family of mechanisms defined in Section 5, there are significant differences. First, note that the mechanism in Definition 16 required three objects and five agents (whereas the mechanism in Section 5 has two objects and four agents). It is not possible to define the mechanism in Definition 16 with two objects. The preferences in discounting combination T in this section are quasilinear preferences and they are different than the discounting combinations specified in Definition 11. Further, unlike property (b) of Definition 11, the discounting combination in this section does not require existence of a price vector $\bar{p}^T$ such that $D(R_i, \bar{p}^T) = L$ for each $R_i \in T$. Finally, note that at a discounting preference R in this section, $p_a^{min}(R) = 2, p_b^{min}(R) = 3, p_c^{min}(R) = 4$, which is unlike in Section 5, where prices of both the objects were the same (this was required for strategy-proofness with non-negative income effects). These differences indicate that the set of desirable mechanisms satisfying no subsidy is quite difficult to characterize in an arbitrary rich domain, and their characteristics vary from quasilinear domain to non-negative income effect domain. Further, in the quasilinear domain, the set of desirable mechanisms satisfying no subsidy is not a singleton, and quite rich.

As we discussed earlier, the mechanism f^T can be extended to a family of mechanisms. Indeed, all we need is a discounting combination which inherits some properties of the discounting combination T such that Claim 6 works. Each such discounting combination defines a desriable mechanism satisfying no subsidy. This will contain as a special case f^T. Each mechanism in this family is desirable and satisfies no subsidy but it is not the MWEP mechanism.

B.2 A modification of first-price auction

Consider an example with a single object and quasilinear preferences. With *symmetric* agents (i.e., agents having independent and identical distribution of values), a symmetric Bayesian Nash equilibrium strategy of the first price auction is increasing and continuous function $b(\cdot)$ of valuations - for an exact expression of this function, see Krishna (2009). Consider the mechanism such that for each valuation profile $v = (v_1, \ldots, v_n)$, the outcome of the bid profile $(b(v_1), \ldots, b(v_n))$ of the first price auction is chosen. Call this mechanism *the first-price based mechanism*. It is Bayesian incentive compatible. Though, the first-price based mechanism satisfies no subsidy, ex-post individual rationality, and no wastage, it fails to satisfy ETE (unless, we break ties using uniform randomization). To see this, if two agents have same value, they bid the same amount in the first-price based mechanism. If there is no randomization to break ties, only one of those agents wins the object at his bid amount, whereas the other agent gets zero payoff. Since bid amount is less than the value in the first-price based mechanism, the winner gets positive payoff, and this violates ETE.

However, this can be rectified in two ways. First, whenever there is tie for the winning bid, all the winning agents get the object with equal probability. This introduces uniform randomization, and ETE is now satisfied. Hence, the *randomized* first-price based mechanism is Bayesian incentive compatible, satisfies ex-post IR, ETE, no wastage, and no subsidy. Obviously, there are profiles of values where such a first-price based mechanism generates more revenue than the Vickrey mechanism - winning bid in the first-price auction may be higher than the second highest value.[24]

An alternate approach to restoring ETE in the first-price based mechanism is to modify it in a deterministic manner whenever there is a tie in the winning bids. Consider a profile of values $(v_1, \ldots, v_n)$ such that more than one agent has bid the highest amount, say, B. Note that this bid B corresponds to value $b^{-1}(B)$. In such a case, we break the winning agent tie deterministically by giving the object (with probability 1) to one of the winning agents. Further, we ask him to pay his value $b^{-1}(B)$. This ensures that the winner and the losing agents all get a payoff of zero, and thus, it restores ETE. More formally, the mechanism corresponding to this *modified first-price based mechanism* is the following.

1. Agents submit their values $(v_1, \ldots, v_n)$.
2. If there is a unique highest valued agent i, he is given the object and he pays $b(v_i)$, where b is the unique symmetric Bayesian equilibrium bidding function of the first-price

[24] It is well known that the expected revenue from both the auctions is the same.

auction.

3. If there are more than one highest valued agents, then *any* one of them is given the object and is asked to pay his value.

Notice that this only modifies the mechanism corresponding to the first-price auction at zero measure profiles of values. Hence, the modified first-price based mechanism is Bayesian incentive compatible. Further, it is deterministic, satisfies ETE, no wastage, no subsidy, and ex-post IR. Because of the same reasons given for first-price auction, there are profiles of values where such a modified first-price based mechanism generates more revenue than the Vickrey mechanism.

This illustrates that we cannot relax strategy-proofness to Bayesian incentive compatibility in our results.