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Demand Uncertainty, Product Differentiation, and Entry Timing under Spatial Competition*

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Abstract

We investigate the entry timing and location decisions under market size uncertainty with Brownian motions in a continuous-time spatial duopoly competition. We show the following results. The entry threshold of the follower *non-monotonically* increases in volatility, implying that the leader's monopoly periods get longer with volatility. However, the leader is more likely to increase the degree of product differentiation as the volatility rises. A larger entry cost asymmetry between the firms places the leader closer to the edge in a preemption equilibrium although such an asymmetry places the leader closer to the center in a sequential equilibrium.

Keywords: Location; Hotelling model; Continuous-time; Entry timing; Real options.

JEL classification: C73, D81, L11, L13,

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1 Introduction

When a firm launches new products and/or services in a market, determining the product characteristics is one of its important decisions. For example, a fast food chain operator opening a new restaurant needs to decide its location considering the demand in the area as well as the competition with other firms. Another example is a manufacturing company that considers the kind of products it should develop while taking into account the future demand. Several studies such as Bronnenberg and Mahajan (2001), and Cleeren et al. (2010) show that product positions influence the pricing decisions of retailing and marketing firms greatly.

Due to the importance of product positioning, many theoretical studies investigate the factors determining the product positions of firms in various contexts by using the Hotelling (1929) type linear city models. These studies include d'Aspremont et al. (1979), Friedman and Thisse (1993), Tabuchi and Thisse (1995), Tyagi (2000, 2001), Kim and Serfes (2006), Matsushima (2009), Sajeesh and Raju (2010), Liu and Tyagi (2011), and Lai and Tabuchi (2012).[1] Although the existing studies offer many interesting insights, they are derived using *static* or *discrete-time* models with consumers purchasing products at most several periods, implying robust results over the markets where the demand is quite stable for a long time. However, if we consider the consumers' repeated purchases in growing or changing markets, entry timing is an important strategic decision for firms.

In actual situations, the uncertainty of future demand for products affects the decisions on location, product positioning, entry timing, and so on. Uncertainty is especially important for firms facing competition because the strategies of entry timing and product positioning with future uncertainty can have a bigger and mutual impact on the strategies of other firms than without. In summary, a firm needs to take uncertainty and competition into account

[1] Several empirical studies also investigate the problems of positioning (e.g., Thomadsen, 2007; Hwang et al., 2010).

while deciding on product positioning and entry timing.[2]

The decision-making of a firm under uncertainty as well as competition is often studied within a real options framework. McDonald and Siegel (1986) is the seminal paper in the literature, and there are many extensions and generalizations after the paper. For example, Décamps and Mariotti (2004) consider a duopoly market in which the irreversible cost of each firm at the market entry is privately observed. Thijssen (2010) studies an investment problem under the assumption that the state variables of the two firms are player-specific in that random terms are not perfectly correlated.

Recently, Huisman and Kort (2015) studied the problem where two asymmetric firms choose not only their investment timing but also their production capacity at investment. Market size is a state variable and follows a geometric Brownian motion as in the real options literature.[3] Huisman and Kort (2015) is novel in that two symmetric firms optimally choose two strategic variables. This approach can be applied to many monopoly/oligopoly problems.[4] They find that the leader firm overinvests in capacity to deter the entry of the follower and that greater uncertainty makes entry deterrence more likely. From their article, uncertainty is of great importance for the decision-making of firms facing competition, and that the multiple dimension of decision-making may lead to different results when compared to a single-dimension case.

In addition, Ebina et al. (2015) extend Lambertini (2002) to study the entry timings of firms facing competition. They construct a continuous-time spatial competition model a la d'Aspremont et al. (1979). In the model, two firms optimally determine their locations and prices as well as the follower's entry timing, but the leader's entry time is exogenously fixed as in Lambertini (2002). The main finding of the study is that the leader has an incentive to

[2] In the context of spatial competition with sequential entry, Neven (1987) and Bonanno (1987) are the pioneering works, which have been extended in several directions (e.g., Loertscher and Muehlheusser, 2011).

[3] A good survey is presented by Chevalier-Roignant et al. (2011)

[4] Lobel et al. (2016) studies a monopolist's optimization problem on product launches within a real options framework. In addition, Nielsen (2002), Weeds (2002), Huisman and Kort (2003), and Pawlina and Kort (2006) study monopoly or duopoly within real options frameworks, but with firms choosing only their timings in the models. Ziedonis (2007) provides a brief overview of the real option literature.

locate close to the center to delay the follower's entry, leading to a non-optimal outcome from the viewpoint of social welfare. The study also sheds light on the importance of multiple dimensions for the firm's decision-making. It is noteworthy that the demand for the product grows at a constant rate, indicating no uncertainty in their model.

In this article, we investigate the entry decisions of firms that endogenously determine their product positions in a market whose size is evolving stochastically. To this end, we substantially extend Ebina et al. (2015), which does not assume uncertainty on the growth path. Furthermore, we endogenize the leader's entry timing; this is also a significant extension of Ebina et al. (2015).[5] We describe uncertainty as the product demand following a geometric Brownian motion, as in Pawlina and Kort (2006) and Huisman and Kort (2015). By doing so, we can analyze the effect of uncertainty on the two firms' entry timings and location strategies in a Hotelling model.

The major findings of the study are as follows. First, the leader firm optimally chooses the center as its location in case of low volatility, and the edge in case of high volatility. In addition, in case of high volatility, the delay of the follower's entry occurs in accordance with the maximum differentiation by the leader, contrasting with the standard intuition in which the leader prefers locating itself closer to the center by anticipating the delay of the follower's entry. The reason for our counterintuitive result is that high volatility makes the leader firm give more importance to the future situation where two firms face price competition, leading to the leader's strategy locating itself as far from the center as possible. Our paper is the first study to construct a model in which demand evolves stochastically forever, and to examine the effect of uncertainty on locations within a continuous-time setting where both firms dynamically optimize their objectives.[6]

Second, the entry threshold of the follower firm is not monotonic in the volatility of the

[5]Ebina and Matsushima (2017) provide another major extension to the work of Ebina et al. (2015) by incorporating exclusive vertical relations. However, the direction of this extension is quite different from ours.

[6] In static frameworks, several studies investigate how uncertainty affects the optimal locations in the literature on industrial organization (e.g., Meagher and Zauner, 2004; Christou and Vettas, 2005).

state variable. In the real options literature, the threshold is always an increasing function of volatility, irrespective of whether the firm is a leader or a follower if there is a negative externality.[7] To the authors' best knowledge, this article is the first to show that greater uncertainty can lead to early entry of the follower firm in some cases. This finding indicates the importance of multiple dimensions for decision-making. We also present some intuitive remarks on why non-monotonicity occurs.

Third, the effect of entry cost asymmetry between the two firms on the leader's location differs, depending on the type of equilibrium. More concretely, in a preemptive equilibrium, a higher entry cost asymmetry makes the leader locate itself closer to the edge, while it makes the leader's location closer to the center in a sequential equilibrium. As we will see later, there are positive and negative effects of the leader's location on its profits, and the magnitude of the two effects is dependent on the type of equilibrium that is realized.

In summary, the analysis shows that multiple dimensions of strategies complicate the problem and may give different results from those in single-dimension cases which are studied by many papers in the real options literature.

The remaining part of the article is organized as follows. Section 2 sets up our model and formulates our problem. Section 3 derives the optimal timing and location of the follower firm. In Section 4, we first categorize the equilibrium type and then derive the optimal timing and location of the follower firm in a sequential equilibrium. We implement numerical calculations to examine how the leader's entry threshold and location are affected by exogenous parameters in Section 5. Finally, some concluding remarks are presented in Section 6.

2 The Model

In this section, we construct our model based on Ebina et al. (2015) and introduce uncertainty into it as in Pawlina and Kort (2006).

[7] Mason and Weeds (2010) show that the threshold of a leader is non-monotonic if there is a positive externality. However, under the assumption of a negative externality, the thresholds of the leader and the follower are increasing in the volatility.

Two firms, indexed by $i \in \{1, 2\}$, produce homogeneous goods. Consumers are uniformly distributed over the unit segment $[0, 1]$, as proposed by Hotelling (1929).[8] The density of consumer distribution at time t is Y_t, which stochastically changes as explained later. Each consumer is indexed by $x \in [0, 1]$ and repeatedly purchases at each instance $[t, t + \mathrm{d}t)$ at most one unit of the good. He chooses the firm to purchase from when he actually decides to purchase.[9] The consumption of a unit of the good entails a positive utility. On the other hand, the consumer at point $x \in [0, 1]$ incurs a quadratic transportation cost $c(x_i - x)^2$ and the price p_{it} at time $t \in [0, \infty)$ while buying a good from firm i located at $x_i \in [0, 1]$.[10] To summarize, the utility of the consumer at point $x \in [0, 1]$ and time $t \in [0, \infty)$ is given by

$$u_t(x; x_1, x_2, p_{1t}, p_{2t}) = \begin{cases} \bar{u} - p_{1t} - c(x_1 - x)^2 & \text{if purchased from firm 1,} \\ \bar{u} - p_{2t} - c(x_2 - x)^2 & \text{if purchased from firm 2,} \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $\bar{u}$ denotes the gross surplus each consumer enjoys from purchasing the good and $c > 0$ is a parameter describing the level of transportation cost or product differentiation.[11] Alternatively, we can regard c as the parameter representing the width of the market given the fixed length of the Hotelling line segment.[12]

The following assumption can make the equilibrium meaningful.

Assumption 1. *$\bar{u} > 3c$.*

Assumption 1 guarantees that at least one of the two firms has an incentive to supply a positive amount of goods after maximizing its profit wherever it is located. It also guarantees that a monopolist prefers serving all consumers regardless of its location.

[8] This setting and the following assumptions are standard in the literature on spatial economics.

[9] In equilibrium, all consumers purchase a unit of the product at all times, according to Assumption 1 presented below.

[10] The problem in which both firms can locate themselves outside the interval $[0, 1]$ is studied in a different paper.

[11] If $c = 0$, the follower can earn no profit after its entry, implying that our setting corresponds to a monopolistic situation because the follower never enters the market. To avoid this simple case, we assume $c > 0$.

[12] See the discussion in Section 5.

Now, we introduce uncertainty into our market model. The density of consumer distribution, or the market size Y_t, is dynamically stochastic. We impose the following assumption on Y.

Assumption 2. *The process Y follows a geometric Brownian motion as*

$$\mathrm{d}Y_t = \alpha Y_t \mathrm{d}t + \sigma Y_t \mathrm{d}W_t,$$

where α is the expected growth rate, σ is the market volatility, and $\{W_t\}_{t\geq 0}$ is a standard Brownian motion. The initial value of the state process, $Y_0 \equiv y_0$, is sufficiently low.

Assumption 2 states that the future profit flow of each firm is uncertain and follows a geometric Brownian motion.[13] The assumption on the initial value y_0 is standard in the real options literature, and means that the market is too small and that neither firm has made an entry into the market at the initial time.

The game in this article proceeds as follows. Firm i chooses the time of entry $T_i \in [0, \infty)$ and the location $x_i \in [0, 1]$ simultaneously. The entry incurs an irreversible cost F_i at T_i. Existing firm(s) simultaneously choose the price p_{it} at each time t, observing all available information such as the realization of Y and their location(s) x_i. Although firm i can vary the price p_{it} at any time, its location is fixed forever after the determination of x_i. We consider the Nash equilibrium in which firm i maximizes its present value of cash flows with respect to $(T_i, x_i, \{p_{it}\}_{t\geq T_i})$ given the other firm's strategy. Here is a remark on our assumption. We assume that firms locate within the line segment $[0, 1]$, which contrasts with the assumption in several previous papers that firms are allowed to locate outside the line segment (e.g., Tyagi, 2000; Sajeesh and Raju, 2010; Liu and Tyagi, 2011). If we follow the latter assumption, we will obtain an outcome in which the leader firm, say firm 1, locates at $x_1 = 1/2$ irrespective of the exogenous parameters.

We present the following assumptions for the entry cost F_i.

[13] Exponential market growth is seen to be valid in many industries; for example, see Lages and Fernandes (2005) on telecommunication services, Victor and Ausubel (2002) on DRAM, and Vakratsas and Kolsarici (2008) on pharmaceuticals.

Assumption 3. $F_1 = F < \kappa F = F_2$, *where* $\kappa > 1$.

Assumption 3 states that firm 1 has an advantage in the entry cost over firm 2. Since asymmetry occurs only in the entry cost, we speculate that firm 1 is always the leader for market entry in our model.

Now, let us describe the present value of the firms at time $t \in [0, \infty)$ given that firm j enters the market at point x_j at time $t = T_j$. Here, T_i^ℓ denotes the entry time of firm i when it is the leader, and T_j^f denotes that of firm j when it is the follower. Superscripts ℓ and f represent the leader and the follower, respectively. As the model is time-homogeneous, the optimal entry time is expressed as the first hitting time; that is,

$$T_i^\ell = T_i(y_i^\ell) \equiv \inf\{t \geq 0; Y_t \geq y_i^\ell\} \quad \text{and} \quad T_j^f = T_j(y_j^f) \equiv \inf\{t \geq 0; Y_t \geq y_j^f\},$$

where y_i^ℓ and y_j^f are the thresholds of the leader and the follower, respectively.

Let $\bar{x} \in [0, 1]$ be a point at which the consumer is indifferent between purchasing from firm 1 and purchasing from 2. We can easily verify from (1) that

$$\bar{x} = \frac{p_{2t} - p_{1t} + c(x_2^2 - x_1^2)}{2c(x_2 - x_1)}, \tag{2}$$

which indicates that the optimal price of each firm depends on the locations of both firms (x_1, x_2). Therefore, the value function of firm j at time t when it is the follower is written with discount rate r as

$$\mathbb{E}_y \left[\int_{T_j(y_j^f)}^{\infty} \mathrm{e}^{-r(s-t)} Y_s \int_{\bar{x}}^{1} p_{js}^f \mathrm{d}x \mathrm{d}s - \mathrm{e}^{-r\left(T_j(y_j^f)-t\right)} F_j \right], \tag{3}$$

where $\mathbb{E}_y$ denotes the expectation operator conditional on $Y_t = y$. We assume that $r > \alpha$ to ensure finiteness of the value function.[14]

On the other hand, the value function of firm i as the leader, denoted by V_i^ℓ, is expressed as

$$\mathbb{E}_y \left[\int_{T_i(y_i^\ell)}^{T_j(y_j^f)} \mathrm{e}^{-r(s-t)} Y_s \int_0^1 p_{is}^\ell \mathrm{d}x \mathrm{d}s + \int_{T_j(y_j^f)}^{\infty} \mathrm{e}^{-r(s-t)} Y_s \int_0^{\bar{x}} p_{is}^\ell \mathrm{d}x \mathrm{d}s - \mathrm{e}^{-r\left(T_i(y_i^\ell)-t\right)} F_i \right]. \tag{4}$$

[14] If $r \leq \alpha$, the integral of equation (3) diverges to positive infinity by choosing a larger time T_2, meaning that waiting for a longer time would always be a better strategy, and an optimal entry timing would not exist.

The first term in the right-hand side of (4) describes the discounted cash flow whereas firm i is the monopolist, and the second term is the discounted cash flow after the other firm's entry. More concretely, if firm i is the leader, it earns a monopoly profit flow for $t \in [T_i, T_j)$ and a duopoly profit flow for $t \geq T_j$.

3 Follower's Value Functions

In this section, we derive the optimal price, location, and timing outcomes of the follower firm. As in the literature, we implement backward induction for the derivation of the solutions.

First, given the locations x_1 and x_2, we consider the problem of optimal prices at each time t before and after the entry of firm 2. From (2), we can conclude that the uncertainty of Y does not affect the equilibrium prices. Therefore, we obtain the following lemma, which describes the equilibrium prices of both firms.

Lemma 1. *We assume the leader firm i's location x_i is $x_i \in [0, 1/2]$ without loss of generality. The prices set by the leader firm i and the follower firm $j(\neq i)$ are respectively,*

$$\hat{p}^{\ell}_{it} = \begin{cases} p^M_i(x_i) = \bar{u} - c(1-x_i)^2, & t \in [T_i, T_j), \\ p^{D\ell}_i(x_i, x_j) = \frac{c}{3}(x_j - x_i)(2 + x_i + x_j), & t \in [T_j, \infty), \end{cases} \tag{5}$$

$$\hat{p}^{f}_{jt} = p^{Df}_j(x_i, x_j) = \frac{c}{3}(x_j - x_i)(4 - x_i - x_j), \quad t \in [T_j, \infty). \tag{6}$$

Proof. See the proof of Lemma 1 in Ebina et al. (2015). □

Firm i has monopolistic power over the price at $t \in [T_i, T_j)$ such that all consumers purchase its goods. Then, the optimal price of firm i before the entry of firm j is the price at which the consumer at location 1 is indifferent between purchasing and not purchasing the good. The optimal monopolistic price p^M_i in Lemma 1 satisfies this condition. The optimal prices for the duopoly $p^{D\ell}_i(x_i, x_j)$ and $p^{Df}_j(x_i, x_j)$ are based on the standard calculation in the context of spatial competition (e.g., d'Aspremont et al., 1979).

With the prices $\hat{p}_{it}^{\ell}$ and $\hat{p}_{jt}^{f}$, the instantaneous profit flows of the two firms are expressed as $Y_t\pi_{it}^{\ell}$ and $Y_t\pi_{jt}^{f}$, respectively, where

$$\pi_{it}^{\ell}(x_i,x_j)=\begin{cases}\pi_i^{M\ell}(x_i)=\bar{u}-c(1-x_i)^2, & t\in[T_i,T_j),\\ \pi_i^{D\ell}(x_i,x_j)=\frac{c}{18}(x_j-x_i)(2+x_i+x_j)^2, & t\in[T_j,\infty),\end{cases} \tag{7}$$

$$\pi_{jt}^{f}(x_i,x_j)=\begin{cases}0, & t\in[0,T_j),\\ \pi_j^{Df}(x_i,x_j)=\frac{c}{18}(x_j-x_i)(4-x_i-x_j)^2, & t\in[T_j,\infty),\end{cases} \tag{8}$$

as $\bar{x}=(2+x_1+x_2)/6$ in equilibrium.

3.1 Optimal location and entry threshold

First, we consider the problem of the follower relating to when it enters and where it locates in the market. We assume that firm i is the leader and has already invested and located at $x_i\in[0,1/2]$. We write the value function of the follower as

$$V_j^{f*}(y;x_i)=\max_{\substack{(y_j^f,x_j)\\ y_j^f\geq 0,x_j\in[0,1]}} V_j^f(y;y_j^f,x_i,x_j),$$

where

$$\begin{aligned}V_j^f(y;y_j^f,x_i,x_j)=&\mathbb{E}_y\left[\int_{T_j(y_j^f)}^{\infty}\mathrm{e}^{-r(s-t)}\pi_j^{Df}(x_i,x_j)Y_s\mathrm{d}s-\mathrm{e}^{-r\left(T_j(y_j^f)-t\right)}F_j\right].\\ =&\left(\frac{\pi_j^{Df}(x_i,x_j)y_j^f}{r-\alpha}-F_j\right)\left(\frac{y}{y_j^f}\right)^{\beta}\end{aligned} \tag{9}$$

and

$$\beta=\frac{1}{2}-\frac{\alpha}{\sigma^2}+\sqrt{\left(\frac{1}{2}-\frac{\alpha}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}}.$$

Note that the function V_j^f depends on the other firm's location as the instantaneous profit flow π_j^{Df} depends on x_i. This implies that the pair of optimal strategies (y_j^f,x_j) also depends on x_i. However, with regard to the location of the follower firm j, we have the following lemma.

Lemma 2. *In equilibrium, the follower firm j always locates at $x_j^f = 1$.*

Proof. Since V_j^f is given by (9), the optimal location of the follower maximizes its instantaneous profit π_j^{Df} in (8) given x_i. The lemma immediately follows. $\square$

Once the optimal location is obtained, the optimal timing can easily be derived as in the following lemma. We omit the proof because it is an easy exercise.

Lemma 3. *The value function of firm j as the follower is given by*

$$V_j^{f*}(y; x_i) = \begin{cases} \left(\frac{\bar{y}_j^f(x_i)\pi_j^{Df}(x_i,1)}{r-\alpha} - F_j\right)\left(\frac{y}{\bar{y}_j^f(x_i)}\right)^{\beta}, & \textit{if } y < \bar{y}_j^f(x_i), \\ \frac{y\pi_j^{Df}(x_i,1)}{r-\alpha} - F_j, & \textit{if } y \geq \bar{y}_j^f(x_i), \end{cases}$$

where the investment threshold of the follower is

$$\bar{y}_j^f(x_i) = \frac{\beta}{\beta - 1}\frac{(r-\alpha)}{\pi_j^{Df}(x_i, 1)}F_j. \tag{10}$$

From Lemma 2, the follower firm always locates as far away from the location of the leader as possible while entering the market. This replicates the results of Lambertini (2002) and Ebina et al. (2015) and seems to be robust to the endogeneity of the follower's entry timing under uncertainty.

From (10), we easily obtain the following corollary:

Corollary 1. *If x_1 is increased, the optimal threshold for the follower to enter $\bar{y}_2^f(x_1)$ is increased.*

The corollary shows that the leader can obstruct the follower's entry by choosing its location close to 1/2. On the other hand, setting x_i close to 1/2 can induce a tougher price competition after the follower's entry. The leader firm faces the trade-off between the two with regard to location.

4 Leader's Strategy and Equilibria

In this section, we derive the outcome of the subgame perfect Nash equilibrium (SPNE). We face two difficulties while deriving the equilibria, when compared to the process used in Pawlina and Kort (2006). The first difficulty arises because the value functions of our model are determined by the firms' entry timings as well as location choices, but in the previous study, these functions were determined by only their entry timings. Thus, the location choice in our model entails complexity, making it difficult to derive an analytical solution.

The second difficulty arises from the asymmetry of the firms' profits. Since the firms' location choices may not be symmetric on the Hotelling interval $[0,1]$, the profits $\pi_i^{D\ell}(x_i,1)$ and $\pi_j^{Df}(x_i,1)$ may have different values, leading to a situation where an equilibrium becomes more complicated.

To derive equilibrium, we need to consider the leader's optimization problem given the follower's strategy. Suppose that firm i is a leader and enters the market with location x_i at $T_i(y_i^\ell)$. Given the other firm's strategy $(y_j, x_j) = (\bar{y}_j^f(x_i), 1)$, the present value of the leader firm i is written as

$$\hat{V}_i^\ell(y; y_i^\ell, x_i) = \mathbb{E}_y\left[\mathrm{e}^{-r\left(T_i(y_i^\ell)-t\right)} V_i^\ell(y_i^\ell; \bar{y}_j^f(x_i), x_i)\right]$$

where

$$\begin{aligned} V_i^\ell(y; \bar{y}_j^f(x_i), x_i) =& \mathbb{E}_y\left[\int_t^{T_j(\bar{y}_j^f(x_i))} \mathrm{e}^{-r(s-t)}\pi_i^{M\ell}(x_i) Y_s \mathrm{d}s + \int_{T_j(\bar{y}_j^f(x_i))}^\infty \mathrm{e}^{-r(s-t)}\pi_i^{D\ell}(x_i) Y_s \mathrm{d}s - F_i\right] \\ =& \begin{cases} \frac{y\pi_i^{M\ell}(x_i)}{r-\alpha} - F_i - \frac{\bar{y}_j^f(x_i)\left(\pi_i^{M\ell}(x_i)-\pi_i^{D\ell}(x_i,1)\right)}{r-\alpha}\left(\frac{y}{\bar{y}_j^f(x_i)}\right)^\beta & \text{for } y \le \bar{y}_j^f(x_i), \\ \frac{y\pi_i^{D\ell}(x_i,1)}{r-\alpha} - F_i & \text{for } y > \bar{y}_j^f(x_i). \end{cases} \end{aligned}$$

If firm i is a leader and can choose its location and the entry threshold irrespective of the other's strategy, the value function is expressed as

$$V_i^{\ell*}(y) = \max_{\substack{(y_i^\ell, x_i) \\ y_i^\ell \ge 0, x_i \in [0,1/2]}} \hat{V}_i^\ell(y; y_i^\ell, x_i). \tag{11}$$

We define $(\bar{y}_i^{S\ell}, x_i^{S\ell})$ to be the entry threshold and the location that attain (11). It is worth noting that $V_i^{\ell *}$ does not take into account the preemptive action by the other firm j. In other words, $V_i^{\ell *}$ may not be the value function of firm i in equilibrium even if it is actually a leader.

Now, we investigate the outcome of the SPNE. Following Pawlina and Kort (2006), three types of equilibria can occur: sequential, preemptive, and simultaneous. The leader's location choice and threshold are key to investigate the equilibrium, because the follower's strategy for location and entry timing is dominant in that the follower chooses its location $x_j = 1$ and threshold $\bar{y}_j^f$ after the leader enters the market.

4.1 Sequential equilibrium

Here, we consider the first type of equilibrium, sequential equilibrium.

In the beginning, we discuss sufficient conditions for a sequential equilibrium to occur. One simple and apparent sufficient condition is that

$$\xi_2(y; x_1, x_2) \leq 0$$

for any $y \in (0, \bar{y}_2^f(x_1))$, $x_1 \in [0, 1/2]$, and $x_2 \in [0, 1/2]$, where

$$\xi_2(y; x_1, x_2) \equiv V_2^\ell(y; \bar{y}_1^f(x_2), x_2, 1) - V_2^{f*}(y; x_1).$$

The inequality means that firm 2 has no incentive to be a leader before it decides to enter the market as a follower in any situation.

Another sufficient condition is as follows. Define the lowest level of the state variable y at which firm 2 becomes willing to enter the market as a leader

$$y_2^\xi(x_1, x_2) = \inf\{y \geq 0; \xi_2(y; x_1, x_2) > 0\}$$

for fixed (x_1, x_2).[15] Suppose on the contrary to the above trivial condition that $\xi_2(y; x_1, x_2) > 0$ for some $(y, x_1, x_2) \in (0, \infty) \times [0, 1/2] \times [0, 1/2]$. If $\bar{y}_1^{S\ell} \leq y_2^\xi(x_1, x_2)$ for any $x_1 \in [0, 1/2]$

[15] We set $y_2^\xi = \infty$ if $\xi_2 \leq 0$ for any (x_1, x_2).

and $x_2 \in [0, 1/2]$ satisfying $\xi_2(y; x_1, x_2) > 0$, firm 1 actually enters the market before firm 2 becomes willing to do so as a leader. Then, a sequential equilibrium occurs.

Intuitively, if the cost asymmetry between the two firms is substantial, there is only a sequential equilibrium. In other words, a sequential equilibrium occurs if $\kappa = F_2/F_1$, the magnitude of cost asymmetry, is sufficiently high.[16] Pawlina and Kort (2006) show the necessary and sufficient condition for a sequential equilibrium to occur in their model where two firms only consider the optimal entry timing, not choosing their locations.

Now, we suppose that a sequential equilibrium occurs. Then, (11) is actually the value function of the leader before the time of its entry, and the optimal strategy is given by $(\bar{y}_1^{S\ell}, x_1^{S\ell})$. To derive $(\bar{y}_1^{S\ell}, x_1^{S\ell})$, we calculate

$$\hat{V}_1^{\ell}(y; y_1^{\ell}, x_1) = \left(\frac{\pi_1^{M\ell}(x_1) y_1^{\ell}}{r - \alpha} - \frac{\left[\pi_1^{M\ell}(x_1) - \pi_1^{D\ell}(x_1, 1)\right] \bar{y}_2^{f}(x_1)}{r - \alpha} \left(\frac{y_1^{\ell}}{\bar{y}_2^{f}(x_1)} \right)^{\beta} - F_1 \right) \left(\frac{y}{y_1^{\ell}} \right)^{\beta}. \tag{12}$$

From (12) with (10), the maximization problem of the leader is formulated as

$$\max_{\substack{(y_1^{\ell}, x_1) \\ y^{\ell} \geq 0, x_1 \in [0,1/2]}} \hat{v}_1^{\ell}(y_1^{\ell}, x_1; \beta),$$

where

$$\hat{v}_1^{\ell}(y_1^{\ell}, x_1; \beta) = [\pi_1^{M\ell}(x_1) y_1^{\ell} - (r - \alpha) F_1](y_1^{\ell})^{-\beta} - [\pi_1^{M\ell}(x_1) - \pi_1^{D\ell}(x_1, 1)] \left(\frac{r - \alpha}{\pi_2^{Df}(x_1)} F_2 \right)^{1-\beta}.$$

Thus, we have the first-order condition with respect to y_1^{ℓ} to obtain

$$\bar{y}_1^{S\ell}(x_1) = \frac{\beta}{\beta - 1} \frac{r - \alpha}{\pi_1^{M\ell}(x_1)} F_1. \tag{13}$$

Now, we are ready to show the following lemma regarding $x_1^{S\ell}$.

Lemma 4. *There are constants $\underline{\beta}$ and $\overline{\beta}$ such that*

(i) $x_1^{S\ell} = 1/2$ *for* $\beta > \overline{\beta}$,

[16] We numerically show in Section 5.3 that this intuition is correct, in the sense that a type of equilibrium switches from a preemptive equilibrium to a sequential one as κ increases.

(ii) $x_1^{S\ell} = 0$ *for* $\beta < \underline{\beta}$.

Proof. After differentiating $\hat{v}_1^\ell$ with respect to x_1, we substitute (13) to obtain

$$
\begin{aligned}
\frac{\partial \hat{v}_1^\ell}{\partial x_1} =& \pi_1^{M\ell\prime}(x_1)(y_1^\ell)^{1-\beta} - [\pi_1^{M\ell\prime}(x_1) - \pi_1^{D\ell\prime}(x_1,1)] \left(\frac{r-\alpha}{\pi_2^{Df}(x_1)}F_2\right)^{1-\beta} \\
& - (1-\beta)[\pi_1^{M\ell}(x_1) - \pi_1^{D\ell}(x_1,1)] \left(\frac{r-\alpha}{\pi_2^{Df}(x_1)}F_2\right)^{-\beta} \frac{(r-\alpha)\pi_2^{Df\prime}(x_1)}{\pi_2^{Df}(x_1)^2}F_2 \\
=& \left(\frac{\beta}{\beta-1}\right)^{1-\beta} \left[\left(\frac{r-\alpha}{\pi_1^{M\ell}(x_1)}F_1\right)^{1-\beta} \pi_1^{M\ell\prime}(x_1) - \left(\frac{r-\alpha}{\pi_2^{Df}(x_1,1)}F_2\right)^{1-\beta} \right. \\
& \left. \times \left([\pi_1^{M\ell\prime}(x_1) - \pi_1^{D\ell\prime}(x_1,1)] + (\beta-1)[\pi_1^{M\ell}(x_1) - \pi_1^{D\ell}(x_1,1)] \frac{\pi_2^{Df\prime}(x_1,1)}{\pi_2^{Df}(x_1,1)} \right) \right].
\end{aligned}
\tag{14}
$$

If $\beta \to 1$, the right-hand side of (14) converges to $\pi_1^{D\ell\prime}(x_1,1)$, which is always negative. On the other hand, consider the ratio of the first and the second terms in the square bracket of (14):

$$
\begin{aligned}
& \frac{\left(\frac{r-\alpha}{\pi_1^{M\ell}(x_1)}F_1\right)^{1-\beta} \pi_1^{M\ell\prime}(x_1)}{\left(\frac{r-\alpha}{\pi_2^{Df}(x_1,1)}F_2\right)^{1-\beta} \left([\pi_1^{M\ell\prime}(x_1) - \pi_1^{D\ell\prime}(x_1,1)] + (\beta-1)[\pi_1^{M\ell}(x_1) - \pi_1^{D\ell}(x_1,1)] \frac{\pi_2^{Df\prime}(x_1,1)}{\pi_2^{Df}(x_1,1)} \right)} \\
=& \frac{\left(\frac{F_2/\pi_2^{Df}(x_1,1)}{F_1/\pi_1^{M\ell}(x_1)}\right)^{\beta-1}}{(\beta-1)\frac{\pi_1^{M\ell}(x_1)-\pi_1^{D\ell}(x_1,1)}{\pi_1^{M\ell\prime}(x_1)} \frac{\pi_2^{Df\prime}(x_1,1)}{\pi_2^{Df}(x_1,1)} + 1 - \frac{\pi_1^{D\ell\prime}(x_1,1)}{\pi_1^{M\ell\prime}(x_1)}}.
\end{aligned}
$$

It follows from L'Hôpital's Theorem that the above equation diverges if $\beta \to \infty$ since $F_1/\pi_1^{M\ell}(x_1) < F_2/\pi_2^{Df}(x_1,1)$. Therefore, we confirm that (14) is positive for any x_1 if β is sufficiently large. Now the lemma follows from noting that all functions to be examined are continuous. □

Note that the volatility σ affects only through the parameter β and that $\partial\beta/\partial\sigma < 0$. The above proposition implies that in a sequential equilibrium, the location of the leader is the center, if the volatility of the market size is sufficiently small. On the other hand, if the uncertainty over the future size of the market is sufficiently large, the optimal location of the leader in a sequential equilibrium moves closer to the edge. The observation is presented in the following proposition.

Proposition 1. *There exist constants $\underline{\sigma}$ and $\overline{\sigma}$ such that*

(a) $x_1^{S\ell} = 1/2$ *for* $\sigma < \underline{\sigma}$,

(b) $x_1^{S\ell} = 0$ *for* $\sigma > \overline{\sigma}$.

As we already showed in Lemma 2, the follower chooses its location at $x_2^f = 1$ to minimize the price competition between the two firms. Given this fact, there are three effects concerning a decision on the location by the leader. First, when the cash flows before the follower enters are considered, it is optimal for the leader to be located at 1/2. This maximizes the leader's monopolistic profit by charging a higher price while still attracting all consumers. Second, being located at 1/2 deters the future entry by the follower, since the price competition becomes severe and causes a delay of the follower's entry. Third, when the cash flows after the follower enters are considered, being located at 1/2 is less attractive for the leader firm because it maximizes price competition in a duopolistic situation.

It is a well known result in the real options literature that when uncertainty is large, a value of waiting with entry seems to be high, leading to the follower's late entry. Then, the leader's monopoly period being likely to last long, the first effect should dominate the third effect and the leader should choose $x_1^\ell = 1/2$. However, Proposition 1 is inconsistent with the above intuition. How can this be understood?

We learn the above interpretation through mathematical analysis. First, we rewrite (12) at $y = y_1^\ell$ as

$$\hat{V}_1^\ell(y_1^\ell; y_1^\ell, x_1) = \frac{\pi_1^{M\ell}(x_1) y_1^\ell}{r-\alpha} \left[1 - \left(\frac{y_1^\ell}{\bar{y}_2^f(x_1)} \right)^{\beta-1} \right] + \frac{\pi_1^{D\ell}(x_1) y_1^\ell}{r-\alpha} \left(\frac{y_1^\ell}{\bar{y}_2^f(x_1)} \right)^{\beta-1} - F_1. \tag{15}$$

Equation (15) shows that the value function of the leader is the weighted average of the monopolistic and duopolistic cash flows with weight $(y_1^\ell/\bar{y}_2^f(x_1))^{\beta-1}$. It should also be noted that

$$\left(\frac{y_1^\ell}{\bar{y}_2^f}\right)^\beta = \mathbb{E}_y\left[\mathrm{e}^{-r\left(T_2^f(\bar{y}_2^f)-T_1^\ell(y_1^\ell)\right)}\right].$$

Therefore, the weight is closely related to the discount factor for the period $T_2^f - T_1^\ell$ with rate r. Thus, the change in the volatility σ has an impact on

(i) the monopolistic profits represented by $\frac{\pi_1^{M\ell}(x_1)y_1^\ell}{r-\alpha}$,

(ii) the follower's entry timing represented by $\left(\frac{y_1^\ell}{\bar{y}_2^f(x_1)}\right)^{\beta-1}$,

(iii) the duopolistic profits represented by $\frac{\pi_1^{D\ell}(x_1)y_1^\ell}{r-\alpha}$.

We can present some intuitive explanation on the negative relationship between σ and $x_1^{S\ell}$ in the following manner. Suppose that σ becomes large. For fixed y_1^ℓ and y_2^f, the weight $(y_1^\ell/y_2^f)^{\beta-1}$ in (15) becomes large and close to 1, since $y_1^\ell/y_2^f < 1$ and β is decreasing in σ. This implies that the leader should take this into more account for the future situation where the two firms face fiercer price competition. To put it differently, if the effect of x_1 on the monopolistic cash flow is rather small, then the leader firm should give more importance to the future cash flow, causing its location to be close to 0.

From our result, we can say that a firm should consider uncertain future cash flows rather than the current ones in the case of higher uncertainty. Our result in Proposition 1 is consistent with the results of the extant papers in this sense, and contributes to the literature by presenting the impact of volatility on the firms' positioning strategies in an effective manner.

In the literature on standard real options, most studies consider the case of the optimal entry timing alone. Then, σ positively affects the value function of the leader alone through the second effect. However, under our setting, where each firm chooses the timing as well as location, the problem is not simple, with results differing from those in the standard model. In fact, Section 5 presents other novel results due to the multi-dimensionality of strategies.

Finally, by substituting $x_1^{S\ell}$ for each case in Proposition 1, we have the following proposition stating the outcomes of SPNE.

Proposition 2. *Let $(y_1^{E\ell}, y_2^{Ef}, x_1^{E\ell}, x_2^{Ef}, p_1^{E\ell}, p_2^{Ef})$ be the equilibrium solution of the locations, thresholds, and prices. Then, we have the following three cases.*

(a) If Equation (14) is positive for any $x_1 \in [0, 1/2]$, the outcome of the subgame perfect equilibrium is expressed as $(y_1^{E\ell}, y_2^{Ef}, x_1^{E\ell}, x_2^{Ef}, p_1^{E\ell}, p_2^{Ef}) = (y_1^*, y_2^*, x_1^*, x_2^*, p_1^*, p_2^*)$, where

$$\begin{aligned}
y_1^* &\equiv \frac{\beta}{\beta - 1}\frac{4(r-\alpha)F}{4\bar{u} - c}, \; y_2^* \equiv \frac{\beta}{\beta - 1}\frac{144(r-\alpha)\kappa F}{25c}, \\
x_1^* &\equiv \frac{1}{2}, \; x_2^* \equiv 1, \\
p_{1t}^* &\equiv \begin{cases} p_1^M(x_1^*) = \bar{u} - \frac{c}{4} & \text{for } t \in [T_1(y_1^*), T_2(y_2^*)) \\ p_1^D(x_1^*, x_2^*) = \frac{7c}{12} & \text{for } t \in [T_2(y_2^*), \infty), \end{cases} \\
p_{2t}^* &\equiv p_2^D(x_1^*, x_2^*) = \frac{5c}{12} \quad \text{for } t \in [T_2(y_2^*), \infty).
\end{aligned}$$

The consumer at $\bar{x}^* \equiv \frac{7}{12}$ is indifferent between purchasing the good from firm 1 and purchasing it from firm 2.

(b) If Equation (14) is negative for any $x_1 \in [0, 1/2]$, the outcome of the subgame perfect equilibrium is expressed as $(y_1^{E\ell}, y_2^{Ef}, x_1^{E\ell}, x_2^{Ef}, p_1^{E\ell}, p_2^{Ef}) = (y_1^{**}, y_2^{**}, x_1^{**}, x_2^{**}, p_1^{**}, p_2^{**})$, where

$$\begin{aligned}
y_1^{**} &\equiv \frac{\beta}{\beta - 1}\frac{(r-\alpha)F}{\bar{u} - c}, \; y_2^{**} \equiv \frac{\beta}{\beta - 1}\frac{2(r-\alpha)\kappa F}{c}, \\
x_1^{**} &\equiv 0, \; x_2^{**} \equiv 1, \\
p_{1t}^{**} &\equiv \begin{cases} p_1^M(x_1^{**}) = \bar{u} - c & \text{for } t \in [T_1(x_1^{**}), T_2(x_2^{**})) \\ p_1^D(x_1^{**}, x_2^{**}) = c & \text{for } t \in [T_2(x_2^{**}), \infty), \end{cases} \\
p_{2t}^{**} &\equiv p_2^D(x_1^{**}, x_2^{**}) = c \quad \text{for } t \in [T_2(x_2^{**}), \infty).
\end{aligned}$$

The consumer at $\bar{x}^{**} = \frac{1}{2}$ is indifferent between purchasing the good from firm 1 and purchasing it from firm 2.

Remark 1. *We cannot analytically prove the presence of an inner solution*

$$(y_1^{E\ell}, y_2^{Ef}, x_1^{E\ell}, x_2^{Ef}, p_1^{E\ell}, p_2^{Ef}) = (y_1^{***}, y_2^{***}, x_1^{***}, x_2^{***}, p_1^{***}, p_2^{***})$$

with $x_1^{***} \in (0, 1/2)$ when $\underline{\beta} < \beta < \overline{\beta}$. Fortunately, from our numerical analysis, there is an inner solution under a wide range of parameter settings if β is intermediate and $\bar{u}$ is large enough.

Proposition 2 with Remark 1 shows three cases for the leader's location in our sequential equilibrium. Which case occurs depends on the parameters describing the dynamics of the market size α, σ, and so on. To grasp the intuition behind the two propositions, we proceed with a numerical analysis in Section 5.

4.2 Preemptive equilibrium

Second, we consider another type of equilibrium, namely, preemptive equilibrium. Pawlina and Kort (2006) define a preemptive equilibrium as the situation in which firm 2, which is disadvantaged in an investment cost, has an incentive to become the leader. In this case, firm 1 needs to note that firm 2 would enter the market before the state variable Y reaches the optimal threshold for firm 1.

Assume that $y_2^{\xi}(x_1^{S\ell}, x_2) < \bar{y}_1^{S\ell}$ for some $x_2 \in [0, 1/2]$. In this case, $\xi_2(y; x_1^{S\ell}, x_2) > 0$ for some $y \in (y_2^{\xi}(x_1^{S\ell}, x_2), y_1^{S\ell})$, meaning that firm 2 has an incentive to become a leader in the interval. Hence, firm 1 should take the following two actions: (i) lowering the entry threshold y_1^{ℓ} to deter the other firm's entry, and (ii) changing its location x_1 to fit with the adjusted entry threshold y_1. If firm 1 chooses y_1^{ℓ} for the entry threshold as a leader, the location should be

$$\tilde{x}_1(y_1^{\ell}) = \operatorname*{argmax}_{x_1 \in [0,1/2]} V_1^{\ell}(y_1^{\ell}; \bar{y}_2^{f}(x_1), x_1). \tag{16}$$

Let $\bar{y}_1^{P\ell}$ and $x_1^{P\ell}$ be the entry threshold and the location of firm 1 in a preemptive equilibrium. With $\bar{y}_1^{P\ell}$ and $x_1^{P\ell}$, the other firm has no incentive to be a leader for $y < \bar{y}_1^{P\ell}$. In other words,

$\bar{y}_1^{P\ell}$ and $x_1^{P\ell}$ satisfy

$$\bar{y}_1^{P\ell} = y_2^{\xi}(x_1^{P\ell}, x_2^{\xi}(x_1^{P\ell})) \quad \text{and} \quad x_1^{P\ell} = \tilde{x}_1(\bar{y}_1^{P\ell}),$$

where $x_2^{\xi}(x_1)$ is defined by the equation

$$\xi_2(y_2^{\xi}(x_1, x_2^{\xi}); x_1, x_2^{\xi}) = 0.$$

From the above observation, a preemptive equilibrium is rather complicated and seems hard to solve, analytically. More concretely, $x_1^{P\ell}$ is the solution of the non-linear equations $x = \tilde{x}_1(y_2^{\xi}(x, x_2^{\xi}(x)))$ and the function $\tilde{x}_1(\cdot)$ includes the maximization problem (16). The difficulty in our case comes from the fact that the payoff of a Hotelling-type model is a cubic function with respect to the state variable.[17]

In the next section, we numerically calculate the optimal strategies, if an equilibrium is of a preemptive-type, and conduct comparative statics to examine how the volatility or other parameters affect both firms' entry thresholds and locations. Note for the numerical calculation that the leader's equilibrium location $x_1^{P\ell}$ is derived from the fact that

$$x_1^{P\ell} = \sup \mathcal{X}_1^{P\ell},$$

where

$$\mathcal{X}_1^{P\ell} = \{0\} \bigcup \left\{ x_1 \in (0, 1/2]; \frac{\partial}{\partial x_1} V_1^{\ell}(y_2^{\xi}(x_1, x_2^{\xi}); \bar{y}_2^{f}(x_1), x_1) > 0 \right\}. \tag{17}$$

4.3 Simultaneous equilibrium

Finally, let us consider a simultaneous equilibrium and show that it cannot occur.

Lemma 5. *A simultaneous equilibrium cannot occur as an outcome of SPNE.*

[17] On the contrary, Huisman and Kort (2015) assume a quadratic payoff function, which allows them to explicitly solve a preemptive equilibrium.

Proof. Let $\hat{V}_i^T$ be the value function of firm i in the case of a simultaneous equilibrium. In a simultaneous equilibrium, the optimal locations are apparently given by $(x_1, x_2) = (0, 1)$. Therefore, if y^T is the entry threshold of both firms, we have

$$\hat{V}_i^T(y; y^T) = \begin{cases} \left(\frac{\pi^T y^T}{r-\alpha} - F_i\right)\left(\frac{y}{y^T}\right)^\beta, & y < y^T, \\ \frac{\pi^T y}{r-\alpha} - F_i, & y \geq y^T, \end{cases}$$

where

$$\pi^T = \frac{c}{2}.$$

Define

$$\begin{aligned} \zeta_1(y, x) =& \hat{V}_1^\ell(y; y^T, x) - \hat{V}_1^T(y; y^T), \\ \zeta_2(y) =& V_2^{f*}(y) - \hat{V}_2^T(y; y^T). \end{aligned}$$

A necessary condition that a simultaneous equilibrium happens is

(i) $\zeta_1 \leq 0$ for all $y \in [\bar{y}_1^{S\ell}, \bar{y}_2^f(0)]$,

and

(ii) $\zeta_2 \leq 0$ for all $y \in [y^T \wedge \bar{y}_2^f(0), y^T \vee \bar{y}_2^f(0)]$.

First, we suppose that $x_1^{S\ell} > 0$. Since $\bar{y}_2^f(0) < \bar{y}_2^f(x_1^{S\ell})$, we have

$$V_2^{T*}(y; y^T) \leq \hat{V}_2^T(y; \bar{y}_2^f(0)) = V_2^f(y; \bar{y}_2^f(0), 0, 1) < V_2^{f*}(y) = V_2^f(y; \bar{y}_2^f(x^{S\ell}), x_1^{S\ell}, 1)$$

for $y < y_2^f(0)$, implying that a sequential equilibrium dominates a simultaneous equilibrium for firm 2.

Suppose next that $x_1^{S\ell} = 0$. Since $(x_1, x_2) = (0, 1)$ for both sequential and simultaneous equilibria, firm 2 always enters the market at $y = \bar{y}_2^f(0)$ irrespective of the type of equilibrium. However,

$$\zeta_1(y; x_1^{S\ell}) = \frac{y\pi_1^{M\ell}(0)}{r-\alpha} - F_1 - \frac{\bar{y}_2^f(0)\left(\pi_1^{M\ell}(0) - \pi_1^{D\ell}(0,1)\right)}{r-\alpha}\left(\frac{y}{\bar{y}_2^f(0)}\right)^\beta$$

$$
\begin{aligned}
&-\left(\frac{\pi_1^{D\ell}(0,1)\bar{y}_2^f(0)}{r-\alpha}-F_1\right)\left(\frac{y}{\bar{y}_2^f(0)}\right)^\beta \\
=&\frac{y\pi_1^{M\ell}(0)}{r-\alpha}-F_1-\left(\frac{\bar{y}_2^f(0)\pi_1^{M\ell}(0)}{r-\alpha}-F_1\right)\left(\frac{y}{\bar{y}_2^f(0)}\right)^\beta.
\end{aligned}
$$

We now show that $\zeta_1(y;x_1^{S\ell})>0$ for some $y<\bar{y}_2^f(0)$. Differentiating $\zeta_1(y;x_1^{S\ell})$ with respect to y, we have

$$
\zeta_1'(y;x_1^{S\ell})=\frac{\pi_1^{M\ell}(0)}{r-\alpha}-\beta\left(\frac{\bar{y}_2^f(0)\pi_1^{M\ell}(0)}{r-\alpha}-F_1\right)\frac{y^{\beta-1}}{\bar{y}_2^f(0)^\beta}.
$$

Therefore,

$$
\zeta_1'(0;x_1^{S\ell})=\pi_1^{M\ell}(0)/(r-\alpha)>0
$$

and

$$
\zeta_1'(\bar{y}_2^f(0);x_1^{S\ell})=\frac{\beta-1}{r-\alpha}\left\{-\pi_1^{M\ell}(0)+\frac{F_1}{F_2}\pi_2^{Df}(0,1)\right\}<0.
$$

Since $\zeta_1(0;x_1^{S\ell})=-F_1<0$ and $\zeta_1(\bar{y}_2^f(0);x_1^{S\ell})=0$, we have $\zeta_1>0$ for y close to $\bar{y}_2^f(0)$. This implies that a sequential equilibrium dominates a simultaneous equilibrium for firm 1 in the case $x_1^{S\ell}=0$.

We can argue for the case of $x_1^{P\ell}$, in a manner similar to $x_1^{S\ell}$, which completes the proof □

Intuitively, when there is no cash flow in advance of the entry, the two firms have no incentive to coordinate under cost asymmetry, and hence, a simultaneous equilibrium cannot occur.

5 Numerical Analysis

In this section, we investigate the underlying properties of our model using numerical analysis in depth.

5.1 The effect of volatility

This subsection numerically examines how the volatility σ affects the leader's location $x_1^{E\ell}$. As we have already seen, the type of equilibrium is mainly determined by the cost asymmetry. That is, a sequential equilibrium occurs if the cost asymmetry is significant, and a preemptive equilibrium occurs otherwise. Therefore, we analyze with two cases, in one of which case the value of κ is large, and in the other of which it is small. Other parameter values are given in Table 1.

$\bar{u}$	c	r	α	F
30	1	0.1	0.099	100

Table 1: Base case parameter values.

Case 1 ($\kappa = 10$): We consider the situation where the cost asymmetry is large. It can be checked using numerical calculations that a sequential equilibrium in fact occurs irrespective of the volatility σ.

The outcomes of SPNE (the locations, entry thresholds, and prices of both firms) are presented in Table 2 for $\sigma \in \{0.0001, 0.1, 0.2 \ldots, 1\}$, confirming the analytical result in Proposition 2. That is, $x_1^{E\ell} = 1/2$ for a small σ and $x_1^{E\ell} = 0$ for a large σ. Further, $x_1^{E\ell}$ is decreasing in σ and is between 0 and 1/2 when σ takes an intermediate value, again confirming the existence of an inner solution.[18]

Several important observations are obtained from the numerical results in Table 2. First, we briefly discuss how the volatility σ influences the thresholds $x_1^{E\ell}$ and x_2^{Ef}. Table 2 shows that the locations of the two firms $x_1^{E\ell}$ is decreasing in σ, whereas $x_2^{Ef} = 1$ always holds. Thus, σ affects only the equilibrium location of firm 1. Numerical calculations confirm the validity of Lemma 2 and Proposition 1.

[18] We conclude with numerical calculations for a wide variety of parameter settings that $x_1^{E\ell}$ is non-increasing in σ, even for an inner solution.

σ	$x_1^{E\ell}$	x_2^{Ef}	$\bar{x}^E$	$y_1^{E\ell}$	y_2^{Ef}	$p_1^{M\ell}$	$p_1^{D\ell}$	p_2^{Df}
0.0001	0.5	1	0.583	0.336	576	29.75	0.583	0.417
0.1	0.5	1	0.583	0.353	605	29.75	0.583	0.417
0.2	0.5	1	0.583	0.404	692	29.75	0.583	0.417
0.3	0.5	1	0.583	0.488	837	29.75	0.583	0.417
0.4	0.5	1	0.583	0.607	1039	29.75	0.583	0.417
0.5	0.241	1	0.540	0.767	703	29.42	0.820	0.698
0.6	0.108	1	0.518	0.961	677	29.20	0.924	0.860
0.7	0.018	1	0.503	1.191	712	29.04	0.988	0.976
0.8	0	1	0.5	1.451	842	29	1	1
0.9	0	1	0.5	1.744	1012	29	1	1
1	0	1	0.5	2.072	1202	29	1	1

Table 2: The outcome of the subgame perfect equilibrium in Case 1 with $\kappa = 10$. In this case, a sequential equilibrium occurs.

Second, we discuss the relationship between $y_1^{E\ell}$ or y_2^{Ef} and σ. To investigate the effect of σ, the following decomposition is useful:

$$\frac{\mathrm{d}y_i^{Ek}}{\mathrm{d}\sigma} = \frac{\mathrm{d}y_i^{Ek}(\beta(\sigma), x_1^{E\ell}(\sigma))}{\mathrm{d}\sigma} = \frac{\partial y_i^{Ek}}{\partial \beta}\frac{\partial \beta}{\partial \sigma} + \frac{\partial y_i^{Ek}}{\partial x_1}\frac{\partial x_1^{E\ell}}{\partial \sigma}, \quad (i,k) = (1,\ell),\ (2,f). \tag{18}$$

We call the first term of (18) the option effect; this is induced through the parameter β. The option effect always appears in a standard real options model, and is monotonically increasing in the volatility σ. The second term of (18), called the location effect, is the effect induced through the leader's location $x_1^{E\ell}$. The location effect is new in the literature and should be considered in the model where the strategy of each firm is multi-dimensional.

The effect of σ on the threshold of the leader firm, $y_1^{E\ell}$ can be explained as follows. Note again that the first term of (18) is strictly positive. For the second term, we easily verify that $\partial y_1^{E\ell}/\partial x_1$ and $\partial x_1^{E\ell}/\partial \sigma$ are both non-positive. In other words, the option and location effects are both positive and the leader's threshold is always increasing with respect to volatility.

However, the story is different for the effect of σ on y_2^{Ef}. Although the option effect is still positive for y_2^{Ef}, the location effect, or the effect of σ via the leader's location, can be negative, because $\partial y_2^{Ef}/\partial x_1 > 0$ and $\partial x_1^{E\ell}/\partial \sigma < 0$ when $x_1^{E\ell}$ is on $(0, 1/2)$. In other words, the option effect is positive but the location effect can be negative for the follower's threshold if the equilibrium has an inner solution. Further, the total effect (18) is negative if the location effect dominates the option effect. Our numerical result shows that this can happen with our parameter setting (see y_2^{Ef} when $\sigma = 0.4, 0.5$, and 0.6 in Table 2).

Now, let us explain why σ has a negative effect on y_2^{Ef} if it takes an intermediate value. If $x_1^{E\ell} = 0$ or $x_1^{E\ell} = 1/2$, the follower knows that the leader would not change its location $x_1^{E\ell}$ for a marginal change in σ and thus considers only the option effect, leading to σ having a positive effect on y_2^{Ef}. On the other hand, if $x_1^{E\ell}$ is between 0 and 1/2, the follower actually takes the marginal change of $x_1^{E\ell}$ that affects the present value of its own future profit flows. Since the leader chooses a location closer to 0, the present value of the follower becomes higher and the follower can enter the market more easily. Hence, the follower optimally chooses a lower y_2^{Ef}.

Mason and Weeds (2010) show that the threshold of a leader is non-monotonic if there is a positive externality. However, to the best of the authors' knowledge, the negative effect of the volatility on a threshold for some parameter settings is new in the real options literature under the assumption of a negative externality. Our model setup in which each firm optimizes the threshold and location, confirms the importance of the analysis of multi-dimensional strategies.

Case 2 ($\kappa = 1.1$): We consider the situation where the cost asymmetry is small. It can be checked using numerical calculations that a preemptive equilibrium in fact occurs irrespective of the volatility σ.

The outcomes of SPNE (the locations, entry thresholds, and prices of both firms) in a preemptive equilibrium are presented in Table 2 for $\sigma \in \{0.0001, 0.1, 0.2 \ldots, 1\}$.

A similar observation to Case 1 is obtained from the above results. First, the location of the leader $x_1^{E\ell}$ is monotonically decreasing in the volatility σ, meaning that the product differentiation widens as the demand uncertainty becomes larger. Second, the effect of σ on

σ	$x_1^{E\ell}$	x_2^{Ef}	$\bar{x}^E$	$y_1^{E\ell}$	y_2^{Ef}	$p_1^{M\ell}$	$p_1^{D\ell}$	p_2^{Df}
0.0001	0.5	1	0.583	0.050	63	29.75	0.583	0.417
0.1	0.5	1	0.583	0.052	67	29.75	0.583	0.417
0.2	0.5	1	0.583	0.059	76	29.75	0.583	0.417
0.3	0.5	1	0.583	0.069	92	29.75	0.583	0.417
0.4	0.5	1	0.583	0.083	114	29.75	0.583	0.417
0.5	0.236	1	0.539	0.117	77	29.42	0.824	0.704
0.6	0.102	1	0.517	0.161	74	29.19	0.929	0.867
0.7	0.011	1	0.502	0.225	77	29.02	0.993	0.985
0.8	0	1	0.5	0.280	93	29	1	1
0.9	0	1	0.5	0.336	111	29	1	1
1	0	1	0.5	0.399	132	29	1	1

Table 3: The outcome of the subgame perfect equilibrium in Case 2 with $\kappa = 1.1$. In this case, a preemptive equilibrium occurs.

the follower's entry threshold y_2^{Ef} is not monotonic (See y_2^{Ef} when $\sigma = 0.4, 0.5, 0.6$ in Table 3). The effect is negative if σ takes an immediate value, or in a situation where an equilibrium has an inner solution $x_1^{E\ell} \in (0, 1/2)$.

As we have already explained, the leader firm should give more importance to the future situation, if uncertainty of the cash flow is higher. Thus, the leader takes the duopoly market into more account with a higher σ, and $x_1^{E\ell}$ is non-increasing as a result. We confirm the analytical result in Proposition 1 with numerical examples.

5.2 The effect of growth rate

In this subsection, we investigate the effect of α on the leader's location, by conducting a comparative statics with numerical calculations. Here, we study Case 1 ($\kappa = 10$), meaning that a sequential equilibrium occurs.[19]

Figure 1 illustrates how the parameters σ and α affect the equilibrium location of the leader. We observe from the figure that the leader's location is decreasing in α. In other words, the leader optimally locates itself at the center if the market grows slowly in average.

[19] We omit the analysis in Case 2, because similar observations are obtained.

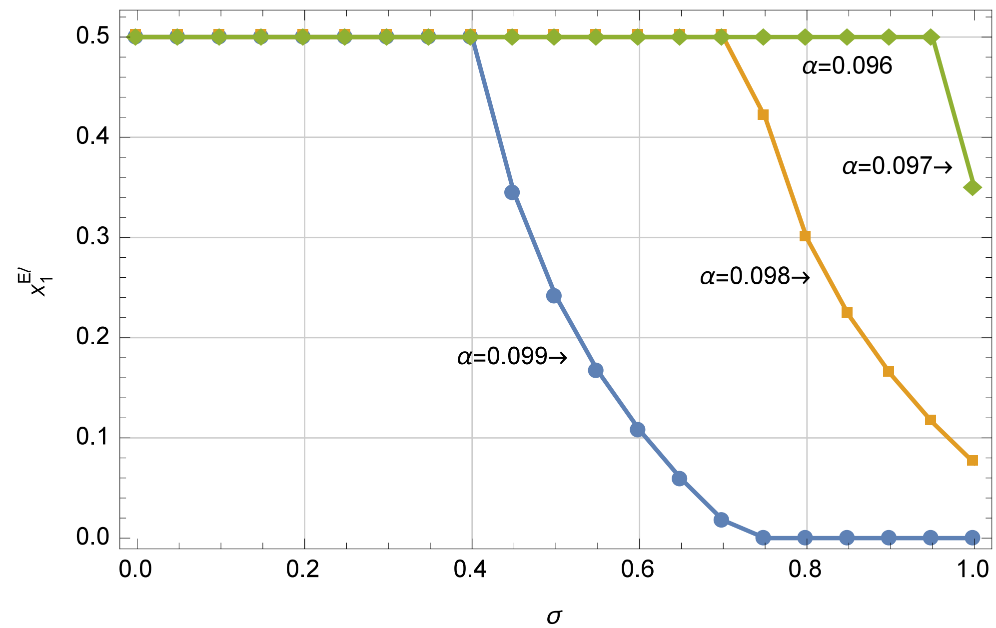

Figure 1: The relationship between $x_1^{E\ell}$ and α with different values of σ.

Especially, $x_1^{E\ell} = 1/2$ ways holds when $\alpha = 0.096$; this is depicted at the upper left-hand side. When $\alpha = 0.099$ the location of the leader depends on the volatility σ; $x_1^{E\ell} = 0.5$ if $\sigma \leq 0.4$ and $x_1^{E\ell} = 0$ if $\sigma \geq 0.75$. Ebina et al. (2015) obtain a similar result with their numerical analysis on pages 908–910.

An intuitive explanation on the above observation can be presented as follows. A large value of α means the rapid market size growth on average, inducing the follower to enter the market earlier. Then, the leader should avoid price competition with the follower. Thus, the leader firm optimally chooses $x_1^{E\ell} = 0$.

5.3 The effect of cost asymmetry

Next, we consider how the degree of asymmetry κ affects the equilibrium outcome. Table 4 shows the effect of κ on the outcome of SPNE when $\sigma = 0.5$.

First, we can see from Table 4 that a preemptive equilibrium occurs for a smaller κ, and a sequential equilibrium for a larger κ. The reason is almost self-evident. If the asymmetry

κ	$x_1^{E\ell}$	x_2^{Ef}	$y_1^{E\ell}$	y_2^{Ef}	Type of Equilibrium
1.1	0.2359	1	0.1166	77	Preemptive
2	0.2341	1	0.2338	99	Preemptive
3	0.2329	1	0.3779	149	Preemptive
4	0.2319	1	0.5336	198	Preemptive
5	0.2311	1	0.6994	248	Preemptive
6	0.2326	1	0.7669	415	Sequential
7	0.2353	1	0.7668	487	Sequential
8	0.2376	1	0.7667	558	Sequential
9	0.2396	1	0.7667	631	Sequential

Table 4: The outcome of the subgame perfect equilibrium with different values of κ. The type of equilibrium changes at κ between 5 and 6.

in costs becomes large, the entry of the follower firm is significantly late in comparison to the leader's entry. Thus, the leader does not need to take the follower's entry timing into consideration and can optimally choose when to enter.

Second, we observe that the leader firm locates itself farther from the center as the cost asymmetry becomes larger in the case of a preemptive equilibrium. On the other hand, if a sequential equilibrium occurs, the leader's location is closer to the center. The above observations imply that the effect of cost asymmetry differs, depending on the type of equilibrium.

The intuition for the effect of κ is as follows. It should be noted that the leader's locating itself closer to the center has two effects on the follower's strategy. The first one is positive for the leader in that it enjoys higher monopolistic profits and deters the follower's entry. The second one is negative in that it lowers its profits in a duopolistic market. In a preemptive equilibrium where a disadvantaged firm has an incentive to become a leader, the leader should deter the other firm's entry by entering the market earlier. The leader's earlier entry to preempt signifies the importance of the discounted sum of instantaneous monopoly profits. To increase the discounted sum, the leader needs to locate closer to the center when preemption occurs. If κ increases from a lower level (e.g., $\kappa = 1.1$ in Table 4), the leader's entry timing gets delayed (see $y_1^{E\ell}$ in Table 4), implying that the discounted sum of instantaneous monopoly profits becomes less important and $x_1^{E\ell}$ decreases. On the other hand, in a sequential equilibrium where κ is large, the leader firm is not forced to enter earlier and sets its threshold satisfactorily. In fact, κ hardly affects the leader's threshold in each sequential equilibrium in Table 4. On the contrary, a higher value of κ means a higher entry cost F_2 in this analysis, leading to a higher entry threshold of the follower. Thus, if the cost asymmetry is large, the leader firm can enjoy a longer monopoly cash flow, and sets its location to be closer to the center.

5.4 The effect of transportation cost

Finally, we investigate the effects of the transportation cost c on the leader's location. In our setting, an agent who is located far from the firm should pay high transportation costs. Therefore, given the fixed length of the Hotelling line, we can interpret that widening the market area corresponds to the increase in the transportation cost parameter c.

Figure 2 illustrates how the parameter c affects the equilibrium location of the leader. Again, we study Case 1 with $\kappa = 10$ and $\sigma = 0.3$, where a sequential equilibrium occurs. One can confirm that the equilibrium location of the leader is decreasing in c, implying that the distance between the two firms is increasing as market areas are widened.

Our theoretical result is closely related to the empirical finding presented by Thomadsen (2007), who computationally investigates the product positioning (location) strategies of two asymmetric firms in the fast food industry, McDonald's and Burger King, in the United States. In his study, McDonald's is advantaged in cost and quality and can firstly choose its positioning (leader in our study), whereas Burger King is disadvantaged and secondly chooses it (follower in our study). The major findings by Thomadsen (2007) are that (1) McDonald's and Burger King should pursue different location strategies, which means that the minimum differentiation never occurs, and that (2) under some conditions, McDonald's prefers locating at or near the center, whereas Burger King tries to locate far away from McDonald's to attract different types of customers. In addition, the prices set by McDonald's and Burger King are

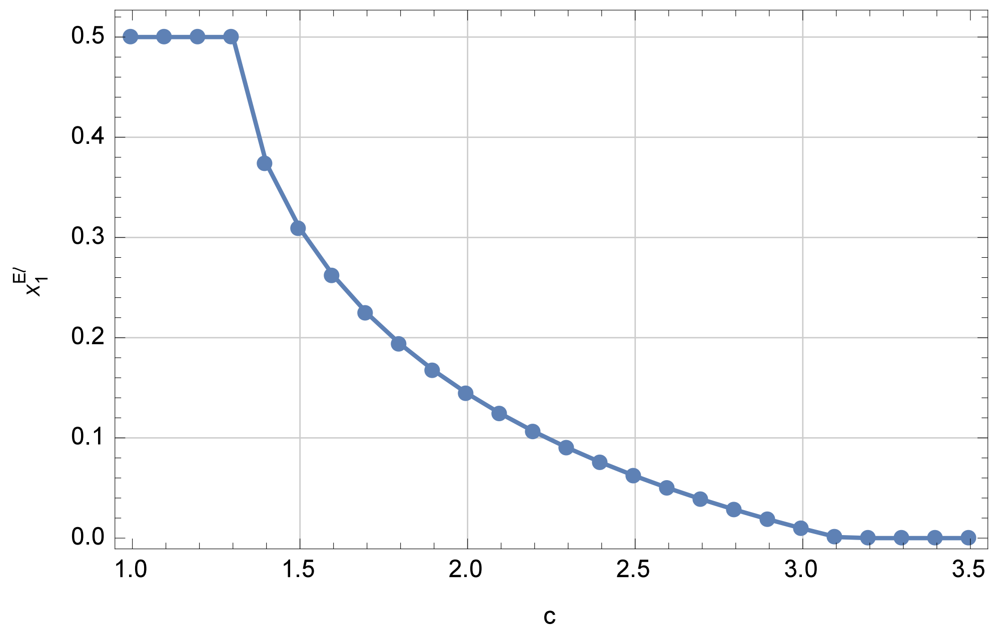

Figure 2: The relationship between the transportation cost c and the equilibrium location of the leader $x_1^{E\ell}$.

increasing as the two firms are located farther apart in most cases. These results seem consistent with ours, as shown in Proposition 2 and Table 2.

Note that Ebina et al. (2015) also obtain similar results as the above with a model where no uncertainty exists. However, our findings strengthen Thomadsen (2007) in the following sense. If the demand evolves stochastically, the volatility signifies the tendency that a leader locates farther away from the center. This means that demand uncertainty is also one of the causes for Thomadsen (2007)'s findings. In summary, our theoretical results give one possible explanation on the empirical results.

6 Conclusion

In this article, we construct a Hotelling-type spatial model with uncertainty of market demand in a real options framework. Contrary to the standard real options literature, the strategy of each firm is multi-dimensional in that each firm maximizes its value function with respect to multiple variables, entry timing, and location. This multi-dimensional approach leads to many interesting results. One important finding is that the threshold of the follower is not monotonically increasing in volatility, which is new in the literature, and can be obtained only with multi-dimensional strategies. Further, although the follower's entry timing tends to be late as the volatility increases, the leader is more likely to locate at the edge as the volatility gets higher.

Finally, we mention three possible directions for future studies. The first is to consider a circular city model, which is another representative model of spatial competition. The second is to study an oligopoly model where more than two firms consider market entry. The third is to generalize the stochastic process Y. Considering these extensions are useful to clarify relationships between firms' locations, their thresholds, and a level of the volatility. However, as a final remark, we argue that the results obtained in this study are not restrictive and can be applied to many actual competitive situations.

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