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MINIMUM PRICE EQUILIBRIUM IN THE ASSIGNMENT MARKET

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Minimum price equilibrium in the assignment market[1]

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Abstract

We investigate an assignment market in which multiple objects are assigned, together with associated payments, to a group of agents with unit demand preferences. Preferences over bundles, the pairs of (object, payment), accommodate income effects. Among all (Walrasian) equilibria in such a market, there is one supported by the coordinate-wise minimum prices, the minimum price equilibrium (MPE). We propose a price adjustment process, "the Serial Vickrey process," that finds an MPE in a finite number of steps. The Serial Vickrey process introduces objects one by one, and on the basis of the structural properties of MPE, the "Serial Vickrey sub-process" sequentially finds an MPE for $k+1$ objects by using an MPE for k objects. In the Serial Vickrey process, instead of revealing the whole preference, each agent only reports finitely many "indifference prices." We also discuss the application of the Serial Vickrey process to calibrate agents' utility functions in the quantitative analysis of housing market research in the assignment model.

Keywords: Assignment market, minimum price equilibrium, income effects, Serial Vickrey process, housing market

JEL Classification: C63, C70, D44

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1 Introduction

It has been long understood that the (Walrasian) equilibrium exists under general conditions, and designing a price adjustment process that converges to the equilibrium is a fundamental tool to discover the market-clearing prices and achieve an efficient allocation. For economies with divisible resources, the study of the price adjustment process, the so-called "tatonnement process," can be dated back to Leon Walras. The tatonnement process provides a powerful tool to achieve efficiency in economies with divisible resources.[1]

Real markets often exhibit indivisibility and heterogeneity in which monetary transfer is also involved, e.g., the labor market, the housing market, and the market for spectrum licenses. Economies with indivisible resources are distinct from those with only divisible resources. The question of how to design the price adjustment process to identify the equilibrium in an economy with indivisibility has attracted intensive attention since the 1980s.

An assignment market with unit-demand agents is one of the most prominent models in this area: Several heterogenous objects are sold to a group of agents, each agent is interested in receiving at most one object, and each transaction involves a monetary transfer. In such a market, if preferences satisfy some standard assumptions, there is a (Walrasian) equilibrium supported by the coordinate-wise minimum prices, among all equilibria, i.e., the minimum price equilibrium (MPE).[2] Any price adjustment process targeting an MPE not only achieves an efficient outcome but also has nice incentive properties.[3]

In their seminal paper, Demange et al. (1986) investigate the above assignment market by assuming that agents have quasi-linear preferences. They propose two price adjustment processes that take the form of ascending auctions, the "exact DGS auction" and "approximate DGS auction," to obtain an MPE. Precisely, the exact DGS auction finds the exact minimum equilibrium price (MEP) in a finite number of steps if the price increment and agents' valuations have the same unit of measure, e.g., both are integers. The approximate DGS auction finds an *approximate* MPE in finite steps in the sense that its outcome prices deviate from the MEP coordinate-wise only within some interval. Motivated by the elegant property of MPE, a sequence of works focus on designing different forms of the price adjustment processes to find an MPE in the quasi-linear environment, e.g., Grigorieva et al. (2007), Mishra and Parkes (2009), Andersson and Erlanson (2013).

Many real-life markets capture the features of the assignment market, and in particular,

[1] See, for example, Scarf (1960), Kamiya (1990), and Herings (2002).

[2] See Facts 1 and 2 for details.

[3] See, e.g., Demange and Gale (1985) and Morimoto and Serizawa (2015) for details.

income effects must be seriously considered. Two stylized examples follow.

Example A (The housing market): Houses are heterogeneous in location and size. Each agent is generally interested in obtaining one house. The high prices of houses encourage agents to reduce their expenditures for complements of houses, which reduces their benefits from houses. In many cases, agents borrow from financial institutions to pay house prices. These factors make agents' preferences non-quasi-linear, which exhibits income effects.

Example B (An auction with large payments): In the 2000 U.K. 3G spectrum licenses auction, each firm was restricted to at most one license. To pay for the winning bids, firms borrowed from financial markets. The borrowing cost was non-linear in the total amount of borrowing. This factor also makes preferences non-quasi-linear.[4]

Existing research has already introduced income effects, in the form of "non-quasi-linear preferences," into the study of housing markets and auctions with large payments, e.g., Andersson and Svensson (2014), Morimoto and Serizawa (2015), Baisa (2016, 2017), and Herings (2018). However, the question of how to design the price adjustment process to achieve efficiency in the assignment market, particularly while accommodating income effects, remains unresolved.

A natural question is whether the exact DGS auction or approximate DGS auction works well when income effects are present. As demonstrated in Section 2, when the price increment unit is larger than the valuation unit, the exact DGS auction substantially overshoots the MEP. When preferences are non-quasi-linear, the approximate DGS auction generates prices outside the interval estimated for the quasi-linear setting.

Our aim is to study the assignment market with unit-demand agents, specifically while allowing for general preferences to accommodate income effects. As mentioned above, in such a general setting, the MPE is well defined. To pursue efficiency and incentive compatibility, our central aim is to propose a price adjustment process that finds an MPE in a finite number of steps. Thus, our first key result is as follows:

For each general preference profile, the "Serial Vickrey process" finds an MPE in a finite number of steps.

The Serial Vickrey process introduces objects one by one and sequentially employs the "**Serial Vickrey sub-process**" to derive the MPE for $k+1$ objects from the MPE for k objects. Due to the uniqueness of MEP, the Serial Vickrey process is unaffected by the order in which we introduce the objects.

The Serial Vickrey sub-process plays a central role in the Serial Vickrey process. When

[4] See Klemperer (2004) for details.

the first object is introduced, it coincides with the Vickrey auction. Since a second object is introduced, the Serial Vickrey sub-process consists of three stages.

Stage 1: We construct an equilibrium for $k+1$ objects, based on the MPE for k objects by the "**E-generating process**."

Stage 2: We check whether the constructed equilibrium in Stage 1 is an MPE for $k+1$ objects by the "N_C-**identifying process**." If so, we terminate the Serial Vickrey sub-process. Otherwise, we identify the objects to be reassigned and the prices to be adjusted and proceed to Stage 3.

Stage 3: We reassign the objects and adjust their corresponding prices by the "**MPE-assignment-finding process**" and eventually obtain an MPE for $k+1$ objects

The novelty of the Serial Vickrey process lies in exploiting two main structural properties of MPE. The first structural property is called "demand connectedness." It states that in an MPE, every object is connected directly or indirectly to a null object or an object with zero price by agents' demand sets, and each agent receives a connected object or null object. We are not the first to investigate this property or similar properties of an MPE,[5] but we are the first to employ this property in the following respects: We use "demand connectedness" to partition the agents and objects into connected and unconnected and characterize an MPE in terms of connected agents and objects (Proposition 1). We then use the "demand connectedness" of an MPE for k objects to construct an equilibrium for $k+1$ objects. The E-generating process in Stage 1 is the application (Propositions 2 and 3). Next, we use this property again to check whether the constructed equilibrium is an MPE. The N_C-identifying process in Stage 2 achieves this goal (Proposition 4).

The second structural property depicts the relation between an arbitrary equilibrium and the MPE by a dynamic price adjustment process, which we call the "I pay others' indifference prices (IPOIP) process" (Theorems 1 and 2). This process is the central component of the MPE-assignment-finding process in Stage 3 and is in the spirit of a Vickrey payment. After identifying the objects to be reassigned and the prices to be adjusted in Stage 2, in Stage 3, we run the MPE-assignment-finding process, via the IPOIP process, to complete the object reassignment and price adjustment. We are the first to explore the this structural property, and it is essentially different from the existing properties.[6] Thus, the our structural characterization requires a novel construction approach and proof techniques. Note that in

[5] This property is also explicitly or implicitly discussed by, e.g., Demange and Gale (1985), Alkan and Gale (1990), and Morimoto and Serizawa (2015).

[6] For example, consider the perturbation lemma in Alkan and Gale (1990), Fact A.1 in the Appendix, and another structural property proposed by Alaei et al. (2016). See also Section 6.

the Serial Vickrey process, instead of revealing all of an agent's preferences, each agent only reports finitely many indifference prices (IPs), each representing the price of an object that makes the agent indifferent between that object and the tentatively assigned bundle. All the adjustment processes are based on reported IPs, represented by finite dimensional vectors. Note that the "Scarf lemma" method is known to be a powerful tool for equilibrium computation in economies with indivisibility, but it requires agents to report their full utility functions (e.g., Quinzii, 1984). For general preference settings, compared with the Scarf lemma method, our information requirement is much smaller.

The housing market is an important research topic in both macroeconomics and urban economics. As suggested by, e.g., Duranton and Puga (2015), "the assignment model" is well suited for studying the housing market (or urban land use), as it accommodates the heterogeneity of both houses and agents, i.e., the assignment-based housing market model. Indeed, existing works employ an assignment-based housing market model to calibrate agents' utility functions and conduct quantitative analysis of government policies' effects on the housing market and citizens' welfare, e.g., Kaneko et al. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015).

However, such models come at the cost of technically complex equilibrium computation. In quantitative analysis, equilibrium computation helps calibrate agents's utility functions. For tractability, existing works assume that (i) agents have the same utility function and are distinguished only by differences in income, and (ii) houses have the "common-ranking feature." These two assumptions activate the "recursive equation system," which is an equilibrium computation method that calibrates agents' utility functions, e.g., Kaneko et al. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015).

Many real-life cases motivate us to go beyond these two assumptions, e.g., the investigation of housing markets in a multicentric city model or agents having common-tiered preferences.[7] However, without these two assumptions, "the recursive equation system" fails to work (Zhou and Serizawa, 2018).

Note that in an assignment-based housing market model, agents can never pay more than their income levels. Due to income constraints, a housing market equilibrium may not always exist (Quinzii, 1984). A mild condition, "indispensability," is imposed to guarantee the existence of housing market equilibrium. Such a mild condition is commonly imposed on agents' preferences with respect to income levels when conducting quantitative analysis. In particular, if such a mild condition is imposed, one of the housing market equilibria is supported by the coordinate-wise minimum prices. Thus, our second key result is to establish

[7] See Section 5 for details.

the connection between the our model and the assignment-based housing market model:

For each well-defined (utility, income) profile in the assignment-based housing market model, there is a general preference profile in the assignment market such that the associated MEPs are the same as the corresponding minimum housing market equilibrium prices.

The construction of the general preference profile suggests a way of using the Serial Vickrey process to obtain the minimum housing market equilibrium prices. Then, following the standard econometric method, e.g., Maattanen and Tervio (2014), agents' utility functions can be estimated. Thus, the Serial Vickrey process also contributes to the calibration of agents' utility functions in scenarios that cannot be analyzed by using existing techniques in quantitative analysis of an assignment-based housing market.

The remainder of the paper is organized as follows: Section 2 describes the difficulties in using DGS auctions to obtain MPEs for general preferences. Section 3 defines the assignment market and MPE. Section 4 defines the Serial Vickrey process and describes its properties. Section 5 discusses its application to the housing market. Section 6 relates our results to the literature. Section 7 concludes the paper. Proofs are relegated to the Appendix.

2 Difficulties with using DSG auctions under general preferences

In the following, we use two examples to show that the "exact DGS auction" and "approximate DGS auction" fail to work when general preferences are considered. Specifically, these two auctions substantially overshoot the MEPs.

By using a similar approach as in the exact DGS auction, we can show that when the price increment is larger than the measurement of agents' valuation, the auction in Mishra and Parkes (2009) substantially undershoots the MEP, and the auction in Andersson and Erlanson (2013) either substantially overshoots or undershoots the MEP. Note also that by using a similar approach as in the approximate DGS auction, we can show that Crawford and Knoer (1981)'s salary adjustment process and Hatfield and Milgrom (2005)'s cumulative offer process also fail to approximate an MPE for general preferences in our setting.[8]

2.1 Exact DGS auction

The exact DGS auction: Starting with reserve prices, agents report their demand sets at the current prices. The auctioneer raises the prices of objects in the "minimum overdemanded

[8]In the assignment market, the approximate DGS auction, Crawford and Knoer (1981)'s salary adjustment process and Hatfield and Milgrom (2005)'s cumulative offer process are essentially the same.

set" (MOD) by one unit or stops the auction if no set is MOD at the current prices.

The exact DGS auction finds the MEP in a finite number of steps if (i) agents have quasi-linear preferences and (ii) the price increment is equal to the measurement unit of agents' valuations, e.g., both are integers (Demange et al. 1986). The following example shows that even if only (ii) fails, i.e., the price increment is larger than the measurement unit of agents' valuations, the exact DGS auction generates an outcome whose prices are higher than the MEP and fail to approximate it. Note that (ii) often fails to hold for general preferences.

Consider the case of two agents, 1 and 2, and two objects, A and B. Receiving object 0 means receiving nothing. Agents have quasi-linear preferences. Let $V^i(x)$ denote agent i's valuation over $x = 0, A, B$. Let

$$\begin{aligned} V_1(0) &= 0, \quad V_1(A) = 9.2, \quad V_1(B) = 9.8, \\ V_2(0) &= 0, \quad V_2(A) = 9.1, \quad V_2(B) = 9.6. \end{aligned}$$

Let p_A and p_B be the prices of A and B, and $p \equiv (p_A, p_B)$. The price of object 0 is zero. Agent i's demand set at p is: $D_i(p) \equiv \{x \in \{0, A, B\} : V_i(x) - p_x \geq V_i(y) - p_y, y \in \{0, A, B\}\}$. Since the MEP for this value profile coincides with the Vickrey payment, the MEP is $p^{\min} = (0, 0.5)$.

The DGS auction starts from $p = (0, 0)$, the reserve prices, with an integer increment. At $p = (0, 0)$, both agents demand only object A. Since only object A is overdemanded (also MOD), then increase only p_A by one unit. At $p = (1, 0)$, both agents demand only object B (B is overdemanded, also MOD), and so increase only p_B by one unit. Again, at $p = (1, 1)$, both agents demand only object A. Similarly, the price of each object alternatively increases at least to $(9, 9)$. Thus, the outcome prices substantially overshoot $p^{\min} = (0, 0.5)$.

Alaei et al. (2010) demonstrate that for general preferences, even if the prices of objects in the MOD are updated at different rates, the modified DGS auction may never converge.

2.2 Approximate DGS auction

The approximate DGS auction: Agents are called bid on objects one by one, according to some exogenously given queue. If an agent bids on an unassigned object, he becomes committed to that object at the reserve price. If an agent bids on an assigned object at some price, the price of that object is increased by a price increment, and the agent becomes committed to that object at the increased price. Simultaneously, the agent to whom the object had been assigned becomes uncommitted and occupies the first position among the remaining uncommitted agents. If an agent bids on no object, he drops out of the auction. The auction terminates when all uncommitted agents drop out.

The approximate DGS auction obtains an outcome where the prices derive from the (exact) MEP, coordinate-wise, by at most $k \cdot \delta$ (δ: the price increment; k: the minimum of the numbers of agents and objects), if agents have quasi-linear preferences (Demange et al. 1986). The following example shows that if agents have general preferences, the outcome prices of an approximate DGS auction lie outside the estimation in the quasi-linear setting.

Consider the case of three agents, 1, 2, and 3, and two objects, A and B. Agents are called in the order 1, 2, and 3. Let $\delta \equiv 1$ and agents' preferences satisfy the standard assumptions (See Section 3), and in addition:
For agent 1, $(0,0)\, I_1\, (A, 0.3)\, I_1\, (B, 20.4)$;
For agent 2, $(0,-20)\, I_2\, (A, 5)\, I_2\, (B, 20.4)$ and $(0,0)\, I_2\, (A, 20.2)\, I_2\, (B, 20.6)$; and
For agent 3, $(0,-21)\, I_3\, (A, 0.5)\, I_3\, (B, 20.4)$ and $(0,0)\, I_3\, (A, 20.6)\, I_3\, (B, 20.8)$,
where I_i denotes agent $i's$ indifference relation.

For the above preference profile, the MEP is $p^{\min} = (0.5, 20.4)$.

The approximate DGS auction starts from $p = (0, 0)$, the reserve prices. First, agent 1 is called on and demands object B, and so agent 1 is committed to B at price 0. Second, agent 2 is called, and since agent 2 demands object A at $p = (0, 1)$, he then bids A and is committed to A at price 0. Third, agent 3 is called on, and since agent 3 demands object B at $p = (1, 1)$, he then bids B and is committed to B at price 1. Then, agent 1 becomes uncommitted. Since he is the only uncommitted bidder, agent 1 is called on, and since he demands object B at $p = (1, 2)$, he then bids B and is committed to B at price 2. Agent 3 thus becomes uncommitted. Since he is the only uncommitted bidder, agent 3 is called on to bid. Note that agents 1 and 3 alternatively bid on object B until its price reaches 20. Since agent 1 is committed to object B at price 20, agent 3 is called on; because agent 3 demands object A at $p = (1, 21)$, he then bids A and is committed to it at price 1. By similar reasoning, agents 2 and 3 alternatively bid on object A until its price reaches 20 but stop bidding at 21. The outcome price of object A, i.e., 20, overshoots its $p_A^{\min} = 0.5$, much more than $k \cdot \delta = 2$. Given k and δ, the set of preference profiles for such undesirable deviations is non-negligible.

3 The assignment market and minimum price equilibrium

3.1 Model and definitions

Consider an economy with $n \geq 1$ agents and $m \geq 1$ objects. Let N and M denote the sets of **agents** and **objects**, respectively. Objects can be identical or heterogenous. Not receiving an object is called receiving a **null** object, object 0. Let $L \equiv M \cup \{0\}$. Each agent receives at most one object. For agent $i \in N$, let $x_i \in L$ denote the object that agent i receives and $t_i \in \mathbb{R}$ the associated payment. The agents' common **consumption set** is $L \times \mathbb{R}$, and a generic **bundle** for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} \equiv (0{,}0)$.

Each agent i has a complete and transitive preference R_i over $L \times \mathbb{R}$. Let P_i and I_i be the associated strict and indifference relations. Assume the following properties of preferences.

Money monotonicity: For each $x_i \in L$ and each pair $t_i, t_j \in \mathbb{R}$, if $t_i < t_i'$, $(x_i, t_i)\, P_i\, (x_i, t_i')$.

Possibility of compensation: For each $t_i \in \mathbb{R}$ and each pair $x_i, x_j \in L$, there is $t_j \in \mathbb{R}$ such that $(x_i, t_i)\, I_i\, (x_j, t_j)$.

Money monotonicity states that for a given object, a lower payment makes the agent better off. The possibility of compensation holds that there is no object that is always good or bad. A preference R_i is **general** if it satisfies the two properties just defined. Let $\mathcal{R}^G$ be the class of general preferences. We call $(\mathcal{R}^G)^n$ the **general domain**. The general domain contains, e.g., the quasi-linear domain and domains exhibiting positive or negative income effects. It could also model various behaviors of agents, e.g., risk aversion or facing distortional taxes.

By money monotonicity and possibility of compensation, for each $R_i \in \mathcal{R}^G$, each $z_i \in L \times \mathbb{R}$, and each $y \in L$, there is a unique amount $V_i(y; z_i) \in \mathbb{R}$ such that $(y, V_i(y; z_i))\, I_i\, z_i$. We interpret $V_i(y_i; z_i)$ as agent $i's$ **indifference price (IP) of y at z_i for R_i**.

Example 1: Figure 1 illustrates a preference R_i, $V_i(0; z_i)$, and $V_i(B; z_i)$ for $M = \{A, B\}$.

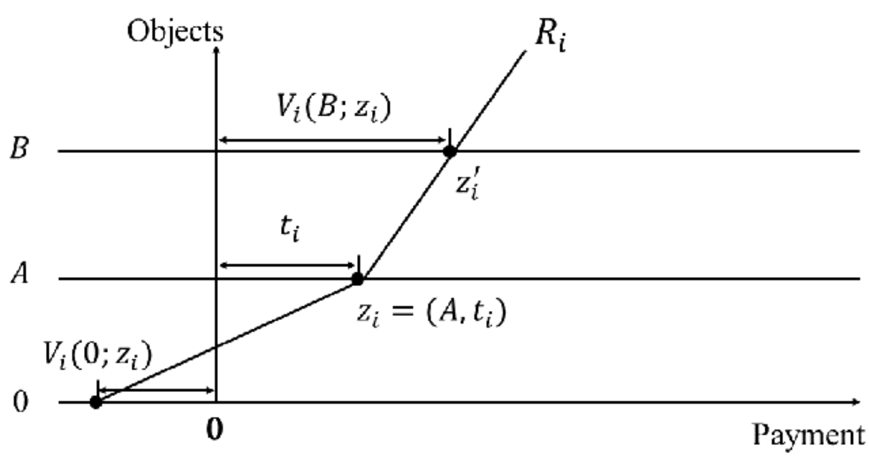

Figure 1: Illustration of a preference R_i, $V_i(0; z_i)$, and $V_i(B; z_i)$

In Figure 1, there are three horizontal lines, corresponding to objects 0, A, and B. The intersection of the vertical line and each horizontal line denotes the bundle of the corresponding object and no payment. For example, the origin $\mathbf{0}$ denotes the bundle of object 0 and no payment. For each point on one of the three lines, the distance between that point and the vertical line denotes the payment. For example, z_i denotes the bundle consisting of object A and payment t_i. By money monotonicity, moving leftward along the same line makes the agent better off, e.g., $(0, A)\, P_i\, z_i$. If bundles are connected by a indifference curve, this means that the agent is indifferent between them. For example, for agent i, z_i and z_i' are connected, and thus, $z_i\, I_i\, z_i'$.

In Figure 1, the IP of object 0 at z_i is $V_i(0; z_i)$ since $(0, V_i(0; z_i))\, I_i\, z_i$. The IP of object B at z_i is $V_i(B; z_i)$ since $(B, V_i(B; z_i))\, I_i\, z_i$.

Let $x \equiv (x_1, \ldots, x_n) \in L^n$ be an **object assignment** such that for each pair $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$. Let X be the set of object assignments. Given $x \in X$, let $t \equiv (t_1, \ldots, t_n) \in \mathbb{R}^n$ denote the associated payment. A (feasible) **allocation** is an n-tuple $z \equiv (x, t) \equiv (z_1, \ldots, z_n) \in [L \times \mathbb{R}]^n$ such that $(x_1, \ldots, x_n) \in X$. We denote the set of feasible allocations by Z. Given $z \in Z$ and $N' \subseteq N$, let $z_{N'} \equiv (z_i)_{i \in N'}$ and $z_{N \setminus N'} \equiv (z_i)_{i \in N \setminus N'}$.

A **preference profile** is an n-tuple $R \equiv (R_i)_{i \in N} \in (\mathcal{R}^G)^n$. Given $R \in (\mathcal{R}^G)^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{N \setminus N'} \equiv (R_i)_{i \in N \setminus N'}$.

Let $p \equiv (p_1, \cdots, p_m) \in \mathbb{R}^m_+$ be a price vector. Given $p \in \mathbb{R}^m_+$ and $M' \subseteq M$, let $p_{M'} \equiv (p_x)_{x \in M'}$ and $p_{M \setminus M'} \equiv (p_x)_{x \in M \setminus M'}$. The price of the null object is assumed to be zero. Without loss of generality, objects' reserve prices are assumed to be zero.

For each $R_i \in \mathcal{R}^G$ and each $p \in \mathbb{R}^m_+$, agent $i's$ **demand set at** p **for** R_i is defined as $D_i(p) \equiv \{x \in L : \text{for each } y \in L,\ (x, p_x)\, R_i\, (y, p_y)\}$.

Definition 1: Let $R \in (\mathcal{R}^G)^n$. A pair $((x, t), p) \in Z \times \mathbb{R}^m_+$ is a (Walrasian) **equilibrium** for R if

$$\text{for each } i \in N,\ x_i \in D_i(p) \text{ and } t_i = p_{x_i}, \tag{E-i}$$

$$\text{for each } y \in M,\ \text{if for each } i \in N,\ x_i \neq y, \text{ then } p_y = 0. \tag{E-ii}$$

(E-i) states that each agent receives an object from his demand set and pays its price. (E-ii) states that the prices of unassigned objects are zero.

Fact 1 (**Existence**)(Alkan and Gale, 1990; Alkan, 1992): For each $R \in (\mathcal{R}^G)^n$, there is an equilibrium.

Given $R \in (\mathcal{R}^G)^n$, let $W(R)$ denote the **set of equilibria** for R. Let $\mathcal{P}(R)$ denote the **set of equilibrium price vectors** for R, respectively, i.e.,

$$\mathcal{P}(R) \equiv \{p \in \mathbb{R}^m_+ : \text{for some } z \in Z, (z, p) \in W(R)\}.$$

Fact 2 (**Lattice property**)(Demange and Gale, 1985; Morimoto and Serizawa, 2015): For each $R \in (\mathcal{R}^G)^n$, $P(R)$ is a complete lattice.

Fact 2 implies that there is a unique price vector $p \in \mathcal{P}(R)$ such that for each $p' \in \mathcal{P}(R)$, $p \leq p'$. A **minimum price equilibrium (MPE)** is an equilibrium whose price vector is minimum. Given $R \in (\mathcal{R}^G)^n$, let $p^{\min}(R)$ denote the minimum equilibirum price vector (**MEP**) for R and $W^{\min}(R)$ the **set of MPEs associated with** $p^{\min}(R)$.

For each preference profile R, $p^{\min}(R)$ is unique, but the indifference in the preferences may result in the multiple MPE allocations. These MPE allocations are indeed welfare-equivalent for each agent, i.e., for each $R \in (\mathcal{R}^G)^n$, each pair $(z, p^{\min}(R)), (z', p^{\min}(R)) \in W^{\min}(R)$, and each $i \in N$, $z_i \, I_i \, z'_i$. To simplify the notation, we write $p^{\min}$ instead of $p^{\min}(R)$.

3.2 Illustration of minimum price equilibria

Consider an economy with four agents and three objects, i.e., $N = \{1, 2, 3, 4\}$ and $M = \{A, B, C\}$. In Subsubsection 3.2.1, we illustrate agents' general preferences, and in Subsubsection 3.2.2., we illustrate three MPEs for four-agent economies with one object, two objects, and three objects. Such illustrations are helpful to understand how the Serial Vickrey process works.

3.2.1 Preference settings

A preference R_i can be represented by a set of IP functions: For each $t \in \mathbb{R}$, let $V_i(\cdot; (0, t))$ is a mapping from M to $\mathbb{R}$, and so $\{V_i(\cdot; (0, t)) : t \in \mathbb{R}\}$ is a representation of R_i. For example, the indifference curve in Figure 1 illustrates an IP function at payment $V_i(0; z_i)$.

In the following, for each R_i, we construct $\{V_i(\cdot; (0, t)) : t \in \mathbb{R}\}$ by specifying some IP functions over several given payments. Then, the IP functions for other payments are either *parallel translations* or *convex combinations* of the specified IP functions.

Agent 1 (R_1): His IP functions at payments $0, -2$ and -4 are given by:
(i) At 0: $V_1(A; \mathbf{0}) = 4$, $V_1(B; \mathbf{0}) = V_1(C; \mathbf{0}) = 5$.
(ii) At -2: $V_1(A; (0, -2)) = 2$, $V_1(B; (0, -2)) = V_1(B; (0, -2)) = 4$.
(iii) At -4: $V_1(A; (0, -4)) = 0$, $V_1(B; (0, -4)) = 2$, $V_1(B; (0, -4)) = 3$.

For each $t > 0$, $V_1(\cdot; (0, t))$ is a parallel translation of $V_1(\cdot; \mathbf{0})$:

$$V_1(A; (0, t))) = 4 + t,\ V_1(B; \mathbf{0}) = V_1(C; \mathbf{0}) = 5 + t.$$

For each $t \in [-2, 0]$, $V_1(\cdot; (0, t)))$ is a convex combination of $V_1(\cdot; \mathbf{0})$ and $V_1(\cdot; (0, -2))$:

$$V_1(\cdot; (0, t))) = (t + 2)/2 \cdot V_1(\cdot; \mathbf{0})) - t/2 \cdot V_1(\cdot; (0, -2)).$$

For each $t \in [-4, -2]$, $V_1(\cdot; (0, t)))$ is a convex combination of $V_1(\cdot; (0, -2))$ and $V_1(\cdot; (0, -4))$:

$$V_1(\cdot;(0,t))) = (t+4)/2 \cdot V_1(\cdot;(0,-2))) + (-2-t)/2 \cdot V_1(\cdot;(0,-4)).$$

For each $t < -4$, $V_1(\cdot;(0,t))$ is a parallel translation of $V_1(\cdot;(0,-4))$:

$$V_1(A;(0,t)) = t+4,\ V_1(B;(0,t)) = t+6,\ V_1(C;(0,t)) = t+7.$$

Figure 2 illustrates R_1, and IP functions of (i), (ii), and (iii) are depicted by bold lines.

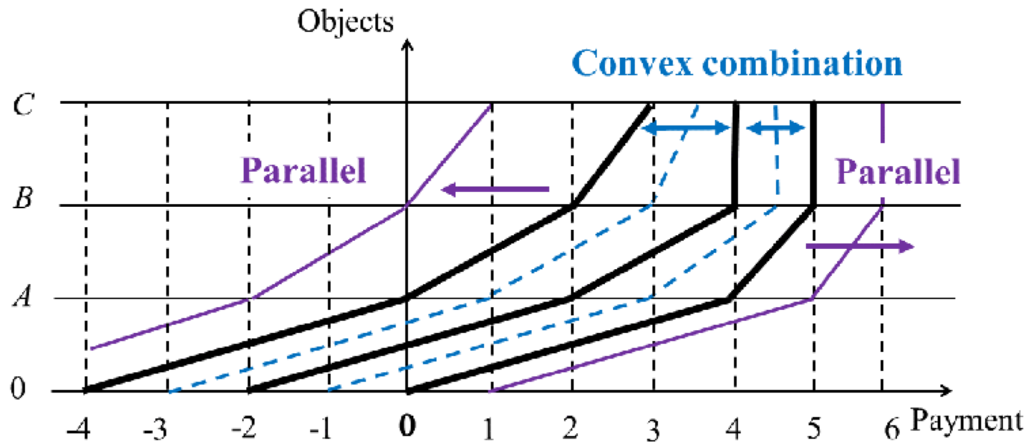

Figure 2: Illustration of R_1

We define R_2, R_3, and R_4 similarly.

Agent 2 (R_2): His IP functions at payments $0, -2$ and -4 are given by:

(i) $V_2(A;\mathbf{0}) = V_2(B;\mathbf{0}) = V_2(C;\mathbf{0}) = 3$,

(ii) $V_2(A;(0,-2)) = 1$, $V_2(B;(0,-2)) = V_2(C;(0,-2)) = 2$, and

(iii) $V_2(A;(0,-4)) = -1$, $V_2(B;(0,-4)) = 0$, $V_2(C;(0,-4)) = 1$.

For each $t > 0$ and each $t < -4$, $V_2(\cdot;(0,t))$ is a parallel translation of $V_2(\cdot;\mathbf{0})$ and $V_2(A;(\cdot,-4))$, respectively. For each $t \in [-2,0]$ and each $t \in [-4,-2]$, $V_2(\cdot;(0,t)))$ is a convex combination of $V_2(\cdot;\mathbf{0})$ and $V_2(\cdot;(0,-2))$ and $V_2(\cdot;(0,-2))$ and $V_2(\cdot;(0,-4))$, respectively.

Agent 3 (R_3): His IP functions at payments 0 and -2 are given by:

(i) $V_3(A;\mathbf{0}) = V_3(B;\mathbf{0}) = 2$, $V_3(C;\mathbf{0}) = 1$

(ii) $V_3(A;(0,-2)) = 0$, $V_3(B;(0,-2)) = 1$, $V_3(C;(0,-2)) = 0$.

For each $t > 0$ and each $t < -2$, $V_3(\cdot;(0,t))$ is a parallel translation of $V_3(\cdot;\mathbf{0})$ and $V_3(A;(\cdot,-2))$, respectively. For each $t \in [-2,0]$, $V_3(\cdot;(0,t)))$ is a convex combination of $V_2(\cdot;\mathbf{0})$ and $V_2(\cdot;(0,-2))$.

Agent 4 (R_4): His IP function at payment 0 is given by $V_4(A;\mathbf{0}) = V_4(B;\mathbf{0}) = 1$, $V_4(C;\mathbf{0}) = 2$. For each $t \neq 0$, $V_4(\cdot;(0,t))$ is a parallel translation of $V_4(\cdot;\mathbf{0})$. Thus, R_4 is quasi-linear.

3.2.2 Illustration of minimum price equilibria

Given the preferences in Subsubsection 3.2.1, we illustrate three MPEs in the four-agent economies with object A, objects A and B, and all the objects.

An MPE of the economy with object A: We use each agent i's IP $V_i(A;\mathbf{0})$ of A from $\mathbf{0}$. Since $V_1(A;\mathbf{0}) = 4$, $V_2(A;\mathbf{0}) = 3$, $V_3(A;\mathbf{0}) = 2$, and $V_4(A;\mathbf{0}) = 1$, $V_1(A;\mathbf{0}) = 4$ is the highest among the IPs. Based on this ranking, we assign A to agent 1 and ask him to pay the second highest IP, i.e., $V_2(A;\mathbf{0}) = 3$, and have the other agents keep $\mathbf{0}$. It is easy to see that $z^{\min}(A) \equiv ((A,3),\mathbf{0},\mathbf{0},\mathbf{0})$ is an MPE allocation and the associated MEP is $p^{\min}(A) = 3$. Note that $z^{\min}(A)$ coincides with the outcome of a Vickrey auction of object A.

An MPE of the economy with objects A and B: The MPE allocation is $z^{\min}(A,B) \equiv (z_1^{\min}, z_2^{\min}, z_3^{\min}, z_4^{\min}) = ((B,2.5),(A,2),\mathbf{0},\mathbf{0})$ and the MEPs are $p^{\min}(A,B) \equiv (p_A^{\min}, p_B^{\min}) = (2,2.5)$. Figure 3 illustrates $(z^{\min}(A,B),\, p^{\min}(A,B))$.

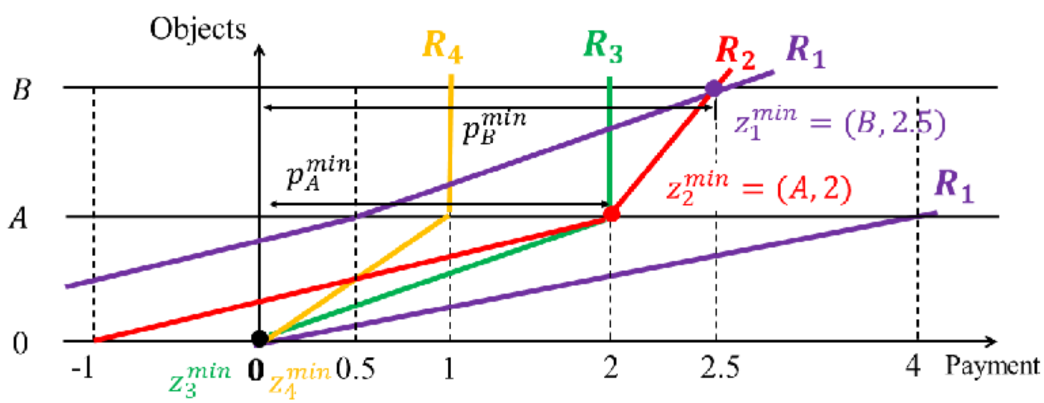

Figure 3: Illustration of $z^{\min}(A,B)$

To see why $z^{\min}(A,B)$ is an MPE allocation, first, note that for each $i = 1,2,3,4$, $z_i^{\min}$ is maximal for R_i in $\{\mathbf{0},(A,p_A^{\min}),(B,p_B^{\min})\}$. In addition, all the objects are assigned. Thus, (E-i) and (E-ii) are satisfied. Thus, $z^{\min}$ is an equilibrium allocation.

Second, let $p = (p_A, p_B)$ be an equilibrium price. We show that $p \geq p^{\min}(A,B)$. Since there are four agents and two objects, by (E-i), at least two agents demand the null object at p. If $p_A < 2$, then $(A,p_A)\,P_i\,\mathbf{0}$ for each $i = 1,2,3$, and thus, only agent 4 may demand the null object, a contradiction. Thus, $p_A \geq 2 = p_A^{\min}$. If $p_B < 2.5$, then by $p_A \geq 2$, $D_1(p) = D_2(p) = \{B\}$, contradicting (E-i). Thus, $p_B \geq 2.5 = p_B^{\min}$.

An MPE of the economy with objects A**,** B**, and** C: By the same reasoning as above, we can show that the MPE allocation is $z^{\min}(A, B, C) = (z_1^{\min}, z_2^{\min}, z_3^{\min}, z_4^{\min}) = ((C, 2), (B, 1.5), (A, 1), \mathbf{0})$ and the MEPs are $p^{\min}(A, B, C) \equiv (p_A^{\min}, p_B^{\min}, p_C^{\min}) = (1, 1.5, 2)$. Figure 4 illustrates $(z^{\min}(A, B, C),\ p^{\min}(A, B, C))$.

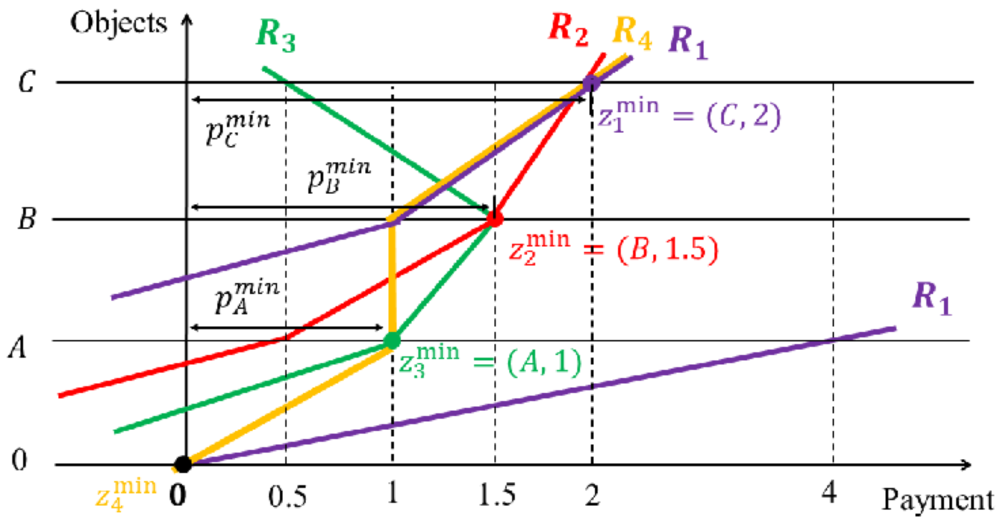

Figure 4: Illustration of $z^{\min}(A, B, C)$

4 Serial Vickrey process

This section proposes a price adjustment process, which we call the "**Serial Vickrey process**," to find an MPE for general preferences in a finite number of steps. The Serial Vickrey process introduces objects one by one and sequentially finds an MPE for $k + 1$ objects, based on an MPE for k objects, by the "**Serial Vickrey sub-process**."

When the first object is introduced, the Serial Vickrey sub-process just coincides with the second-price auction. In general, the Serial Vickrey sub-process consists of three stages. **Stage 1** is an "**E-generating process**," which constructs an equilibrium for $k + 1$ objects, based on an MPE for k objects. **Stage 2** is a "N_C-**identifying process**," which identifies whether the constructed equilibrium for $k + 1$ objects is an MPE for $k + 1$ objects. If not, in **Stage 3**, we apply an "**MPE-assignment-finding (MPEAF) process**," to obtain an MPE for $k + 1$ objects.

We use Subsections 4.1 to 4.4 to establish the Serial Vickrey sub-process.

Subsection 4.1: We introduce the central concepts, "demand connected path," "connected agents and objects," and accordingly, "unconnected agents and objects."
Subsection 4.2 (Stage 1): We define a demand-connectedness-path-finding process and use it to formalize the E-generating process.
Subsection 4.3 (Stage 2): We define the N_C-identifying process. This process identifies whether an arbitrary equilibrium is an MPE, i.e., whether all the agents are connected.
Subsection 4.4 (Stage 3): We define the MPEAF process. The MPEAF process is used to provide a price adjustment process to match the unconnected agents to unconnected objects, supported by the MEPs. We first define the "IPOIP process" and use it to provide some structural characterizations. Then, based on these characterizations, we formalize the MPEAF process. Finally, we offer a complete description of the Serial Vickrey sub-process.

Subsection 4.5 defines the Serial Vickrey process, describes its properties, and offers some further discussions.

4.1 Demand connectedness

In the following, we introduce "demand connectedness," which plays an important role in Stages 1 and 2 of the Serial Vickrey sub-process.

Recall the second case in Subsubsection 3.2.2. First, $D_3(p^{\min}(A)) = \{A, 0\}$ and $z_3^{\min} = \mathbf{0}$. We say that agent 3's demand "connects" objects A to 0. Second, $D_3(p^{\min}(A, B)) = \{A, 0\}$, $D_2(p^{\min}(A, B)) = \{A, B\}$, $z_3^{\min} = \mathbf{0}$, and $z_2^{\min} = (B, 2.5)$. Thus, agent 3's demand connects objects A to 0, and agent 2's demand connects objects B to A. This is what we call a "demand connected path." We formalize these concepts.

Definition 2: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in Z \times \mathbb{R}_+^m$. An object $x \in M$ is **connected** if there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents that forms a **demand connectedness path (DCP)** such that
(i) $1 \leq \Lambda \leq \min\{m + 1, n\}$,
(ii) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(iii) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(iv) $x_{i_\Lambda} = x$, and
(v) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

Remark 1: (i) If $(z, p) \in W(R)$, then by (E-ii), unassigned objects are connected.
(ii) Since the connectedness is defined only for $x \in M$, the null object is not connected.

Definition 3: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in Z \times \mathbb{R}_+^m$. An agent $i \in N$ is **connected** if x_i is connected or $x_i = 0$.

Let N_C and M_C be the sets of connected agents and objects, respectively. Let $N_U \equiv N \backslash N_C$ and $M_U \equiv M \backslash M_C$ be the sets of unconnected agents and objects, respectively.

Remark 2: (i) By Remark 1(i), $N_C = \emptyset$ does not imply $M_C = \emptyset$.
(ii) By Definition 3, an agent who receives the null object is connected. Thus, $M_C = \emptyset$ does not imply $N_C = \emptyset$.
(iii) For each $x \in M$, if $p_x = 0$, then $x \in M_C$.[9]
(iv) $N_C \neq \emptyset$ if and only if there is some agent $i \in N$ such that $p_{x_i} = 0$.[10]

Based on the concepts of connected agents and objects, we provide a new characterization of MPE.

Proposition 1 (**Characterization of MPE by connectedness**): Let $R \in (\mathcal{R}^G)^n$, $(z,p) \in W(R)$, and N_U and M_U be defined at (z,p). Then, the following statements are equivalent: (i) $p = p^{\min}$, (ii) $N = N_C$, and (iii) $M = M_C$.

4.2 Stage 1 of the Serial Vickrey sub-process

4.2.1 Demand-connectedness-path-finding process

Since DCPs play important roles in Stage 1 of the Serial Vickrey sub-process, we explain how to find DCPs. Note that the demand sets used to identify the DCPs can be derived from the reported IPs.

Definition 4: Demand-connectedness-path-finding (DCPF) process Let $R \in (\mathcal{R}^G)^n$ and $(z,p) \in Z \times \mathbb{R}^m_+$. Let $x \in M$ be connected in (z,p), and let $i \in N$ be such that $x_i = x$.

Phase 1: *Round 1:* Let $N_1 \equiv \{i\}$.
If $p_x = 0$, then stop the DCPF process.
If $p_x > 0$, then let $N_2 \equiv \{j \in N \backslash N_1 : x \in D_j(p)\} \neq \emptyset$, and go to Round 2.
Round $s(\geq 2)$*:* Let $L(N_s) \equiv \{y \in L : x_j = y \text{ for some } i \in N_t\}$.
If there is $y \in L(N_s)$ such that $y = 0$ or $p_y = 0$, then stop Phase 1 and go to Phase 2.
Otherwise, let $N_{s+1} \equiv \{j \in N \backslash \cup_{k=1}^{s} N_k : D_j(p) \cap L(N_s) \neq \emptyset\} \neq \emptyset$, and go to Round $s+1$.

[9]To see this, let $x \in M$ be such that $p_x = 0$. If x is unassigned, then by Definition 2, $x \in M_C$. If x is assigned to some $i \in N$, then $\{i\}$ constitutes a demand sequence satisfying (i) to (v), and so, $x \in M_C$.

[10]The if part comes from Definition 3. To understand the only if part, by contradiction, suppose that for each $i \in N$, $p_{x_i} > 0$. Since $N_C \neq \emptyset$, then for each $i \in N_C$, by $p_{x_i} > 0$, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of distinct agents satisfying Definition 2, with $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$. This contradicts that for each $i \in N$, $p_{x_i} > 0$.

Phase 2: Let S be the final round of the DCPF process. Then, construct a sequence $\{i_s\}_{s=1}^{S}$ of distinct agents as follows: (i) Choose $i_1 \in N_S$ such that $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, and (ii) for each $j \in \{2, \cdots, S\}$, choose $i_j \in N_{S+1-j}$ such that $x_{i_j} \in L(N_{S+1})$ and $x_{i_j} \in D_{i_{j-1}}(p)$.

Proposition 2: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in Z \times \mathbb{R}^m_+$. Let $x \in M$ be connected in (z, p), and let $i \in N$ be such $x_i = x$. Then, we have the following:
(i) Phase 1 of DCPF process stops in a finite number of steps, i.e., $S < +\infty$.
(ii) The sequence $\{i_s\}_{s=1}^{S}$ of distinct agents of Phase 2 is a DCP w.r.t. x.

By the finiteness of N, Proposition 2(i) is obvious. It is also straightforward that the sequence $\{i_s\}_{s=1}^{S}$ constructed in Proposition 2(ii) satisfies (i)-(v) of Definition 2.

Example 2: We demonstrate a DCPF process for object B in $(z^{\min}(A, B, C), p^{\min}(A, B, C))$ of Figure 4. Let $x = B$. Then, $i = 2$.

Phase 1: At Round 1, $N_1 = \{2\}$. Since $p^{\min}_B > 0$ and $N_2 = \{j \in N \backslash N_1 : x \in D_j(p)\} = \{3\} \neq \emptyset$, then go to Round 2.
At Round 2, $L(N_2) = \{A\}$ and $N_3 = \{j \in \{1, 4\} : D_j(p^{\min}) \cap \{A\} \neq \emptyset\} = \{4\}$. Since $p^{\min}_A > 0$ and $N_3 \neq \emptyset$, then go to Round 3.
At Round 3, $L(N_3) = \{0\}$. Since $0 \in L(N_3)$, we stop Phase 1.

Phase 2 constructs a DCP consisting of a sequence of agents $\{4, 3, 2\}$, i.e., $i_1 = 4$, $i_2 = 3$, and $i_3 (= i_\Lambda) = 2$.

4.2.2 E-generating process

Given $N' \subseteq N$, $x \in L$ and $z \in [L \times \mathbb{R}]^n$, let $\pi^x \equiv (\pi^x_1, \cdots, \pi^x_{|N'|})$ be the permutation on N' such that $V_{\pi^x_1}(x; z_{\pi^x_1}) \geq \cdots \geq V_{\pi^x_{|N'|}}(x; z_{\pi^x_{|N'|}})$, e.g., π^x_1 is the agent whose IP for x at $z_{\pi^x_1}$ is highest in N'. For each $h \in N'$, let $C^h(R_{N'}, x; z) \equiv V_{\pi^x_h}(x; z_{\pi^x_h})$ be the h-th highest IP for x from z for R among N'. Let $C^h_+(R_{N'}, \cdot; \cdot) \equiv \max\{0, C^h(R_{N'}, \cdot; \cdot)\}$. If $N' = \emptyset$, let $C^h_+(R_{N'}, \cdot; \cdot) \equiv 0$, and if $N' = N$, let $C^h_+(R_{N'}, \cdot; \cdot) \equiv C^h_+(R, \cdot; \cdot)$.

Let $M(k) \equiv \{1, \ldots, k\} (1 \leq k \leq m)$. Let $W(k, R)$ and $W^{\min}(k, R)$ be the sets of equilibria and MPEs, respectively, for the economy with objects $M(k)$ and the preference profile R. In the following, we propose a process that generates an equilibrium $(z, p) \in W(k+1, R)$ from $(z^*, p^{\min}) \in W^{\min}(k, R)$.

Definition 5: **E-generating process** Let $(z^*, p^{\min}) \in W^{\min}(k, R)$. Introduce object $k+1$.
Phase 1: Each agent i reports $V_i(k+1; z^*_i)$, and compute $C^1(R, k+1; z^*)$.
If $C^1(R, k+1; z^*) \leq 0$, then let (z, p) be such that
(a) $p_{k+1} = 0$, and $p = (p^{\min}, p_{k+1})$, and (b) for each $i \in N$, $z_i = z^*_i$.

Otherwise, go to Phase 2.

Phase 2: Select an agent i such that $V_i(k+1; z_i^*) = C^1(R, k+1; z^*)$. Run the DCPF process w.r.t. x_i^* for $(z^*, p^{\min})$, and obtain a DCP $\{i_\lambda\}_{\lambda=1}^{\Lambda}$. Let (z, p) be such that

(a) $p_{k+1} = C_+^2(R, k+1; z^*)$, and $p = (p^{\min}, p_{k+1})$,

(b) $z_{i_\Lambda} = (k+1, p_{k+1})$,

(c) for each $i_l \in \{i_\lambda\}_1^{\Lambda-1}$, $z_{i_l} = z^*_{i_{l+1}}$, and

(d) for each $j \in N \backslash \{i_\lambda\}_1^{\Lambda}$, $z_j = z_j^*$.

Notice that in Phase 2 of the E-generating process, the agent with the highest IP from $(z^*, p^{\min}) \in W^{\min}(k, R)$ obtains the additionally introduced object and pays the second-highest IP. In this respect, this process is similar to a Vickrey auction.

Proposition 3 (**Property of E-generating process**): Let $R \in (\mathcal{R}^G)^n$ and $(z^*, p^{\min}) \in W^{\min}(k, R)$. The E-generating process finds an equilibrium $(z, p) \in W(k+1, R)$ in a finite number of steps.

Example 3: Let $R \in (\mathcal{R}^G)^4$ be the preference profile specified in Subsubsection 3.2.1. We demonstrate that the E-generating process indeed generates an equilibrium allocation $z(A, B, C)$ of the economy with objects A, B and C based on $z^{\min}(A, B)$ in Figure 3.

Recall that $z^{\min}(A, B) = ((B, 2.5), (A, 2), \mathbf{0}, \mathbf{0})$. In Phase 1, each agent i reports his IP for C from $z_i^{\min}(A, B)$, i.e., $V_1(C; (B, 2.5)) = 3.25$, $V_2(C; (A, 2)) = 2.5$, and $V_3(C; \mathbf{0}) = 1$, and $V_4(C; \mathbf{0}) = 2$. Since $C^1(R, C; z^{\min}(A, B)) = V_1(C; (A, 3)) = 3.25 > 0$, go to Phase 2.

Since $C_+^2(R, k+1; z^{\min}(A, B)) = V_2(C; (A, 2)) = 2.5$, $p = (p^{\min}, p_{k+1}) = (2, 2.5, 2.5)$, then $V_1(C; (B, 2.5))$ is the highest among the four IPs from $z^{\min}(A, B)$, and $V_2(C; (A, 2)) = 2.5$ is the second highest. In the E-generating process, we have agent 1, who has the highest IP, obtain C and pay the second-highest IP, i.e., $z_1(A, B, C) = (C, 2.5)$. Now, object B is unassigned. Remember that there is a DCP, $i_1 = 4$, $i_2 = 3$, and $i_3 = 2$, from object B to object 0 (Example 2). We move agents' bundles along this path. By $i_3 = 2$, $z_2(A, B, C) = z_1^{\min}(A, B) = (B, 2.5)$. By $i_2 = 3$, $z_3(A, B, C) = z_2^{\min}(A, B) = (A, 2)$. By $i_1 = 4$, $z_4(A, B, C) = z_4^{\min}(A, B) = \mathbf{0}$.

Let $p \equiv (p_A^{\min}(A, B), p_B^{\min}(A, B), 2.5)$. Then, $D_1(p) = \{C\}$, $D_2(p) = \{A, B\}$, $D_3(p) = \{0, A\}$, $D_4(p) = \{0\}$. Thus, $z(A, B, C) \equiv ((C, 2.5), (B, 2.5), (A, 2), \mathbf{0})$ is an equilibrium allocation of the economy with objects A, B and C.

Figure 5 illustrates an E-generating process from $z^{\min}(A, B)$ to $z(A, B, C)$.

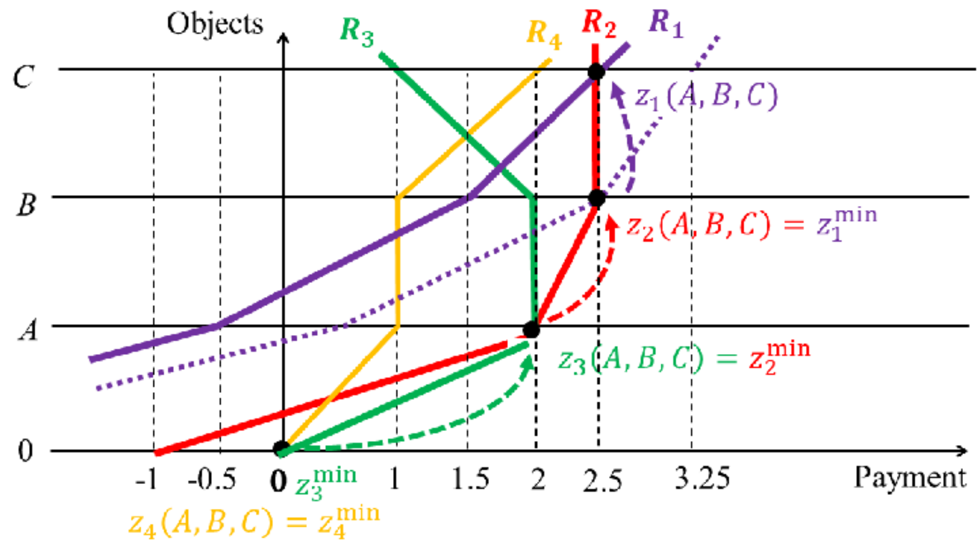

Figure 5: Illustration of an E-generating process from $z^{\min}(A,B)$ to $z(A,B,C)$.

We summarize Stage 1 of the Serial Vickrey sub-process as follows:

Stage 1: Given $(z^*, p^{\min}) \in W^{\min}(k,R)$ in Step k, run the E-generating process to obtain an equilibrium $(z,p) \in W(k+1,R)$.

4.3 Stage 2 of the Serial Vickrey sub-process

In Stage 2, we judge whether an equilibrium (z,p) obtained in Stage 1 is an MPE or not, and identify the connected agents in (z,p) if (z,p) is not an MPE. Note that by Proposition 1, once the set N_C of connected agents in $(z,p) \in W(R)$ is identified, it is straightforward to judge whether (z,p) is an MPE or not, i.e, if $N_C = N$, then (z,p) is an MPE, and is not otherwise. We propose a process to identify the set N_C of connected agents in an equilibrium.

Definition 6: **N_C-identifying process** Let $R \in (\mathcal{R}^G)^n$ and $(z,p) \in W(R)$.

Round 1: Let $N_1 \equiv \{i \in N : p_{x_i} = 0\}$. If $N_1 = \emptyset$, then let $N^* = \emptyset$ and stop the process. Otherwise, go to Round 2.

Round $s(\geq 2)$: Let

$$M(N_{t-1}) \equiv \{y \in M \backslash \{x_i : \cup_{k=1}^{s-1} N_k\} : p_y > 0, \text{ and } y \in D_i(p) \backslash \{x_i\} \text{ for some } i \in N_{t-1}\}.$$

If $M(N_{t-1}) = \emptyset$, then let $N^* = \cup_{k=1}^{s-1} N_k$ and stop the process.

Otherwise, let $N_s \equiv \{i \in N : x_i = y \text{ for some } y \in M(N_{s-1})\}$, and go to Round $s+1$.

Proposition 4 (**Property of N_C-identifying process**): Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in W(R)$. Let S be the total round of the N_C-identifying process. Then (i) the N_C-identifying process at (z, p) terminates in a finite number of steps, i.e., $S < +\infty$ and (ii) $N_C = N^*$.

Example 4: We illustrate N_C-identifying process for $(z^{\min}(A, B, C), p^{\min}(A, B, C))$ in Figure 4. At Round 1, $N_1 = \{4\}$. Since $N_1 \neq \emptyset$, go to Round 2. At Round 2, $M(N_1) = \{A, C\}$ and $N_3 = \{1, 3\}$. Since $M(N_1) \neq \emptyset$, go to Round 3. At Round 3, $M(N_2) = \{B\}$ and $N_3 = \{2\}$. Since $M(N_2) \neq \emptyset$, go to Round 4. At Round 4, $M(N_3) = \emptyset$. Since $M(N_3) = \emptyset$, then the set of connected agents is $N_C = N_1 \cup N_2 \cup N_3$.

Since $N_C = N$, Proposition 1 verifies that $(z^{\min}(A, B, C), p^{\min}(A, B, C)) \in W^{\min}(3, R)$.

Example 5: We illustrate N_C-identifying process for $z(A, B, C)$ in Figure 5. At Round 1, $N_1 = \{4\}$. Since $N_1 \neq \emptyset$, go to Round 2. At Round 2, $M(N_1) = \emptyset$. Since $M(N_1) = \emptyset$, $N_C = N_1 = \{4\}$.

Since $N_C \neq N$, Proposition 1 verifies that $z(A, B, C) \notin W^{\min}(3, R)$.

Note that M_C is the set of real objects assigned to N_C. Thus, once N_C is identified, M_C can be immediately obtained and so do M_U and N_U.

We summarize Stage 2 of Serial Vickrey sub-process as follows:

Stage 2: Given $(z, p) \in W(k+1, R)$ obtained in Stage 1, run N_C-identifying process to identify the set N_C of connected agents at (z, p). If $N_C = N$, then (z, p) is an MPE. Then go to the next step by introducing object $k + 2$. Otherwise, identify M_C, M_U, and N_U from N_C and go to Stage 3.

4.4 Stage 3 of the Serial Vickrey sub-process

Based on the concepts of connected agents and objects, this subsection provides new characterizations of MPEs, in terms of their structural properties. These characterizations are essential to Stage 3 of the Serial Vickrey sub-process.

4.4.1 Definition of the IPOIP process and its illustration

Lemma 1 below shows that to obtain an MPE from $(z, p) \in W(R)$, we only need to reassign objects in M_U to agents in N_U.

Lemma 1: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and $(z^{\min}, p^{\min}) \in W^{\min}(R)$. Let N_U and M_U be defined at (z, p). Then,
(i) $|N_U| = |M_U|$,
(ii) for each $x \in M_U$, $C^1_+(R_{N_C}, x; z) \leq p^{\min}_x < p_x$, and

(iii) for each $i \in N_U$, $x_i^{\min} \in M_U$.

The following process is the key to conduct the reassignment process to assign objects in M_U to agents in N_U. Let μ be an assignment of M_U to N_C, and let $\mu(i)$ be agent i's assigned object at μ. Note that μ is a bijection from N_U to M_U. Let $\Omega(M_U)$ be the set of all bijections from N_U to M_U.

Definition 7: **"I pay others' indifference prices" (IPOIP) process** Let $R \in (\mathcal{R}^G)^n$, $(z,p) \in W(R)$, and N_U and M_U be defined at (z,p). Let $\mu \in \Omega(M_U)$. The $k-$**"I pay others' indifference prices" (IPOIP) process for** μ is defined as follows: For each $x \in M_U$ and each $i \in N_U$,
(i) $\overline{p}_x^0 \equiv C_+^1(R_{N_C}, x; z)$ and $\overline{z}_i^0(\mu) \equiv (\mu(i), \overline{p}_{\mu(i)}^0)$, and
(ii) for each $s = 1, \cdots k$,

$$\overline{p}_x^s(\mu) \equiv C_+^1(R, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C})) \text{ and } \overline{z}_i^s(\mu) \equiv (\mu(i), \overline{p}_{\mu(i)}^s(\mu)).$$

The intuition behind the IPOIP process is as follows: First, set the staring price of each unconnected object $x \in M_U$ by $\overline{p}_x^0 = C_+^1(R_{N_C}, x; z)$. Given an assignment $\mu \in \Omega(M_U)$, each unconnected agent $i \in N_U$ is tentatively assigned a bundle $\overline{z}_i^0(\mu) = (\mu(i), \overline{p}_{\mu(i)}^0)$. For each $x \in M_U$, agent i reports his IP for x from $\overline{z}_i^0(\mu)$, i.e., $V_i(x; \overline{z}_i^0(\mu))$. Then, the price of x is updated to $\overline{p}_x^1(\mu) = C_+^1(R, x; (\overline{z}_{N_U}^0(\mu), z_{N_C}))$. Each unconnected agent $i \in N_U$ is assigned the same object but with an updated price, i.e., $\overline{z}_i^1(\mu) \equiv (\mu(i), \overline{p}_{\mu(i)}^1)$. Similarly, the price of each $x \in M_U$ is further updated to $\overline{p}_x^2(\mu) = C_+^1(R, x; (\overline{z}_{N_U}^1(\mu), z_{N_C}))$, and so forth. In each $s = 1, \cdots k$, the formation of each agent's payment $\overline{p}_x^s(\mu)$ takes the feature of a Vickrey–Clarke–Groves payment.

Example 6: In Figure 5, $z(A, B, C)$ is not an MPE allocation of the economy with objects A, B and C. Moreover, at $z(A, B, C)$, $N_U = \{1, 2, 3\}$ and $M_U = \{A, B, C\}$. Thus, there are six possible assignments of the three unconnected objects to three unconnected agents such that $\mu_a \equiv (A, B, C)$, $\mu_b \equiv (A, C, B)$, $\mu_c \equiv (B, A, C)$, $\mu_d \equiv (B, C, A)$, $\mu_e \equiv (C, B, A)$, and $\mu_f \equiv (C, A, B)$ where $\mu_\lambda = (\mu_\lambda(1), \mu_\lambda(2), \mu_\lambda(3))$ for each $\lambda = a, \ldots, f$. Take $\mu_a = (A, B, C)$ as an example. Figure 6 illustrates an IPOIP process for μ_a.

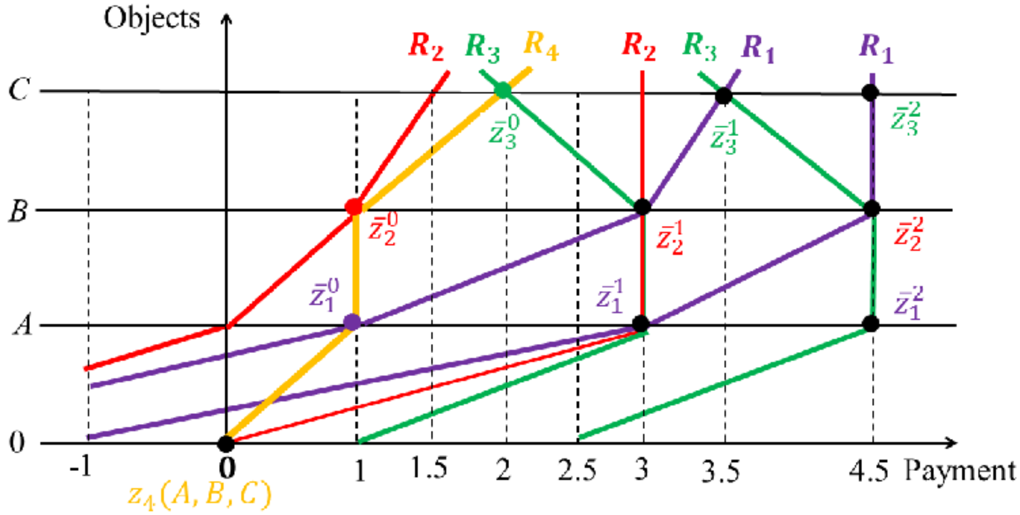

Figure 6: Illustration of the IPOIP process for μ_1

Let $p^0 \equiv (p_A^0, p_B^0, p_C^0) = (V_4(A; \mathbf{0}), V_4(B; \mathbf{0}), V_4(B; \mathbf{0})) = (1, 1, 2)$. For each $i = 1, 2, 3$, let $\bar{z}_i^0(\mu_1) = (\mu_1(i), p^0_{\mu_1(i)})$, i.e., $\bar{z}_1^0(\mu_a) = (A, 1)$, $\bar{z}_2^0(\mu_a) = (B, 1)$ and $\bar{z}_3^0(\mu_a) = (\mu_a(3), 2)$.

Each unconnected agent i reports $V_i(x; \bar{z}_i^0(\mu_a))$ for each unconnected object x. Then, for each unconnected object x, let

$$p_x^1(\mu_a) \equiv \max\{p_x^0, V_1(x; \bar{z}_1^0(\mu_a)), V_2(x; \bar{z}_2^0(\mu_a)), V_3(x; \bar{z}_3^0(\mu_a))\}.$$

Then, $p_A^1(\mu_a) = \max\{1, 1, 0, 3\} = 3$, $p_B^1(\mu_a) = \max\{1, 3, 1, 3\} = 3$ and $p_C^1(\mu_a) = \max\{2, 3.5, 3, 2\} = 3.5$. For each $i = 1, 2, 3$, let $\bar{z}_i^1(\mu_a) = (\mu_a(i), p^1_{\mu_a(i)}(\mu_a))$, i.e., $\bar{z}_1^0(\mu_a) = (A, 3)$, $\bar{z}_2^0(\mu_a) = (B, 3)$ and $\bar{z}_3^0(\mu_a) = (\mu_1(3), 3.5)$. Again, each unconnected agent i reports $V_i(x; \bar{z}_i^1(\mu_a))$ for each unconnected object x. Then, for each unconnected object x, let

$$p_x^2(\mu_a) \equiv \max\{p_x^1(\mu_a), V_1(x; \bar{z}_1^1(\mu_a)), V_2(x; \bar{z}_2^1(\mu_a)), V_3(x; \bar{z}_3^1(\mu_a))\}.$$

Then, $p_A^2(\mu_a) = \max\{3, 3, 3, 4.5\} = 4.5$, $p_B^2(\mu_a) = 4.5$ and $p_B^2(\mu_a) = 4.5$.

We repeat this process $|M_U|$ times, where $|M_U|$ is the number of unconnected objects. Since the unconnected objects are A, B and C, $|M_U| = 2$. Thus, we stop at $s = 2$.

Given $\mu \in \Omega(M_U)$, let $\bar{p}^0 \equiv (\bar{p}_x^0)_{x \in M_U}$, and for each $s \leq k$, let $\bar{p}^s(\mu) \equiv (\bar{p}_x^s(\mu))_{x \in M_U}$ be the prices of the $k-$IPOIP process for μ at round s. In this process, prices are non-decreasing.

Fact 3 (**Monotonicity of IPOIP process**): Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$. In the $k-$IPOIP process for μ, for each $x \in M_U$ and each $s = 1, \cdots, k$,

$$\bar{p}^0 \leq \bar{p}^1(\mu) \leq \cdots \leq \bar{p}^s(\mu).$$

4.4.2 Structural characterizations

Based on the IPOIP process, we give the following new structural characterizations of MPE.

Theorem 1 characterizes the structure of MPE by IPOIP processes. This theorem enables us to obtain an MPE from an equilibrium.

Theorem 1 (**Structural characterization of MPE by IPOIP processes**): Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and $(z^{\min}, p^{\min}) \in W^{\min}(R)$. Let N_C, M_C, N_U, and M_U be defined at (z, p). Then, the followings hold.
(i) For each $x \in M_C$, $p_x = p_x^{\min}$, and for each $i \in N_C$, $z_i = z_i^{\min}$.
(ii) There is $\mu \in \Omega(M_U)$ such that
(ii-1) μ is an MPE object assignment over N_U, and
(ii-2) for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu) = \min_{\mu' \in \Omega(M_U)} \overline{p}_x^{|M_U|-1}(\mu')$.

Example 7: Based on Example 6, we apply this IPOIP process to $\mu_b, \ldots, \mu_f$. The outcomes of this process for six assignments are summarized as follows:

$\mu_a = (A, B, C)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	3	3	3.5
2	4.5	4.5	4.5

$\mu_b = (A, C, B)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	1	3	3.5
2	3.5	3.5	3.5

$\mu_c = (B, A, C)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	3	3	2
2	3	3	3.5

$\mu_d = (B, C, A)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	1	2	2
2	1	2	3

$\mu_e = (C, B, A)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	1	1.5	2
2	1	1.5	2

$\mu_f = (C, A, B)$

s	p_A^s	p_B^s	p_C^s
0	1	1	2
1	1	2	2
2	2	2	2

For each unconnected object x, $p_x^{\min}(A, B, C) = \min\{p_x^2(\mu_\lambda) : \lambda = a, \ldots, f\}$, i.e., $p^{\min}(A, B, C) = (1, 1.5, 2)$. Note that $p^2(\mu_e) = p^{\min}(A, B, C)$, and so $\mu_e = (C, A, B)$ is an MPE assignment. Thus, $z^{\min}(A, B, C) = ((C, 2), (B, 1.5), (A, 1), \mathbf{0})$, as shown in Figure 4.

Remark 3: (i) In Theorem 1, if $M_U \neq \emptyset$, then the existence of an MPE object assignment $\mu \in M_U$ is guaranteed by Facts 1 and 2 and Lemma 1(iii).
(ii) Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and $\mu \in \Omega(M_U)$ be an MPE object assignment over N_U. For a given $\mu \in \Omega(M_U)$, if $\overline{p}^s(\mu) = p_{M_U}^{\min}$ for some round $s \leq k$ in the $k-$IPOIP process w.r.t. μ, then $\overline{p}^s = \cdots = \overline{p}^k(\mu)$.
(iii) Since $\Omega(M_U)$ is finite, Theorem 1 ensures that the IPOIP process finds the MEPs in finitely many rounds.

When $M_U = \emptyset$, Theorem 1 is equivalent to Proposition 1. The novelty of Theorem 1 consists in characterizing the MEPs and assignments of unconnected objects when $M_U \neq \emptyset$.

We further characterize the relation between an MPE and the price sequence generated by an IPOIP process for the "right" object assignment, i.e., the MPE object assignment. This characterization suggests the possibility that we can find an MPE without exhausting all assignments in $\Omega(M_U)$ as in Theorem 1.

Theorem 2 (Structural characterization of MPE by an IPOIP process): Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, $(z^{\min}, p^{\min}) \in W^{\min}(R)$, and N_U and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$ and $s \leq |M_U|$. In the $|M_U|-$IPOIP process for μ, the following two statements are equivalent:

(i) $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$;

(ii) $\overline{p}^{s-1}(\mu) = p^{\min}_{M_U}$, and μ is an MPE object assignment of N_U.

Since the cardinality of $|\Omega(M_U)|$ is the number of the permutations of M_U, when M_U is large, a long time is required to exhaust all assignments in $\Omega(M_U)$ to compute the MEPs. Theorem 2 states that if $\overline{p}^{s-1}(\mu) = \overline{p}^s(\mu)$ in the $|M_U|-$IPOIP process for some $\mu \in \Omega(M_U)$ and $s \leq |M_U|$, then $\overline{p}^{s-1}(\mu)$ and μ are identified as the MEPs and assignment before exhausting all assignments in $\Omega(M_U)$. This fact motivates the following definition.

Definition 8: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$ and $p' \in \mathbb{R}^{|M_U|}$. Then, μ **succeeds in** the $k-$IPOIP process at round $s \leq k$ if $\overline{p}^s(\mu) = \overline{p}^{s-1}(\mu)$.

Example 8: In the table for Example 7, $\overline{p}^1(\mu_e) = (1, 1.5, 2) = \overline{p}^2(\mu_e)$. Thus, μ_e succeeded in the 3$-$IPOIP process at round 2. By Theorem 2, this already verifies that $\overline{p}^1(\mu_e) = p^{\min}_{M_U}$ and μ_e is an MPE object assignment of N_U without running an IPOIP process for μ_f.

Lemma 1 and Theorems 1 and 2 form the theoretical foundations of the Serial Vickrey process, together with the following by-products. By Lemma 1, we have the following

Corollary 1: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$ be an MPE object assignment of N_U. If for some $i \in N_U$ and some $x \in M_U$, $V_i(x; z) < C^1_+(R_{N_C}, x; z)$, then $\mu(i) \neq x$.

Corollary 1 states that an agent $i \in N_U$ never receives an object $x \in M_U$ in an MPE allocation if his IP $V_i(x; z)$ for that object in equilibrium is less than the maximum value $C^1_+(R_{N_C}, x; z)$ of the IPs of connected agents for x. In that case, an assignment μ such that $\mu(i) \neq x$ is disqualified as a candidate MPE assignment without running an IPOIP process for μ. Let **the initially qualified set** $\Omega'(M_U)$ be such that

$$\Omega'(M_U) \equiv \{\mu \in \Omega(M_U) : \forall i \in N_U, V_i(\mu(i); z) \geq \overline{p}^0_{\mu(i)}\},$$

where $\overline{p}_x^0 = C_+^1(R_{N_C}, x; z)$ for each $x \in M_U$. To find an MPE assignment, we have to run an IPOIP process for the object permutation only in $\Omega'(M_U)$.

Suppose that $|M_U| = 5$. Then, $|\Omega(M_U)| = 5! = 120$. If $V_i(x; z) < \overline{p}_x^0$ for some agent $i \in N_U$ and object $x \in M_U$; then, we can dispense with running an IPOIP process for $4! = 24$ assignments. This fact reduces the number of computations needed to obtain the MEPs for certain cases.

Example 9: Consider $(z(A, B, C), p) \equiv ((C, 2.5), (B, 2.5), (A, 2), \mathbf{0}), (2.5, 2.5, 2)) \in W(R)$ constructed in Figure 5. From Example 6, $N_C = N_1 = \{4\}$. Thus, $N_U = \{1, 2, 3\}$ and $M_U = \{A, B, C\}$. Note that $V_1(A; z(A, B, C)) = -0.5 < 1 = C_+^1(R_{N_C}, A; z(A, B, C))$. Thus, by Corollary 1, we do not need to run an IPOIP process for $\mu_a = (A, B, C)$ and $\mu_b = (A, C, B)$. Note further that $V_3(C; z(A, B, C)) = 1 < 2 = C_+^1(R_{N_C}, C; z(A, B, C))$. Thus, also by Corollary 1, we do not need to run an IPOIP process for $\mu_c = (B, A, C)$. Accordingly, we need to run an IPOIP process only for assignments in $\Omega'(M_U) = \{\mu_d, \mu_e, \mu_f\}$ to compute the MEPs.

When running the $|M_U|$ −IPOIP process, in some cases, $\overline{p}^{|M_U|}(\mu)$ needs to be computed. To do so, each agent $i \in N_U$ needs to report $V_i(x; \overline{z}_i^{|M_U|-1}(\mu))$ for each object $x \in M_U$. By Theorem 1, we have the following.

Corollary 2: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$ be an MPE object assignment of N_U. Let $\mu' \in \Omega(M_U)$. If for some $i \in N_U$ and some $x \in M_U$, $V_i(x; \overline{z}_i^{|M_U|-1}(\mu')) < C_+^1(R_{N_C}, x; z)$, then $\mu(i) \neq x$.

Corollary 2 states that an agent $i \in N_U$ never receives an object $x \in M_U$ in an MPE allocation if his IP $V_i(x; \overline{z}_i^{|M_U|-1}(\mu))$ for x at the output of the$|M_U|$ −IPOIP process for μ is less than the maximum value of the IPs of connected agents for x. In that case, an assignment μ such that $\mu(i) = x$ is also disqualified as a candidate MPE assignment without running an IPOIP process for μ. Given $\mu \in \Omega(M_U)$, let **the set** $EX(\mu)$ **of disqualified assignments by** μ be such that

$$EX(\mu) \equiv \{\mu' \in \Omega(M_U) : \exists i \in N_U, \exists x \in M_U \text{ s.t. } \mu'(i) = x \text{ and } V_i(x; \overline{z}_i^{|M_U|-1}(\mu)) < \overline{p}_x^0\},$$

where $\overline{p}_x^0 = C_+^1(R_{N_C}, x; z)$ for each $x \in M_U$. Then, there is no need to run an IPOIP process for any object permutation in $EX(\mu)$.

Example 10: Take $\mu_d = (B, C, A)$ in Example 7. In the table, $\overline{p}^{|M_U|-1}(\mu_4) = (1, 2, 3)$. Thus, $\overline{z}_1^{|M_U|-1}(\mu_d) = (B, 2)$, $\overline{z}_2^{|M_U|-1}(\mu_d) = (C, 3)$ and $\overline{z}_3^{|M_U|-1}(\mu_d) = (A, 1)$. Thus, $V_1(A; \overline{z}_1^{|M_U|-1}(\mu_d)) = 0 < 1 = \overline{p}_A^0$ and $V_3(C; \overline{z}_3^{|M_U|-1}(\mu_d) = 0.5 < 2 = \overline{p}_C^0$. Thus, $EX(\mu_d) = \{\mu_a, \mu_b, \mu_c\}$.

Definition 9: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and M_U be defined at (z, p). Let $\mu \in \Omega(M_U)$ and $p' \in \mathbb{R}^{|M_U|}$. Then, μ **survives** the $k-$IPOIP process against p' if $\overline{p}^k(\mu) \leq p'$.

By Theorems 1 and 2, we have the following.

Corollary 3: Let $R \in (\mathcal{R}^G)^n$, $(z,p) \in W(R)$, and M_U be defined at (z,p). Let $\mu \in \Omega(M_U)$ be an MPE object assignment of N_U and $\mu' \in \Omega(M_U)$. Then, μ survives the $|M_U|-$IPOIP process against $\overline{p}^{|M_U|-1}(\mu')$ and p.

Corollary 3 states that the MPE object assignment can always survive in the $|M_U|-$IPOIP process against any other assignment in $\Omega(M_U)$.

Example 11: Take $\mu_e = (B, C, A)$ in Example 7. Recall that $|M_U| = 3$. Since $\overline{p}^{|M_U|}(\mu_e) = (1, 1.5, 2)$, μ_e survives $the 3-$IPOIP process against $\overline{p}^{|M_U|-1}(\mu_\lambda)$ for each $\lambda \neq e$, and $p = (2, 2, 2.5)$ in Figure 5, and also succeeds in the $3-$IPOIP process against $\overline{p}^{|M_U|-1}(\mu_\lambda)$ for each $\lambda \neq e$, and $p = (2, 2, 2.5)$.

4.4.3 MPE-assignment-finding process

Based on the results obtained in Subsubsection 4.4.2, we are ready to propose the key adjustment process in Stage 3. The basic idea of Stage 3 is as follows: Using the above facts, we propose a process to find an MPE assignment for a $k+1$-object economy. In this process, we first restrict the assignments to $\Omega'(M_U)$. Then, we check assignments in $\Omega'(M_U)$ one by one by running IPOIP processes. Once some assignment $\mu \in \Omega'(M_U)$ succeeds in the $|M_U|-$IPOIP process, μ is identified as an MPE assignment. If an assignment $\mu \in \Omega'(M_U)$ does not succeed in the $|M_U|-$IPOIP process, we check whether it survives against the output price of the IPOIP process for the previous assignment. If the current assignment survives, it disqualifies some of the remaining assignments, and they are removed from the qualified assignments, and so forth.

Definition 10: MPE-assignment-finding (MPEAF) process Let $(z,p) \in W(k+1, R)$ be generated in Stage 2. Collect $\Omega'(M_U)$ for (z,p).

Session 0: Set $\mu^{*0} \equiv x_{M_U}$, $p^{*0} \equiv p_{M_U}$, and $\Omega^{*0}(M_U) \equiv \Omega'(M_U)$. Choose $\mu_1 \in \Omega^{*0}(M_U)$ and go to Session 1.

Session $s(\geq 1)$: Run the $|M_U|-$IPOIP process for μ_s.

Phase $s-1$: If μ_s succeeds at some round $r \leq |M_U|$, then terminate the MPEAF process by setting (z^*, p^*) as $\mu^{*s} \equiv \mu_s$ and $p^{*s} \equiv \overline{p}^r(\mu_s)$.

If μ_s does not succeed at any round $r \leq |M_U|$, then go to *Phase* $s-2$.

Phase $s-2$: If μ_s survives against $\overline{p}^{|M_U|-1}(\mu_{s-1})$, go to *Phase* $s-2-1$. Otherwise, go to *Phase* $s-2-2$.

Phase $s-2-1$: Set $\mu^{*s} \equiv \mu_s$, and $p^{*s} \equiv \overline{p}^{|M_U|-1}(\mu_s)$. Collect

$$EX(\mu_s) \equiv \{\mu \in \Omega^{*s-1}(M_U) : \exists i \in N_U, \exists x \in M_U \text{ s.t. } \mu(i) = x \text{ and } V_i(x; \overline{z}_i^{|M_U|-1}(\mu_s)) < \overline{p}_x^0\},$$

and set $\Omega^{*s}(M_U) \equiv \Omega^{*s-1}(M_U)\backslash(EX(\mu_s) \cup \{\mu_s\})$. Then, choose $\mu_{s+1} \in \Omega^{*s}(M_U)$, and go to Session $s+1$.

Phase $s-2-2$: Set $\mu^{*s} \equiv \mu^{*s-1}$, $p^{*s} \equiv p^{*s-1}$, and $\Omega^{*s}(M_U) \equiv \Omega^{*s-1}(M_U)\backslash\{\mu_s\}$. Then, choose $\mu_{s+1} \in \Omega^{*s}(M_U)$, and go to Session $s+1$.

Proposition 5 guarantees that the MWEAF process finds an MPE allocation in a finite number of steps.

Proposition 5: Let $R \in (\mathcal{R}^G)^n$ and $(z,p) \in W(k+1,R)$ be generated in Stage 2.
(i) Let $\mu^{*0} \equiv x_{M_U}$ and $p^{*0} \equiv p_{M_U}$. The MPEAF process generates a finite sequence of allocations $\{(\mu^{*i}, p^{*i}\}_{i=0}^T$ whose price sequence is nonincreasing, i.e., $T < +\infty$, and for each $t = 1, \cdots, T$, $p^t \leq p^{t-1}$.
(ii) Let (z^*, p^*) be such that
(a) for each $x \in M_U$, $p^*_x = p^{*T}_x$, and for each $i \in N_U$, $z^*_i = (\mu^{*T}(i), p^{*T}_{\mu^{*T}(i)})$, and
(b) $p^{*T}_{M_C} = p_{M_C}$ and $z^{*T}_{N_C} = z_{N_C}$.
Then, $(z^*, p^*) \in W^{\min}(k+1,R)$.

Example 12: We illustrate the MPEAF process given the equilibrium depicted by Figure 5. Recall that $(z(A,B,C),p) = ((C,2.5),(B,2.5),(A,2),\mathbf{0}),(2,2.5,2.5)) \in W(R)$. In Example 5, we identified that $N_C = \{4\}$, $N_U = \{1,2,3\}$, $M_U = \{A,B,C\}$.

In Session 0, $\mu^{*0} = (C,B,A)$ and $p^{*0} \equiv (2,2.5,2.5)$. Using the result of Example 9, we set $\Omega^{*0}(M_U) = \Omega'(M_U) = \{\mu_d, \mu_e, \mu_f\}$. Choose $\mu_1 = \mu_d \in \Omega^{*0}(M_U)$, and go to Session 1.

In Session 1, run the 3−IPOIP process for μ_1. As the table in Example 7 indicates, $\overline{p}^2_C(\mu_d) = 3 > p^{*0}_C$. Thus, μ_1 does not survive, and set $\mu^{*1} \equiv \mu^{*0}$, $p^{*1} \equiv p^{*0}$, and $\Omega^{*1}(M_U) \equiv \Omega^{*0}(M_U)\backslash\{\mu_1\} = \{\mu_e, \mu_f\}$. Choose $\mu_2 = \mu_f \in \Omega^{*1}(M_U)$, and go to Session 2.

In Session 2, run the 3−IPOIP process for μ_2. Based on the table for μ_f in Example 7, we further have $\overline{p}^3_A(\mu_2) = 2$, $\overline{p}^3_B(\mu_2) = \overline{p}^3_C(\mu_2) = 2.5$. Since $\overline{p}^3(\mu_2) \leq p^{*1}$, then μ_2 survives in the 3−IPOIP process against p^{*1}. Set $\mu^{*2} \equiv \mu_2$, and $p^{*2} \equiv \overline{p}^2(\mu_2) = (2,2,2)$. In such a case, $EX(\mu_2) = \emptyset$. Thus, set $\Omega^{*2}(M_U) \equiv \Omega^{*1}(M_U)\backslash(EX(\mu_2) \cup \{\mu_2\}) = \{\mu_e\}$. Choose $\mu_3 = \mu_e \in \Omega^{*2}(M_U)$, and go to Session 3.

In Session 3, run the 3−IPOIP process forμ_3. As the table for μ_e in Example 7 indicates, $\overline{p}^1(\mu_e) = \overline{p}^2(\mu_e)$. Thus, μ_e succeeds in the 3−IPOIP process. Thus, set $\mu^{*3} \equiv \mu_3$, $p^{*3} \equiv \overline{p}^2(\mu_3)$, and terminate at (μ^{*3}, p^{*3}).

As we discuss in Example 7, indeed $p^{*3} = (1,1.5,2)$ is the MEP for A, B, and C. The MPEAF process generates the sequence $\{(\mu^{*i}, p^{*i})\}_{i=0}^3$ of allocations. As Proposition 5 states, the generated price sequence is nonincreasing, $p^{*0} \geq p^{*1} \geq p^{*2} \geq p^{*3}$.

We summarize Stage 3 of the Serial Vickrey sub-process as follows:

Stage 3: Given N_C, M_C, M_U, and N_U generated in Stage 2, run the MPEAF process to obtain an MPE for a $k+1$-object economy. Then, go to the next step by introducing object $k+2$.

4.4.4 Summary of the Serial Vickrey sub-process

Since the Serial Vickrey sub-process has many elements, we summarize it.

Serial Vickrey sub-process for $M(k+1)$

Stage 1: Given $(z^*, p^{\min}) \in W^{\min}(k, R)$ in Step k, run the E-generating process to obtain an equilibrium $(z, p) \in W(k+1, R)$. Then, go to Stage 2.
Stage 2: Run the N_C-identifying process to identify the set N_C of connected agents in (z, p). If $N_C = N$, then $(z, p) \in W^{\min}(k+1, R)$, and go to the next step for $M(k+2)$. Otherwise, identify M_C, M_U, and N_U from N_C, and go to Stage 3.
Stage 3: Run the MPEAF process to obtain an MPE for a $k+1$-object economy, and go to the next step for $M(k+2)$.

By Propositions 2, 3, 4 and 5, we have the following.

Proposition 6: Let $R \in (\mathcal{R}^G)^n$, $1 \leq k \leq m$, $(z, p) \in W(k, R)$, and the Serial Vickrey sub-process generates $(z^*, p^*) \in W^{\min}(k+1, R)$ in a finite number of steps.

Proposition 6 is a direct outcome of Proposition 5.

4.5 Summary of Serial Vickrey process

Since the Serial Vickrey process also has many elements, we summarize it.

Serial Vickrey process

Let $R \in (\mathcal{R}^G)^n$. Initialize the allocation for each agent as $\mathbf{0}$. Introduce the object into the economy sequentially by its index, $1, 2, \cdots$.
Step $k (\geq 1)$: Introduce object k. Call back the agents and objects tentatively exiting the market. Run the Serial Vickrey sub-process with respect to the economy with k objects. If $k = m$, stop at the output of the Serial Vickrey sub-process. Otherwise, go to **Step** $k+1$.

By Proposition 6, we have the following.

Theorem 3: Let $R \in (\mathcal{R}^G)^n$. The Serial Vickrey process converges to an MPE in a finite number of steps.

Remark 4: The order of introducing objects into the economy is independent of the computation of $p^{\min} \in \mathcal{P}(m, R)$ since $p^{\min}$ is unique for the m-object economy (Fact 2).

We add two remarks to conclude. The first shows that for some "well-structured" preference settings, the Serial Vickrey process obtains an MPE with a one-shot IPOIP process in each Serial Vickrey sub-process. The second discusses the application of Theorems 1 and 2 in the task assignment model.

Remark 5 (i): In the multi-item auction model under quasi-linear preferences, the integer value assumption plays an important role in running the existing auctions, as discussed in the introduction. If the Serial Vickrey process is applied, this assumption can be dropped. Notably, quasi-linearity simplifies Stage 3 of the Serial Vickrey sub-process, and so does the Serial Vickrey process. Let $(z, p) \in W(k+1, R)$ be the output of Stage 1 of the Serial Vickrey sub-process. Quasi-linearity implies that x_{M_U} is an MPE object assignment of unconnected objects at (z, p). Thus, in Stage 3, we only run one IPOIP process with respect to x_{M_U}, and the obtained prices are the MPE prices for M_U.
(ii) In an Alonso-type discrete housing market, agents' utility profile satisfies "identical common-ranking," and so the object assignment satisfies positive assortative matching, i.e., agents with higher incomes obtain houses of better quality.[11] Such a property simplifies Stage 3 of the Serial Vickrey sub-process. The reasoning is the same as that in (i).

Remark 6: The the multi-task assignment model of Sun and Yang (2003) and Svensson (2007) describes an assignment market without outside options. In such a model, each task is endowed with limited compensation, and each agent must take a task with certain amount of compensation, even if the obtained bundle is worse than $\mathbf{0}$. It is known that there is an envy-free allocation, the "fair and optimal allocation," supported by coordinate-wise minimum compensation among all the envy-free allocations. Theorems 1 and 2, via Stage 3 of the Serial Vickrey sub-process, yield a fair and optimal allocation.

5 Application of the Serial Vickrey process to the housing market

The housing market exhibits indivisibility and heterogeneity, and each agent generally has unit-demand preferences with income effects. The housing market accords with the main features of our model. As suggested by macroeconomists and urban economists, e.g., Duranton and Puga (2015), when the assignment model is applied to the housing market (or urban land use), its merit is that it can accommodate heterogeneity in both houses and agents. For example, existing works employ the assignment model to conduct quantitative analysis

[11] See Section 5 below or Zhou and Serizawa (2018) for details.

of government policies and their effects on the housing market and citizens' welfare, e.g., Kaneko et al. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015).

On the other hand, the cost of using the assignment model to study the housing market is its substantially greater technical complexity than classical methods. The complexity comes from the equilibrium computation. Equilibrium computation is a central aspect of quantitative analysis, as it is used to calibrate the agents's utility functions. For tractability, the above works additionally assume that (i) agents have the same utility function and differ only in their income levels and (ii) houses have the "common-ranking feature." Owing to these two assumptions, equilibria can be computed by a "recursive equation system."

However, these two assumptions are not suitable except in specific environments, for example, in the case of the monocentric city, i.e., "an Alonso-type discrete housing market." In other words, the assumptions are not suitable in the case of multicentric city. Even if a real-life situation resembles the monocentric city model, the distribution of housing prices and agents' locations differ from theoretical predictions (Tabuchi, 2018). Moreover, in the housing market, common-tiered preferences may prevail (Zhou and Serizawa, 2018).

Without these two assumptions, "the recursive equation system" fails to work (Zhou and Serizawa, 2018). Going beyond these two assumptions introduces further heterogeneity in the housing market, which is an important area for research on the housing market (Duranton and Puga, 2015). A natural question is whether we have an alternative way to conduct the equilibrium computation for quantitative analysis. The Serial Vickrey process provides us with a solution.

For completeness, we first review the housing market model. Let N, M, and L be the same notations defined in Section 3. Each agent $i \in N$ is endowed with an income level $I_i \in \mathbb{R}_{++}$. For agent $i \in N$, let $x_i \in L$ denote the object that agent i receives and $r_i \equiv I_i - p_{x_i}$ as the *residual money* that agent i retains after paying the house price p_{x_i}. A generic bundle for agent i is a pair $z_i^h \equiv (x_i, r_i) \in L \times \mathbb{R}$. Agents have preferences on $L \times \mathbb{R}$, with a utility representation u_i. Assume that u_i satisfies the following property.

Monotonicity: For each $x_i \in L$ and each pair $r_i, r_i' \in \mathbb{R}$, if $r_i < r_i'$, $u_i(x_i, r_i) < u_i(x_i, r_i')$.

Finiteness: For each $r_i \in \mathbb{R}$ and each pair $x_i, x_i' \in L$, there is $r \in \mathbb{R}$ such that $u_i(x_i, r_i) = u_i(x_i', r)$.

Continuity: For each $x_i \in L$ and each $r_i \in \mathbb{R}$, $U_i(x_i, r_i)$ is continuous with respect to r_i.

Indispensability: For each $x_i \in M$, $u_i(0, I_i) > u_i(x_i, 0)$.

A utility function u_i represents **a general housing market preference** if it satisfies the four properties just defined. Let $\mathcal{U}^H$ be the class of such utility functions. Let $u \equiv (u_i)_{i \in N}$ be a utility profile. A (feasible) **allocation** is an n-tuple $z^h \equiv (x, r) \equiv (z_1^h, \ldots, z_n^h) \equiv$

$((x_1,r_1),\ldots,(x_n,r_n)) \in [L\times\mathbb{R}]^n$ such that $(x_1,\ldots,x_n)\in X$. We denote the set of feasible allocations by Z^h. Let agent $i's$ **demand set with budget** I_i **at** p **for** u_i be defined as $D_i(p,I_i) \equiv \{x\in L : p_x \le I_i$ and $(x,p_x)\,R_i\,(y,p_y)$, for each $y\in L$ such that $p_x\le I_i\}$. Let $I\equiv (I_i)_{i\in N}$ be an income profile.

Definition 11: Let $u\in(\mathcal{U}^H)^n$ and $I\in(\mathbb{R}_{++})^n$. A pair $(x,p)\in Z^h\times\mathbb{R}^m_+$ is a **housing market equilibrium** for (u,I) if

$$\text{for each } i \in N,\ x_i\in D_i(p,I_i) \text{ and } r_i = I_i - p_{x_i}, \quad \text{(E'-i)}$$

$$\text{for each } y \in M,\ \text{if for each } i\in N,\ x_i\ne y, \text{ then } p_y=0. \quad \text{(E'-ii)}$$

Let $W(u,I)$ denote the **set of housing equilibria** for (u,I). Fact 4 is a parallel result of Facts 1 and 2 in our model.

Fact 4 (Quinzii, 1984): For each $u\in(\mathcal{U}^H)^n$ and each $I\in(\mathbb{R}_{++})^n$, there is a housing market equilibrium, and one of the available equilibria is supported by the coordinate-wise minimum prices, $p^{\min}(u,I)$.

The technical difficulties of equilibrium computation on $u\in(\mathcal{U}^H)^n$ require the following additional assumptions on the utility functions, e.g., Kaneko et al. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015).

Specifically, let $\pi\equiv(\pi(1),\ldots,\pi(m+1))$ be a permutation of objects or houses in L, where $\pi(1)$ denotes the house with the highest quality, $\pi(2)$ denotes the house with the second-highest quality, and so forth. W.o.l.g., let $\pi\equiv(m,\ldots,0)$. Assume that $u_i\in\mathcal{U}^H$ additionally satisfies the following assumptions.

Common-quality-ranking: For each $i\in N$ and each $r\in\mathbb{R}_+$, $u_i(m,r) > \cdots > u_i(0,r)$.

Normality: For each pair $x,y\in L$, each pair $r,r'\in\mathbb{R}_+$ with $r<r'$, if $u_i(x,r)=u_i(y,r')$ and $d>0$, then $u_i(x,r+d)>u_i(y,r'+d)$.

An utility function $u_i\in\mathcal{U}^H$ represents **a common-ranking preference** if it satisfies the above two properties. Let $\mathcal{U}^{RH}$ be the class of utility functions representing common-ranking preferences. Moreover, assuming that agents have identical utility functions is also important. A utility profile $u^{ir}\equiv(u_i)^n\in(\mathcal{U}^{RH})^n$ is an **identical common-ranking utility profile** if for each pair $i,j\in N$, $u_i(\cdot,\cdot)=u_j(\cdot,\cdot)$. Let $(\mathcal{U}^{IRH})^n$ be the set of identical common-ranking utility profiles. A tuple (N,M,u^{ir},I) describes the Alonso-type discrete housing market.

An Alonso-type discrete housing market is explicitly and implicitly used to conduct quantitative analysis of the housing market, e.g., Kaneko et el. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015). The following fact forms the central theoretical foundation of such quantitative analysis. Note that in those works, Fact 5 may take different expressions, but its essence is the same.

Fact 5 (Kaneko et el. 2006; Maattanen and Tervio, 2014; Zhou and Seriazawa, 2018): Let $u \in (U^{IRH})^n$ and $I \in (R_{++})^n$. In an equilibrium $(x, p) \in W(u, I)$,
(ii) (**Positive assortative matching**) (ii-1) if $m + 1 \leq n$, $x_n = m, \cdots, x_{n-m+1} = 1$, and $x_{n-m} = \cdots = x_1 = 0$, and
(ii-2) if $m + 1 > n$, $x_n = m, \cdots, x_1 = m - n + 1$ and
(iii) (**Recursive equation system**) (iii-1) if $m + 1 \leq n$, $p_{n-m} = 0$, and for each $k \in \{n - m + 1, \cdots, m - 1\}$, $u_{n-m+k}(k, I'_{n-m+k} - p_k) = u_{n-m+k}(k + 1, I'_{n-m+k} - p_{k+1})$, and
(iii-2) if $m + 1 > n$, $p_1 = \cdots = 0 \leq p_{m-n+1}$ and for each $k \in \{m - n + 1, \cdots, m - 1\}$, $u_{n-m+k}(k, I'_{n-m+k} - p_k) = u_{n-m+k}(k + 1, I'_{n-m+k} - p_{k+1})$.
(iv) in the MPE, $p_1 = \cdots = p_{m-n+1} = 0$.

Among all the equilibria, the MPE is of particular interest and used to calibrate the agents' utility functions, e.g., Maattanen and Tervio (2014). The following example illustrates the calibration process.

Example 13: Let $M = \{h_0, h_1, h_2\}$ and $N = \{1, 2, 3\}$. Let $\sigma : L \to R_{++}$ be a scoring function such that $\sigma(0) < \sigma(h_1) < \sigma(h_2) < \sigma(h_3)$. Each score represents the quality of the corresponding house. Let $I_1 < I_2 < I_3$ and $\alpha \in (0, 1)$. For each $i \in N$, let $u_i(\sigma(h), I_i - p_h) = \sigma(h) + (I_i - p_h)^{1-\alpha}$. Suppose that the income data I, the house quality data $\sigma(M)$, and the housing price data p are available. Given an arbitrary α, insert the income data I and house quality data $\sigma(M)$ into the recursive equation system; we can then obtain the estimated prices $\widehat{p}$ (the estimated MEPs). By changing the values of α, we can obtain a sequence of estimated prices. Based on the comparison of $\widehat{p}$ and p, we calibrate α.

If different agents have different α values, then the corresponding utility profile does not satisfy the identical common-ranking property, and the recursive equation system fails to work. However, if we can construct the relation between the MPE in our model and the MPE in the housing market, by running the Serial Vickrey process, we can indirectly obtain the MEPs in the housing market.

Theorem 4: For each $u \in (\mathcal{U}^H)^n$ and each $I \in (\mathbb{R}_{++})^n$, there is an $R \in (\mathcal{R}^G)^n$ such that $p^{\min}(u, I) = p^{\min}(R)$.

The proof of Theorem 4 is derived by constructing R. The construction is intuitive: Let $R \in (\mathcal{R}^G)^n$ be such that for each $x, y \in L$, each $t, t' \in \mathbb{R}$, $(x, t)\, R_i\, (y, t')$ if and only if $u_i(x, I_i - t) \geq u_i(y, I_i - t')$. Thus, for each $(x, p_x) \in L \times \mathbb{R}$ and each $y \in L$, $V_i(y; (x, p_x)) = I_i - r$ where r is the solution to $u_i(y, I_i - r) = u_i(x, I_i - p_x)$. Recall that the Serial Vickrey process obtains the MEPs by using IPs. Thus, we could use the equation $u_i(y, I_i - r) = u_i(x, I_i - p_x)$ to collect required IPs $V_i(y; (x, p_x))$ for R. Whenever we have $p^{\min}(R)$, $p^{\min}(u, I)$ is also obtained. Note that the computation process for the IPs can be completed by computer programming. Thereafter, by employing the same econometric method as in, e.g., Maattanen and Tervio (2014), we can calibrate the utility function.

6 Related literature

6.1 Properties and characterizations of MPE and MPE rules

Researchers study MPE in an assignment market with unit-demand agents. When agents have quasi-linear preferences, Leonard (1983) demonstrates the equivalence between MPE and Vickrey allocations. Mishra and Talman (2010) characterize MPE by using "overdemanded sets" and "underdemanded sets." Although when agents have general preferences, the equivalence relation between the MPE and Vickrey allocations does not hold, Alkan and Gale (1990) and Morimoto and Serizawa (2015) show that the characterizations of the MPE by overdemanded sets and underdemanded sets still hold.

The MPE rule, a mapping assigning to each preference profile an MPE for the profile, has also attracted considerable attention. The MPE rule satisfies desirable properties such as efficiency, strategy-proofness, fairness, anonymity, and individual rationality. Moreover, the MPE rule is a unique rule satisfying properties for the class of quasi-linear preferences by Holmstrom (1979) and Ashlagi and Serizawa (2012) and for the class of general preferences by Sakai (2008), Saitoh and Serizawa (2008), Morimoto and Serizawa (2015), Kazumura et al. (2017), and Zhou and Serizawa (2018).

Our structural characterizations demonstrate the dynamic properties of the MPE and are thus different from the static properties of the equilibria addressed in the above literature. Note that the MPE rule is simply a mapping, not a step-by-step process. Although the results in the above literature demonstrate how attractive the MPE rule is, they are silent on how to compute MPE allocations, which is the focus of the present paper.

6.2 Price adjustment process in the assignment market

Researchers also study how to compute MPE in an assignment market with unit-demand agents. Since Crawford and Knoer (1981) and Demange et al. (1986) developed processes to compute the MPE, many subsequent authors, such as Alkan (1992), Hatfield and Milgrom (2005), Mishra and Parkes (2009), and Andersson and Erlanson (2013), have proposed variants. Although these processes converge to the MPE in some environments, none of them compute the exact MPE in our general model. This fact is demonstrated by the examples in Section 2.

Andersson and Svensson (2018) additionally introduce price controls into our model and propose a finite ascending-price sequence that terminates at a “minimum rationing price equilibrium.” Without price controls, their proposed equilibrium coincides with the MPE. Although their proposed price sequence is finite, they do not specify how to identify two adjacent prices in the sequence in a finite number of steps. Our result is a complement of theirs in the sense that the Serial Vickrey process fills this theoretical gap.

An exchange economy (with indivisibility and money) and a two-sided matching market for general preferences contain our model as a special case. Thus, our research is also related to Quinzii (1984) and Herings (2018). In the exchange economy, Quinzii (1984) uses the Scarf Lemma (Scarf, 1967) to constructively demonstrate the existence of the core. This implies that the Scarf method can find an equilibrium in our model. The Scarf method differs from the Serial Vickrey process in the following respects: First, the Scarf method does not imply any structural characterizations of the equilibria, i.e., Theorems 1 and 2. By contrast, Theorems 1 and 2 are the theoretical basis of the our process. Second, the Serial Vickrey process contains an iterative way to obtain the equilibrium, instead of exhausting all the possible object assignments and prices as in the Scarf method. Third, as stated in the introduction, our approach has considerably less demanding information requirements than the Scarf method.

Herings (2018) extends Shapley and Shubik (1975)’s matching market by allowing two-sided object price controls. A price adjustment process, in the spirit of Crawford and Knoer (1981), is proposed to demonstrate the existence of stable outcomes in the discrete market, the limit result of which implies the existence result in the continuous market. When applying such a process in our settings, it coincides with the approximate DGS auction. As discussed in Section 2, this is different from the Serial Vickrey process.

Using the same model as ours, Caplin and Leahy (2014) establish a formula to compute the MEP using abstract graph structures. Given an object assignment and a graph of DCPs, they require agents to report their IPs along the path to compute a price vector. First, given an assignment of m objects, the formula maximizes the sum of prices over the graphs, the total number of which is $\sum_{k=1}^{m} \binom{m}{k} k \cdot m^{m-1-k}$. Then, the formula minimizes the sum over object assignments to obtain the MEP. Since the number of possible object assignments is ${}_{\max\{m,n\}}P_{\min\{m,n\}}$, the total number of cases that needs to be examined in their formula is extremely large if n or m is large, as is the number of rounds that agents report their IPs. Note that the number of computations of the IPOIP process in the last step is the dominant factor in the counterpart of the Serial Vickrey process. On the other hand, given an assignment of m objects, the IPOIP process is conducted m rounds. Moreover, the maximal number of possible object assignments in an IPOIP process is $(\min\{m,n\})!$. These numbers are not so large if n or m is a moderate number.[12] This comparison illustrates the merits of the Serial Vickrey process, which introduces objects one by one and makes use of structural properties of equilibria.

In the two-sided matching model, Alaei et al. (2016)'s results also suggest a process to compute the exact MPE in our model. Similar to ours, in the process of Alaei et al. (2016), an MPE of an economy is computed based on the equilibria of its subeconomies, and agents report their IPs from the equilibria of subeconomies. However, their process requires that one compute the equilibria of all the subeconomies, while our process requires the computation of the equilibria of a much small number of subeconomies.[13] As a result, the number of rounds of reporting and the total number of IPs agents report in our process are much smaller. This is also a merit of the Serial Vickrey process, which introduces objects one by one and makes use of structural properties of equilibria. For some applications, e.g., in Remark 5, our process has the merit of convergent speed. Furthermore, our process contains new applications not covered by Alaei et al. (2016), e.g., in Remark 6.[14]

[12]For example, if $n = 10$ and $m = 3$,

$$\begin{aligned} P(\max\{m,n\},\min\{m,n\}) \cdot \sum_{k=1}^{m} \binom{m}{k} k \cdot m^{m-1-k} &= 720 \cdot 16 = 11520 \\ (\min\{m,n\})! \cdot m &= 3! \cdot 3 = 18 \end{aligned}$$

[13]For instance, consider the case in which there are more agents than objects, i.e., $n > m$. In the worst-case scenario, the number of subeconomies our process needs to compute is $m + 2! + \cdots + m!$. It is independent of n. On the other hand, the number of subeconomies their process needs to compute is $m \cdot n + \sum_{k=1}^{m-1} m! \cdot n! \backslash (m-k)! \cdot (n-k)!$. For example, if $n = 10$ and $m = 3$, our process involves at most 11 subeconomies, while their process entails 4890 subeconomies to be considered.

[14]The starting point of Alaei et al. (2016)'s method is the null bundle, and in their process, no agent receives a tentative bundle worse than the null. However, this is not the case in Remark 6.

6.3 Housing market in the assignment model

The assignment model has advantages in quantitative analysis, e.g., the calibration of agents' utility functions, of the housing market. Of the existing approaches, the Alonso-type housing market and similar ones receive particular attention, e.g., Kaneko et el. (2006), Maattanen and Tervio (2014), and Landvogit et al. (2015). As discussed in Section 4, the recursive equation system plays an important role in quantitative analysis but requires the imposition of several assumptions, which hold only in specific environments.

The Serial Vickrey process can be applied to more general environments where the recursive equation system fails to work. Thus, it contributes a methodological development in the quantitative analysis of the housing market described by the assignment model.

7 Concluding remarks

The assignment market with unit-demand agents covers many real-life applications. The MPE rule, which has several attractive properties, is often suggested as a desirable candidate to solve allocation problems, especially when agents' preferences exhibit income effects. However, the MPE rule is difficult to implement.[15] We provide a price adjustment process, the Serial Vickrey process, which implements the MPE rule. Thus, the Serial Vickrey process can be used to solve allocation problems whenever MPE rules are desirable candidates.

We also relate the Serial Vickrey process to its potential application to quantitative housing market research using the assignment model. Our process helps researchers to introduce greater heterogeneity in both houses and agents and study issues that cannot be addressed by existing techniques. For example, the Serial Vickrey process can be used to investigate the housing market in a metropolitan area with multiple city centers and calibrate agents' utility functions.

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Appendix

Let $|\cdot|$ denote the cardinality of a set.

Definition: (i) A non-empty set $M' \subseteq M$ of objects is **overdemanded at** p for R if $|\{i \in N : D_i(p) \subseteq M'\}| > |M'|$.
(ii) A non-empty set $M' \subseteq M$ of objects is (**weakly**) **underdemanded at** p for R if

$$[\forall x \in M', \, p_x > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \, (\leq) < |M'| \, .$$

Fact A.1 (Mishra and Talman, 2010).[16] For each $R \in (\mathcal{R}^G)^n$, p is an equilibrium price for R if and only if no set is overdemanded and no set is underdemanded at p for R.

Fact A.2 (Alkan and Gale, 1990; Morimoto and Serizawa, 2015). For each $R \in (\mathcal{R}^G)^n$, p is an MEP for R if and only if no set is overdemanded and no set is weakly underdemanded at p for R.

Fact A.3: Let $R \in (\mathcal{R}^G)^n$, $(z,p) \in W(R)$, and M_C be defined at (z,p). Let a non-empty set $M' \subseteq M_C$ be such that for each $x \in M'$, $p_x > 0$. Then, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|$.
Proof: Since $(z,p) \in W(R)$, and for each $x \in M'$, $p_x > 0$, then by Fact A.1, $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \geq |M'|$. To show ">", we proceed by contradiction. Suppose that $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$. Then, by $M' \subseteq M_C$,

$$\text{for each } i \in N \text{ such that } D_i(p) \cap M' \neq \emptyset, \, x_i \in M', \text{ and } i \in N_C. \qquad (*)$$

Let $i \in N$ such that $x_i \in M'$. Then by $(*)$, $i \in N_C$. By $x_i \in M'$, $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents $(2 \leq \Lambda \leq \min\{m+1, n\})$ such that
(a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,
(b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,
(c) $x_{i_\Lambda} = x_i$, and
(d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.
Claim: Let $l = 1, \cdots, \Lambda - 1$ and $N(l) \equiv \{i_{\Lambda-1}, \cdots, i_{\Lambda-l}\}$. Then, for each $j \in N(l)$, $x_j \in M'$.
Step 1: The Claim holds for $l = 1$.

By (c), $x_{i_\Lambda} = x_i \in M'$. By (d), $D_{i_{\Lambda-1}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-1}} \in M'$.
Induction hypothesis: The Claim holds for s such that $1 \leq s < \Lambda - 1$.
Step 2: The Claim holds for $l = s + 1$.

By induction hypothesis, $x_{i_{\Lambda-s}} \in M'$. By (d), $x_{i_{\Lambda-s}} \in D_{i_{\Lambda-(s+1)}}(p)$. Thus $D_{i_{\Lambda-(s+1)}}(p) \cap M' \neq \emptyset$. Thus by $(*)$, $x_{i_{\Lambda-(s+1)}} \in M'$.

[16] Mishra and Talman (2010)'s result also holds for general preferences.

Let $l = \Lambda - 1$. The above Claim implies that for each $j \in \{i_1, \cdots, i_{\Lambda-1}\}$, $x_j \in M'$. If $|\Lambda| > |M'|$, then the feasibility condition is violated. If $|\Lambda| \leq |M'|$, then by $x_{i_1} \in M'$, $p_{x_{i_1}} > 0$ and (a) is violated. Thus $|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| = |M'|$ does not hold. **Q.E.D.**

Proof of Proposition 1

Proof: We prove Proposition 3 by showing (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii)$\Longrightarrow$(i).

Step 1: (i)$\Longrightarrow$(ii), i.e., $p = p^{\min} \Longrightarrow N = N_C$

It is straightforward that $N_C \subseteq N$. For each $i \in N$, if $p_{x_i} = 0$, then by Remark 2(iii), $i \in N_C$ and if $p_{x_i} > 0$, by Corollary 2 in Morimoto and Serizawa (2015) and Definition 3, $i \in N_C$. Thus $N \subseteq N_C$. Thus $N = N_C$.

Step 2: (ii)$\Longrightarrow$(iii), i.e., $N = N_C \Longrightarrow M = M_C$.

It is straightforward that $M_C \subseteq M$. For each $x \in M$, if there is some $i \in N$ such that $x_i = x$, then by Definition 2, $x \in M_C$. Otherwise, x is unassigned and by Definition 2, $x \in M_C$. Thus $M \subseteq M_C$. Thus $M = M_C$.

Step 3: (iii)$\Longrightarrow$(i), i.e., $M = M_C \Longrightarrow p = p^{\min}$.

Since $(z, p) \in W(R)$, then $p \geq p^{\min}$. To prove $p = p^{\min}$, we proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M$ such that for each $x \in M'$, $p_x > p_x^{\min} \geq 0$. Since $M = M_C$, then $M' \subseteq M^C$. Since $(z, p) \in W(R)$, then by Fact A.3,

$$|\{i \in N : D_i(p) \cap M' \neq \emptyset\}| > |M'|.$$

Thus, for each $i \in N$ such that $D_i(p) \cap M' \neq \emptyset$, by $p_{M'}^{\min} < p_{M'}$, $D_i(p^{\min}) \subseteq M'$. Thus

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M'\}\right| > |M'|.$$

Thus M' is overdemanded at $p^{\min}$, violating Fact A.2. Thus $p = p^{\min}$. **Q.E.D.**

Proof of Proposition 3

Proof: The E-generating process terminates either at Phase 1 or Phase 2. We prove that z generated in either case is an equilibirum.

The Case of Phase 1: Let z be the output of Phase 1. In this case, $C^1(R, k+1; z^*) \leq 0$ and $z = z^*$. Since $z = z^*$, then for each $i \in N$ and each $x \in M(k) \cup \{0\}$,

$$z_i = z_i^* \underset{\text{Def of Equilibirum}}{R_i} (x, p_x^{\min}) = (x, p_x)$$

and for object $k+1$,

$$z_i = z_i^* \underset{C^1(R,k+1;z^*) \leq 0}{R_i} (k+1, p_{k+1}).$$

Thus, (z,p) satisfies (E-i). It is straightforward that (z,p) satisfies (E-ii).
The Case of Phase 2: Let z be the output of Phase 2. In this case, $C^1(R,k+1;z^*)>0$, and there is $i \in N'$ such that $z_i^* = (k+1, C^2(R,k+1;z^*))$.

For each $x \in M(k+1) \cup \{0\}$,

$$z_i \underset{C_+^2(R,k+1;z^*) \leq C^1(R,k+1;z^*)}{R_i} z_i^* \underset{\text{Def of Equilibirum}}{R_i} (x, p_x^{\min}) = (x, p_x).$$

For each $i_l \in \{i_\lambda\}_1^{\Lambda-1}$ and each $x \in M(k) \cup \{0\}$,

$$z_{i_l} = z_{i_{l+1}}^* \underset{\text{(ii)}}{I_{i_l}} z_{i_l}^* \underset{\text{Def of Equilibirum}}{R_{i_l}} (x, p_x^{\min}) = (x, p_x),$$

and for object $k+1$,

$$z_{i_l} = z_{i_{l+1}}^* I_{i_l} z_{i_l}^* \underset{V_{i_l}(k+1;z_{i_l}^*) \leq C_+^2(R,k+1;z^*)}{R_{i_l}} (k+1, p_{k+1}).$$

For each $j \in N \backslash \{i_\lambda\}_1^\Lambda$ and each $x \in M(k) \cup \{0\}$,

$$z_j = z_j^* \underset{\text{Def of Equilibirum}}{R_j} (x, p_x^{\min}) = (x, p_x)$$

and for object $k+1$,

$$z_j = z_j^* \underset{V_j(k+1;z_j^*) \leq C_+^2(R,k+1;z^*)}{R_j} (k+1, p_{k+1}).$$

Thus, (z,p) satisfies (E-i). Unassigned objects at $M(k)$ remain unassigned with zero prices, and $p_{x_{i_1}} = p_{x_{i_1}}^{\min} = 0$. Thus (z,p) satisfies (E-ii). **Q.E.D.**

Proof of Proposition 4

Case 1: $N_C = \emptyset$. By Remark 2(iii), there is no agent $i \in N$ such that $p_{x_i} = 0$. Thus, the FDA process stops at $N_1' = \emptyset$, i.e., $N_C = \emptyset$.
Case 2: $N_C \neq \emptyset$. Let T be the final round of the process. Obviously $T < +\infty$.

First, we show that $\overset{T}{\underset{k=1}{\cup}} N_k' \subseteq N_C$. By Remark 2(iii), there is some $i \in N$ such that $p_{x_i} = 0$. Thus, $N_1' \neq \emptyset$ and $N_1' \subseteq N_C$. If $T = 2$, i.e., $N_T' = \emptyset$, then $\overset{2}{\underset{k=1}{\cup}} N_k' \subseteq N_C$.

Let $T > 2$. Thus $N_2' \neq \emptyset$. By the definition of N_2', for each $i \in N_2'$, $p_{x_i} > 0$ and there is $j \in N_1$ such that $x_i \in D_j(p)$. By Definition 3, $N_2' \subseteq N_C$. By induction argument, for each $t = 1, \cdots, T-1$, $N_t' \neq \emptyset$ and $N_t' \subseteq N_C$. Recall that $N_T' = \emptyset$. Thus $\overset{T}{\underset{k=1}{\cup}} N_k' \subseteq N_C$.

Then, we show that $\overset{T}{\underset{k=1}{\cup}} N'_k = N_C$. We proceed by contradiction. Suppose that there is $i \in N_C \backslash \overset{T}{\underset{k=1}{\cup}} N'_k$. Then $i \notin N'_1$ and $p_{x_i} > 0$. By Definition 2, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of $\Lambda(\Lambda \geq 2)$ distinct agents such that (a) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$, (b) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$, (c) $x_{i_\Lambda} = x_i$, and (d) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \in D_{i_\lambda}(p)$.

By Definition 6, (a) implies $i_1 \in N'_1$. By (d), there is $i \in \{i_2, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_1$, e.g., $i = i_2$. Thus, $i \in N'_2$ and $N'_2 \neq \emptyset$. Let $i_{l_1} \in \{i_2, \cdots, i_\Lambda\}$ be such that there is no $l' > l_1$ such that $i_{l'} \in N'_2$, i.e., agent i_{l_1} is the agent who belongs to N'_2 with the largerst index in $\{i_\lambda\}_{\lambda=2}^{\Lambda}$. By (d), there $i \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $x_i \in D_j(p)$ for some $j \in N'_2$, e.g., $i = i_{l_1+1}$. Thus $i \in N'_3$ and $N'_3 \neq \emptyset$. By same reasoning, we can select $i_{l_2} \in \{i_{l_1+1}, \cdots, i_\Lambda\}$ such that $i_{l_2} \in N'_3$ with the largest index in $\{i_\lambda\}_{\lambda=l_1+1}^{\Lambda}$. Repeating such argument, we can show $i = i_\Lambda \in \overset{T}{\underset{k=1}{\cup}} N'_k$, contradicting $i \in N_C \backslash \overset{T}{\underset{k=1}{\cup}} N'_k$. **Q.E.D.**

Proof of Lemma 1

Proof: (i) By Remark 2(ii), for each $x \in M_U$, $p_x > 0$. By $(z, p) \in W(R)$, for each $x \in M_U$, there is $i \in N$ such that $x_i = x$. By Definition 3, $i \in N \backslash N_C = N_U$. Thus $|M_U| \leq |N_U|$.

If there is $i \in N_U$ such that $x_i = 0$, then by Definition 3, $i \in N_C$, a contradiction. Thus, for each $i \in N_U$, $x_i \in M$. By Definition 2, for each $i \in N_U$, $x_i \notin M_C$, and thus $x_i \in M \backslash M_C = M_U$. Thus $|N_U| \leq |M_U|$. Thus $|M_U| = |N_U|$.

(ii) **Step 1**: For each $x \in M_U$, $p_x^{\min} < p_x$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_U$ such that for each $x \in M'$, $p_x^{\min} = p_x$. By Remark 2(ii), for each $x \in M'$, $p_x^{\min} = p_x > 0$.

If there is $i \in N_C$ such that $D_i(p) \cap M' \neq \emptyset$, then by Definition 3, for $j \in N$ such that $x_j \in D_i(p) \cap M'$, $j \in N_C$. Thus, for each $i \in N_C$, $D_i(p) \cap M' = \emptyset$. Thus by Definition 2, $x_j \in M_C$, contradicting $x_j \in M' \subseteq M_U$. Thus, by $p \geq p^{\min}$ and $p_{M'}^{\min} = p_{M'}$,

$$\text{for each } i \in N_C,\ D_i(p^{\min}) \cap M' = \emptyset. \quad (*)$$

Since $p_{M_U \backslash M'}^{\min} < p_{M_U \backslash M'}$ and $p_{M'}^{\min} = p_{M'}$, then

$$\text{for each } i \in N_U \text{ such that } x_i \in M_U \backslash M',\ D_i(p^{\min}) \cap M' = \emptyset. \quad (**)$$

Thus,

$$\begin{aligned}
&\left|\{i \in N : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\
= \; & \left|\{i \in N \backslash N_C : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| && \text{by } (*) \\
= \; & \left|\{i \in N_U : D_i(p^{\min}) \cap M' \neq \emptyset\}\right| \\
\leq \; & |N_U| - |\{i \in N_U : x_i \in M_U \backslash M'\}| && \text{by } (**) \\
= \; & |\{i \in N_U : x_i \in M'\}| = |M'|.
\end{aligned}$$

Thus M' is weakly underdemanded, violating Fact A.2.

Step 2: For each $x \in M_U$, $p_x^{\min} \geq C_+^1(R_{N_C}, x; z)$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_U$ such that for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(R_{N_C}, x; z)$.

Case 1: For each $x \in M_C$, $p_x^{\min} = p_x$.

For each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibirum}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1}}{P_i} z_i \underset{\text{Def of Equilibirum}}{R_i} (x, p_x) = (x, p_x^{\min}).$$

Thus, for each $i \in N_U$, $D_i(p^{\min}) \cap (M_C \cup \{0\}) = \emptyset$ and thus $D_i(p^{\min}) \subseteq M_U$.

Since for each $x \in M'$, $0 \leq p_x^{\min} < C_+^1(R_{N_C}, x; z)$, then there is $i \in N_C$ such that $V_i(x; z_i) = C_+^1(R_{N_C}, x; z) > 0$, and so by $p_{M_C} = p_{M_C}^{\min}$, $D_i(p^{\min}) \subseteq M_U$. Thus,

$$\left|\{i \in N : D_i(p^{\min}) \subseteq M_U\}\right| \geq 1 + |N_U| \underset{\text{(i)}}{>} |M_U|.$$

Thus M' is overdemanded, violating Fact A.2.

Case 2: There is a non-empty set $M'' \subseteq M_C$ such that $0 \leq p_x^{\min} < p_x$.

For each $i \in N_U$, since $p_{M_C \backslash M''}^{\min} = p_{M_C \backslash M''}$, then by the same reasoning in Case 1, $D_i(p^{\min}) \cap (M_C \cup \{0\} \backslash M'') = \emptyset$ and thus $D_i(p^{\min}) \subseteq M_U \cup M''$. Thus

Since $M'' \subseteq M_C$ and for each $x \in M''$, $p_x > 0$, by Fact A.3,

$$|\{i \in N_C : D_i(p) \cap M'' \neq \emptyset\}| > |M''|.$$

For each $i \in N_C$ with $D_i(p) \cap M'' \neq \emptyset$, by $p_{M_C \backslash M''}^{\min} = p_{M_C \backslash M''}$ and for each $x \in M''$, $p_x > p_x^{\min}$, $D_i(p^{\min}) \subseteq M'' \cup M_U$. Thus,

$$\begin{aligned}
\left|\{i \in N : D_i(p^{\min}) \subseteq M'' \cup M_U\}\right| \;&\geq\; |N_U| + |\{i \in N_C : D_i(p) \cap M'' \neq \emptyset\}| \\
&>\; |N_U| + |M''| \underset{\text{(i)}}{=} |M_U| + |M''| = |\{M'' \cup M_U\}|
\end{aligned}$$

Thus $M'' \cup M_U$ is overdemanded, violating Fact A.2.

(iii) First, we show the following claim:

Claim: For each $x \in M_C$, $p_x = p_x^{\min}$.

We proceed by contradiction. Suppose that there is a non-empty set $M' \subseteq M_C$ such that for each $x \in M'$, $p_x > p_x^{\min} \geq 0$.

For each $i \in N_C$ and each $x \in M_U$, $z_i \, R_i \, (x, C_+^1(R_{N_C}, x; z)) \underset{\text{Lemma 1(ii)}}{R_i} (x, p_x^{\min})$. Thus, for each $i \in N_C$ with $D_i(p) \cap M' \neq \emptyset$, by $p_{M'} > p_{M'}^{\min}$, $D_i(p^{\min}) \cap M_U = \emptyset$.

By Fact A.3,

$$|\{i \in N_C : D_i(p) \cap M' \neq \emptyset\}| > |M'| \, .$$

Thus, for each $i \in N_C$ with $D_i(p) \cap M' \neq \emptyset$, since $D_i(p^{\min}) \cap M_U = \emptyset$, $p_{M'} < p_{M'}^{\min}$, and $p_{M_C \setminus M'} = p_{M_C \setminus M'}^{\min}$, then $D_i(p^{\min}) \subseteq M'$. Thus

$$\left|\{i \in N_C : D_i(p^{\min}) \subseteq M'\}\right| > |M'| \, .$$

Thus M' is overdemanded at $p^{\min}$, violating Fact A.2.

Thus, for each $i \in N_U$ and each $x \in M_C \cup \{0\}$,

$$z_i^{\min} \underset{\text{Def of Equilibirum}}{R_i} (x_i, p_{x_i}^{\min}) \underset{\text{Step 1 in (ii)}}{P_i} z_i \underset{\text{Def of Equilibirum}}{R_i} (x, p_x) \underset{\text{Claim}}{=} (x, p_x^{\min}) \, .$$

Thus, for each $i \in N_U$, $x_i^{\min} \in M_U$. **Q.E.D.**

Proof of Fact 3

Proof: We inductively prove Fact 3.

First, we show that for each $x \in M_U$, $\overline{p}_x^0 \leq \overline{p}_x^1(\mu)$. For each $x \in M_U$,

$$\overline{p}_x^0 \equiv C_+^1(R_{N_C}, x; z) \underset{N_C \subseteq N}{\leq} C_+^1(R, x; (\overline{z}_{N_U}^0(\mu), z_{N_C})) \equiv \overline{p}_x^1(\mu).$$

Thus, for each $x \in M_U$, $\overline{p}_x^0 \leq \overline{p}_x^1(\mu)$.

Induction hypothesis: For each $x \in M_U$ and each $s = 1, \cdots, l$, $\overline{p}_x^0 \leq \cdots \leq \overline{p}_x^l(\mu)$.

We show that for each $x \in M_U$, $\overline{p}_x^l(\mu) \leq \overline{p}_x^{l+1}(\mu)$. For each $x \in M_U$,

$$\overline{p}_x^l(\mu) \equiv C_+^1(R, x; (\overline{z}_{N_U}^{l-1}(\mu), z_{N_C})) \underset{\text{induction hypothesis, } \overline{p}^{l-1}(\mu) \leq \overline{p}^l(\mu)}{\leq} C_+^1(R, x; (\overline{z}_{N_U}^l(\mu), z_{N_C})) \equiv \overline{p}_x^{l+1}(\mu),$$

Thus, for each $x \in M_U$, $\overline{p}_x^l(\mu) \leq \overline{p}_x^{l+1}(\mu)$. **Q.E.D.**

Lemma 2: Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and $(z^{\min}, p^{\min}) \in W^{\min}(R)$. Let N_C and M_U be the sets of connected agents and unconnected objects at (z, p). Then there is $x \in M_U$ such that $p_x^{\min} = C_+^1(R_{N_C}, x; z)$.

Proof: Let M_C and N_U be the sets of connected objects and unconnected agents at (z,p), respectively. We proceed by contradiction. Suppose that for each $x \in M_U$, $p_x^{\min} > C_+^1(R_{N_C}, x; z)$. By Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap M_U = \emptyset$. Thus,

$$\begin{aligned} & \left|\{i \in N : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ = & \left|\{i \in N_U : D_i(p^{\min}) \cap M_U \neq \emptyset\}\right| \\ \leq & \left|N_U\right| \underset{\text{Lemma 1(i)}}{=} \left|M_U\right|. \end{aligned}$$

Thus, M_U is weakly underdemanded, contradicting Fact A.2. Thus there is $x \in M_U$ such that $p_x^{\min} = C_+^1(R_{N_C}, x; z)$. **Q.E.D.**

Lemma 3: Let μ be an MPE object assignment over N_U. For each $x \in M_U$ and each $s = 1, 2, \cdots$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$.

Proof: We inductively prove Lemma 3.

Step 1: For each $x \in M_U$, $\overline{p}_x^1(\mu) \leq p_x^{\min}$.

For each $x \in M_U$, by Lemma 1(ii), $\overline{p}_x^0 \equiv C_+^1(R_{N_C}, x; z^0) \leq p_x^{\min}$. Thus, for each $x \in M_U$ and each $i \in N_C$, $V_i(y; z_i) \leq \overline{p}_x^0 \leq p_x^{\min}$. For each $x \in M_U$ and each $j \in N_U$,

$$V_j(x; \overline{z}_j^0(\mu)) \underset{\overline{p}_{\mu(j)}^0 \leq p_{\mu(j)}^{\min}}{\leq} V_j(x; (\mu(j), p_{\mu(j)}^{\min})) \underset{\text{Def of Equilibirum}}{\leq} p_x^{\min}.$$

Thus, for each $x \in M_U$, $\overline{p}_x^1(\mu) = C_+^1(R, x; (\overline{z}_{N_U}^0(\mu), z_{N_C})) \leq p_x^{\min}$.

Induction hypothesis: For some $s \geq 1$, and for each $x \in M_U$, $\overline{p}_x^s(\mu) \leq p_x^{\min}$.

Step 2: For each $x \in M_U$, $\overline{p}_x^{s+1}(\mu) \leq p_x^{\min}$.

For each $x \in M_U$, by Lemma 1(ii), $\overline{p}_x^0 = C_+^1(R_{N_C}, x; z^0) \leq p_x^{\min}$. Thus, for each $x \in M_U$ and each $i \in N_C$, $V_i(y; z_i) \leq \overline{p}_x^0 \leq p_x^{\min}$.

For each $x \in M_U$ and each $j \in N_U$,

$$V_j(x; \overline{z}_j^s(\mu)) \underset{\overline{p}_{\mu(j)}^s \leq p_{\mu(j)}^{\min}}{\leq} V_j(x; (\mu(j), p_{\mu(j)}^{\min})) \underset{\text{Def of Equilibirum}}{\leq} p_x^{\min}.$$

Thus, for $x \in M_U$, $\overline{p}_x^{s+1}(\mu) \equiv C_+^1(R, x; (\overline{z}_{N_U}^s(\mu), z_{N_C})) \leq p_x^{\min}$. **Q.E.D.**

Lemma 4: Let μ be an MPE object assignment over N_U. For each $x \in M_U$ and each $s = 1, \cdots$, if $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$, then $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$.

Proof: Let $x \in M_U$ and $s \in \mathbb{N}^+$ be such that $\overline{p}_x^{s-1}(\mu) = p_x^{\min}$. By Lemma 3 and Fact 3, $\overline{p}_x^{s-1}(\mu) \leq \overline{p}_x^s(\mu) = p_x^{\min}$. Thus, $\overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu)$. **Q.E.D.**

Proof of Theorem 1

(i) By the Claim in the proof of Lemma 1(iii), to complete the proof of Theorem 1(i), we only need to show that for each $i \in N_C$, $z_i = z_i^{\min}$.

For each $i \in N_C$ and each $x \in M_U$,

$$z_i \, R_i \, (x, C_+^1(R_{N_C}, x; z)) \underset{\text{Lemma 1(ii)}}{R_i} (x, p_x^{\min}).$$

and for each $y \in M_C \cup \{0\}$,

$$z_i \underset{\text{Def of Equilibirum}}{R_i} (y, p_y) = (y, p_y^{\min}).$$

Thus for each $i \in N_C$ and each $x \in L$, $z_i \, R_i \, (x, p_x^{\min})$ and thus $z_i \, R_i \, z_i^{\min}$. Also note that

$$z_i^{\min} \, R_i \, (x_i, p_{x_i}^{\min}) \, I_i \, (x_i, p_{x_i}) = z_i$$

Thus $z_i^{\min} \, I_i \, z_i$. By Lemma 1(iii), for each $i \in N_C$, $x_i^{\min} \in M_C \cup \{0\}$. Thus, for each $i \in N_C$, we can set $z_i = z_i^{\min}$ while let unassigned objects at (z, p) remain unassigned at $(z^{\min}, p^{\min})$.
(ii) **Step 1:** Let μ be an MPE object assignment over N_U. Then, for each $x \in M_U$, $\overline{p}_x^{|M_U|-1}(\mu) = p_x^{\min}$.

First, we show the following claim.
Claim: For each $s = 0, 1, \cdots$, let $M_s \equiv \{x \in M_U : \overline{p}_x^s(\mu) = p_x^{\min}\}$ and $N_s \equiv \{i \in N_U : \mu(i) \in M_s\}$. Then, for each $s = 0, 1, \cdots$,
(a) $|M_s| = |N_s|$, and (b) if $M_U \backslash M_s \neq \emptyset$, then $M_{s+1} \supsetneq M_s$.

By Definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $M_U \backslash M_s \neq \emptyset$. By Step 1-2, $M_{s+1} \supseteq M_s$. Suppose that $M_{s+1} = M_s$. By Lemma 3, for each $x \in M_U \backslash M_s$, $p_x^{\min} > \overline{p}_x^s(\mu) \geq \overline{p}_x^0$. Thus, by $M_{s+1} = M_s$, for each $x \in M_U \backslash M_s$, $\overline{p}_x^0 \leq \overline{p}_x^{s+1}(\mu) < p_x^{\min}$. By Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

If $i \in N_s$, then for each $x \in M_U \backslash M_s$,

$$\begin{aligned} V_i(x, z_i^{\min}) &= V_i(x, \overline{z}_i^s(\mu)) && \text{by } i \in N_s \\ &\leq C_+^1(R_N, x; (\overline{z}_{N_U}^s(\mu), z_{N_C})) \\ &= \overline{p}_x^{s+1}(\mu) < p_x^{\min}. \end{aligned}$$

Thus, for each $i \in N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Since for each $i \in N_C \cup N_s$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$, then

$$\begin{aligned} \{i \in \;& N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \\ = \{i \in \;& N_U \backslash N_s : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \\ \subseteq \;& N_U \backslash N_s. \end{aligned}$$

Thus,

$$\begin{aligned} & \left|\{i \in N_U \backslash N_s : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\}\right| \\ \leq & \; |N_U \backslash N_s| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U \backslash M_s| . \end{aligned}$$

Thus, $M_U \backslash M_s$ is weakly underdemanded, contradicting Fact A.2. Thus, $M_{s+1} \supsetneq M_s$.

Now we complete the proof of Step 1. If $M_U \backslash M_0 = \emptyset$, then for each $x \in M_U$, $\overline{p}^0_x(\mu) = p^{\min}_x$, and so Lemma 4 implies that for each $x \in M_U$, $\overline{p}^{|M_U|-1}_x(\mu) = p^{\min}_x$. Thus, assume $M_U \backslash M_0 \neq \emptyset$. Then, the above claim says that as s increases, M_s expands strictly until $\{x \in M_U : \overline{p}^s_x(\mu) = p^{\min}_x\} = M_U$. Since Lemma 2 implies $M_0 \neq \emptyset$, M_s expands strictly at most $|M_U| - 1$ times. Thus, $\{x \in M_U : p^{\min}_x = \overline{p}^{|M_U|-1}_x(\mu)\} = M_U$, i.e., for $x \in M_U$, $\overline{p}^{|M_U|-1}_x(\mu) = p^{\min}_x$.

Step 2: Let $\mu' \in \Omega(M_U)$ be a non-MPE object assignment over N_U. Then, for each $x \in M_U$, $p^{\min}_x \leq p^{|M_U|-1}_x(\mu')$.

Definition: Let $x^{\min}_{N_U}$ be an MPE object assignment over N_U, $\mu' \in \Omega(M_U)$, and $i \in N_U$. A sequence $\{\sigma^l(i)\}^d_{l=1}$ of distinct agents ($1 \leq d \leq n$) is called a **trading cycle from i in μ'** if (i) $\sigma^1(i) = i$, and (ii) for each $l \in \{1, \cdots, d-1\}$, $\mu'(\sigma^{l+1}(i)) = x^{\min}_{\sigma^l(i)}$ and $\mu'(\sigma^1(i)) = x^{\min}_{\sigma^d(i)}$.

Step 2-1: Let $i \in N_U$, and $\{\sigma^l(i)\}^d_{l=1}$ be a trading cycle from i in μ' and $s \geq 0$. If $\overline{p}^s_{x^{\min}_{\mu'(i)}}(\mu') \geq p^{\min}_{x^{\min}_{\mu'(i)}}$, then for each $j \in \{\sigma^1(i), \cdots, \sigma^d(i)\}$, $\overline{p}^{s+d}_{x^{\min}_j}(\mu') \geq p^{\min}_{x^{\min}_j}$.

If $d = 1$, then Step 2-1 trivially holds. In the following, let $d \geq 2$. In the proof, without loss of generality, we assume that $i = \sigma^1(i) = 1, \sigma^2(i) = 2, \cdots, \sigma^d(i) = d$. Then, $\mu'(2) = x^{\min}_1$, $\mu'(3) = x^{\min}_2, \cdots, \mu'(d) = x^{\min}_{d-1}$, $\mu'(1) = x^{\min}_d$ and $\overline{p}^s_{x^{\min}_d}(\mu') \geq p^{\min}_{x^{\min}_d}$. We inductively show that for each $j \in \{1, \cdots, d\}$, $\overline{p}^{s+d}_j(\mu') \geq p^{\min}_j$. Note that

$$\begin{aligned} & \overline{p}^{s+1}_{x^{\min}_1}(\mu') \\ = & \; C^1_+(R, x^{\min}_1; (\overline{z}^s_{N_U}(\mu'), z_{N_C})) \\ \geq & \; V_1(x^{\min}_1, \overline{z}^s_1(\mu')) \\ = & \; V_1(x^{\min}_1, (x^{\min}_d, \overline{p}^s_{x^{\min}_d}(\mu'))) && \text{by } \mu'(1) = x^{\min}_d \\ \geq & \; V_1(x^{\min}_1, (x^{\min}_d, p^{\min}_{x^{\min}_d}(\mu'))) && \text{by } \overline{p}^s_{x^{\min}_d}(\mu') \geq \overline{p}^{\min}_{x^{\min}_d}(\mu') \\ \geq & \; p^{\min}_{x^{\min}_1}. && \text{Def of Equilibrium} \end{aligned}$$

Thus, $\overline{p}^{s+1}_{x^{\min}_1}(\mu') \geq p^{\min}_{x^{\min}_1}$.

Let $j \in \{1, \cdots, d\}$, and assume that $\overline{p}^{s+j}_j(\mu') \geq p^{\min}_j$. Then, by similar reasoning as above but replacing $\mu'(1) = x^{\min}_d$ and $\overline{p}^s_{x^{\min}_d}(\mu') \geq p^{\min}_{x^{\min}_d}$ by $\mu'(j+1) = x^{\min}_j$ and $\overline{p}^{s+j}_j(\mu') \geq p^{\min}_j$, respectively, $\overline{p}_{j+1}^{s+j+1}(\mu') \geq p_{j+1}^{\min}$ holds. Thus, for each $k \in \{1, \cdots, d\}$, $\overline{p}_{x_k^{\min}}^{s+k}(\mu') \geq p_{x_k^{\min}}^{\min}$. Thus, by Fact 3, for each $j \in \{1, \cdots, d\}$, $\overline{p}_j^{s+d}(\mu') \geq p_j^{\min}$.

Step 2-2: Let $x_{N_U}^{\min}$ be an MPE object assignment over N_U, and $\mu' \in \Omega(M_U)$. Let $\{N_l(\mu')\}_{l \in K}$ be a partition of N_U such that $K \equiv \{1, \cdots, k\}$, and for each $l \in K$, agents in $N_l(\mu')$ form a trading cycle, i.e., there is $i \in N_l(\mu')$ such that there is a sequence $\{\sigma^l(i)\}_{l=1}^{|N_l(\mu')|}$ of distinct agents forming a trading cycle from i in μ'. Let $L_0 \equiv \{l \in K$: there is $i \in N_l(\mu')$ s.t. $p_{x_i^{\min}}^{\min} = \overline{p}_{x_i^{\min}}^0\}$ and $M_0 \equiv \{x \in M_U$:there is $i \in \underset{r \in L_0}{\cup} N_r(\mu')$ s.t. $\mu'(i) = x\}$. For each $s = 1, 2, \cdots$, let $L_s \equiv \{l \in K$:there are $i \in N_l(\mu')$ and $j \in \underset{r \in L_{s-1}}{\cup} N_r(\mu')$ s.t. $z_j^{\min} \, I_j \, z_i^{\min}\}$ and $M_s \equiv \{x \in M_U$:there is $i \in \underset{r \in L_s}{\cup} N_r(\mu')$ s.t. $\mu'(i) = x\}$. Then, for each $s = 0, 1, \cdots$,

$$\text{(a) } \left| \underset{r \in L_s}{\cup} N_r(\mu') \right| = |M_s| \text{, and (b) if } K \backslash L_s \neq \emptyset \text{, then } L_{s+1} \supsetneq L_s.$$

By Definition, for each $s = 0, 1, \cdots$, (a) holds. Thus, we show only (b).

Let $K \backslash L_s \neq \emptyset$. By Definition, $L_{s+1} \supseteq L_s$. Suppose that $L_{s+1} = L_s$. By $K \backslash L_s \neq \emptyset$, and Lemma 1(i), $M_U \backslash M_s \neq \emptyset$. For each $x \in M_U \backslash M_s$, by $L_{s+1} = L_s$, $x \notin M_0$ and so by Lemma 1(ii), $p_x^{\min} > \overline{p}_x^0$. Thus, by Lemma 1(ii), for each $i \in N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Let $i \in \underset{r \in L_s}{\cup} N_r(\mu')$ and $x \in M_U \backslash M_s$. Let $j \in N_U$ be such that $\mu'(j) = x$, i.e., $z_j^{\min} = (x, p_x^{\min})$. By $x \in M_U \backslash M_s$ and $L_{s+1} = L_s$, $j \notin \underset{r \in L_{s+1}}{\cup} N_r(\mu')$. Thus, by $i \in \underset{r \in L_s}{\cup} N_r(\mu')$, $z_i^{\min} \, I_i \, z_j^{\min}$ does not hold. By the definition of equilibrium, $z_i^{\min} \, R_i \, z_j^{\min}$ and so $z_i^{\min} \, P_i \, z_j^{\min}$, i.e., $x \notin D_i(p^{\min})$. Thus for each $i \in \underset{r \in L_s}{\cup} N_r(\mu')$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$.

Since for each $i \in \underset{r \in L_s}{\cup} N_r(\mu') \cup N_C$, $D_i(p^{\min}) \cap (M_U \backslash M_s) = \emptyset$, then

$$\begin{aligned} &\{i \in N : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \\ = &\{i \in N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \\ \subseteq &\; N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu'). \end{aligned}$$

Thus

$$\begin{aligned} &\left| \{i \in N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') : D_i(p^{\min}) \cap (M_U \backslash M_s) \neq \emptyset\} \right| \\ \leq &\left| N_U \backslash \underset{r \in L_s}{\cup} N_r(\mu') \right| \underset{\text{Lemma 1(i) and (a)}}{=} |M_U \backslash M_s| \,. \end{aligned}$$

Thus, $M_U \backslash M_s$ is weakly underdemanded, contradicting Fact A.2. Thus, (b) $L_{s+1} \supsetneq L_s$ holds.

Now we complete the proof of Step 2. By the finiteness of N_U, Step 2-2 implies that there is $q \in \{0, \cdots, k\}$ such that $L_q = K$. Let $d_0 \equiv \max_{l \in L_0} |N_l(\mu')|$ and for each $r = 1, \cdots, q$, $d_r \equiv \max_{l \in L_r \backslash L_{r-1}} |N_l(\mu')|$. By Fact A.4, $L_0 \neq \emptyset$.

If $q = 0$, then $d_0 \leq |N_U| = |M_U|$. Thus, by Step 2-1 and Fact 3, at round $d_0 - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. If $q > 0$, by Step 2-1, at round $d_0 - 1$, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu')$ and there is $y \in M_1 \backslash M_0$ such that $p_y^{\min} \leq \overline{p}_y^{d_0-1}(\mu')$. By Step 2-1, at round $d_0 + d_1 - 1$, for each $x \in M_1 \backslash M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By Fact 3, for each $x \in M_0$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. Thus for each $x \in M_1$, $p_x^{\min} \leq \overline{p}_x^{d_0+d_1-1}(\mu')$. By induction argument, at round $D \equiv \sum_{i=0}^{q} d_i - 1$, for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{D}(\mu')$. Since $D \equiv \sum_{i=0}^{q} d_i \leq |N_U| = |M_U|$, then for each $x \in M_U$, $p_x^{\min} \leq \overline{p}_x^{d_0-1}(\mu') \leq \overline{p}_x^{|M_U|-1}(\mu')$. Thus Step 2 holds.

Step 3: Completion of the proof

By Remark 3(i), there is $\mu \in \Omega(M_U)$ such that μ is an MPE object assignments over N_U. By Step 1, for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu)$. By Step 2, for each $\mu' \in \Omega(M_U) \backslash \{\mu\}$ and each $x \in M_U$, $p_x^{\min} \leq p_x^{|M_U|-1}(\mu')$. Thus for each $x \in M_U$, $p_x^{\min} = \overline{p}_x^{|M_U|-1}(\mu) = \min_{\mu' \in \Omega(M_U)} \overline{p}_x^{|M_U|-1}(\mu')$.

Proof of Theorem 2

It is straightforward that (ii) implies (i). Thus, we only show that (i) implies (ii).

Let $\mu \in \Omega(M_U)$ and $s \leq |M_U|$ be such that $\overline{p}_{M_U}^{s-1}(\mu) = \overline{p}_{M_U}^{s}(\mu)$.

Step 1: Let $i \in N_U$, $x \in M_U$, $s' \leq s-1$, and $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s-1}(\mu)$. Then $\overline{p}_{\mu(i)}^{s'-1}(\mu) = \overline{p}_{\mu(i)}^{s-1}(\mu)$.

Note that

$$\begin{aligned} \overline{p}_x^{s-1}(\mu) &= V_i(x, \overline{z}_i^{s'-1}(\mu)) \\ &\leq V_i(x, \overline{z}_i^{s-1}(\mu)) && \text{by Fact 3 and } s' \leq s-1 \\ &\leq C_+^1(R, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C})) && \text{by } i \in N_U \\ &= \overline{p}_x^{s}(\mu). \end{aligned}$$

Thus, by $\overline{p}^{s-1}(\mu) = \overline{p}^{s}(\mu)$, $V_i(x, \overline{z}_i^{s'-1}(\mu)) = V_i(x, \overline{z}_i^{s-1}(\mu))$. Since $\overline{z}_i^{s'-1}(\mu) = (\mu(i), \overline{p}_{\mu(i)}^{s'-1}(\mu))$ and $\overline{z}_i^{s-1}(\mu) = (\mu(i), \overline{p}_{\mu(i)}^{s-1}(\mu))$, then $\overline{p}_{\mu(i)}^{s'-1}(\mu) = \overline{p}_{\mu(i)}^{s-1}(\mu)$.

Before proving Step 2, we introduce a weak variant of connectedness.

Definition: Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in Z \times \mathbb{R}^m$. An agent $i \in N$ is **weakly connected** at p if there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents such that

(i) $1 \leq \Lambda \leq \min\{m+1, n\}$,

(ii) $x_{i_1} = 0$ or $p_{x_{i_1}} = 0$,

(iii) for each $\lambda \in \{2, \cdots, \Lambda\}$, $x_{i_\lambda} \neq 0$ and $p_{x_{i_\lambda}} > 0$,

(iv) $x_{i_\Lambda} = x_i$, and

(v) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

The weak connectedness of agent is weaker than Definition 3 since the weak connectedness does not require that for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $\{x_{i_\lambda}, x_{i_{\lambda+1}}\} \subseteq D_{i_\lambda}(p)$, but instead only $z_{i_\lambda} \, I_{i_\lambda} \, z_{i_{\lambda+1}}$.

Definition : Let $R \in (\mathcal{R}^G)^n$ and $(z, p) \in Z \times \mathbb{R}^m$. An object $x \in M$ is **weakly connected** at p if (i) there is some weakly connected agent $i \in N$ such that $x_i = x$ or (ii) for each $i \in N$, $x_i \neq x$.

Let $R \in (\mathcal{R}^G)^n$, $(z, p) \in W(R)$, and N_C and M_C be defined at (z, p). Then agents in N_C and objects in M_C are all weakly connected.

Step 2: For each $x \in M_U$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$.

Let M' be the set of weakly connected objects in M_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$. To prove $M' = M_U$, we proceed by contradiction. Suppose that $M_U \backslash M' \neq \emptyset$. Let $N' \equiv \{i \in N_U : \mu(i) \in M'\}$. Then, N' is the set of weakly connected agents in N_U at $(\overline{p}^{s-1}(\mu), p_{M_C})$, and $|M'| = |N'|$. Then by Lemma 1(i) and $|M'| = |N'|$, $N_U \backslash N' \neq \emptyset$.

If there is $x \in M_U \backslash M'$ such that $\overline{p}_x^{s-1}(\mu) = C_+^1(R_{N_C}, x; z)$, then either $\overline{p}_x^{s-1}(\mu) = 0$ or there is some $j \in N_C$ such that $(x, \overline{p}_x^{s-1}(\mu)) \, I_j \, z_j$, contradicting $x \in M_U \backslash M'$. Thus, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) \neq C_+^1(R_{N_C}, x; z)$. Thus, by Fact 3, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) \geq C_+^1(R_{N_C}, x; z)$ and so $\overline{p}_x^{s-1}(\mu) > C_+^1(R_{N_C}, x; z) \geq 0$.

Let $x \in M_U \backslash M'$. Note that $\overline{p}_x^{s-1}(\mu) \equiv C_+^1(R, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C})) \geq C^1(R_{N'}, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C}))$. Suppose $\overline{p}_x^{s-1}(\mu) = C^1(R_{N'}, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C}))$. Then, there is $i \in N'$ such that $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$. By $i \in N'$ and $\overline{p}_x^{s-1}(\mu) = V_i(x; \overline{z}_i^{s-1}(\mu))$, x is a weakly connected object at $(\overline{p}^{s-1}(\mu), p_{M_C})$, contradicting $x \in M_U \backslash M'$. Thus, for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(R_{N'}, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C}))$.

Let s' be the earliest round in the IPOIP process such that there is $x \in M_U \backslash M'$ such that $\overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Then, by Fact 3 and $s' \leq s - 1$,

$$\text{for each } s'' < s' \text{ and each } y \in M_U \backslash M', \; \overline{p}_y^{s''}(\mu) < \overline{p}_y^{s'}(\mu) \leq \overline{p}_y^{s-1}(\mu). \qquad (*)$$

Since for each $y \in M_U \backslash M'$, $\overline{p}_y^{s-1}(\mu) > C_+^1(R_{N_C}, x; z)$, then $s' \geq 1$.

By the definition of IPOIP process and $s' \geq 1$, there is $i \in N_U$ such that $V_i(x, \overline{z}_i^{s'-1}(\mu)) = \overline{p}_x^{s'}(\mu) = \overline{p}_x^{s-1}(\mu)$. Note that for each $x \in M_U \backslash M'$, $\overline{p}_x^{s-1}(\mu) > C^1(R_{N'}, x; (\overline{z}_{N_U}^{s-1}(\mu), z_{N_C}))$. By Fact 3, for each $s'' \leq s - 1$, $\overline{p}_x^{s-1}(\mu) > C^1(R_{N'}, x; (\overline{z}_{N_U}^{s''}(\mu), z_{N_C}))$. Thus, $i \notin N'$ and so $i \in N_U \backslash N'$, and $\mu(i) \in M_U \backslash M'$. By Step 1, $\overline{p}_{\mu(i)}^{s'-1}(\mu) = \overline{p}_{\mu(i)}^{s-1}(\mu)$, contradicting $(*)$.

Thus $M_U \backslash M' \neq \emptyset$ fails to hold, i.e., $M_U = M'$.

Step 3: Let $M_0 \equiv \{x \in M_U : \overline{p}_x^{s-1}(\mu) = C_+^1(R_{N_C}, x; z)\}$. Then $M_0 \neq \emptyset$.

By Definitions of weakly connected objects and agents and Step 2, there is no $x \in M_U \backslash M_0$ that is weakly connected to some $y \in M_C$ directly at $(\overline{p}^{s-1}(\mu), p_{M_C})$. Thus, $M_0 \neq \emptyset$ just follows Step 2.

Step 4: For each $x \in M_U$, $\overline{p}_x^{s-1}(\mu) \leq p_x^{\min}$.

If $M_U = M_0$, then by Lemma 1(ii), Step 4 trivially holds. Thus, let $M_U \backslash M_0 \neq \emptyset$.

Let $M' \equiv \{x \in M_U : \forall\ x \in M', \overline{p}_x^{s-1}(\mu) > p_x^{\min}\}$. To show $M' = \emptyset$, we proceed by contradiction. Suppose that $M' \neq \emptyset$.

Let $N' \equiv \{i \in N_U : \mu(i) \in M'\}$. By Definition, $|N'| = |M'|$. By Lemma 1(i) and $|M'| = |N'|$, $|N_U \backslash N'| = |M_U \backslash M'| \neq \emptyset$. By Step 3, $M_U \backslash M' \supseteq M_0 \neq \emptyset$ and so $N_U \backslash N' \neq \emptyset$.

For each $i \in N_U$ and each $x \in L \backslash M_U$,

$$z_i^{\min} \, R_i \, (x_i, p_{x_i}^{\min}) \underset{\text{Lemma 1(ii)\&(iii)}}{P_i} (x_i, p_{x_i}) = z_i \underset{\text{Def of Equilibirum}}{R_i} (x, p_x) \underset{\text{Theorem 1(i)}}{=} (x, p_x^{\min}).$$

and so $D_i(p^{\min}) \subseteq M_U$.

For each $i \in N_U$ and each $y \in M_U \backslash M'$,

$$\begin{aligned} \overline{z}_i^{s-1}(\mu) \quad & R_i \, (y, \overline{p}_y^{s-1}(\mu)) && \text{by } \overline{p}_y^{s-1}(\mu) = \overline{p}_y^{s}(\mu) \geq V_i(y, \overline{z}_i^{s-1}(\mu)) \\ & R_i \, (y, p_y^{\min}). && \text{by } \overline{p}_y^{s-1}(\mu) \leq p_y^{\min} \end{aligned}$$

For each $i \in N'$ and each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibirum}}{R_i} (\mu(i), p_{\mu(i)}^{\min}) \underset{p_{\mu(i)}^{\min} < \overline{p}_{\mu(i)}^{s-1}(\mu)}{P_i} \overline{z}_i^{s-1}(\mu) \underset{N' \subseteq N_U}{R_i} (y, p_y^{\min}).$$

Thus, for each $i \in N'$, by $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$. Thus,

$$\left| \{i \in N_U : D_i(p^{\min}) \subseteq M'\} \right| \geq |N'| = |M'| \, .$$

Since by Step 2, for each $x \in M'$, x is weakly connected at $(\overline{p}^{s-1}(\mu), p_{M_C})$ and $\overline{p}_x^{s-1}(\mu) > p_x^{\min} \geq 0$, then by $N_U \backslash N' \neq \emptyset$, there are $i \in N_U \backslash N'$ and $x' \in M'$ such that $\overline{z}_i^{s-1}(\mu) \, I_i \, (x', \overline{p}_{x'}^{s-1}(\mu))$. Thus for each $y \in M_U \backslash M'$,

$$z_i^{\min} \underset{\text{Def of Equilibirum}}{R_i} (x', p_{x'}^{\min}) \underset{p_{x'}^{\min} < \overline{p}_{x'}^{s-1}(\mu)}{P_i} (x', \overline{p}_{x'}^{s-1}(\mu)) \, I_i \, \overline{z}_i^{s-1}(\mu) \underset{i \in N_U \backslash N'}{R_i} (y, p_y^{\min}),$$

and so $y \notin D_i(p^{\min})$. By $D_i(p^{\min}) \subseteq M_U$, $D_i(p^{\min}) \subseteq M'$. Thus,

$$\left| \{i \in N_U : D_i(p^{\min}) \subseteq M'\} \right| \geq |N'| + 1 > |M'| \, .$$

Thus M' is overdemanded, contradicting Fact A.2.

Step 5: For each $i \in N_U$ and each $x \in L \backslash M_U$, $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Let $i \in N_U$ and $x \in L \backslash M_U$. By Lemma 1(iii), $x_i^{\min} \in M_U$. Thus,

$$V_i(x_i^{\min}; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(R, x_i^{\min}; (\overline{z}^{s-1}(\mu), z_{N_C})) = \overline{p}^s_{x_i^{\min}}(\mu) \underset{\overline{p}^{s-1}(\mu)=\overline{p}^s(\mu)}{=} \overline{p}^{s-1}_{x_i^{\min}}(\mu).$$

Thus, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}^{s-1}_{x_i^{\min}}(\mu))$. Note

$$(x_i^{\min}, \overline{p}^{s-1}_{x_i^{\min}}(\mu)) \underset{\text{Step 4}}{R_i} (x_i^{\min}, p^{\min}_{x_i^{\min}}) = z_i^{\min} \underset{\text{Def. of Equilibirum}}{R_i} (x, p_x^{\min}).$$

Thus, by $\overline{z}_i^{s-1}(\mu)\, R_i\, (x_i^{\min}, \overline{p}^{s-1}_{x_i^{\min}}(\mu))$, $\overline{z}_i^{s-1}(\mu)\, R_i\, (x, p_x^{\min})$, i.e., $V_i(x; \overline{z}_i^{s-1}(\mu)) \leq p_x^{\min}$.

Step 6: $((\overline{z}^{s-1}(\mu), z_{N_C}), (\overline{p}^{s-1}(\mu), p_{M_C})) \in W^{\min}(R)$

By Lemma 1(ii) and Theorem 1(i), for each $i \in N_C$, (E-i) holds. For each $i \in N_U$ and each $x \in M_U$,

$$V_i(x; \overline{z}_i^{s-1}(\mu)) \leq C_+^1(R, x; (\overline{z}^{s-1}(\mu), z_{N_C})) = \overline{p}_x^s(\mu) = \overline{p}_x^{s-1}(\mu),$$

and for each $x \in L \backslash M_U$,

$$V_i(x; \overline{z}_i^{s-1}(\mu)) \underset{\text{Step 5}}{\leq} p_x^{\min} \underset{\text{Theorem 1(i)}}{=} p_x.$$

Thus for each $i \in N_U$, (E-i) holds. (E-ii) holds obviously. Thus $((\overline{z}^{s-1}_{N_U}(\mu), z_{N_C}), (\overline{p}^{s-1}_{M_U}(\mu), p_{M_C})) \in W(R)$. By Theorem 1(i), $p_{M_C} = p^{\min}_{M_C}$. By Step 4 and Fact 2, $\overline{p}^{s-1}(\mu) = p^{\min}_{M_U}$. Thus Step 6 holds. **Q.E.D.**

Proof of Proposition 5

If $M_C = M(k+1)$, then by Proposition 1, the Serial Vickrey sub-process terminates at $z \in W^{\min}(k+1, R)$, and Proposition 5 trivially holds. In the following, let $M_C \subsetneq M(k+1)$.

To see (i), note that $T < +\infty$ comes from the finiteness of M_U and $\Omega(M_U)$. By the construction of Serial Vickrey sub-process, for each $t = 1, \cdots, T$, $p^{*t} \leq p^{*t-1}$.

To see (ii), we first show the following claim.

Claim: (a) for each $t < T$, μ^{*t} is not an MPE object assignment of N_U and (b) μ^{*T} is an MPE object assignment of N_U.

To see (a), by contradiction, suppose not, i.e., there is $t < T$ such that μ^{*t} is an MPE object assignment of N_U. w.o.l.g. assume that there is no $t' < t$ such that $\mu^{*t'}$ is an MPE object assignment of N_U. By Corollary 3, for each $t' < t$, μ^{*t} can succeed in the $|M_U|$ −IPOIP process w.r.t. μ^{*t} for $p^{*t'}$. Thus the Serial Vickrey sub-process terminates at t, contradicting that T is the final step. (b) is a direct outcome of Theorem 2.

By Theorem 1 and 2, it is straightforward that (ii) holds. **Q.E.D.**

Proof of Theorem 3

Proof: For each $k \in M$ such that $k < m$, given $(z^*, p^{\min}) \in W^{\min}(k, R)$, Stages 1 and 2 of the Serial Vickrey sub-process take a finite number of steps bounded by some polynomial with respect to m for both FDCP process to construct $(z, p) \in W(k+1, R)$ and FCA process to identify M_C at (z, p). By Proposition 5, Stage 3 in the Serial Vickrey sub-process finds $z^{\min} \in Z(k+1, R)$ in a finite number of steps. Thus, the Serial Vickrey process finds an MPE in a finite number of steps. **Q.E.D.**